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Singular invariant hyperfunctions on the space of complex and quaternion Hermitian matrices

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Abstract. Singular invariant hyperfunctions on the space of $n \times n$ complex and quaternion matrices are discussed in this paper. Following a parallel method employed in the author's paper on invariant hyperfunctions on the symmetric matrix spaces, we give an algorithm to determine the orders of poles of the complex power of the determinant function and to determine exactly the support of singular invariant hyperfunctions, i.e., invariant hyperfunctions whose supports are contained in the set of points of rank strictly less than n, obtained as negative-order-coefficients of the Laurent expansions of the complex powers.

1. Introduction.

In the preceding paper [**6**], the author has determined the exact orders of pole of the complex power of the determinant function on the $n \times n$ real symmetric matrix space $\boldsymbol{V} := \mathrm{Sym}_n(\boldsymbol{R})$ and the exact supports of the Laurent expansion coefficients. In this paper we shall deal with the same problem on two similar vector spaces—the space of complex and quaternion Hermitian matrices.

For a given homogeneous polynomial $P(x)$ on the vector space $\boldsymbol{V}$, we can define the hyperfunctions as a complex power of $P(x)$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s, & \text{if } x \in \boldsymbol{V}_i, \\ 0, & \text{if } x \notin \boldsymbol{V}_i. \end{cases}$$

with $s \in \boldsymbol{C}$ where $\boldsymbol{V}_i$'s are connected components of the set $\boldsymbol{V} - \{P(x) = 0\}$. Each $|P(x)|_i^s$ is well defined as a continuous function on $\boldsymbol{V}$ when the real part of $s \in \boldsymbol{C}$ is sufficiently large and meromorphically continued to the whole complex plane. The locations of poles are described by the b-function of $P(x)$ and the possible orders of pole are calculated by the roots of the b-function. But the order of pole of the linear combination $\sum a_i |P(x)|_i$ is fully depend on the

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coefficients a_i and it is not determined only by the b-function. It is rather easy to verify that the support of the Laurent expansion coefficients of $|P(x)|_i$ is contained in the set $\{P(x)=0\}$ if the degree of the coefficient is negative. But the exact determination of the support seems to be not so easy.

The purpose of this paper is to give the complete answer of these problems when $P(x)$ is the determinant function on the space of complex or quaternion Hermitian matrix space. First, we give the exact order of the poles of the linear combination of the complex powers of the determinant function on the space of complex and quaternion Hermitian matrices (Theorem 4.2) and we determine the exact supports of hyperfunctions appearing as Laurent expansion coefficients of the complex power of the relative invariant (Theorem 4.3).

The method of the proof is similar to the case of the symmetric matrix space—the microlocal method. We only give the outline of the proof because it is easily constructed from the proof of that in the case of the symmetric matrix space. However, the results are different from those in the case of symmetric matrices. The reason for the obvious difference seems to be a consequence of the difference of the structure of the roots of b-functions. Namely, we can find half integers in the roots of b-function of the determinant of real symmetric matrix, but the roots of b-functions of the determinants of complex or quaternion matrix are all integers.

We list here some related works on this topic. Similar results has been obtained by Blind [**1**] and [**2**] by a functional analytic method. Raïs [**7**] treated invariant distributions from his original view point. Ricci and Stein [**8**] considered the invariant distributions on the complex Hermitian matrix space. Sato-Shintani [**9**] dealt with the zeta functions associated to the complex Hermitian matrix space, which is closely related to the hyperfunctions treated here.

2. The Hermitian matrix space over the complex and the quaternion field.

Let $\boldsymbol{V} := \mathrm{Her}_n(\boldsymbol{C})$ be the space of $n \times n$ Hermitian matrices over the complex field $\boldsymbol{C}$, i.e., $x \in \mathrm{M}_n(\boldsymbol{C})$ such that $x = {}^t\bar{x}$, and let $\boldsymbol{G} := GL_n(\boldsymbol{C})$ be the general linear group over $\boldsymbol{C}$. By regarding $GL_n(\boldsymbol{C})$ as a real algebraic group and considering $\boldsymbol{V} = \mathrm{Her}_n(\boldsymbol{C})$ as an n^2-dimensional real vector space, $GL_n(\boldsymbol{C})$ operates algebraically on the real vector space $\boldsymbol{V}$ by

$$g : x \longmapsto g \cdot x \cdot {}^t\bar{g}, \tag{1}$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. Here, $\bar{g}$ means the complex conjugate matrix of g and ${}^tg$ is the transposition of g. This is a linear representation of $\boldsymbol{G}$ on $\boldsymbol{V}$.

Let $P(x) := \det(x)$. Then $P(x)$ is an irreducible polynomial with real coefficients on $\boldsymbol{V}$, and it is relatively invariant under the action of $\boldsymbol{G}$ corresponding to the character $|\det(g)|^2$, i.e., $P(g \cdot x) = |\det(g)|^2 P(x)$. We put $\boldsymbol{S} :=$

$\{x \in \boldsymbol{V} \mid P(x) = 0\}$ and call it the singular set. The complement set $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ open $\boldsymbol{G}$-orbits

$$\boldsymbol{V}_i := \{x \in \mathrm{Her}_n(\boldsymbol{C}) \mid \mathrm{sgn}(x) = (2i, 2(n-i))\} \tag{2}$$

with $i = 0, 1, \dots, n$. Here, $\mathrm{sgn}(x)$ for $x \in \mathrm{Her}_n(\boldsymbol{C})$ stands for the signature of the quadratic form $q_x(\vec{v}) := {}^t\bar{\vec{v}} \cdot x \cdot \vec{v}$ on $\vec{v} \in \boldsymbol{C}^n$ when we consider $q_x(\cdot)$ as a quadratic form on $\boldsymbol{R}^{2n} \simeq \boldsymbol{C}^n$. The map

$$\chi : g \longmapsto \det(g \cdot {}^t\bar{g}) = |\det(g)|^2 \tag{3}$$

is a continuous homomorphism from $GL_n(\boldsymbol{C})$ to $\boldsymbol{R}^{\times}_{>0}$. The kernel of χ is denoted by $\boldsymbol{G}^1$ and it is a connected closed subgroup of $\boldsymbol{G}$. The singular set $\boldsymbol{S}$ consists of a finite number of $\boldsymbol{G}^1$-orbits. Namely we have

$$\boldsymbol{S} := \bigcup_{1 \le i \le n, 0 \le j \le n-i} \boldsymbol{S}_i^j \tag{4}$$

where $\boldsymbol{S}_i^j := \{x \in \boldsymbol{V} \mid \mathrm{sgn}(x) = (2j, 2(n-i-j))\}$ is a $\boldsymbol{G}^1$-orbit with $\mathrm{sgn}(\cdot)$ defined as above.

In the same way, by putting $\boldsymbol{V} := \mathrm{Her}_n(\boldsymbol{H})$ to be the space of $n \times n$ Hermitian matrices over the Hamilton's quaternion field $\boldsymbol{H}$, and by putting $GL_n(\boldsymbol{H})$ to be the general linear group over $\boldsymbol{H}$, we can consider the same situation. The group $\boldsymbol{G} := GL_n(\boldsymbol{H})$ acts on $\boldsymbol{V}$ in the same manner as (1) where $\bar{g}$ means the quaternion conjugate matrix of g. We use the same notations as in the complex case. Let $P(x) := \det(x)$, put $\boldsymbol{S} := \{x \in \boldsymbol{V} \mid P(x) = 0\}$ and call it the *singular set*. Then $P(x)$ is an irreducible polynomial on $\boldsymbol{V}$, and it is a relatively invariant polynomial under the action of $\boldsymbol{G}$ corresponding to the character $\chi(g) := |\det(g)|^2$, i.e., $P(g \cdot x) = \chi(g)P(x)$. The non-singular subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ open $\boldsymbol{G}$-orbits

$$\boldsymbol{V}_i := \{x \in \mathrm{Her}_n(\boldsymbol{H}); \mathrm{sgn}(x) = (4i, 4(n-i))\} \tag{5}$$

with $i = 0, 1, \dots, n$. Here, $\mathrm{sgn}(x)$ for $x \in \mathrm{Her}_n(\boldsymbol{H})$ stands for the signature of the quadratic form $q_x(\vec{v}) := {}^t\bar{\vec{v}} \cdot x \cdot \vec{v}$ on $\vec{v} \in \boldsymbol{H}^n$ when we consider $q_x(\cdot)$ as a quadratic form on $\boldsymbol{R}^{4n} \simeq \boldsymbol{H}^n$. The continuous homomorphism χ from $\boldsymbol{G}$ to $\boldsymbol{R}^{\times}_{>0}$ is defined as in the same way as (3). The kernel of χ is denoted by $\boldsymbol{G}^1$ and it is a connected closed subgroup of $\boldsymbol{G} = GL_n(\boldsymbol{H})$. The singular set $\boldsymbol{S}$ consists of a finite number of $\boldsymbol{G}^1$-orbits. Namely we have (4) where $\boldsymbol{S}_i^j := \{x \in \boldsymbol{V} \mid \mathrm{sgn}(x) = (4j, 4(n-i-j))\}$ is a $\boldsymbol{G}^1$-orbit with $\mathrm{sgn}(\cdot)$ defined as above.

Remark 2.1. We give here a definition of the determinant of a quaternion Hermitian matrix. Let $\mathrm{M}_n(\boldsymbol{H})$ be the space of $n \times n$ quaternion matrices. Since the quaternion field $\boldsymbol{H}$ is non-commutative, the determinant of $x \in \mathrm{M}_n(\boldsymbol{H})$ is not defined in the ordinary way. However we can define the determinant for a quaternion Hermitian matrix and we give the definition here.

Let $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a basis of the quaternion field $\boldsymbol{H}$ over $\boldsymbol{R}$. Here, 1 is the identity element and

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{ki} = \boldsymbol{j}. \tag{6}$$

An element z of $\boldsymbol{H}$ is given by $z = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ with $a, b, c, d \in \boldsymbol{R}$ and the quaternion conjugate of z is given by $\bar{z} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$. In particular, when $c = d = 0$, z is a complex number.

Note that we can write

$$z = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} = (a + b\boldsymbol{i}) + (c + d\boldsymbol{i})\boldsymbol{j} = \alpha + \beta\boldsymbol{j}$$

with $\alpha = a + b\boldsymbol{i}$ and $\beta = c + d\boldsymbol{i}$. Then we can regard $\boldsymbol{H}$ as the algebra $\boldsymbol{C} \oplus \boldsymbol{C}\boldsymbol{j}$. Consider the algebra homomorphism $\imath$ from $\boldsymbol{H}$ to $\mathrm{M}_2(\boldsymbol{C})$ by

$$\imath : z = \alpha + \beta\boldsymbol{j} \longmapsto \begin{bmatrix} \alpha, & \bar{\beta} \\ -\beta, & \bar{\alpha} \end{bmatrix}. \tag{7}$$

Let $X = (z_{i,j}) \in \mathrm{Her}_n(\boldsymbol{H})$ be an $n \times n$ quaternion Hermitian matrix. By the homomorphism $\imath$ in (7), X is mapped in $\mathrm{M}_{2n}(\boldsymbol{C})$ by

$$X \longmapsto \imath(X \cdot \boldsymbol{j}) = (\imath(z_{i,j} \cdot \boldsymbol{j})). \tag{8}$$

Since $-{}^t(\imath(X \cdot \boldsymbol{j})) = \imath(X \cdot \boldsymbol{j})$, we see that $\imath(X \cdot \boldsymbol{j})$ is an alternating matrix. Then by putting

$$\det(X) = \mathrm{Pf}(\imath(X \cdot \boldsymbol{j})) \tag{9}$$

we can define the determinant for the quaternion Hermitian matrix X. Here $\mathrm{Pf}(A)$ means the Pfaffian of an alternating matrix A. It is easily checked that $\det(X)$ is an irreducible polynomial with real coefficients on $\mathrm{Her}_n(\boldsymbol{H})$.

3. The complex powers of the determinant.

Let $\boldsymbol{V}$ be $\mathrm{Her}_n(\boldsymbol{C})$ or $\mathrm{Her}_n(\boldsymbol{H})$. We define the complex power of the determinant function for a complex number $s \in \boldsymbol{C}$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in \boldsymbol{V}_i, \\ 0 & \text{if } x \notin \boldsymbol{V}_i \end{cases} \tag{10}$$

and consider a linear combination of $|P(x)|_i^s$

$$P^{[\vec{a}, s]}(x) := \sum_{i=0}^{n} a_i |P(x)|_i^s, \tag{11}$$

with $s \in \boldsymbol{C}$ and an $n+1$-dimensional vector $\vec{a} := (a_0, a_1, \dots, a_n) \in \boldsymbol{C}^{n+1}$.

Let $\mathscr{S}(\boldsymbol{V})$ be the Schwartz's space of rapidly decreasing functions on $\boldsymbol{V}$ and let $\mathscr{S}(\boldsymbol{V})'$ be the space of tempered distributions on $\boldsymbol{V}$. For a function $f(x) \in \mathscr{S}(\boldsymbol{V})$, the integral

$$\int P^{[\vec{a},s]}(x) f(x)\, dx \tag{12}$$

is absolutely convergent for a complex number s with sufficiently large real part. Then (12) is holomorphic with respect to $s \in \boldsymbol{C}$ if the real part $\Re(s)$ is sufficiently large and

$$f(x) \longmapsto \int P^{[\vec{a},s]}(x) f(x)\, dx \tag{13}$$

defines a tempered distribution on $\boldsymbol{V}$ with a holomorphic parameter $s \in \boldsymbol{C}$. From the general theory of b-functions of prehomogeneous vector spaces, (12) is meromorphically extended to the whole complex plane with respect to s. Then (13) is defined for all $s \in \boldsymbol{C}$ as a tempered distribution with a meromorphic parameter s. Since the space of tempered distributions $\mathscr{S}(\boldsymbol{V})'$ on $\boldsymbol{V}$ is naturally embedded into the space of hyperfunctions $\mathscr{B}(\boldsymbol{V})$ on $\boldsymbol{V}$, $P^{[\vec{a},s]}(x)$ is well defined as a hyperfunction with a meromorphic parameter $s \in \boldsymbol{C}$.

Let λ be a fixed complex number. If $P^{[\vec{a},s]}(x)$ does not have pole at $s = \lambda$, $P^{[\vec{a},\lambda]}(x)$ is well defined and

$$P^{[\vec{a},\lambda]}(g \cdot x) = \chi(g) P^{[\vec{a},\lambda]}(x)$$

for all $g \in \boldsymbol{G}$. Hence $P^{[\vec{a},\lambda]}(x)$ is invariant under the action of $\boldsymbol{G}^1$. The support of $P^{[\vec{a},\lambda]}(x)$ is given by

$$\mathbf{Supp}(P^{[\vec{a},\lambda]}(x)) = \bigcup_{i \in \{i \in \boldsymbol{Z} \,|\, a_i \neq 0\}} \overline{\boldsymbol{V}}_i.$$

If $P^{[\vec{a},s]}(x)$ has pole at $s = \lambda$, then $P^{[\vec{a},\lambda]}(x)$ does not have a meaning itself. But the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ are $\boldsymbol{G}^1$-invariant hyperfunctions on $\boldsymbol{V}$. Let

$$\sum_{j \in \boldsymbol{Z}} P_j^{[\vec{a},\lambda]}(x)(s-\lambda)^j$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = \lambda$. Then we have

$$\mathbf{Supp}(P_j^{[\vec{a},\lambda]}(x)) \subset \boldsymbol{S}$$

if $j < 0$. In this way, we see that all the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ of negative degree are $\boldsymbol{G}^1$-invariant and their supports are contained in $\boldsymbol{S}$. We call them *singular invariant hyperfunctions*.

Conversely, we have the following proposition.

Proposition 3.1 ([**4**], [**5**]). *Any singular invariant hyperfunction on* $\boldsymbol{V}$ *is given as a linear combination of some negative-order coefficients of Laurent expansions of* $P^{[\vec{a},s]}(x)$ *at various poles and for some* $\vec{a} \in \boldsymbol{C}^{n+1}$.

Proof. The prehomogeneous vector spaces

$$(\boldsymbol{G}, \boldsymbol{V}) := (GL_n(\boldsymbol{C}), \mathrm{Her}_n(\boldsymbol{C}))$$

and

$$(\boldsymbol{G}, \boldsymbol{V}) := (GL_n(\boldsymbol{H}), \mathrm{Her}_n(\boldsymbol{H}))$$

satisfy the sufficient conditions stated in [**4**] and [**5**]. One is the finite-orbit condition and the other is that the dimension of the space of relatively invariant hyperfunctions coincides with the number of open orbits. Then we have the result. □

4. Main results.

The order of $P^{[\vec{a},s]}(x)$ fully depends on the vector $\vec{a}$. In fact, even if $P^{[\vec{a},s]}(x)$ has a possible pole at $s = \lambda$, we can take $\vec{a}$ so that $P^{[\vec{a},s]}(x)$ is holomorphic at $s = \lambda$. On the other hand, by taking another $\vec{a}$, we can make the order of pole of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ to be the possibly highest order. We have seen that the support of the Laurent expansion coefficients of $P^{[\vec{a},s]}(x)$ of negative degree is contained $\boldsymbol{S}$. But the exact support may be a proper subset of $\boldsymbol{S}$. The exact determination of the support is not deduced from the general theory. The purpose of this paper is to give a complete answer to these problems.

We give some definitions to state the main results of this paper. We define the coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$ in the same way as the case of symmetric matrix space in the following. $\boldsymbol{d}^{(k)}[s_0]$ is an $n-k+1$-tuple of linear forms on $\boldsymbol{C}^{n+1}$ given by

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^{n-k+1}$$

with $k = 0, 1, \dots, n$, where $(\boldsymbol{C}^{n+1})^*$ means the dual vector space of $\boldsymbol{C}^{n+1}$. Each element of $\boldsymbol{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \boldsymbol{C}^{n+1}$, i.e., a linear map from $\boldsymbol{C}$ to $\boldsymbol{C}^{n+1}$,

$$d_i^{(k)}[s_0] : \boldsymbol{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a} \rangle \in \boldsymbol{C}.$$

We denote

$$\langle \boldsymbol{d}^{(k)}[s_0], \vec{a} \rangle := (\langle d_0^{(k)}[s_0], \vec{a} \rangle, \langle d_1^{(k)}[s_0], \vec{a} \rangle, \dots, \langle d_{n-k}^{(k)}[s_0], \vec{a} \rangle) \in \boldsymbol{C}^{n-k+1}.$$

Definition 4.1 (Coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$). We define the *coefficient vectors* $\boldsymbol{d}^{(k)}[s_0]$ $(k = 0, 1, \dots, n)$ by induction on k in the following way. First, we set

$$\boldsymbol{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \dots, d_n^{(0)}[s_0])$$

such that $\langle d_i^{(0)}[s_0], \vec{a}\rangle := a_i$ for $i = 0, 1, \dots, n$. Next, by induction on k, we define all the coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$ for $k = 0, 1, \dots, n$ by

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^{n-k+1},$$

with $d_j^{(k)}[s_0] := d_j^{(k-1)}[s_0] + (-1)^{s_0+1} d_{j+1}^{(k-1)}[s_0]$.

The following proposition is trivial by the definition.

Proposition 4.1. *Let s_0 be an integer. For an integer i in $0 \le i \le n-2$ and $\vec{a} \in \boldsymbol{C}^{n+1}$, if $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle = 0$, then $\langle \boldsymbol{d}^{(i+1)}[s_0], \vec{a}\rangle = 0$. In other words, if $\langle \boldsymbol{d}^{(i+1)}[s_0], \vec{a}\rangle \ne 0$, then $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle \ne 0$.*

Using the above mentioned vectors $\boldsymbol{d}^{(k)}[s_0]$, we can determine the exact orders of $P^{[\vec{a},s]}(x)$ at poles in the following theorem.

Theorem 4.2. *The exact order of the poles of $P^{[\vec{a},s]}(x)$ is computed by the following algorithm.*

1. (*In the complex case.*) *The exact order $P^{[\vec{a},s]}(x)$ at $s = -m$ $(m = 1, 2, \dots)$ is computed by the following algorithm.*
   (a) *If $1 \le m \le n$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than m.*
   - *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(1)}[-m], \vec{a}\rangle = 0$.*
   - *For integers p in $1 \le p < m$, then $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(p)}[-m], \vec{a}\rangle \ne 0$.*
   - *$P^{[\vec{a},s]}(x)$ has pole of order m if and only if $\langle \boldsymbol{d}^{(m)}[-m], \vec{a}\rangle \ne 0$.*

   (b) *If $m > n$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than n.*
   - *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(1)}[-m], \vec{a}\rangle = 0$.*
   - *For integers p in $1 \le p < n$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(p)}[-m], \vec{a}\rangle \ne 0$.*
   - *$P^{[\vec{a},s]}(x)$ has pole of order n if and only if $\langle \boldsymbol{d}^{(n)}[-m], \vec{a}\rangle \ne 0$.*
2. (*In the quaternion case.*) *The exact order $P^{[\vec{a},s]}(x)$ at $s = -m$ $(m = 1, 2, \dots)$ is computed by the following algorithm.*
   (a) *If $1 \le m \le 2n-1$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than $\lfloor (m+1)/2 \rfloor$.*
   - *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(1)}[-m], \vec{a}\rangle = 0$.*
   - *For integers p in $1 \le p < \lfloor (m+1)/2 \rfloor$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(p)}[-m], \vec{a}\rangle \ne 0$.*
   - *$P^{[\vec{a},s]}(x)$ has pole of order $\lfloor (m+1)/2 \rfloor$ if and only if $\langle \boldsymbol{d}^{(\lfloor (m+1)/2 \rfloor)}[-m], \vec{a}\rangle \ne 0$.*

   (b) *If $m > 2n$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than n.*
   - *$P^{[\vec{a},s]}(x)$ is holomorphic if and only if $\langle \boldsymbol{d}^{(1)}[-m], \vec{a}\rangle = 0$.*

- *For integers p in $1 \le p < n$, $P^{[\vec{a},s]}(x)$ has pole of order p if and only if $\langle \boldsymbol{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(p)}[-m], \vec{a}\rangle \neq 0$.*
- *$P^{[\vec{a},s]}(x)$ has pole of order n if and only if $\langle \boldsymbol{d}^{(n)}[-m], \vec{a}\rangle \neq 0$.*

Next we consider the exact support of the Laurent expansion coefficients of the complex powers of the relative invariant. Remember that the structures of the singular orbits given by

$$\boldsymbol{V} := \bigsqcup_{\substack{0 \le i \le n \\ 0 \le j \le n-i}} \boldsymbol{S}_i^j \tag{14}$$

where we denote $\boldsymbol{S}_0^j = \boldsymbol{V}_j$. It is easily seen that $\boldsymbol{S} := \bigsqcup_{1 \le i \le n} \boldsymbol{S}_i$ with $\boldsymbol{S}_i = \bigsqcup_{0 \le j \le n-i} \boldsymbol{S}_i^j$. Each singular orbit is a stratum which not only is a $\boldsymbol{G}$-orbit but is a $\boldsymbol{G}^1$-orbit. The strata $\boldsymbol{S}_i^j$ $(1 \le i \le n, 0 \le j \le n-i)$ have the closure inclusion relation

$$\overline{\boldsymbol{S}_i^j} \supset \boldsymbol{S}_{i+1}^{j-1} \cup \boldsymbol{S}_{i+1}^j. \tag{15}$$

They have the same structure as the case of symmetric matrix space and we use the same notation here.

The support of a singular invariant hyperfunction is a closed set consisting of a union of several strata $\boldsymbol{S}_i^j$. Since the support is a closed $\boldsymbol{G}$-invariant subset, we can express the support of a singular invariant hyperfunction as a closure of a union of the highest rank strata, which is easily rewritten by a union of singular orbits. The exact support of the Laurent coefficients of $P^{[\vec{a},s]}(x)$ is given by the following theorem.

THEOREM 4.3 (Support of the singular invariant hyperfunctions). *Let m be a positive integer and suppose that $P^{[\vec{a},s]}(x)$ has pole of order p at $s = -m$. Let*

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},-m]}(x)(s+m)^j \tag{16}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = -m$. The support of the coefficients $P_j^{[\vec{a},-m]}(x)$ is contained in $\boldsymbol{S}$ if $j < 0$. Recall that the coefficient vectors

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \ldots, d_{n-k}^{(k)}[s_0]) \in ((\boldsymbol{C}^{n+1})^*)^{n-k+1},$$

are defined in Definition 4.1 *at $s = -m$ $(m = 1, 2, \ldots)$. In both the complex case and the quaternion case, the support of $P_w^{[\vec{a},-m]}(x)$ $(w = -1, -2, \ldots, -p)$ is contained in the closure $\overline{\boldsymbol{S}_{-w}}$ and it is given by*

$$\mathbf{Supp}(P_w^{[\vec{a},-m]}(x)) = \overline{\left(\bigcup_{j \in \{0 \le j \le n+w \,|\, \langle d_j^{(-w)}[-m], \vec{a}\rangle \neq 0\}} \boldsymbol{S}_{-w}^j \right)}. \tag{17}$$

5. Outline of the proof of the main results.

The rest of this paper is devoted to give outlines of proof of Theorem 4.2 and Theorem 4.3.

We consider invariant hyperfunctions on $\boldsymbol{V}$ under the action of $\boldsymbol{G}$ as solutions to a holonomic system. Let $f(x)$ be a hyperfunction on $\boldsymbol{V}$. We say that $f(x)$ is a χ^s-*invariant hyperfunction* if

$$f(\rho(g)x) = \chi(g)^s f(x), \tag{18}$$

for all $g \in \boldsymbol{G}$ with $s \in \boldsymbol{C}$ and $\chi(g) := \det(g)^2$. Then it is a hyperfunction solution to the following system of linear differential equations $\mathscr{M}_s$ obtained by taking infinitesimal actions of $\boldsymbol{G}$,

$$\mathscr{M}_s : \left(\left\langle d\rho(A)x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A)\right)u(x) = 0 \quad \text{for all } A \in \mathfrak{G}. \tag{19}$$

Here, $\mathfrak{G}$ is the Lie algebra of $\boldsymbol{G}$; $d\rho$ is the infinitesimal representation of ρ; $\delta\chi$ is the infinitesimal character of χ. The system of linear differential equation (19) is a regular holonomic system and hence the solution space is finite dimensional. See for detail [**5**].

The characteristic subvariety of the holonomic system (19) is denoted by $\mathbf{ch}(\mathscr{M}_s)$. It is given by

$$\mathbf{ch}(\mathscr{M}_s) := \{(x, y) \in T^*\boldsymbol{V} \mid \langle d\rho(A)x, y\rangle = 0 \text{ for all } A \in \mathfrak{G}\}. \tag{20}$$

The characteristic variety has the following irreducible component decomposition,

$$\mathbf{ch}(\mathscr{M}_s) := \bigcup_{i=0}^{n} \Lambda_i, \tag{21}$$

with $\Lambda_i = \overline{T^*_{\boldsymbol{S}_i}\boldsymbol{V}}$ where $T^*_{\boldsymbol{S}_i}\boldsymbol{V}$ stands for the conormal bundle of the rank $(n-i)$-orbit $\boldsymbol{S}_i$. It is a well known result that the singular support of the hyperfunction solution to $\mathscr{M}_s$ is contained in $\mathbf{ch}(\mathscr{M}_s)$.

We define the subset Λ_i° by

$$\Lambda_i^\circ := \Lambda_i - \bigcup_{i \neq j} \Lambda_j \tag{22}$$

It is an open-dense subset of Λ_i. The open subset Λ_i° consists of several open connected components, each of which is a $\boldsymbol{G}$-orbit. Furthermore, Λ_i° is a non-singular algebraic subvariety and an open dense subset in Λ_i.

$$\Lambda_i^\circ = \bigsqcup_{\substack{0 \le j \le n-i \\ 0 \le k \le i}} \Lambda_i^{j,k}, \tag{23}$$

with

$$\Lambda_i^{j,k} := \boldsymbol{G} \cdot \left(\begin{pmatrix} I_{n-i}^{(j)} & \\ & 0_i \end{pmatrix}, \begin{pmatrix} 0_{n-i} & \\ & I_i^{(k)} \end{pmatrix} \right). \tag{24}$$

Here, $I_p^{(q)} := \begin{pmatrix} I_q & \\ & -I_{p-q} \end{pmatrix}$ and I_p is an identity matrix of size p. Each orbit $\Lambda_i^{j,k}$ is a connected component in Λ_i°.

Hyperfunction solutions $u(s,x)$ to $\mathscr{M}_s$ that we consider in this paper are the linear combinations

$$u(s,x) = P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s, \tag{25}$$

with $\vec{a} = (a_0, a_1, \ldots, a_n) \in \boldsymbol{C}^{n+1}$ introduced in (11). Since $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \boldsymbol{C}$, the microfunction $\mathbf{sp}(P^{[\vec{a},s]}(x))$ and its principal symbols $\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x))$ depend on $s \in \boldsymbol{C}$ meromorphically. See [**6**, §3.3] for details on principal symbols.

We define the coefficient functions of $P^{[\vec{a},s]}(x)$ on the Lagrangian connected component $\Lambda_i^{j,k}$ as a function of $\vec{a}$ and s in the same way as [**6**, Proposition 3.3]. Let

$$\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x)) := c_i^{j,k}(\vec{a},s)\Omega_i^{j,k}(s)/\sqrt{|dx|}, \tag{26}$$

with $c_i^{j,k}(\vec{a},s)$ being a meromorphic function in $s \in \boldsymbol{C}$. We call $c_i^{j,k}(\vec{a},s)$ a *coefficient function* or simply a *coefficient* of $P^{[\vec{a},s]}(x)$ on $\Lambda_i^{j,k}$ with respect to the *canonical basis*,

$$\Omega_i^{j,k}(s)/\sqrt{|dx|}. \tag{27}$$

The canonical basis (27) is defined in the same way as [**6**, Proposition 3.3]. Then the coefficient functions $c_i^{j,k}(\vec{a},s)$ depend on $\vec{a} \in \boldsymbol{C}^{n+1}$ linearly and on $s \in \boldsymbol{C}$ meromorphically.

Then we have Proposition 5.1 and Proposition 5.10.

Proposition 5.1. *The following three conditions are equivalent.*
1. $P^{[\vec{a},s]}(x)$ *has pole of order* p *at* $s = s_0$.
2. $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\bigcup_{i=1}^n \Lambda_i^{\circ}}$ *has pole of order* p *at* $s = s_0$.
3. *All the coefficient functions in* $\{c_i^{j,k}(\vec{a},s) \mid 0 \le i \le n, 0 \le j \le n-i, 0 \le k \le i\}$ *have pole of order not greater than* p *at* $s = s_0$ *and at least one coefficient of them has pole of order* p *at* $s = s_0$.

The proof of Proposition 5.1 can be carried out in the same way as [**6**, Proposition 3.7]. Then we have only to compute the orders of pole or the supports of Laurent expansion coefficients of the coefficient functions $c_i^{j,k}(\vec{a},s)$ instead of those of $P^{[\vec{a},s]}(x)$. We use the following two relations (28) and (29) in the proof of the main theorem.

Proposition 5.2. *The coefficient functions on Λ_i° and Λ_{i+1}° have the following relations. These relations depend on $s \in \boldsymbol{C}$ meromorphically.*

1. (*the complex case*)

$$\begin{bmatrix} c_{i+1}^{j,k+1}(\vec{a},s) \\ c_{i+1}^{j,k}(\vec{a},s) \end{bmatrix} = \frac{\Gamma(s+i+1)}{\sqrt{2\pi}} \begin{bmatrix} \exp(-(\pi/2)\sqrt{-1}(s+i+1)) & \exp(+(\pi/2)\sqrt{-1}(s+i+1)) \\ \exp(+(\pi/2)\sqrt{-1}(s+i+1)) & \exp(-(\pi/2)\sqrt{-1}(s+i+1)) \end{bmatrix} \times \begin{bmatrix} \exp(+(\pi/2)\sqrt{-1}(i-2k)) & 0 \\ 0 & \exp(-(\pi/2)\sqrt{-1}(i-2k)) \end{bmatrix} \times \begin{bmatrix} c_i^{j+1,k}(\vec{a},s) \\ c_i^{j,k}(\vec{a},s) \end{bmatrix} \tag{28}$$

2. (*the quaternion case*)

$$\begin{bmatrix} c_{i+1}^{j,k+1}(\vec{a},s) \\ c_{i+1}^{j,k}(\vec{a},s) \end{bmatrix} = \frac{\Gamma(s+2i-1)}{\sqrt{2\pi}} \begin{bmatrix} \exp(-(\pi/2)\sqrt{-1}(s+2i-1)) & \exp(+(\pi/2)\sqrt{-1}(s+2i-1)) \\ \exp(+(\pi/2)\sqrt{-1}(s+2i-1)) & \exp(-(\pi/2)\sqrt{-1}(s+2i-1)) \end{bmatrix} \times \begin{bmatrix} \exp(\pi\sqrt{-1}(i-2k)) & 0 \\ 0 & \exp(-\pi\sqrt{-1}(i-2k)) \end{bmatrix} \times \begin{bmatrix} c_i^{j+1,k}(\vec{a},s) \\ c_i^{j,k}(\vec{a},s) \end{bmatrix} \tag{29}$$

Proof. See [**3**, Theorem 2.13]. The above relations are the cases of $\mathrm{Her}_n(\boldsymbol{C})$ and $\mathrm{Her}_n(\boldsymbol{H})$. □

By the relation formulas in Proposition 5.2, we have the following three propositions in the complex (resp. quaternion) case.

Proposition 5.3. *Let $s_0 := -m$ $(m = 1, 2, \ldots)$. For a fixed integer p satisfying $0 \le p \le \min\{m, n\}$ (resp. $0 \le p \le \min\{(m+1)/2, n\}$), we suppose that $\langle \boldsymbol{d}^{(p)}[s_0], \vec{a}\rangle \neq 0$. Then $\boldsymbol{c}_q^{\bullet,\bullet}(\vec{a},s)$ has pole of order q for $q = 0, 1, \ldots, p$ at $s = s_0$.*

PROPOSITION 5.4. *Under the same condition of* Proposition 5.3, $\boldsymbol{c}^{\bullet,0}_{q,(\vec{a},s_0),-q} = (\mathrm{nzc})_q \times \langle \boldsymbol{d}^{(q)}[s_0], \vec{a}\rangle$ *for* $q = 0, 1, \ldots, p$ *where* $(\mathrm{nzc})_q$ *is a non-zero constant depending on* q.

PROPOSITION 5.5. *Let* $s_0 := -m$ $(m = 1, 2, \ldots)$. *For a fixed integer* p *satisfying* $0 \le p \le \min\{m, n\}$ (*resp.* $0 \le p \le \min\{(m+1)/2, n\}$), *we suppose that* $\langle \boldsymbol{d}^{(p+1)}[s_0], \vec{a}\rangle = 0$. *Then* $\boldsymbol{c}^{\bullet,\bullet}_q(\vec{a}, s)$ *has pole of order at most* p *for* $q = p, p+1, \ldots$ *at* $s = s_0$.

Then we have the following proposition.

PROPOSITION 5.6 (Exact orders of coefficient functions). *Let* $s_0 := -m$ $(m = 1, 2, \ldots)$. *The order of the pole of the coefficient function* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ *at* $s = s_0$ *has the following property. There exists an integer* p *in* $0 \le p \le \min\{m, n\}$ (*resp.* $0 \le p \le \min\{(m+1)/2, n\}$) *such that the orders of the pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_q(\vec{a}, s)$ *coincide with* q *for* q *in* $0 \le q \le p$ *and the orders of the pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_q(\vec{a}, s)$ *do not exceed* p *for* q *in* $p \le q \le n$

Now we can give the proof of Theorem 4.2 in the complex (resp. quaternion) case. We do not have to prove the converses since if we establish all the statements, then the converses are automatically true since all the possible cases are proved. We give here proof of part (a) of Theorem 4.2.

LEMMA 5.7. *Let* $s_0 := -m$ $(m = 1, 2, \ldots)$. *If* $\langle \boldsymbol{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic at* $s = s_0$.

PROOF. By applying Proposition 5.5 in the case of $p = 0$, all the coefficients $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ $(0 \le i \le n)$ are holomorphic at $s = s_0$. Thus, by Proposition 5.1, $P^{[\vec{a},s]}(x)$ is holomorphic at $s = s_0$. □

LEMMA 5.8. *Let* $s_0 := -m$ $(m = 1, 2, \ldots)$. *For a fixed integer* p *satisfying* $1 \le p < \min\{m, n\}$ (*resp.* $0 \le p \le \min\{(m+1)/2, n\}$), *if* $\langle \boldsymbol{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \boldsymbol{d}^{(p)}[-m], \vec{a}\rangle \ne 0$, *then* $P^{[\vec{a},s]}(x)$ *has pole of order* p *at* $s = s_0$.

PROOF. From the conditions $\langle \boldsymbol{d}^{(p)}[-m], \vec{a}\rangle \ne 0$ and $\langle \boldsymbol{d}^{(p+1)}[-m], \vec{a}\rangle = 0$, we see that all the coefficients $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ have pole of order at most p, and that the coefficient $\boldsymbol{c}^{\bullet,\bullet}_p(\vec{a}, s)$ has pole of order p by applying Proposition 5.3 and Proposition 5.5. Then all the coefficients $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ $(0 \le i \le n)$ have pole of order at most p, and at least one of them has pole of order p. Thus, by Proposition 5.1, $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. □

LEMMA 5.9. *Let* $s_0 := -m$ $(m = 1, 2, \ldots)$ *and suppose that* $m \le n$ (*resp.* $m \le 2n$). *If* $\langle \boldsymbol{d}^{(m)}[-m], \vec{a}\rangle \ne 0$ (*resp.* $\langle \boldsymbol{d}^{(\lfloor (m+1)/2 \rfloor)}[-m], \vec{a}\rangle \ne 0$), *then* $P^{[\vec{a},s]}(x)$ *has pole of order* m (*resp.* $\lfloor (m+1)/2 \rfloor$) *at* $s = s_0$.

PROOF. Since $\langle \boldsymbol{d}^{(m)}[-m], \vec{a}\rangle \neq 0$ (resp. $\langle \boldsymbol{d}^{(\lfloor (m+1)/2\rfloor)}[-m], \vec{a}\rangle \neq 0$), we see that the coefficient $\boldsymbol{c}_m^{\bullet;\bullet}(\vec{a}, s)$ (resp. $\boldsymbol{c}_{\lfloor (m+1)/2\rfloor}^{\bullet;\bullet}(\vec{a}, s)$) has pole of order m by applying Proposition 5.3 for the case $p = m$. On the other hand, all the coefficients of $P^{[\vec{a},s]}(x)$ have pole of order at most m since we are considering the poles at $s = -m$ with $m \leq n$ (resp. $m \leq 2n$) by the theory of b-functions. Then all the coefficients $\boldsymbol{c}_i^{\bullet;\bullet}(\vec{a}, s)$ $(0 \leq i \leq n)$ have pole of order at most m (resp. $\lfloor (m+1)/2\rfloor$), and at least one of them has pole of order m (resp. $\lfloor (m+1)/2\rfloor$). Thus, by Proposition 5.1, $P^{[\vec{a},s]}(x)$ has pole of order m (resp. $\lfloor (m+1)/2\rfloor$) at $s = s_0$. $\square$

By Lemma 5.7, Lemma 5.8 and Lemma 5.9, we have part (a) of Theorem 4.2 in the complex (resp. quaternion) case. The proof of part (b) is almost the same, so we omit them.

Next we go to the proof of Theorem 4.3.

PROPOSITION 5.10. *Suppose that $P^{[\vec{a},s]}(x)$ has pole of order p at $s = s_0$. We give the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = s_0$ by*

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},s_0]}(x)(s - s_0)^w. \tag{30}$$

Then we have

$$\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x)) = \overline{\left(\bigcup_{\left\{(i,j)\in \boldsymbol{Z}^2 \,\middle|\, \begin{array}{l} c_i^{j,k}(\vec{a},s) \text{ has pole of} \\ \text{order } \geq -w \text{ for some } k \\ \text{in } 0 \leq k \leq i \text{ at } s=s_0 \end{array}\right\}} \boldsymbol{S}_i^j\right)} \tag{31}$$

The proof of Proposition 5.10 can be carried out in the same way as [**6**, Proposition 3.8]. By Proposition 5.10 and the argument in [**6**, §6], we can prove Theorem 4.3.

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