J. Math. Soc. Japan
Vol. 60, No. 2 (2008) pp. 583–602
doi: 10.2969/jmsj/06020583

Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent

By Yoshihiro Mizuta and Tetsu Shimomura

(Received Feb. 2, 2007)
(Revised Aug. 13, 2007)

Abstract. Our aim in this paper is to deal with Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent.

1. Introduction.

The space introduced by Morrey [**14**] in 1938 has become a useful tool for studying the existence and regularity of partial differential equations. In recent years, the generalized Lebesgue spaces and the corresponding Sobolev spaces of variable exponent have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with variable growth; see Růžička [**18**]. Our aim in this paper is to establish Sobolev's inequality for generalized Morrey spaces of variable exponent; the borderline case is concerned with Trudinger's inequality.

In the n-dimensional Euclidean space $\boldsymbol{R}^n$, we consider the Riesz potential of order α for a locally integrable function f on $\boldsymbol{R}^n$, which is defined by

$$U_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)dy.$$

Here $0 < \alpha < n$ and it is natural to assume that $U_\alpha|f| \not\equiv \infty$, which is equivalent to

$$\int_{\boldsymbol{R}^n} (1+|y|)^{\alpha-n}|f(y)|dy < \infty; \tag{1.1}$$

for this fact, see [**12**, Theorem 1.1, Chapter 2]. If f is a locally integrable function

2000 *Mathematics Subject Classification.* Primary 46E35; Secondary 31B15.

Key Words and Phrases. Riesz potentials, maximal functions, Sobolev's embedding theorem of variable exponent, Morrey space.

on $\boldsymbol{R}^n$ satisfying (1.1) and the Morrey condition

$$\sup_{x \in \boldsymbol{R}^n, r>0} r^{-\nu} \int_{B(x,r)} |f(y)|^p dy < \infty,$$

then it is shown (see Adams [**1**] and Peetre [**17**]) that $U_\alpha f$ satisfies Sobolev's inequality, where $1 < p < \infty$ and $B(x,r)$ denotes the open ball centered at x of radius r. Further, in case $\nu = n - \alpha p$, we know Trudinger's type inequality for $U_\alpha f$ due to the paper by Nakai [**15**, Theorem 2.2].

Following Kováčik and Rákosník [**11**], we consider a positive continuous function $p(\cdot)$ on $\boldsymbol{R}^n$, which is called a variable exponent. To extend those results mentioned above, we consider the $L^{p(\cdot),\nu,\beta}$ norm by

$$\|f\|_{p(\cdot),\nu,\beta,\boldsymbol{R}^n} = \inf\left\{\lambda > 0 : \sup_{x \in \boldsymbol{R}^n, r>0} r^{-\nu}(\log(2+r^{-1}))^\beta \int_{B(x,r)} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}$$

and denote by $L^{p(\cdot),\nu,\beta}(\boldsymbol{R}^n)$ the space of all measurable functions f on $\boldsymbol{R}^n$ with $\|f\|_{p(\cdot),\nu,\beta,\boldsymbol{R}^n} < \infty$. This space $L^{p(\cdot),\nu,\beta}(\boldsymbol{R}^n)$ is referred to as a Morrey space of variable exponent. For related results about Lebesgue or Sobolev spaces of variable exponent, see also Edmunds-Rákosník [**7**] and Růžička [**18**].

In this paper we are concerned with $p(\cdot)$ satisfying the following log-Hölder condition

$$|p(x) - p(y)| \le \frac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b}{\log(1/|x-y|)}$$

whenever $|x-y| < 1/4$, where a and b are nonnegative constants. A typical example is given by

$$p(x) = p_0 + \frac{a \log(\log(1/|x_0 - x|))}{\log(1/|x_0 - x|)} + \frac{b}{\log(1/|x_0 - x|)}$$

for $x \in B(x_0, r_0)$, $0 < r_0 < 1/4$. Sobolev's theorem consists of three different aspects, that is, Sobolev's inequality, Trudinger's inequality and continuity. The border between Sobolev's inequality and Trudinger's inequality is caused by the first parameter p_0, and the border between Trudinger's inequality and continuity is caused by the second parameter a. In the present paper, we deal with the case that a is small; when a is large, the continuity property has been discussed in the coming paper by the authors [**13**].

Recently Diening [**6**] has established embedding results for Riesz potentials in the case $a = 0$. Our first task is then to establish the boundedness of Hardy-Littlewood maximal functions from $L^{p(\cdot),\nu,\beta}$ to some Orlicz classes, as an extension of Chiarenza and Frasca [**4**, Theorem 1] in the constant case and the authors' [**9**, Theorem 2.4] with $\nu = \beta = 0$. As an application of the boundedness of maximal functions, we establish Sobolev's inequality as well as Trudinger's inequality for Riesz potentials of functions in Morrey spaces of variable exponent, as an extension of Adams [**1**], Cruz-Uribe, Fiorenza and Neugebauer [**3**], Chiarenza and Frasca [**4**], Futamura and the first author [**8**], Futamura and the authors [**9**] and Nakai [**15**].

2. Preliminary results for constant exponents.

Throughout this paper, let C denote various constants independent of the variables in question.

For an open set G in $\boldsymbol{R}^n$, $1 < p < \infty$, $0 \le \nu \le n$ and a real number β, following Nakai [**15**], we consider the family $L^{p,\nu,\beta}(G)$ of all measurable functions f on G such that

$$\|f\|_{p,\nu,\beta,G}^p = \sup_{x \in G, r>0} r^{-\nu}(\log(2 + r^{-1}))^{\beta} \int_{G \cap B(x,r)} |f(y)|^p dy < \infty,$$

where $B(x, r)$ denotes the open ball centered at x of radius $r > 0$. In what follows we assume that $f = 0$ outside G. Under this assumption, we easily see that if $f \in L^{p,\nu,\beta}(G)$, then $f \in L^{p,\nu,\beta}(\boldsymbol{R}^n)$ and

$$\|f\|_{p,\nu,\beta,\boldsymbol{R}^n} \le C\|f\|_{p,\nu,\beta,G}.$$

We sometimes write $\|f\|_{p,\nu,\beta}$ instead of $\|f\|_{p,\nu,\beta,G}$ for simplicity.

We recall the notion of maximal functions of locally integrable functions f on $\boldsymbol{R}^n$, which are in fact defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $|E|$ denotes the n-dimensional Lebesgue measure of a measurable set $E \subset \boldsymbol{R}^n$.

First we present the boundedness of maximal functions in the Morrey space $L^{p,\nu,\beta}$ due to Nakai [**15**, Theorem 2.1].

LEMMA 2.1. *If* $0 \le \nu < n$, *then*

$$\|Mf\|_{p,\nu,\beta} \le C\|f\|_{p,\nu,\beta}$$

for all $f \in L^{p,\nu,\beta}(\boldsymbol{R}^n)$.

For reader's convenience we give a proof of Lemma 2.1.

Proof of Lemma 2.1. Let $\|f\|_{p,\nu,\beta} \le 1$, $x \in \boldsymbol{R}^n$ and $r > 0$. Write $A_0 = B(x, 2r)$ and $A_j = B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ for each positive integer j. We set

$$f_j = f\chi_{A_j},$$

where χ_E denotes the characteristic function of E. Note that

$$\begin{aligned}\int_{B(x,r)} Mf(z)^p dz &\le 2^{p-1}\left(\int_{B(x,r)} Mf_0(z)^p dz + \int_{B(x,r)} Mg_0(z)^p dz\right)\\ &\equiv 2^{p-1}(I_1 + I_2),\end{aligned}$$

where $g_0 = \sum_{j=1}^{\infty} |f_j|$. We have

$$I_1 \le \int Mf_0(z)^p dz \le C \int |f_0(z)|^p dz = C \int_{B(x,2r)} |f(z)|^p dz \le Cr^{\nu}(\log(2+r^{-1}))^{-\beta}.$$

Next we see that for $z \in B(x, r)$

$$\begin{aligned}Mf_j(z) &\le C(2^j r)^{-n} \int_{B(x,2^{j+1}r)} |f(y)| dy\\ &\le C\left((2^j r)^{-n} \int_{B(x,2^{j+1}r)} |f(y)|^p dy\right)^{1/p}\\ &\le C(2^j r)^{(\nu-n)/p}(\log(2 + (2^j r)^{-1}))^{-\beta/p},\end{aligned}$$

so that

$$\begin{aligned}Mg_0(z) &\le \sum_{j=1}^{\infty} Mf_j(z)\\ &\le C\sum_{j=1}^{\infty}(2^j r)^{(\nu-n)/p}(\log(2 + (2^j r)^{-1}))^{-\beta/p}\\ &\le Cr^{(\nu-n)/p}(\log(2 + r^{-1}))^{-\beta/p}.\end{aligned}$$

Hence it follows that

$$I_2 \le Cr^{\nu-n}(\log(2+r^{-1}))^{-\beta}\int_{B(x,r)} dz = Cr^{\nu}(\log(2+r^{-1}))^{-\beta}.$$

Thus we obtain

$$\int_{B(x,r)} Mf(z)^p dz \le Cr^{\nu}(\log(2+r^{-1}))^{-\beta},$$

which proves the lemma. □

REMARK 2.2. When $\nu = n$ and $\beta \le p$, Theorem 2.1 by Nakai [**15**] implies that

$$\|Mf\|_{p,n,\beta-p} \le C\|f\|_{p,n,\beta}.$$

For $0 < \alpha < n$ and a locally integrable function f on $\boldsymbol{R}^n$ we define the Riesz potential $U_\alpha f$ by

$$U_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)dy;$$

note that $U_\alpha|f| \not\equiv \infty$ if and only if

$$\int (1+|y|)^{\alpha-n}|f(y)|dy < \infty; \tag{2.1}$$

for this fact, see [**12**, Theorem 1.1, Chapter 2].

LEMMA 2.3. *Let $0 \le \nu < n-\alpha p$. If f is a nonnegative measurable function on $\boldsymbol{R}^n$ such that $\|f\|_{p,\nu,\beta} \le 1$, then*

$$\int_{\boldsymbol{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{-(n-\nu)/p^\sharp}(\log(1/\delta))^{-\beta/p}$$

for $x \in \boldsymbol{R}^n$ and $0 < \delta < 1/4$, where $1/p^\sharp = 1/p - \alpha/(n-\nu) > 0$.

This lemma will be proved later (in Lemma 4.1) in variable exponent setting; For constant exponent case, we refer the reader to the book by Adams and Hedberg [**2**].

With Lemmas 2.1 and 2.3 we can apply Hedberg's trick (see [**10**]) to obtain Sobolev type inequality for Riesz potentials due to Adams [**1**, Theorem 3.1], Chiarenza and Frasca [**4**, Theorem 2] and Nakai [**15**, Theorem 2.2].

Theorem 2.4. *Let $0 \le \nu < n - \alpha p$. If f is a nonnegative measurable function on $\boldsymbol{R}^n$ with $\|f\|_{p,\nu,\beta} \le 1$, then*

$$\|U_\alpha f(\log(2 + U_\alpha f))^{\alpha\beta/(n-\nu)}\|_{p^\sharp,\nu,\beta} \le C.$$

Proof. Suppose $\|f\|_{p,\nu,\beta} \le 1$. For $x \in \boldsymbol{R}^n$ and $0 < \delta < 1/4$, write

$$U_\alpha f(x) = \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{\boldsymbol{R}^n \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy.$$

We see from Lemma 2.3 that

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C\delta^{-(n-\nu)/p^\sharp}(\log(2+\delta^{-1}))^{-\beta/p}.$$

Now, taking $\delta = Mf(x)^{-p/(n-\nu)}(\log(2 + Mf(x)))^{-\beta/(n-\nu)}$ when $Mf(x)$ is large enough, we obtain

$$U_\alpha f(x)(\log(2 + U_\alpha f))^{\alpha\beta/(n-\nu)} \le CMf(x)^{p/p^\sharp}.$$

Therefore it follows from Lemma 2.1 that

$$\begin{aligned}\int_{B(z,r)} \left\{U_\alpha f(x)(\log(2 + U_\alpha f))^{\alpha\beta/(n-\nu)}\right\}^{p^\sharp} dx &\le C\int_{B(z,r)} Mf(x)^p dx \\ &\le Cr^\nu(\log(2 + r^{-1}))^{-\beta}\end{aligned}$$

for $z \in \boldsymbol{R}^n$ and $r > 0$, which yields the required property. □

In case $\nu = n - \alpha p > 0$, we modify Lemma 2.3 and Theorem 2.4 as follows:

Lemma 2.5. *Let $\nu = n - \alpha p > 0$, $\beta \le p$ and G be a bounded open set in $\boldsymbol{R}^n$. Let f be a nonnegative measurable function on G such that $\|f\|_{p,\nu,\beta} \le 1$. If $\beta < p$, then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C(\log(1/\delta))^{1-\beta/p}$$

and if $\beta = p$, *then*

$$\int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C \log(\log(1/\delta))$$

for $x \in G$ *and* $0 < \delta < 1/4$.

In view of Lemma 2.5, we have the following exponential integrability of Trudinger type; see Nakai [**15**, Theorem 2.2]. We also refer the reader to the paper by Sawano, Sobukawa and Tanaka [**19**, Theorem 3.1].

THEOREM 2.6. *Let* $\nu = n - \alpha p > 0$, $\beta \le p$ *and* G *be a bounded open set in* $\boldsymbol{R}^n$. *Then there exist positive constants* c_1 *and* c_2 *such that in case* $\beta < p$,

$$r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{G \cap B(z,r)} \exp(c_1 U_\alpha f(x)^{p/(p-\beta)})dx \le c_2$$

and in case $\beta = p$,

$$r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{G \cap B(z,r)} \exp(\exp(c_1 U_\alpha f(x)))dx \le c_2$$

for all $z \in G$ *and* $r > 0$, *whenever* f *is a nonnegative measurable function on* G *satisfying* $\|f\|_{p,\nu,\beta,G} \le 1$.

REMARK 2.7. In the conclusion of Theorem 2.6 for $\beta = 0$ we can not add an exponent $q > 1$ such that

$$r^{-\nu} \int_{G \cap B(z,r)} \exp(c_1 U_\alpha f(x)^q)dx \le c_2.$$

For this, consider the function $f(y) = |y|^{-\alpha}\chi_G(y)$, where $G = B(0,1)$. If $\nu = n - \alpha p > 0$, then

$$r^{-\nu} \int_{B(x,r)} |f(y)|^p dy \le r^{-\nu} \int_{B(x,r)} |x-y|^{-\alpha p} dy \le C r^{-\nu} r^{n-\alpha p} = C$$

for all $x \in G$ and $r > 0$, so that $f \in L^{p,\nu}(G)$. On the other hand, we see that

$$U_\alpha f(x) \geq \int_{G \setminus B(x,|x|/2)} |x-y|^{\alpha-n} f(y)dy$$
$$\geq 3^{-\alpha} \int_{G \setminus B(x,|x|/2)} |x-y|^{-n} dy \geq C \log(2/|x|)$$

for $x \in G$, and hence

$$r^{-\nu} \int_{G \cap B(0,r)} \exp(cU_\alpha f(x)^q) dx = \infty$$

for $r > 0$, $c > 0$ and $q > 1$.

Theorem 2.6 is somewhat different from the usual Trudinger's inequality when $f \in L^p(G)$ as in Remark 2.7. Here we will modify Theorem 2.6 when $\nu = n - \alpha p > 0$.

Lemma 2.8. *Let $\nu = n - \alpha p > 0$ and R_G denote the diameter of G. If f is a nonnegative measurable function on G such that*

$$|||f|||_{p,\nu}^p = \sup_{x \in G} \int_0^{R_G} \left(t^{-\nu} \int_{B(x,t)} f(y)^p dy \right) \frac{dt}{t} \leq 1, \tag{2.2}$$

then

$$\int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \leq C(\log(1/\delta))^{1/p'}$$

for $x \in G$ and $0 < \delta < 1/4$.

Remark 2.9. Note that

$$\|f\|_{p,\nu,0} \leq C |||f|||_{p,\nu} \tag{2.3}$$

and

$$|||f|||_{p,\nu} \leq C \|f\|_{p,\nu,\beta} \qquad \text{when } \beta > 1. \tag{2.4}$$

Further, we see from condition (2.2) that

$$\lim_{r \to 0} r^{-\nu} \int_{B(x,r)} f(y)^p dy = 0. \tag{2.5}$$

PROOF OF LEMMA 2.8. In view of Hölder's inequality and (2.2), we have

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
&\le \left(\int_{G\setminus B(x,\delta)} |x-y|^{(\alpha-n+\nu/p)p'}dy\right)^{1/p'} \left(\int_{G\setminus B(x,\delta)} |x-y|^{-\nu} f(y)^p dy\right)^{1/p} \\
&\le C\left(\int_\delta^{R_G} t^{-1}dt\right)^{1/p'} \left(\int_\delta^{R_G} \left(t^{-\nu}\int_{B(x,t)} f(y)^p dy\right)\frac{dt}{t}\right)^{1/p} \\
&\le C(\log(1/\delta))^{1/p'},
\end{aligned}$$

as required. □

THEOREM 2.10. *Let $\nu = n - \alpha p > 0$ and G be a bounded open set in $\boldsymbol{R}^n$. Then there exist positive constants c_1 and c_2 such that*

$$r^{-\nu}\int_{G\cap B(z,r)} \exp(c_1 U_\alpha f(x)^{p'})dx \le c_2$$

for all $z \in G$ and $r > 0$, whenever f is a nonnegative measurable function on G satisfying $\|\!|f|\!\|_{p,\nu} \le 1$.

PROOF. Suppose $\nu = n - \alpha p > 0$ and $\|\!|f|\!\|_{p,\nu} \le 1$. We have by Lemma 2.8

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C(\log(1/\delta))^{1/p'}$$

for $x \in G$ and $0 < \delta < 1/4$. Now, considering $\delta = Mf(x)^{-1/\alpha}(\log(2 + Mf(x)))^{1/(\alpha p')}$ when $Mf(x)$ is large enough, we obtain

$$U_\alpha f(x) \le C(\log(2 + Mf(x)))^{1/p'},$$

which yields

$$\begin{aligned}
\int_{G\cap B(z,r)} \exp(C_1 p U_\alpha f(x)^{p'})dx &\le \int_{G\cap B(z,r)} (2 + Mf(x))^p dx \\
&\le C_2\int_{G\cap B(z,r)} dx + C_2\int_{G\cap B(z,r)} Mf(x)^p dx
\end{aligned}$$

for $z \in G$ and $r > 0$. Hence Lemma 2.1 gives

$$r^{-\nu} \int_{G \cap B(z,r)} \exp(C_1 p U_\alpha f(x)^{p'}) dx \le C_3$$

for such z and r, as required. □

Remark 2.11. Let $\nu = n - \alpha p > 0$ and G be a bounded open set in $\boldsymbol{R}^n$. Then we can find a positive constant c such that

$$r^{-\nu} \int_{G \cap B(z,r)} \{\exp(c U_\alpha f(x)^{p'}) - 1\} dx \le 1$$

for all $z \in G$ and $r > 0$, whenever f is a nonnegative measurable function on G satisfying $\|\|f\|\|_{p,\nu} \le 1$.

3. Variable exponents and boundedness of maximal operators.

In what follows we assume that G is a bounded open set in $\boldsymbol{R}^n$, and consider a positive continuous function $p(\cdot)$ on G.

In this section let us assume that:

(p1) $1 < p_*(G) = \inf_G p(x) \le \sup_G p(x) = p^*(G) < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \dfrac{b}{\log(1/|x-y|)}$

whenever $|x-y| < 1/4$, $x \in G$ and $y \in G$.

Let $1/p'(x) = 1 - 1/p(x)$.

We know the following result.

Lemma 3.1 ([**9**, Lemma 2.1]). *There exists a positive constant C such that*

$$|p'(x) - p'(y)| \le \omega(|x-y|) \qquad \textit{whenever } x \in G \textit{ and } y \in G,$$

where $\omega(r) = \omega(r;x) = \frac{a}{(p(x)-1)^2} \frac{\log(\log(1/r))}{\log(1/r)} + \frac{C}{\log(1/r)}$ *for* $0 < r \le r_0$ $(\le 1/4)$ *and* $\omega(r) = \omega(r_0)$ *for* $r \ge r_0$*; here* r_0 *is chosen so small that* $\omega(r)$ *is nondecreasing for fixed* $x \in G$.

For $0 \le \nu \le n$ and a real number β, we define the family $L^{p(\cdot),\nu,\beta}(G)$ of all measurable functions f on G such that

$$\|f\|_{p(\cdot),\nu,\beta,G}$$
$$= \inf\left\{\lambda > 0 : \sup_{x\in G, r>0} r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{G\cap B(x,r)} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\} < \infty.$$

LEMMA 3.2 (cf. [**9**, Lemma 2.2]). *Suppose $0 \le \nu < n$. If f is a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$, then*

$$\{Mf(x)\}^{p(x)} \le C\big\{Mg(x)(\log(2+Mg(x)))^{A(x)p(x)} + 1\big\},$$

where $g(y) = f(y)^{p(y)}$ and $A(x) = a(n-\nu)/p(x)^2$; we set $f = 0$ and $g = 0$ outside G as before.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$. Then note that

$$r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{B(x,r)} f(y)^{p(y)} dy \le 1 \tag{3.1}$$

whenever $x \in G$ and $r > 0$. If $r \ge 1/2$, then we see that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le \frac{1}{|B(x,r)|}\int_{B(x,r)} \{1 + f(y)^{p(y)}\}dy \le C$$

by our assumption. For $0 < k \le 1$ and $0 < r < 1/2$, we have by Lemma 3.1

$$\begin{aligned}
&\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \\
&\le k\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)} (1/k)^{p'(y)}dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy\right\} \\
&\le k\big\{(1/k)^{p'(x)+\omega(r)} + F\big\},
\end{aligned}$$

where $F = |B(x,r)|^{-1}\int_{B(x,r)} f(y)^{p(y)}dy$. Here, considering

$$k = F^{-1/\{p'(x)+\omega(r)\}} = F^{-1/p'(x)+\eta(x)}$$

with $\eta(x) = \omega(r)/\{p'(x)(p'(x)+\omega(r))\}$ when $F \ge 1$, we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le 2F^{1/p(x)}F^{\omega(r)/p'(x)^2};$$

if $F < 1$, then we can take $k = 1$ to obtain

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le 2.$$

Hence it follows that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le 2\big\{F^{1/p(x)}F^{\omega(r)/p'(x)^2} + 1\big\}.$$

If $r \le F^{-1/(n-\nu)}$, then we see that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le C\big\{F^{1/p(x)}(\log(2+F))^{A(x)} + 1\big\}.$$

Next we treat the case where $F^{-1/(n-\nu)} < r$. Then we have

$$\begin{aligned}
&F^{1/p(x)+\omega(r)/p'(x)^2} \\
&\quad\le Cr^{(\nu-n)/p(x)}r^{-(n-\nu)\omega(r)/p'(x)^2}(\log(2+r^{-1}))^{-\beta(1/p(x)+\omega(r)/p'(x)^2)} \\
&\qquad\times\left\{r^{-\nu}(\log(2+r^{-1}))^{\beta}\int_{B(x,r)} f(y)^{p(y)}dy\right\}^{1/p(x)+\omega(r)/p'(x)^2} \\
&\quad\le Cr^{(\nu-n)/p(x)}(\log(1/r))^{A(x)-\beta/p(x)} \\
&\qquad\times\left\{r^{-\nu}(\log(2+r^{-1}))^{\beta}\int_{B(x,r)} f(y)^{p(y)}dy\right\}^{1/p(x)+\omega(r)/p'(x)^2} \\
&\quad\le Cr^{(\nu-n)/p(x)}(\log(1/r))^{A(x)-\beta/p(x)} \\
&\qquad\times\left\{r^{-\nu}(\log(2+r^{-1}))^{\beta}\int_{G\cap B(x,r)} f(y)^{p(y)}dy\right\}^{1/p(x)} \\
&\quad\le C(\log(1/r))^{A(x)}\left\{r^{-n}\int_{G\cap B(x,r)} f(y)^{p(y)}dy\right\}^{1/p(x)},
\end{aligned}$$

in view of (3.1). Since $F^{-1/(n-\nu)} < r$ implies that $(\log(1/r))^{A(x)} \le C(\log F)^{A(x)}$, we find

$$F^{1/p(x)+\omega(r)/p'(x)^2} \le C(\log F)^{A(x)}\left\{r^{-n}\int_{G\cap B(x,r)} f(y)^{p(y)}dy\right\}^{1/p(x)}$$
$$\le CF^{1/p(x)}(\log F)^{A(x)}.$$

Now we have established

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le C\big\{F^{1/p(x)}(\log(2+F))^{A(x)}+1\big\} \tag{3.2}$$

for all $x \in G$ and $r > 0$, which completes the proof. □

THEOREM 3.3. *For $0 \le \nu < n - \alpha p^*(G)$, let $A' > A^*(G) = \sup_{x\in G} A(x)$ when $a > 0$ and $A' = 0$ when $a = 0$. Then there exists $c > 0$ such that*

$$r^{-\nu}(\log(2+r^{-1}))^{\beta}\int_{G\cap B(z,r)}\big\{Mf(x)(\log(2+Mf(x)))^{-A'}\big\}^{p(x)}dx \le c$$

for $z \in G$ and $r > 0$, whenever f is a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$. Write

$$f = f\chi_{\{y:f(y)\ge 1\}} + f\chi_{\{y:f(y)<1\}} = f_1 + f_2.$$

Let $p_0(x) = p(x)/p_0$ with $1 < p_0 < p_*(G)$. Then we have

$$\int_{G\cap B(x,r)} f_1(y)^{p_0(y)}dy \le \int_{G\cap B(x,r)} f_1(y)^{p(y)}dy,$$

which implies that $\|f_1\|_{p_0(\cdot),\nu,\beta,G} \le 1$. Hence it follows from Lemma 3.2 that

$$\{Mf_1(x)\}^{p_0(x)} \le C\big\{Mg(x)(\log(2+Mg(x)))^{a_0(n-\nu)/p_0(x)}+1\big\}$$

for $x \in G$, where $g(y) = f(y)^{p_0(y)}$ and $a_0 = a/p_0$. Since $Mf_2 \le 1$ on G, we establish

$$\{Mf(x)\}^{p_0(x)} \le C\big\{Mg(x)(\log(2+Mg(x)))^{a_0(n-\nu)/p_0(x)}+1\big\}.$$

If $a > 0$, then we can find $p_0 > 1$ such that $a_0(n-\nu)/p_0(x) < A' p_0(x)$, so that

$$\{Mf(x)\}^{p(x)} \le C\big\{Mg(x)(\log(2+Mg(x)))^{A'p(x)/p_0} + 1\big\}^{p_0},$$

which yields

$$\big\{Mf(x)(\log(2+Mf(x)))^{-A'}\big\}^{p(x)} \le C(Mg(x)+1)^{p_0}.$$

Note from (3.1) that

$$r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{B(z,r)} g(y)^{p_0}\,dy \le 1$$

for $z \in G$ and $r > 0$. Hence we see from Lemma 2.1 that

$$\begin{aligned}
&\int_{G\cap B(z,r)} \big\{Mf(x)(\log(2+Mf(x)))^{-A'}\big\}^{p(x)} dx \\
&\quad \le C \int_{G\cap B(z,r)} Mg(x)^{p_0}\,dx + C\int_{G\cap B(z,r)} dx \\
&\quad \le C r^{\nu}(\log(2+r^{-1}))^{-\beta}
\end{aligned}$$

for $z \in G$ and $r > 0$. □

Remark 3.4. Set $\Phi(r,x) = (r(\log(2+r))^{-A'})^{p(x)}$ for $r \ge 0$ and $x \in G$. Then Theorem 3.3 assures the existence of $c_0 > 0$ such that

$$r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{G\cap B(z,r)} \Phi(c_0 Mf(x), x)\,dx \le 1$$

for all $z \in G$ and $r > 0$, whenever $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$. As in Edmunds and Rákosník [**7**], we define

$$\begin{aligned}
&\|f\|_{\Phi,\nu,\beta,G} \\
&\quad = \inf\left\{\lambda > 0 : \sup_{z\in G, r>0} r^{-\nu}(\log(2+r^{-1}))^{\beta} \int_{G\cap B(z,r)} \Phi(|f(x)|/\lambda, x)\,dx \le 1\right\};
\end{aligned}$$

then it follows that

$$\|Mf\|_{\Phi,\nu,\beta,G} \le {c_0}^{-1}\|f\|_{p(\cdot),\nu,\beta,G} \qquad \text{for } f \in L^{p(\cdot),\nu,\beta}(G).$$

When $\nu = \beta = 0$, Theorem 3.3 was proved by the authors [**9**], which is an extension of Diening [**5**] (when $a = 0$).

4. Riesz potentials.

In this section, if $0 < \nu < n - \alpha p(x)$, then we define the Sobolev exponent of $p(\cdot)$ by

$$\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n-\nu}.$$

For a function $q(x)$ on G, we set $q_*(G) = \inf_G q(x)$ as in (p1).

LEMMA 4.1 (cf. [**9**, Lemma 3.1]). *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$.*

(1) *If $0 < \nu < n - \alpha p^*(G)$, then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{-(n-\nu)/p^\sharp(x)}(\log(1/\delta))^{A(x)-\beta/p(x)};$$

(2) *if $\nu \ge n - \alpha p(x) > 0$ and $(A(\cdot) - \beta/p(\cdot))_*(G) > -1$, then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C(\log(1/\delta))^{A(x)-\beta/p(x)+1};$$

(3) *if $\nu \ge n - \alpha p(x) > 0$ and $A(x) - \beta/p(x) + 1 \le 0$, then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\log(\log(1/\delta))$$

for $x \in G$ and $0 < \delta < 1/4$, where $A(x) = a(n-\nu)/p(x)^2$ as before.

PROOF. For a nonnegative measurable function f on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$, we see from (3.1) and (3.2) that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le Cr^{-(n-\nu)/p(x)}(\log(2+1/r))^{A(x)-\beta/p(x)}$$

for $x \in G$ and $r > 0$. Letting R_G denote the diameter of G, we obtain

$$\begin{aligned} I(\delta) &= \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le \int_\delta^{R_G} \left(\int_{B(x,r)} f(y)dy \right) d(-r^{\alpha-n}) + R_G^{\alpha-n} \int_{B(x,R_G)} f(y)dy \\ &\le C \int_\delta^{R_G} r^{n-(n-\nu)/p(x)} (\log(2+r^{-1}))^{A(x)-\beta/p(x)} d(-r^{\alpha-n}) \\ &\le C \int_\delta^{R_G} r^{\alpha-(n-\nu)/p(x)} (\log(2+r^{-1}))^{A(x)-\beta/p(x)} \frac{dr}{r} \end{aligned}$$

for $0 < \delta < R_G/2$. If $n-\nu-\alpha p(x) \ge n-\nu-\alpha p^*(G) > 0$, then

$$\begin{aligned} I(\delta) &\le C\delta^{\alpha-(n-\nu)/p(x)} (\log(1/\delta))^{A(x)-\beta/p(x)} \\ &= C\delta^{-(n-\nu)/p^\sharp(x)} (\log(1/\delta))^{A(x)-\beta/p(x)}; \end{aligned}$$

if $n-\nu-\alpha p(x) \le 0$ and $A(x)-\beta/p(x)+1 \ge (A(\cdot)-\beta/p(\cdot))_*(G)+1 > 0$, then

$$I(\delta) \le C(\log(1/\delta))^{A(x)-\beta/p(x)+1};$$

and if $n-\nu-\alpha p(x) \le 0$ and $A(x)-\beta/p(x)+1 \le 0$, then

$$I(\delta) \le C \log(\log(1/\delta))$$

for small $\delta > 0$, say $0 < \delta \le \delta_0$. Since $I(\delta) \le I(\delta_0)$ for $\delta > \delta_0$, we complete the proof of the present lemma. $\square$

LEMMA 4.2. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$. If $x \in G$ and $0 < \nu < n - \alpha p^*(G)$, then*

$$\rho(U_\alpha f(x), A_\beta(x))^{p^\sharp(x)} \le C\{\rho(Mf(x), A(x))^{p(x)} + 1\},$$

where $\rho(t,y) = t\,(\log(2+t))^{-y}$ and $A_\beta(x) = A(x) - \alpha\beta/(n-\nu)$.

PROOF. For $0 < \delta < 1/4$ we have by Lemma 4.1

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C\delta^{-(n-\nu)/p^\sharp(x)} (\log(1/\delta))^{A(x)-\beta/p(x)}.$$

Considering $\delta = Mf(x)^{-p(x)/(n-\nu)}(\log(2 + Mf(x)))^{p(x)(A(x)-\beta/p(x))/(n-\nu)}$ when $Mf(x)$ is large enough, we see that

$$U_\alpha f(x) \le C\big\{Mf(x)^{p(x)/p^\sharp(x)}(\log(2 + Mf(x)))^{\alpha p(x)(A(x)-\beta/p(x))/(n-\nu)} + 1\big\}.$$

Hence it follows that

$$\rho(U_\alpha f(x), A_\beta(x))^{p^\sharp(x)} \le C\big\{\rho(Mf(x), A(x))^{p(x)} + 1\big\},$$

as required. □

Set $A'_\beta > A^*_\beta(G) = A^*(G) - \alpha\beta/(n-\nu)$ when $a > 0$ and $A'_\beta = -\alpha\beta/(n-\nu)$ when $a = 0$. In view of Theorem 3.3 and Lemma 4.2, we have the following result, which gives an extension of Diening [**6**] together with Futamura and the authors [**9**].

THEOREM 4.3. *Suppose $0 \le \nu < n - \alpha p^*(G)$. Then there exists a positive constant c_0 such that*

$$r^{-\nu}(\log(2 + r^{-1}))^\beta \int_{G\cap B(z,r)} \big\{U_\alpha f(x)(\log(2 + U_\alpha f(x)))^{-A'_\beta}\big\}^{p^\sharp(x)} dx \le c_0$$

for $z \in G$ and $r > 0$, whenever f is a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$.

When $n - \alpha p_*(G) \le \nu < n$, we discuss the exponential integrability of variable exponent Riesz potentials.

THEOREM 4.4. *Suppose $n - \alpha p_*(G) \le \nu < n$ and $(A(\cdot) - \beta/p(\cdot))_*(G) > -1$, that is, $\beta < \inf_{x\in G} p(x)(A(x) + 1)$. Then there exist positive constants c_1 and c_2 such that*

$$r^{-\nu}(\log(2 + r^{-1}))^\beta \int_{G\cap B(z,r)} \exp\left(c_1 U_\alpha f(x)^{1/(A(x)-\beta/p(x)+1)}\right) dx \le c_2$$

for $z \in G$ and $r > 0$, whenever f is a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$.

PROOF. Suppose $n - \alpha p_*(G) \le \nu < n$ and $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$. For $0 < \delta < 1/4$, we have by Lemma 4.1

$$U_\alpha f(x) \le C\delta^\alpha Mf(x) + C(\log(1/\delta))^{A(x)-\beta/p(x)+1}.$$

Now, considering $\delta = Mf(x)^{-1/\alpha}(\log(2+Mf(x)))^{(A(x)-\beta/p(x)+1)/\alpha}$ when $Mf(x)$ is large enough, we obtain

$$U_\alpha f(x) \le C_1(\log(2+Mf(x)))^{A(x)-\beta/p(x)+1}.$$

Taking p_0 such that $1 < p_0 < p_*(G)$, we find

$$\begin{aligned}
&\int_{G\cap B(z,r)} \exp\left(p_0(C_1^{-1}U_\alpha f(x))^{1/(A(x)-\beta/p(x)+1)}\right)dx \\
&\quad \le \int_{G\cap B(z,r)} (2+Mf(x))^{p_0}\,dx \\
&\quad \le C\int_{G\cap B(z,r)} dx + C\int_{G\cap B(z,r)} \left\{Mf(x)(\log(2+Mf(x)))^{-A'}\right\}^{p(x)}dx
\end{aligned}$$

for $z \in G$ and $r > 0$. Hence Theorem 3.3 gives

$$\int_{G\cap B(z,r)} \exp\left(p_0(C_1^{-1}U_\alpha f(x))^{1/(A(x)-\beta/p(x)+1)}\right)dx \le Cr^\nu(\log(2+r^{-1}))^{-\beta}$$

for such z and r, which proves the present theorem. □

Theorem 4.5. *Suppose $n-\alpha p_*(G) \le \nu < n$ and $\beta \ge \sup_{x\in G} p(x)(A(x)+1)$. Then there exist positive constants c_1 and c_2 such that*

$$r^{-\nu}(\log(2+r^{-1}))^\beta \int_{G\cap B(z,r)} \exp(\exp(c_1U_\alpha f(x)))dx \le c_2$$

for $z \in G$ and $r > 0$, whenever f is a nonnegative measurable function on G with $\|f\|_{p(\cdot),\nu,\beta,G} \le 1$.

Remark 4.6. Let $p(\cdot)$ satisfy

$$p(x) \ge p_0 + \omega(|x_0 - x|),$$

where $\omega(r) = a(\log(\log(1/r)))/\log(1/r) + b/\log(1/r)$ is increasing on $(0, r_0)$. Suppose f is a nonnegative measurable function on $B_0 = B(x_0, r_0)$ satisfying $\|f\|_{p(\cdot),\nu,\beta} \le 1$ for $\nu = n - \alpha p_0 > 0$ and $\beta > p_0 - 1 - a\alpha$. Then, in view of [**13**], we see that $U_\alpha f$ is continuous on B_0 and

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o((\log(1/|x_0 - x|))^{-A})$$

as $x \to x_0$, where $A = (a\alpha + \beta + 1)/p_0 - 1 > 0$.

References

[1] D. R. Adams, A note on Riesz potentials, Duke Math. J., **42** (1975), 765–778.

[2] D. R. Adams and L. I. Hedberg, Function spaces and potential theory, Springer-Verlag, Berlin, 1996.

[3] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Math., **28** (2003), 223–238, **29** (2004), 247–249.

[4] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. Apple. (7), **7** (1987), 273–279.

[5] L. Diening, Maximal functions on generalized $L^{p(\cdot)}$ spaces, Math. Inequal. Appl., **7** (2004), 245–253.

[6] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr., **263** (2004), 31–43.

[7] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, II, Math. Nachr., **246–247** (2002), 53–67.

[8] T. Futamura and Y. Mizuta, Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent, Math. Inequal. Appl., **8** (2005), 619–631.

[9] T. Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for Riesz potential spaces of variable exponent, Math. Nachr., **279** (2006), 1463–1473.

[10] L. I. Hedberg, On certain convolution inequalities, Proc. Amer. Math. Soc., **36** (1972), 505–510.

[11] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J., **41** (1991), 592–618.

[12] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[13] Y. Mizuta and T. Shimomura, Continuity properties for Riesz potentials of functions in Morrey spaces of variable exponent, preprint.

[14] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., **43** (1938), 126–166.

[15] E. Nakai, Generalized fractional integrals on Orlicz-Morrey spaces, Banach and function spaces, Yokohama Publ., Yokohama, 2004, pp. 323–333.

[16] W. Orlicz, Über konjugierte Exponentenfolgen, Studia Math., **3** (1931), 200–211.

[17] J. Peetre, On the theory of $L_{p,\lambda}$ spaces, J. Funct. Anal., **4** (1969), 71–87.

[18] M. Růžička, Electrorheological fluids: modeling and mathematical theory, Lecture Notes in Math., **1748**, Springer-Verlag, Berlin, 2000.

[19] Y. Sawano, T. Sobukawa and H. Tanaka, Limiting case of the boundedness of fractional integral operators on non-homogeneous space, J. Inequal. Appl., **2006** (2006), Art. ID 92470, 16 pages.

Yoshihiro Mizuta
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail: mizuta@mis.hiroshima-u.ac.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail: tshimo@hiroshima-u.ac.jp