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Kirchhoff elastic rods in three-dimensional space forms

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Abstract. The Kirchhoff elastic rod is one of the mathematical models of thin elastic rods, and is characterized as a critical point of the energy functional obtained by adding the effect of twisting to the bending energy. In this paper, we investigate Kirchhoff elastic rods in three-dimensional space forms. In particular, we give explicit formulas of Kirchhoff elastic rods in the three-sphere and in the three-dimensional hyperbolic space in terms of Jacobi sn function and the elliptic integrals.

1. Introduction.

The most famous model of thin elastic rods is probably Euler's elastica, which is a critical curve for the energy with the effect of bending only. The uniform symmetric case of the Kirchhoff elastic rod is a generalization of the elastica and is the simplest model with the effect of bending and twisting. (In this paper, we call it a Kirchhoff elastic rod for short.)

Such mathematical models of thin elastic rods in the Euclidean space have been extensively studied since the days of Euler in the 1730s (see, e.g., [**1**], [**23**]). Meanwhile, the elastica or its certain generalizations in Riemannian manifolds, except the Euclidean space, have been investigated since the 1980s not only for their own interests but also for applications to constructing Willmore surfaces, and so on (see, e.g., [**2**], [**3**], [**4**], [**8**], [**16**], [**19**], [**20**], [**26**]).

In this paper, we consider Kirchhoff elastic rods in the simply-connected three-dimensional space forms, $\boldsymbol{R}^3$, S^3 and H^3. In [**21**], by using Pontryagin's maximum principle, Langer and Singer derived the Hamiltonian systems associated to a class of variational problems, including that of Kirchhoff elastic rods in the three-dimensional space forms, and proved their Liouville integrability. These Hamiltonian systems are defined on the cotangent bundle of the orthonormal frame bundle (or its certain enlargement) of the space form. Also, Jurdjevic ([**11**]) considered the complexified Hamiltonian equations induced by the variational problems of generalized Kirchhoff elastic rods (see also [**8**], [**9**], [**10**]). Jurdjevic classified the

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integrable cases, including that of Kirchhoff elastic rods in the three-dimensional space forms, and showed the integration procedures. However, it seems to be difficult to visualize immediately the global shapes of the centerlines of Kirchhoff elastic rods in the non-Euclidean space forms. To make the visualization easier, we would like to take a system of coordinates on the space form itself and obtain simple explicit formulas of Kirchhoff elastic rods by well-known special functions.

In the case of the three-dimensional Euclidean space $\boldsymbol{R}^3$, many authors have been studying explicit expressions of Kirchhoff elastic rods, or their relations with the Lagrange top, the vortex filament equation or the DNA molecule (e.g., [**6**], [**7**], [**14**], [**15**], [**22**], [**25**], [**27**], [**28**]). Langer-Singer ([**22**]) and Shi-Hearst ([**27**]) obtained explicit formulas of the centerlines of Kirchhoff elastic rods by Jacobi sn function and the elliptic integrals in terms of cylindrical coordinates.

It is natural to ask if we can get such explicit expressions as [**22**], [**27**], even in the three-sphere S^3 or the three-dimensional hyperbolic space H^3. In the case of the three-sphere, the author obtained explicit formulas of the centerlines of Kirchhoff elastic rods by Jacobi sn function and the elliptic integrals in terms of a system of coordinates analogous to the cylindrical coordinates (Theorem 6.1 of [**13**]). However, we cannot apply the same method as [**13**] to the case of the three-dimensional hyperbolic space.

In this paper, by using an approach not depending on the signature of the sectional curvature of the space form, we prove that an analogous result also holds for the three-dimensional hyperbolic space H^3.

Let $\mathscr{M}$ be a smooth $n\,(\geq 2)$-dimensional Riemannian manifold with metric $\langle\, , \rangle$. Let $\gamma = \gamma(t) : [0, l] \to \mathscr{M}$ be a smooth unit-speed curve, and $T(t) = \gamma'(t)$ the tangent vector to γ. We denote by $T\mathscr{M}$ the tangent bundle of $\mathscr{M}$ and by ∇ the Levi-Civita connection in $T\mathscr{M}$.

To describe how the elastic rod is twisted, we utilize a smooth orthonormal frame field $M = (M_1, M_2, \ldots, M_{n-1})$ in the normal bundle $T^{\perp}\mathscr{M}$ along γ. We consider the pair $\{\gamma, M\}$ of γ and M. In this paper, we call such a pair $\{\gamma, M\}$ a *unit-speed curve with adapted orthonormal frame*, and γ the *centerline* of $\{\gamma, M\}$. Note that $(T(t), M_1(t), \ldots, M_{n-1}(t))$ is an orthonormal basis of the tangent space $T_{\gamma(t)}\mathscr{M}$ for each t. Now, let ν be a fixed positive constant, which is determined by the material of a given rod. (Throughout the paper, this constant ν is always fixed.) We define the energy $\mathfrak{T}$ as follows:

$$\mathfrak{T}(\{\gamma, M\}) = \int_0^l |\nabla_T T|^2 dt + \nu \sum_{i=1}^{n-1} \int_0^l |\nabla_T^{\perp} M_i|^2 dt,$$

where $\nabla^{\perp}$ denotes the normal connection in $T^{\perp}\mathscr{M}$, so that, $\nabla_T^{\perp} M_i = \nabla_T M_i - \langle \nabla_T M_i, T\rangle T$. The first term of $\mathfrak{T}(\{\gamma, M\})$ expresses the energy of bending, and the second term that of twisting. We call $\{\gamma, M\}$ a *Kirchhoff elastic rod* if $\{\gamma, M\}$ is a critical point of $\mathfrak{T}$ with respect to the variations of unit-speed curves with adapted orthonormal frames which preserve the frames $(\gamma(t), (T(t), M(t)))$ at the both end points.

Let $\mathscr{M}$ be S^3 or H^3 of constant sectional curvature G. In Section 2, according to [**13**], we give explicit expressions of the curvature and torsion of the centerline γ of a Kirchhoff elastic rod $\{\gamma, M\}$ in $\mathscr{M}$, and then parametrize the space of the congruence classes of Kirchhoff elastic rods by four real numbers, which we will write as α, η, p, w (Proposition 2.2).

To obtain the explicit formulas for γ itself, we use a similar method to that of Langer and Singer (see e.g., [**7**], [**12**], [**13**], [**18**], [**20**], [**22**]). In Section 3, we construct two Killing vector fields $\tilde{J}$ and $\tilde{H}$ on $\mathscr{M}$ associated to the Kirchhoff elastic rod $\{\gamma, M\}$ (Lemma 3.2), and prove that $\tilde{J}$ and $\tilde{H}$ commute (Lemma 3.4).

In Section 4, by using these two lemmas, we show that the matrix representations of $\tilde{J}$ and $\tilde{H}$ can be simultaneously canonicalized (Proposition 4.2 and Proposition 4.5), and we construct a system of coordinates suitable for $\{\gamma, M\}$ (the last part of Section 4).

In Section 5, we first express various constants by α, η, p and w. (A part of these calculations is written in the appendix (Section 6).) Then, we give the explicit formulas of the coordinate components of γ in terms of α, η, p, w, Jacobi sn function and the incomplete elliptic integral of the third kind (Theorem 5.3).

Let

$$\varepsilon = \begin{cases} 1 & \text{if } \mathscr{M} = S^3, \\ -1 & \text{if } \mathscr{M} = H^3, \end{cases} \qquad I(\mathscr{M}) = \begin{cases} O(4) & \text{if } \mathscr{M} = S^3, \\ O^+(3,1) & \text{if } \mathscr{M} = H^3. \end{cases}$$

In the case where $\mathscr{M} = S^3$, we embed $\mathscr{M}$ isometrically into the four-dimensional Euclidean space $\boldsymbol{R}^4$, with the canonical coordinates (x_1, x_2, x_3, x_4), as the standard three-sphere of radius $1/\sqrt{G}$. In the case where $\mathscr{M} = H^3$, we embed $\mathscr{M}$ isometrically into $\boldsymbol{R}^4$ with the Lorentzian metric $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$ as the hyperboloid $\{{}^t(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4 \,;\, x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1/G,\ x_4 > 0\}$. We denote by $\iota : \mathscr{M} \to \boldsymbol{R}^4$ the isometric embedding. Let ${}^t(x_1, x_2, x_3, x_4) \in \iota(\mathscr{M})$. We take a local coordinate system (r, θ, ψ), called the *cylindrical coordinates with respect to* ι, on $\mathscr{M}$ by the following relations:

$$\text{In the case of } S^3, \quad x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = \bar{r}\cos\psi, \quad x_4 = \bar{r}\sin\psi.$$

$$\text{In the case of } H^3, \quad x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = -\bar{r}\sinh\psi,\ x_4 = \bar{r}\cosh\psi.$$

Here, $0 < r < 1/\sqrt{G}$ in the case of S^3, and $r > 0$ in the case of H^3. Also,

$\bar{r} = \sqrt{\varepsilon(1/G - r^2)}$. This coordinate system (r, θ, ψ) is analogous to a system of cylindrical coordinates in $\boldsymbol{R}^3$.

Theorem 1.1 (cf. Theorem 5.3). *Let $\{\gamma, M\}$ be a Kirchhoff elastic rod in $\mathscr{M} = S^3$ or H^3. When $\mathscr{M} = H^3$, we assume that the associated Killing vector fields $\tilde{J}$ and $\tilde{H}$ are not parabolic. Then, there exists $P \in I(\mathscr{M})$ satisfying the following: Let (r, θ, ψ) denote the coordinates as above with respect to the isometric embedding $P \circ \iota : \mathscr{M} \to \boldsymbol{R}^4$ instead of ι, and let $r(t)$, $\theta(t)$, $\psi(t)$ denote the r, θ, ψ components of γ. Then,*

$$r(t) = \sqrt{c_1 \operatorname{sn}^2(c_2 t, c_3) + c_4}.$$

Moreover, if there exist no points where $r(t) = 0$ or $\bar{r}(t) = 0$, then

$$\theta(t) = c_5 t + c_6 \mathit{\Pi}(c_2 t, c_7, c_3),$$
$$\psi(t) = c_8 t + c_9 \mathit{\Pi}(c_2 t, c_{10}, c_3),$$

where sn *and $\mathit{\Pi}$ denote Jacobi sn function and the incomplete elliptic integral of the third kind, respectively. Also, $c_1, \ldots, c_{10}$ are real constants, which are explicitly expressed by (α, η, p, w) and G.*

We note that if there exists a point where $r(t) = 0$ or $\bar{r}(t) = 0$, then the explicit formulas of $\theta(t)$, $\psi(t)$ are also obtained. Even when $\mathscr{M} = H^3$ and $\tilde{J}$ or $\tilde{H}$ is parabolic, the explicit formulas for γ are obtained in terms of another coordinate system. Consequently, in all cases, we obtain the explicit formulas of the components of γ (see Theorem 5.3).

The point of the proof of the above theorem is the simultaneous canonicalization of the matrix representations of $\tilde{J}$ and $\tilde{H}$. The main difference between the approach of this paper and that of [**13**] is as follows: In [**13**], in order to prove this simultaneous canonicalizability, the author did not use the commutativity of $\tilde{J}$ and $\tilde{H}$ directly, but exploited the fact that the two vector fields $\tilde{J} \pm 2\sqrt{G}\tilde{H}$ are Killing vector fields on S^3 of constant lengths. Thus, the same method cannot apply to the case of H^3. On the other hand, in this paper, we first prove the commutativity of $\tilde{J}$ and $\tilde{H}$, which holds not depending on the signature of G (Lemma 3.4). Due to this commutativity, we can prove the simultaneous canonicalizability of the matrix representations of $\tilde{J}$ and $\tilde{H}$, and then obtain the explicit formulas for γ even in the case where $\mathscr{M} = H^3$.

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2. The space of Kirchhoff elastic rods.

In this section, according to [**13**], we give explicit expressions of the curvature and torsion of the centerline γ of a Kirchhoff elastic rod $\{\gamma, M\}$ in $\mathscr{M} = \boldsymbol{R}^3$, S^3, H^3, and then parametrize the space of the congruence classes of Kirchhoff elastic rods by four real numbers (Proposition 2.2). Unless otherwise specified, all curves, vector fields, etc., are assumed to be C^∞.

Let $\mathscr{M}$ be $\boldsymbol{R}^3$, S^3 or H^3 of constant sectional curvature G. We fix an orientation of $\mathscr{M}$, and denote by $\times$ the vector product. Let $\{\gamma, M\}$ be a unit-speed curve with adapted orthonormal frame in $\mathscr{M}$. We call $\{\gamma, M\}$ a *Kirchhoff elastic rod* if $\{\gamma, M\}$ is a critical point of the energy $\mathfrak{T}$ with respect to the variations which preserve the frames $(\gamma(t), (T(t), M(t)))$ at the both end points. More precisely, a Kirchhoff elastic rod is defined to be a solution of the associated Euler-Lagrange equation, whose derivation is discussed in detail in Section 2 of [**13**].

Definition 2.1. A unit-speed curve with adapted orthonormal frame $\{\gamma, M\}$ is called a Kirchhoff elastic rod if the following equations hold for some real constants a and μ.

$$\nabla_T\big[2(\nabla_T)^2 T + (3|\nabla_T T|^2 - \mu + 2G + 2\nu a^2)T - 4\nu a T \times \nabla_T T\big] = 0,$$
$$\langle \nabla_T^{\perp} M_1, T \times M_1 \rangle = a.$$

The constant a is uniquely determined for each $\{\gamma, M\}$, and is called the *twist rate* of $\{\gamma, M\}$. Except the case where γ is a geodesic, the constant μ is also uniquely determined, and is called the *Lagrange multiplier* of $\{\gamma, M\}$. In the rest of the paper, we assume that the centerline of a Kirchhoff elastic rod is not a geodesic. (A Kirchhoff elastic rod whose centerline is a geodesic is a relatively trivial object. For details, see page 212 of [**13**].) Note also that a Kirchhoff elastic rod $\{\gamma, M\}$ is real analytic in t.

For a while, we assume the curvature of γ is positive everywhere. By writing down the first equation of Definition 2.1 in terms of the Frenet frame (T, N, B) along γ, we obtain the following equations of the curvature k and torsion τ of γ.

$$2k'' + k^3 + (2\nu a^2 - \mu + 2G)k - 2k\tau(\tau - 2\nu a) = 0, \tag{2.1}$$
$$k^2(\tau - \nu a) = b, \tag{2.2}$$

where b is a constant. Using the substitution $\tau = b/k^2 + \nu a$ and multiplication by k' and integration, we obtain

$$(k')^2 + \frac{k^4}{4} + \frac{1}{2}(2\nu a^2 - \mu + 2G + 2\nu^2 a^2)k^2 + \frac{b^2}{k^2} = c, \tag{2.3}$$

where c is a constant.

Let $\mathrm{sn}(x,p)$, $\mathrm{cn}(x,p)$, $\mathrm{dn}(x,p)$ and $K(p)$ denote Jacobi sn, cn, dn functions and the complete elliptic integral of the first kind, respectively (cf. [**5**], [**20**]). The solution $k(t)$ of (2.3) is expressed by Jacobi sn function, and the space of all congruence classes of Kirchhoff elastic rods, including the case where γ has inflection points, is parametrized by four real numbers (Proposition 3.1 of [**13**]). For the proof, see pages 212–215 of [**13**].

In this paper, instead of the parameter $(a_+, \alpha_1, \alpha_2, \alpha_3)$ in Proposition 3.1 of [**13**], we use another parameter (α, η, p, w), which is defined as follows:

$$\alpha = \alpha_3, \quad \eta = \frac{a_+}{\sqrt{\alpha_3}}, \quad p = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1}}, \quad w = \sqrt{\frac{\alpha_3}{\alpha_3 + \alpha_1}}.$$

Then Proposition 3.1 of [**13**] is rewritten as the following proposition. (As for the definition of a congruence class of unit-speed curves with adapted orthonormal frames, see pages 211–212 of [**13**].)

Proposition 2.2. *The space of all congruence classes of Kirchhoff elastic rods* (*except geodesics*) *defined on* $\boldsymbol{R}$ *in* $\mathscr{M} = \boldsymbol{R}^3$, S^3 *or* H^3 *corresponds to the parameter space* $\mathscr{P} = \tilde{\mathscr{P}}/\sim$, *where*

$$\tilde{\mathscr{P}} = \{(\alpha, \eta, p, w)\ ;\ \alpha > 0,\ -\infty < \eta < \infty,\ 0 \le p \le w \le 1,\ w \ne 0\} (\subset \boldsymbol{R}^4),$$

and the equivalence relation $\sim$ *is defined as follows*: *If* $p = w$ *or* $w = 1$, *then* $(\alpha, \eta, p, w) \sim (\alpha, -\eta, p, w)$.

An element $[(\alpha, \eta, p, w)]$ *of* $\mathscr{P}$ *corresponds to the congruence class of Kirchhoff elastic rods with twist rate* $\pm\eta\sqrt{\alpha}$, *whose curvature* $k(t)$ *and torsion* $\tau(t)$ *are expressed as follows*:

$$k(t) = \sqrt{\alpha(1 - q^2 \,\mathrm{sn}^2(y(t - t_0), p))}, \tag{2.4}$$

$$\tau(t) = \pm\left(\frac{\alpha^{3/2}\sqrt{(1 - w^2)(w^2 - p^2)}}{2w^2 k(t)^2} + \nu\eta\sqrt{\alpha}\right), \tag{2.5}$$

where $q = p/w$, $y = \sqrt{\alpha}/(2w)$ *and* $t_0 \in \boldsymbol{R}$. *Also, the double sign of* $\eta\sqrt{\alpha}$ *and that of the right hand side of* (2.5) *are in the same order.*

We note that the parameters μ, a and b are expressed by (α, η, p, w) as follows:

$$\mu = \frac{\alpha}{2w^2}[-Y + 2w^2(1 + 2\nu\eta^2)], \quad a = \pm\eta\sqrt{\alpha}, \quad b = \pm\frac{\alpha^{3/2}}{2w^2}VX, \tag{2.6}$$

where

$$V = \sqrt{1 - w^2}, \quad X = \sqrt{w^2 - p^2}, \quad Y = 1 + p^2 - (1 + 4\nu^2\eta^2)w^2 - 4Gw^2/\alpha. \tag{2.7}$$

The expressions (2.6) follow from (3.13), (3.14) and (3.15) of [**13**]. For details, see the proof of Proposition 3.1 of [**13**]. The expression of the parameter c in terms of (α, η, p, w) can be obtained as well, but we omit it, because we need not use it below.

We give some relations between (α, η, p, w) and the shape of γ. If the elliptic modulus $p = 0$, then γ is a helix, that is, both k and τ are constant. If $0 < p < 1$, then k is a periodic function with primitive period $2K(p)/y$, which attains the maximum (resp. minimum) value $\sqrt{\alpha}$ (resp. $\sqrt{\beta}$) precisely when $t = 2mK(p)/y + t_0$ (resp. $(2m+1)K(p)/y + t_0$), where $\beta = \alpha(w^2 - p^2)/w^2$ and m is an arbitrary integer. Also, τ is a periodic function with primitive period $2K(p)/y$ or a constant function. If $p = 1$ (which implies $w = 1$), then $k = \sqrt{\alpha}\,\mathrm{sech}(y(t - t_0))$, which is not periodic and attains the maximum value $\sqrt{\alpha}$ at t_0, and converges to the infimum value $\sqrt{\beta}$ $(= 0)$ as $t \to \pm\infty$. In this case, $\tau = \pm\nu\eta\sqrt{\alpha}$.

Also, γ has inflection points if and only if $p = w \neq 1$. In this case, $k = \sqrt{\alpha}|\mathrm{cn}(y(t - t_0), p)|$, which vanishes precisely when $t = (2m + 1)K(p)/y + t_0$ $(m \in \boldsymbol{Z})$, and $\tau = \pm\nu\eta\sqrt{\alpha}$ except at the periodic inflection points.

3. Construction of Killing vector fields.

In this section, we construct two commuting Killing vector fields associated to a Kirchhoff elastic rod (Lemmas 3.2 and 3.4), which will be used in the following sections to construct a system of coordinates and obtain explicit formulas of the components of the centerline of the Kirchhoff elastic rod.

In the rest of the paper, let $\{\gamma, M\}$ be a Kirchhoff elastic rod in $\mathscr{M}$. Without loss of generality, we may assume that t_0 in (2.4) is zero. Now, we define two vector fields J and H along γ by setting

$$\begin{aligned} J &= 2(\nabla_T)^2 T + (3|\nabla_T T|^2 - \mu + 2\nu a^2)T - 4\nu a T \times \nabla_T T, \\ H &= 2\nu a T + T \times \nabla_T T. \end{aligned}$$

Note that in terms of the Frenet frame along γ, these are expressed as follows:

$$J = (k^2 - \mu + 2\nu a^2)T + 2k'N + 2k(\tau - 2\nu a)B, \tag{3.1}$$

$$H = 2\nu aT + kB. \tag{3.2}$$

Before stating the main claims of this section, we give some useful formulas with respect to J and H. By using the formula $X_1 \times (X_2 \times X_3) = \langle X_1, X_3\rangle X_2 - \langle X_1, X_2\rangle X_3$, where X_1, X_2 and X_3 are tangent vectors at a point in $\mathscr{M}$, we see

$$\nabla_T(-T) = T \times H, \tag{3.3}$$

$$\nabla_T H = T \times \frac{1}{2}J. \tag{3.4}$$

We should mention that the relations (3.3) and (3.4) correspond to the $n = 1, 2$ cases of the filament model recursion scheme (1) of [**17**] starting with $-T$. Vector fields $-T$, H and $(1/2)J$ are the first three of the sequence derived from this scheme.

Since the first equation of Definition 2.1 is equivalent to $\nabla_T J = -2G\nabla_T T$, it follows from (3.3) that

$$\nabla_T J = 2GT \times H. \tag{3.5}$$

Using these formulas, we obtain the following two first integrals, which are viewed as the space form versions of the $n = 3$ case of (11) of [**17**].

Proposition 3.1. *The functions $\langle J, H\rangle$ and $|J|^2 + 4G|H|^2$ are constant.*

Proof. By (3.4) and (3.5), it follows that

$$\frac{d}{dt}\langle J, H\rangle = 2G\langle T \times H, H\rangle + \frac{1}{2}\langle J, T \times J\rangle = 0,$$
$$\frac{d}{dt}(|J|^2 + 4G|H|^2) = 4G\langle T \times H, J\rangle + 4G\langle T \times J, H\rangle = 0. \qquad \square$$

We state the main claims of this section, that is, Lemmas 3.2 and 3.4.

Lemma 3.2 (Proposition 4.1 of [**13**]). *The vector fields J, H along γ extend uniquely to Killing vector fields on $\mathscr{M}$.*

The key lemma of the proof of Lemma 3.2 is the following:

Lemma 3.3 ([**18**], [**20**]). *Let $\mathscr{M}$ be $\boldsymbol{R}^3$, S^3, or H^3 of constant sectional curvature G. Let $\gamma = \gamma(t)$ be a unit-speed C^∞ curve in $\mathscr{M}$ whose curvature $k(t)$*

is positive everywhere. Let Λ be a C^∞ vector field along γ. Then Λ extends to a Killing vector field on $\mathscr{M}$ if and only if Λ satisfies the following system of differential equations.

$$\langle \nabla_T \Lambda, T\rangle = 0,$$
$$\big\langle (\nabla_T)^2 \Lambda + G\Lambda, N\big\rangle = 0,$$
$$\left\langle (\nabla_T)^3 \Lambda - \frac{k'}{k}(\nabla_T)^2\Lambda + (G+k^2)\nabla_T\Lambda - \frac{k'}{k}G\Lambda, B\right\rangle = 0,$$

where (T, N, B) is the Frenet frame along γ. Moreover, the Killing vector field is uniquely determined. (Such a vector field Λ is said to be a Killing vector field along γ.)

By using Lemma 3.3 and the Euler-Lagrange equations (2.1) and (2.2), we can prove Lemma 3.2, but we omit it. For details, see [**13**].

LEMMA 3.4. *Let $\tilde{J}$, $\tilde{H}$ denote the unique extensions of J, H as Killing vector fields on $\mathscr{M}$, respectively. Then, $[\tilde{J}, \tilde{H}] = 0$.*

PROOF. The set of all zeros of a Killing vector field on $\mathscr{M}$ is either the empty set, the whole $\mathscr{M}$, or one geodesic in $\mathscr{M}$. Thus it is sufficient to verify that $[\tilde{J}, \tilde{H}] = 0$ on γ, because the image of γ is not contained in any geodesic. Since $[\tilde{J}, \tilde{H}] = \nabla_{\tilde{J}}\tilde{H} - \nabla_{\tilde{H}}\tilde{J}$, it is sufficient to show that $\nabla_{\tilde{H}}\tilde{J} = 0$ and $\nabla_{\tilde{J}}\tilde{H} = 0$ on γ.

First, we show $\nabla_{\tilde{H}}\tilde{J} = 0$ on γ. Since $\tilde{J}$ is not expressed explicitly except at the points on γ, it is difficult to compute $\nabla_{\tilde{H}}\tilde{J}$ directly. And so, we replace $\nabla_{\tilde{H}}\tilde{J}$ by another expression. Let φ^λ ($\lambda \in \boldsymbol{R}$) denote the one-parameter group of isometries generated by $\tilde{J}$, and φ^λ_* the differential map of φ^λ for each λ. We write $(\varphi^\lambda \circ \gamma)(t)$ as $\hat{\gamma}(\lambda, t) = \gamma^\lambda(t)$, and let $\hat{T}(\lambda, t) = \partial\hat{\gamma}/\partial t$ and $\hat{J}(\lambda, t) = \partial\hat{\gamma}/\partial\lambda\,(= \tilde{J}(\hat{\gamma}(\lambda, t)))$. We denote the induced connection by $\nabla^{\hat{\gamma}^{-1}T\mathscr{M}}$, and write $\nabla^{\hat{\gamma}^{-1}T\mathscr{M}}_{\partial/\partial t}$ and $\nabla^{\hat{\gamma}^{-1}T\mathscr{M}}_{\partial/\partial\lambda}$ as $\nabla_{\hat{T}}$ and $\nabla_{\hat{J}}$, respectively. Now, let $\hat{H}$ be the vector field along $\hat{\gamma}$ defined by $\hat{H}(\lambda, t) = 2\nu a\hat{T} + \hat{T} \times \nabla_{\hat{T}}\hat{T}$. The formulas obtained by replacing T, J and H in (3.3), (3.4) and (3.5) by $\hat{T}$, $\hat{J}$ and $\hat{H}$ are valid, because $\hat{T}(\lambda, t) = \varphi^\lambda_*(T(t))$, $\hat{J}(\lambda, t) = \varphi^\lambda_*(J(t))$ and $\hat{H}(\lambda, t) = \varphi^\lambda_*(H(t))$.

Since $\tilde{H}$ coincides with $\hat{H}$ on γ, it suffices to calculate $\nabla_{\hat{H}}\tilde{J}$. Also, since $\hat{H}$ is invariant under the flow φ^λ, $\nabla_{\hat{H}}\tilde{J} = \nabla_{\hat{J}}\hat{H}$ holds. Hence it suffices to show $\nabla_{\hat{J}}\hat{H} = 0$. We write $\hat{T}$, $\hat{J}$, $\hat{H}$, $\nabla_{\hat{J}}$, etc. as T, J, H, ∇_J, etc., unless confusions could occur. We can verify that $\nabla_J T = \nabla_T J$ and

$$\nabla_J\nabla_T X = \nabla_T\nabla_J X + G(\langle T, X\rangle J - \langle J, X\rangle T), \tag{3.6}$$

where X is an arbitrary vector field along $\hat{\gamma}$. These formulas together with $\nabla_T J = -2G\nabla_T T$ and $\nabla_T H = \dfrac{1}{2}T \times J$ yield

$$\begin{aligned}\nabla_J H &= 2\nu a\nabla_T J + \nabla_T J \times \nabla_T T + T \times ((\nabla_T)^2 J + GJ) \\ &= -2G(2\nu a\nabla_T T + T \times (\nabla_T)^2 T) + GT \times J \\ &= -2G\left(\nabla_T H - \frac{1}{2}T \times J\right) = 0. \end{aligned} \tag{3.7}$$

Hence $\nabla_{\tilde{H}} \tilde{J} = 0$ on γ.

Next, we show $\nabla_{\tilde{J}} \tilde{H} = 0$ on γ. In the same way as above, we replace $\nabla_{\tilde{J}} \tilde{H}$ by another expression. Let ψ^λ ($\lambda \in \boldsymbol{R}$) be the one-parameter group of isometries generated by $\tilde{H}$. We write $(\psi^\lambda \circ \gamma)(t)$ as $\bar{\gamma}(\lambda, t) = \gamma^\lambda(t)$, and let $\bar{T}(\lambda, t) = \partial\bar{\gamma}/\partial t$, $\bar{H}(\lambda, t) = \partial\bar{\gamma}/\partial\lambda\,(= \tilde{H}(\bar{\gamma}(\lambda, t)))$, $\nabla_{\bar{T}} = \nabla^{\bar{\gamma}^{-1}T\mathscr{M}}_{\partial/\partial t}$ and $\nabla_{\bar{H}} = \nabla^{\bar{\gamma}^{-1}T\mathscr{M}}_{\partial/\partial \lambda}$. Let $\bar{J}$ be the vector field along $\bar{\gamma}$ defined by

$$\bar{J}(\lambda, t) = 2(\nabla_{\bar{T}})^2\bar{T} + \big(3|\nabla_{\bar{T}}\bar{T}|^2 - \mu + 2\nu a^2\big)\bar{T} - 4\nu a\bar{T} \times \nabla_{\bar{T}}\bar{T}.$$

Since $\tilde{J}$ coincides with $\bar{J}$ on γ, it suffices to calculate $\nabla_{\bar{J}}\tilde{H}$. Also, $\bar{J}(\lambda, t) = \psi^\lambda_*(J(t))$ yields $\nabla_{\bar{J}}\tilde{H} = \nabla_{\bar{H}}\bar{J}$. Thus, it suffices to show $\nabla_{\bar{H}}\bar{J} = 0$. By a calculation similar to that of $\nabla_{\hat{J}}\hat{H}$, we obtain

$$\begin{aligned}\nabla_{\bar{H}}\bar{J} =\,& 2(\nabla_T)^3 H + \big(3|\nabla_T T|^2 - \mu + 2G + 2\nu a^2\big)\nabla_T H \\ & - 4\nu a\big(\nabla_T H \times \nabla_T T + T \times (\nabla_T)^2 H + GT \times H + G\nabla_T T\big)\end{aligned}$$

on γ. It is sufficient to show that the right hand side is equal to zero except at the inflection points of γ. By using $(k^2(\tau - \nu a))' = k(2k'\tau + k\tau' - 2\nu a k') = 0$, we see that the right hand side of the above expression becomes

$$\begin{aligned}&- (\tau - 2\nu a)[2k'' + k^3 + (2\nu a^2 - \mu + 2G)k - 2k\tau(\tau - 2\nu a)]N \\ &+ [2k'' + k^3 + (2\nu a^2 - \mu + 2G)k - 2k\tau(\tau - 2\nu a)]'B,\end{aligned}$$

which is equal to zero by (2.1). Hence $\nabla_{\tilde{J}}\tilde{H} = 0$ on γ, which completes the proof. □

For a later convenience, we need the following lemma, which follows from Lemma 3.3 together with a straightforward calculation.

Lemma 3.5. *Let $\{\gamma, M\}$ be a Kirchhoff elastic rod such that γ is a helix. Then the tangent vector T of γ extends uniquely to a Killing vector field on $\mathscr{M}$.*

4. Construction of coordinates.

In this section, we construct a system of coordinates suitable for a Kirchhoff elastic rod $\{\gamma, M\}$ by using the associated commuting Killing vector fields $\tilde{J}$ and $\tilde{H}$ constructed in the previous section.

Let

$$\varepsilon = \begin{cases} 0 & \text{if } \mathscr{M} = \boldsymbol{R}^3, \\ 1 & \text{if } \mathscr{M} = S^3, \\ -1 & \text{if } \mathscr{M} = H^3. \end{cases}$$

We embed $\mathscr{M}$ isometrically into $\boldsymbol{R}^4 = \{{}^t(x_1, x_2, x_3, x_4);\, x_1, x_2, x_3, x_4 \in \boldsymbol{R}\}$ with the Euclidean metric as

$$\begin{array}{ll} \{{}^t(x_1, x_2, x_3, 1)\,;\, x_1, x_2, x_3 \in \boldsymbol{R}\} & \text{if } \mathscr{M} = \boldsymbol{R}^3, \\ \{{}^t(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4\,;\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1/G\} & \text{if } \mathscr{M} = S^3. \end{array}$$

Also, we embed $\mathscr{M}$ isometrically into $\boldsymbol{R}^4$ with the Lorentzian metric $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$ as the hyperboloid

$$\{{}^t(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4\,;\, x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1/G,\, x_4 > 0\} \quad \text{if } \mathscr{M} = H^3.$$

We denote by $\iota : \mathscr{M} \to \boldsymbol{R}^4$ the above isometric embedding, and we often identify $\mathscr{M}$ with $\iota(\mathscr{M})$. The Euclidean or Lorentzian metric on $\boldsymbol{R}^4$ and the Riemannian metric on $\mathscr{M}$ are denoted by the same notation $\langle\ ,\ \rangle$. Let

$$I(\mathscr{M}) = \begin{cases} E(3) & \text{if } \mathscr{M} = \boldsymbol{R}^3, \\ O(4) & \text{if } \mathscr{M} = S^3, \\ O^+(3,1) & \text{if } \mathscr{M} = H^3, \end{cases} \quad \mathrm{Lie}(I(\mathscr{M})) = \begin{cases} \mathfrak{e}(3) & \text{if } \mathscr{M} = \boldsymbol{R}^3, \\ \mathfrak{o}(4) & \text{if } \mathscr{M} = S^3, \\ \mathfrak{o}(3,1) & \text{if } \mathscr{M} = H^3, \end{cases}$$

where

$$E(3) = \left\{ \begin{pmatrix} R & \boldsymbol{b} \\ 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R});\ R \in O(3),\, \boldsymbol{b} \in \boldsymbol{R}^3 \right\},$$

$$O^+(3,1) = \{P \in GL(4,\boldsymbol{R})\,;\, {}^tPBP = B,\ P_{44} \geq 1\},$$

$$\mathfrak{e}(3) = \left\{ \begin{pmatrix} A & \boldsymbol{b} \\ 0 & 0 \end{pmatrix} \in M(4,\boldsymbol{R})\,;\, A \in \mathfrak{o}(3), \boldsymbol{b} \in \boldsymbol{R}^3 \right\},$$

$$\mathfrak{o}(3,1) = \left\{ \begin{pmatrix} A & \boldsymbol{b} \\ {}^t\boldsymbol{b} & 0 \end{pmatrix} \in M(4,\boldsymbol{R})\,;\, A \in \mathfrak{o}(3), \boldsymbol{b} \in \boldsymbol{R}^3 \right\},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and $O(4)$ is the Lie group of all 4×4 orthogonal matrices and $\mathfrak{o}(4)$ is the Lie algebra of all 4×4 skew-symmetric matrices. That is, $I(\mathscr{M})$ is the isometry group of $\mathscr{M}$, and $\mathrm{Lie}(I(\mathscr{M}))$ is its Lie algebra. For a later use, we define $E_1,\ E_2 \in \mathrm{Lie}(I(\mathscr{M}))$ and $F_1,\ F_2 \in \mathfrak{o}(3,1)$ as follows:

$$E_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \varepsilon & 0 \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For any Killing vector field Y on $\mathscr{M}$, there exists a unique 4×4 matrix $A_Y \in \mathrm{Lie}(I(\mathscr{M}))$, called the *matrix representation of Y with respect to ι*, satisfying $(\iota_* Y)(\boldsymbol{x}) = A_Y\boldsymbol{x}$, where $\boldsymbol{x} = {}^t(x_1,x_2,x_3,x_4) \in \iota(\mathscr{M})$. Note that if $P \in I(\mathscr{M})$, then the matrix representation of Y with respect to $P\circ\iota$ is equal to PAP^{-1}.

Let ${}^t(x_1,x_2,x_3,x_4) \in \iota(\mathscr{M})$. We define a coordinate system (r,θ,ψ), called the *cylindrical coordinates with respect to ι*, on $\mathscr{M} = \boldsymbol{R}^3$, S^3, H^3 by the following relations:

In the case of $\boldsymbol{R}^3$, $\quad x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = -\psi.$

In the case of S^3, $\quad x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = \bar{r}\cos\psi,\quad x_4 = \bar{r}\sin\psi.$

In the case of H^3, $\quad x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = -\bar{r}\sinh\psi,\ x_4 = \bar{r}\cosh\psi.$

Here, $r > 0$ in the case of $\boldsymbol{R}^3$ or H^3, and $0 < r < 1/\sqrt{G}$ in the case of S^3. Also, $\bar{r} = \sqrt{\varepsilon(1/G - r^2)}$. The coordinate fields $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\psi$ are orthogonal. Note that $\partial/\partial\theta$ (resp. $\partial/\partial\psi$) is not defined on the geodesic $r = \sqrt{x_1^2 + x_2^2} = 0$ (resp. $\bar{r} = \sqrt{\varepsilon x_3^2 + x_4^2} = 0$) in the case of $\boldsymbol{R}^3$, S^3, H^3 (resp. S^3), but naturally extends to a smooth vector field on the whole $\mathscr{M}$, which is also denoted by the same notation $\partial/\partial\theta$ (resp. $\partial/\partial\psi$). Then $\partial/\partial\theta$ (resp. $\partial/\partial\psi$) corresponds to the Killing vector field whose matrix representation with respect to ι is E_1 (resp. E_2). In the case of $\mathscr{M} = S^3$, H^3, the following holds:

$$\begin{aligned}
&\left|\frac{\partial}{\partial r}\right| = \frac{1}{\sqrt{\varepsilon G}\,\bar{r}}, \quad \left|\frac{\partial}{\partial \theta}\right| = r, \quad \left|\frac{\partial}{\partial \psi}\right| = \bar{r}, \\
&\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = -\varepsilon G r \bar{r}^2 \frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\psi} = \nabla_{\frac{\partial}{\partial\psi}}\frac{\partial}{\partial\theta} = 0, \quad \nabla_{\frac{\partial}{\partial\psi}}\frac{\partial}{\partial\psi} = G r \bar{r}^2 \frac{\partial}{\partial r}, \\
&\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \nabla_{\frac{\partial}{\partial\psi}}\frac{\partial}{\partial r} = \frac{-\varepsilon r}{\bar{r}^2}\frac{\partial}{\partial\psi}.
\end{aligned} \tag{4.1}$$

When $\mathscr{M} = H^3$, we also need another coordinate system. We consider the upper half-space $\boldsymbol{R}^3_+ = \{{}^t(w_1, w_2, w_3) \in \boldsymbol{R}^3;\ w_3 > 0\}$ with the Poincaré metric $(dw_1^2 + dw_2^2 + dw_3^2)/(-Gw_3^2)$. Let w_j $(j = 1, 2, 3)$ be the functions on $\iota(H^3)$ into $\boldsymbol{R}^3_+$ defined by

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{x_4 - x_3}\begin{pmatrix} x_1 \\ x_2 \\ 1/\sqrt{-G} \end{pmatrix},$$

where ${}^t(x_1, x_2, x_3, x_4) \in \iota(H^3)$. Then, the map ${}^t(w_1, w_2, w_3) : \iota(H^3) \to \boldsymbol{R}^3_+$ is an isometry between $\iota(H^3)$ and $\boldsymbol{R}^3_+$. By identifying H^3 with $\iota(H^3)$, we can view (w_1, w_2, w_3) as a coordinate system on the whole H^3, called the *upper half-space coordinates with respect to* ι. The coordinate field $\partial/\partial w_1$ (resp. $\partial/\partial w_2$) on H^3 coincides with the Killing vector field whose matrix representation with respect to ι is F_1 (resp. F_2).

In what follows, we give an appropriate $P \in I(\mathscr{M})$ and take the cylindrical or upper half-space coordinates with respect to the isometric embedding $P \circ \iota$. First, we consider the transformation of an element of $\mathrm{Lie}(I(\mathscr{M}))$ into the canonical form. We can check the following lemma, whose proof is omitted.

LEMMA 4.1.

(1) *If* $A \in \mathfrak{e}(3)$ (*resp.* $\mathfrak{o}(4)$), *then there exist* $P \in E(3)$ (*resp.* $O(4)$) *and* σ_1,

$\sigma_2 \in \boldsymbol{R}$ *satisfying* $PAP^{-1} = \sigma_1 E_1 + \sigma_2 E_2$.

(2) *If* $A \in \mathfrak{o}(3,1)$, *then the only one of the following* (i) *and* (ii) *holds*:

(i) *There exist* $P \in O^+(3,1)$ *and* σ_1, $\sigma_2 \in \boldsymbol{R}$ *satisfying* $PAP^{-1} = \sigma_1 E_1 + \sigma_2 E_2$.

(ii) *There exists* $P \in O^+(3,1)$ *satisfying* $PAP^{-1} = F_1$.

A matrix $A \in \mathfrak{o}(3,1)$ is said to be *semi-simple* (resp. *parabolic*) if (i) (resp. (ii)) holds. It should be noted that through the Lie algebra isomorphism between $\mathfrak{sl}(2,\boldsymbol{C})$ and $\mathfrak{o}(3,1)$ derived from the spinor map $SL(2,\boldsymbol{C}) \to SO^+(3,1)$, a semi-simple (resp. parabolic) element of $\mathfrak{o}(3,1)$ corresponds to a diagonalizable (resp. non-diagonalizable) element of $\mathfrak{sl}(2,\boldsymbol{C})$. For details about the spinor map, see [**24**].

In this paper, a Killing vector field Y on H^3 is said to be *semi-simple* (resp. *parabolic*) if the matrix representation of Y with respect to ι is semi-simple (resp. parabolic). (It is clear that if Y is semi-simple (resp. parabolic), then the matrix representation of Y with respect to $P \circ \iota$ is also semi-simple (resp. parabolic) for any $P \in I(\mathscr{M})$.) We can check that if a Killing vector field Y is parabolic, then any integral curve of Y is a horocycle, that is, a curve with curvature $\sqrt{-G}$ and torsion 0. Also, if Y is semi-simple, then any integral curve of Y is a helix which is not a horocycle.

Now, Lemma 4.1 immediately yields the following: If $\mathscr{M} = \boldsymbol{R}^3$, S^3 or $\mathscr{M} = H^3$ and $\tilde{J}$ is semi-simple, then there exist $P \in I(\mathscr{M})$ and σ_1, $\sigma_2 \in \boldsymbol{R}$ such that the matrix representation of $\tilde{J}$ with respect to $P \circ \iota$ is $\sigma_1 E_1 + \sigma_2 E_2$. However, due to Lemma 3.4, we have the following stronger assertion.

Proposition 4.2. *Suppose that* $\mathscr{M} = \boldsymbol{R}^3$, S^3 *or* $\mathscr{M} = H^3$ *and* $\tilde{J}$ *is semi-simple. Then there exist* $P \in I(\mathscr{M})$ *and* σ_1, σ_2, ρ_1, $\rho_2 \in \boldsymbol{R}$ *such that the matrix representations of* $\tilde{J}$ *and* $\tilde{H}$ *with respect to* $P \circ \iota$ *are* $\sigma_1 E_1 + \sigma_2 E_2$ *and* $\rho_1 E_1 + \rho_2 E_2$, *respectively. In particular, if* $\mathscr{M} = H^3$ *and* $\tilde{J}$ *is semi-simple, then* $\tilde{H}$ *is again semi-simple.*

Proof. We first consider the case where $\mathscr{M} = H^3$ and $\tilde{J}$ is semi-simple. If $\tilde{J} = 0$, then the assertion is obvious. We assume that $\tilde{J} \neq 0$. Let $A_{\tilde{J}}$ and $A_{\tilde{H}}$ denote the matrix representations of $\tilde{J}$ and $\tilde{H}$ with respect to ι, respectively. Then there exist $P \in I(\mathscr{M})$ and σ_1, $\sigma_2 \in \boldsymbol{R}$ such that the matrix representation of $\tilde{J}$ with respect to $P \circ \iota$ is $PA_{\tilde{J}}P^{-1} = \sigma_1 E_1 + \sigma_2 E_2$. Set

$$PA_{\tilde{H}}P^{-1} = \begin{pmatrix} 0 & -h_1 & -h_2 & h_4 \\ h_1 & 0 & -h_3 & h_5 \\ h_2 & h_3 & 0 & h_6 \\ h_4 & h_5 & h_6 & 0 \end{pmatrix}, \tag{4.2}$$

where $h_1, \ldots, h_6 \in \boldsymbol{R}$. It suffices to show that $h_2 = h_3 = h_4 = h_5 = 0$. Now, it follows from Lemma 3.4 that $(PA_{\tilde{J}}P^{-1})(PA_{\tilde{H}}P^{-1}) = (PA_{\tilde{H}}P^{-1})(PA_{\tilde{J}}P^{-1})$. Thus, a straightforward calculation yields

$$\begin{pmatrix} h_3 & h_4 \\ h_4 & -h_3 \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -h_5 & -h_2 \\ -h_2 & h_5 \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(\sigma_1, \sigma_2) \neq (0, 0)$, we obtain $h_2 = h_3 = h_4 = h_5 = 0$.

Next we consider the case where $\mathscr{M} = S^3$. Suppose that γ is not a helix. There exist $P \in I(\mathscr{M})$ and ρ_1, $\rho_2 \in \boldsymbol{R}$ such that the matrix representation of $\tilde{H}$ with respect to $P \circ \iota$ is $PA_{\tilde{H}}P^{-1} = \rho_1 E_1 + \rho_2 E_2$. Since γ is not a helix, $|H|$ is not a constant function, and hence $|\rho_1| \neq |\rho_2|$. Set

$$PA_{\tilde{J}}P^{-1} = \begin{pmatrix} 0 & -j_1 & -j_2 & -j_4 \\ j_1 & 0 & -j_3 & -j_5 \\ j_2 & j_3 & 0 & -j_6 \\ j_4 & j_5 & j_6 & 0 \end{pmatrix},$$

where $j_1, \ldots, j_6 \in \boldsymbol{R}$. It suffices to prove $j_2 = j_3 = j_4 = j_5 = 0$. By a similar calculation to that of the case of H^3, we see

$$\begin{pmatrix} \rho_1 & \rho_2 \\ \rho_2 & \rho_1 \end{pmatrix}\begin{pmatrix} j_3 & -j_2 \\ j_4 & j_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from $|\rho_1| \neq |\rho_2|$ that $j_2 = j_3 = j_4 = j_5 = 0$.

Suppose that γ is a helix. By Lemma 3.5, the tangent vector T extends uniquely to a Killing vector field $\tilde{T}$ on $\mathscr{M}$. Thus, there exist $P \in I(\mathscr{M})$ and ξ_1, $\xi_2 \in \boldsymbol{R}$ such that the matrix representation of $\tilde{T}$ with respect to $P \circ \iota$ is $\xi_1 E_1 + \xi_2 E_2$. Let (r, θ, ψ) be the cylindrical coordinates with respect to $P \circ \iota$. Then, $\tilde{T} = \xi_1(\partial/\partial\theta) + \xi_2(\partial/\partial\psi)$. It suffices to prove that both of $\tilde{J}$ and $\tilde{H}$ are linear combinations of $\partial/\partial\theta$ and $\partial/\partial\psi$. To prove this, we express the Frenet frame along γ by $\gamma^*(\partial/\partial r)$, $\gamma^*(\partial/\partial\theta)$ and $\gamma^*(\partial/\partial\psi)$, where γ^* denotes the pull-back by γ. For a later convenience in the appendix, we give a calculation valid for both cases of $\mathscr{M} = S^3$ and H^3. Let $r(t)$, $\theta(t)$ and $\psi(t)$ be the r, θ and ψ components of $\gamma(t)$. Since $T = \xi_1\gamma^*(\partial/\partial\theta) + \xi_2\gamma^*(\partial/\partial\psi)$, we obtain

$$r(t) = r_0, \quad \theta'(t) = \xi_1, \quad \psi'(t) = \xi_2, \tag{4.3}$$

where r_0 is a constant satisfying $r_0 \neq 0$ and $\bar{r}_0 := \sqrt{\varepsilon(1/G - r_0^2)} \neq 0$. Using (4.1), we get

$$\nabla_T T = -(\xi_1^2 - \varepsilon\xi_2^2)\varepsilon G r_0 \bar{r}_0^2 \, \gamma^* \frac{\partial}{\partial r}, \quad \sqrt{\alpha} = \sqrt{\varepsilon G}\, r_0 \bar{r}_0 |\xi_1^2 - \varepsilon\xi_2^2|. \tag{4.4}$$

Set $\delta = 1$ (resp. -1) if $\xi_1^2 - \varepsilon\xi_2^2$ is positive (resp. negative). Then,

$$N = -\delta\sqrt{\varepsilon G}\, \bar{r}_0 \, \gamma^* \frac{\partial}{\partial r}, \quad B = \mp\delta\left(\frac{\xi_2 \bar{r}_0}{r_0}\, \gamma^* \frac{\partial}{\partial \theta} - \frac{\xi_1 r_0}{\bar{r}_0}\, \gamma^* \frac{\partial}{\partial \psi}\right), \tag{4.5}$$

where the upper (resp. lower) sign is taken when the orientation of the frame $(\partial/\partial r, \partial/\partial\theta, \partial/\partial\psi)$ is positive (resp. negative). Hence (3.1) and (3.2) yield that J (resp. H) is a linear combination of $\gamma^*(\partial/\partial\theta)$ and $\gamma^*(\partial/\partial\psi)$. Since $\tilde{J}$ (resp. $\tilde{H}$) is the unique extension of J (resp. H) as a Killing vector field on $\mathscr{M}$, we see that $\tilde{J}$ (resp. $\tilde{H}$) is a linear combination of $\partial/\partial\theta$ and $\partial/\partial\psi$.

Finally, we consider the case where $\mathscr{M} = \boldsymbol{R}^3$. In this case, the first equation of Definition 2.1 yields $\nabla_T J = 0$, and so $\tilde{J}$ is a constant vector field. Now, suppose that $\tilde{H}$ is not a constant vector field. Then there exist $P \in I(\mathscr{M})$ and $\rho_1 (\neq 0), \rho_2 \in \boldsymbol{R}$ such that the matrix representation of $\tilde{H}$ with respect to $P \circ \iota$ is $PA_{\tilde{H}}P^{-1} = \rho_1 E_1 + \rho_2 E_2$. Since $\tilde{J}$ is a constant vector field, $PA_{\tilde{J}}P^{-1}$ is expressed as follows:

$$PA_{\tilde{J}}P^{-1} = \begin{pmatrix} 0 & 0 & 0 & j_4 \\ 0 & 0 & 0 & j_5 \\ 0 & 0 & 0 & j_6 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $j_4, j_5, j_6 \in \boldsymbol{R}$. By $\rho_1 \neq 0$ and a similar calculation to those of the cases of H^3, S^3, we obtain $j_4 = j_5 = 0$, and hence $PA_{\tilde{J}}P^{-1} = -j_6 E_2$.

Suppose that $\tilde{H}$ is a constant vector field. We assume that $\tilde{J}$ and $\tilde{H}$ are linearly independent. Since $|H|^2$ is constant, γ is a helix. By (3.1) and (3.2) together with the assumption that $\tilde{J}$ and $\tilde{H}$ are linearly independent, we see T is expressed as a linear combination of J and H. Thus $\nabla_T T = 0$, which contradicts the assumption that γ is not a geodesic. Consequently, $\tilde{J}$ and $\tilde{H}$ are constant vector fields which are linearly dependent. Hence the assertion is obvious. This completes the proof. □

Before examining the case where $\mathscr{M} = H^3$ and $\tilde{J}$ is parabolic, we investigate another exceptional case. In this case, in spite of $\mathscr{M} = \boldsymbol{R}^3$ or S^3, we need to take

$P \in I(\mathscr{M})$ in a different way from Proposition 4.2 (see Case 3 in page 570). We set

$$f = 2\langle J, H\rangle, \quad h = |J|^2 + 4G|H|^2, \tag{4.6}$$

which are constants independent of t by Proposition 3.1.

LEMMA 4.3. *The following are equivalent.*

(1) $J \times H = 0$ *for all* $t \in \boldsymbol{R}$.
(2) $\mathscr{M} = \boldsymbol{R}^3$, S^3, γ *is a helix and* $(\tau - 2\nu a)^2 = G$.
(3) $\mathscr{M} = \boldsymbol{R}^3$, S^3, $p = 0$ *and*

$$\eta = \frac{1}{\nu}\left(\frac{V}{2w} + \sqrt{\frac{G}{\alpha}}\right) \quad \text{or} \quad \eta = \frac{1}{\nu}\left(\frac{V}{2w} - \sqrt{\frac{G}{\alpha}}\right).$$

(4) $\mathscr{M} = \boldsymbol{R}^3$, S^3 *and* $h^2 - 4Gf^2 = 0$.

PROOF. First, we show that (1) and (2) are equivalent. Since

$$\langle J \times H, T\rangle = \langle T \times J, H\rangle = \langle 2\nabla_T H, H\rangle = (|H|^2)' = (k^2)',$$

we see $J \times H \neq 0$ at a point where $(k^2)'(t) \neq 0$. Thus, if γ is not a helix, then $J \times H \neq 0$ everywhere except at periodic points or one point. Next, let γ be a helix. Noting that (2.1) yields $\mu = \alpha + 2\nu a^2 - 2\tau(\tau - 2\nu a) + 2G$, we have

$$J \times H = \sqrt{\alpha}[-(\alpha - \mu + 2\nu a^2) + 4\nu a(\tau - 2\nu a)]N = 2\sqrt{\alpha}(G - (\tau - 2\nu a)^2)N. \tag{4.7}$$

Thus, (1) holds if and only if γ is a helix and $(\tau - 2\nu a)^2 = G$. Hence (1) and (2) are equivalent. Also, by using (2.2) and (2.6), we see that (2) and (3) are equivalent.

We show that (1) and (4) are equivalent. By (4.6),

$$h^2 - 4Gf^2 = (4G|H|^2 - |J|^2)^2 + 16G|J \times H|^2. \tag{4.8}$$

Hence (1) follows from (4). Next, suppose that (1) holds. Then the above argument yields that $\mathscr{M} = \boldsymbol{R}^3$, S^3, γ is a helix and $\alpha - \mu + 2\nu a^2 = 4\nu a(\tau - 2\nu a)$. By (3.1) and (3.2), we have $J = 2(\tau - 2\nu a)H$, which implies $|J|^2 = 4G|H|^2$. Hence (4) holds. □

Finally, we examine the case where $\mathscr{M} = H^3$ and $\tilde{J}$ is parabolic. First, we express by f and h the condition that $\tilde{J}$ is parabolic.

Lemma 4.4. *Let $\mathscr{M} = H^3$. Then the Killing vector field $\tilde{J}$ is parabolic if and only if $f = h = 0$.*

Proof. First, we note that there exists at least one point $t \in \boldsymbol{R}$ such that $\langle J \times H, \nabla_T T \rangle \neq 0$. Indeed, let t_0 be a point satisfying $k(t_0) = k_{\max}(= \sqrt{\alpha})$, for example, let $t_0 = 0$. Then, by a calculation similar to (4.7),

$$\langle J \times H, \nabla_T T \rangle|_{t=t_0} = 2\alpha\left(\frac{k''(t_0)}{\sqrt{\alpha}} + G - (\tau(t_0) - 2\nu a)^2\right) < 0.$$

Let $\hat{\gamma}(\lambda, t)$, $\hat{J}$, $\hat{H}$, $\nabla_{\hat{J}}$, etc. be the same as in Lemma 3.4. We fix $t \in \boldsymbol{R}$ satisfying $\langle J \times H, \nabla_T T \rangle \neq 0$, and let $\Sigma : \boldsymbol{R} \to H^3$ be the curve defined by $\Sigma(\lambda) = \hat{\gamma}(\lambda, t)$. Since $\tilde{J}$ is a Killing vector field, Σ is a helix with constant speed $|J(t)|(> 0)$. The curvature k_Σ and torsion τ_Σ of Σ are calculated as follows:

$$k_\Sigma = \sqrt{-G\left(\frac{Gf^2}{|J|^4} - \frac{h}{|J|^2} + 1\right)} > 0, \tag{4.9}$$

$$\tau_\Sigma = \frac{-Gf}{|J|^2}. \tag{4.10}$$

(The proof of these expressions are written below.) If $\tilde{J}$ is parabolic, then Σ is a horocycle, that is, $k_\Sigma = \sqrt{-G}$ and $\tau_\Sigma = 0$. Hence (4.9) and (4.10) yield $f = h = 0$. Conversely, if $f = h = 0$, then Σ is a horocycle, and hence $\tilde{J}$ is parabolic.

Now, we show (4.9) and (4.10). In the same way as in Lemma 3.4, we write $\hat{J}$, $\hat{H}$, etc. as J, H, etc. First, we show $\nabla_J J = 2GJ \times H$. It follows from $\partial|J|^2/\partial\lambda = 0$ that $\langle \nabla_J J, J \rangle = 0$. Also, $\partial\langle J, H \rangle/\partial\lambda = 0$ and (3.7) yield $\langle \nabla_J J, H \rangle = -\langle J, \nabla_J H \rangle = 0$. Hence $\nabla_J J$ is parallel to $J \times H$. It follows from (3.6), (3.5) and (3.4) that

$$\begin{aligned}
\langle \nabla_J J, \nabla_T T \rangle &= -\langle J, \nabla_J \nabla_T T \rangle = -\langle J, \, (\nabla_T)^2 J + G(J - \langle J, T \rangle T) \rangle \\
&= -G\langle J, \, 2\nabla_T T \times H + 2T \times \nabla_T H + J - \langle J, T \rangle T \rangle \\
&= -G\langle J, 2\nabla_T T \times H \rangle = \langle 2GJ \times H, \nabla_T T \rangle.
\end{aligned}$$

Hence, by $\langle J \times H, \nabla_T T \rangle \neq 0$, we obtain $\nabla_J J = 2GJ \times H$. Thus,

$$k_\Sigma = \frac{2(-G)|J \times H|}{|J|^2} = \frac{-2G\sqrt{|J|^2|H|^2 - \langle J, H \rangle^2}}{|J|^2}.$$

Therefore, by using (4.6), we have (4.9). It follows from $J \times H \neq 0$ that $k_\Sigma > 0$ and τ_Σ is well-defined. Since $(\nabla_J)^2 J = 4G^2(J \times H) \times H$, we see that

$$\tau_\Sigma = \frac{\langle (\nabla_J)^2 J, J \times \nabla_J J\rangle}{(k_\Sigma)^2 |J|^6} = \frac{-2G\langle J, H\rangle}{|J|^2} = \frac{-Gf}{|J|^2}. \qquad \square$$

PROPOSITION 4.5. *Suppose that $\mathscr{M} = H^3$ and $\tilde{J}$ is parabolic. Then there exists $P \in I(\mathscr{M})$ such that the matrix representations of $\tilde{J}$ and $\tilde{H}$ with respect to $P \circ \iota$ are F_1 and $(1/(2\sqrt{-G}))F_2$, respectively. In particular, $\tilde{H}$ is again parabolic.*

PROOF. Since $\tilde{J}$ is parabolic, there exists $P \in I(\mathscr{M})$ such that the matrix representation of $\tilde{J}$ with respect to $P \circ \iota$ is F_1. Let the matrix representation of $\tilde{H}$ with respect to $P \circ \iota$ be as the right hand side of (4.2). By Lemma 3.4 and a straightforward calculation, we have $h_5 = h_3$, $h_4 = h_2$ and $h_1 = h_6 = 0$. Thus,

$$\begin{aligned} \big\langle \tilde{J}(\boldsymbol{x}), \tilde{H}(\boldsymbol{x})\big\rangle &= h_2(-x_3 + x_4)^2, \\ \big|\tilde{J}(\boldsymbol{x})\big|^2 + 4G\big|\tilde{H}(\boldsymbol{x})\big|^2 &= \big(1 + 4G(h_2^2 + h_3^2)\big)(-x_3 + x_4)^2, \end{aligned}$$

for $\boldsymbol{x} = {}^t(x_1, x_2, x_3, x_4) \in H^3$. It follows from Lemma 4.4 that $\langle \tilde{J}, \tilde{H}\rangle = 0$ and $|\tilde{J}|^2 + 4G|\tilde{H}|^2 = 0$ on γ. Hence $h_2 = 0$ and $h_3 = \pm 1/(2\sqrt{-G})$. In the case of $h_3 = -1/(2\sqrt{-G})$, we consider $P_1 \circ P$, where $P_1 (\in O^+(3,1))$ is the transformation sending ${}^t(x_1, x_2, x_3, x_4)$ to ${}^t(x_1, -x_2, x_3, x_4)$, and rewrite $P_1 \circ P$ as P. Then the matrix representations of $\tilde{J}$ and $\tilde{H}$ with respect to $P \circ \iota$ are equal to F_1 and $(1/(2\sqrt{-G}))F_2$, respectively.

Since $QF_2Q^{-1} = F_1$, where $Q \in O^+(3,1)$ is the transformation sending ${}^t(x_1, x_2, x_3, x_4)$ to ${}^t(x_2, x_1, x_3, x_4)$, F_2 is parabolic. Hence $\tilde{H}$ is parabolic. $\square$

Now, we introduce the system of coordinates suitable for $\{\gamma, M\}$. We note that by (4.8), $h^2 - 4Gf^2 \geq 0$ always holds.

THE COORDINATES SUITABLE FOR $\{\gamma, M\}$.

CASE 1. $h^2 - 4Gf^2 > 0$.

By Lemma 4.4, if $\mathscr{M} = H^3$, then $\tilde{J}$ is semi-simple. Thus, there exists $P \in I(\mathscr{M})$ as in Proposition 4.2. We take the cylindrical coordinates (r, θ, ψ) with respect to $P \circ \iota$.

CASE 2. $\mathscr{M} = H^3$ and $h^2 - 4Gf^2 = 0$.

By Lemma 4.4, this case corresponds to the case where $\mathscr{M} = H^3$ and $\tilde{J}$ is parabolic. We take the upper half-space coordinates (w_1, w_2, w_3) with respect to $P \circ \iota$, where P is as in Proposition 4.5.

Case 3. $\mathscr{M} = \boldsymbol{R}^3$, S^3 and $h^2 - 4Gf^2 = 0$.

It follows from Lemma 4.3 that γ is a helix. By Lemma 3.5 and Lemma 4.1, there exist $P \in I(\mathscr{M})$ and ξ_1, $\xi_2 \in \boldsymbol{R}$ such that the matrix representation of $\tilde{T}$ with respect to $P \circ \iota$ is $\xi_1 E_1 + \xi_2 E_2$. We take the cylindrical coordinates (r, θ, ψ) with respect to $P \circ \iota$. In the case of $\mathscr{M} = S^3$, we may assume that $\xi_2 \geq |\xi_1|$ and the orientation of the frame $(\partial/\partial r, \partial/\partial\theta, \partial/\partial\psi)$ is negative (cf. Proposition 5.5 of [**13**]).

5. Explicit formulas.

In this section, we first express various constants by α, η, p and w. (A part of these calculations is written in the appendix (Section 6).) Then, we give explicit formulas of the coordinate components of γ in terms of α, η, p, w, Jacobi sn function and the incomplete elliptic integral of the third kind (Theorem 5.3).

From now on, we always assume $\mathscr{M} = S^3$ or H^3. By using an argument similar to those stated below, we can obtain the $\boldsymbol{R}^3$ version of Theorem 5.3, but we omit it for the sake of simplicity. For details about the $\boldsymbol{R}^3$ case, see also [**7**], [**12**], [**22**] and [**27**].

We express various quantities in the previous sections by (α, η, p, w). In order to simplify expressions of (α, η, p, w), we introduce the following notation:

$$R = VX - 2\nu\eta w^2, \quad S = Y - 2(1 + 4\nu^2\eta^2)w^2, \quad U_1 = Y - 4\nu\eta R,$$
$$U_2 = (1 - p^2 - (1 - 4\nu^2\eta^2 + 4G/\alpha)w^2)X/w - 4\nu\eta w V,$$
$$L_1 = -\alpha(Y^2 + 4R^2) + 16Gw^4(1 + 4\nu^2\eta^2),$$
$$L_2 = -\alpha((Y - 2p^2)^2 + 4w^2(V - 2\nu\eta X)^2) + 16Gw^4(X^2/w^2 + 4\nu^2\eta^2),$$

where V, X and Y are defined by (2.7).

First, we calculate f. It follows from (3.1), (3.2) and (2.2) that

$$f = \pm\frac{2\alpha^{3/2}}{w^2}(\nu\eta Y + R), \tag{5.1}$$

where the upper sign is taken when $b \geq 0$, while the lower sign is taken when $b < 0$.

Next, we calculate h. We denote the values of $|J(t)|$, $|H(t)|$, $|J(t) \times H(t)|$ at a point t satisfying $k(t)^2 = \alpha$ (resp. β) by $|J|_\alpha$, $|H|_\alpha$, $|J \times H|_\alpha$ (resp. $|J|_\beta$, $|H|_\beta$, $|J \times H|_\beta$). (We can check that these values are determined not depending on the choice of t satisfying $k(t)^2 = \alpha$ (resp. β). Also, when $p = 1$, there are no points t satisfying $k(t)^2 = \beta$. In this case, $|J|_\beta$ etc. are not defined.) By (3.4), we see

$|J|^2 = \langle J, T\rangle^2 + 4|\nabla_T H|^2$, which yields

$$|J|_\alpha^2 = (\alpha - \mu + 2\nu a^2)^2 + 4\alpha(b/\alpha - \nu a)^2 = \frac{\alpha^2}{4w^4}(Y^2 + 4R^2).$$

Since $|H|_\alpha^2 = \alpha(1 + 4\nu^2\eta^2)$, we obtain

$$h = \frac{\alpha}{4w^4}\big[\alpha(Y^2 + 4R^2) + 16Gw^4(1 + 4\nu^2\eta^2)\big]. \tag{5.2}$$

A calculation similar to that of $|J|_\alpha^2$ yields

$$|J|_\beta^2 = \frac{\alpha^2}{4w^4}\big[(Y - 2p^2)^2 + 4w^2(V - 2\nu\eta X)^2\big].$$

Also, by a straightforward computation, we have

$$|J \times H|_\alpha^2 = \frac{\alpha^3}{4w^4}U_1^2, \quad 4G|H|_\alpha^2 - |J|_\alpha^2 = \frac{\alpha L_1}{4w^4}, \tag{5.3}$$

$$|J \times H|_\beta^2 = \frac{\alpha^3}{4w^4}U_2^2, \quad 4G|H|_\beta^2 - |J|_\beta^2 = \frac{\alpha L_2}{4w^4}. \tag{5.4}$$

Now, we consider Case 1, that is, $h^2 - 4Gf^2 > 0$. By Proposition 4.2,

$$\tilde{J} = \sigma_1\frac{\partial}{\partial\theta} + \sigma_2\frac{\partial}{\partial\psi}, \qquad \tilde{H} = \rho_1\frac{\partial}{\partial\theta} + \rho_2\frac{\partial}{\partial\psi}. \tag{5.5}$$

We seek for expressions of σ_1, σ_2, ρ_1 and ρ_2 in terms of (α, η, p, w). By virtue of (5.1) and (5.2), it suffices to express them by f and h.

Without loss of generality, we may assume $\rho_1 \geq 0$. (Because if $\rho_1 < 0$, then it suffices to take (r, θ, ψ) with respect to $P_1 \circ P \circ \iota$ instead of $P \circ \iota$, where $P_1 (\in I(\mathscr{M}))$ is the transformation sending ${}^t(x_1, x_2, x_3, x_4)$ to ${}^t(-x_1, x_2, x_3, x_4)$.) In addition, we may assume $\rho_2 \geq 0$. Next, by $h^2 - 4Gf^2 > 0$ and Lemma 4.3, $J \times H \neq 0$ for some t. Thus, if $\rho_1 = 0$, then $\sigma_1 \neq 0$. Hence we may assume, without loss of generality, that if $\rho_1 = 0$, then $\sigma_1 > 0$. Similarly, we may assume that if $\rho_2 = 0$, then $\sigma_2 > 0$. In addition, in the case of $\mathscr{M} = S^3$, we may assume $\rho_1^2 - \varepsilon\rho_2^2 = \rho_1^2 - \rho_2^2 \geq 0$.

Lemma 5.1.

$$\sigma_1^2 = \frac{G}{2}\left(h - \sqrt{h^2 - 4Gf^2}\right), \quad \sigma_2^2 = \frac{\varepsilon G}{2}\left(h + \sqrt{h^2 - 4Gf^2}\right),$$

$$\rho_1^2 = \frac{1}{8}\left(h + \sqrt{h^2 - 4Gf^2}\right), \quad \rho_2^2 = \frac{\varepsilon}{8}\left(h - \sqrt{h^2 - 4Gf^2}\right).$$

Proof. To begin with, we show that the above expressions of σ_1^2, σ_2^2, ρ_1^2 and ρ_2^2 follow from the following relations:

$$\sigma_1\rho_1 = Gf/2, \tag{5.6}$$

$$\sigma_2\rho_2 = \varepsilon Gf/2, \tag{5.7}$$

$$\sigma_1^2 + 4G\rho_1^2 = Gh, \tag{5.8}$$

$$\sigma_2^2 + 4G\rho_2^2 = \varepsilon Gh. \tag{5.9}$$

Let $\mathscr{M} = S^3$. We first show $\rho_1^2 > \rho_2^2$. Seeking the contradiction, we suppose $\rho_1^2 = \rho_2^2$. Then $\rho_1 = \rho_2$. Since $|H|^2 = k(t)^2 + 4\nu^2 a^2$ is not identically zero, $(\rho_1, \rho_2) \neq (0,0)$ holds. Thus, (5.6) and (5.7) yield $\sigma_1 = \sigma_2$, which implies that $\tilde{J}$ and $\tilde{H}$ are linearly dependent. By Lemma 4.3, this contradicts $h^2 - 4Gf^2 > 0$. Hence $\rho_1^2 > \rho_2^2$. Now, by (5.6), (5.7), (5.8) and (5.9), both of ρ_1^2 and ρ_2^2 satisfy the quadratic equation $4x^2 - hx + Gf^2/4 = 0$ of x. Hence we have $\rho_1^2 = (h + \sqrt{h^2 - 4Gf^2})/8$ and $\rho_2^2 = (h - \sqrt{h^2 - 4Gf^2})/8$. Next, let $\mathscr{M} = H^3$. It follows from (5.8) and (5.6) that $\rho_1^2 = (h \pm \sqrt{h^2 - 4Gf^2})/8$. We show that the double sign is $+$. Seeking the contradiction, we suppose that it is $-$. Then $\rho_1 = 0$, $f = 0$ and $h \geq 0$. By (5.7), $\sigma_2 = 0$ or $\rho_2 = 0$. If $\sigma_2 = 0$, then (5.9) and $h \geq 0$ yield $\rho_2 = 0$. Thus, $\rho_1 = \rho_2 = 0$, which contradicts that $|H|^2$ is not identically zero. Hence the double sign is $+$. By a similar argument, we obtain $\rho_2^2 = (-h + \sqrt{h^2 - 4Gf^2})/8$. Also, σ_1^2 and σ_2^2 follow immediately from (5.8) and (5.9).

We show (5.6), (5.7), (5.8) and (5.9). Let $r(t)$, $\theta(t)$, $\psi(t)$ be the r, θ, ψ components of γ. By $|\partial/\partial\theta| = r$, $|\partial/\partial\psi| = \bar{r}$ and (5.5), we have

$$f/2 = \langle J, H\rangle = (\sigma_1\rho_1 - \varepsilon\sigma_2\rho_2)r(t)^2 + \varepsilon\sigma_2\rho_2/G, \tag{5.10}$$

$$h = |J|^2 + 4G|H|^2 = (\sigma_1^2 + 4G\rho_1^2 - \varepsilon(\sigma_2^2 + 4G\rho_2^2))r(t)^2 + \varepsilon(\sigma_2^2 + 4G\rho_2^2)/G. \tag{5.11}$$

Thus, in order to show (5.6), (5.7), (5.8) and (5.9), it suffices to verify

$$\sigma_1\rho_1 - \varepsilon\sigma_2\rho_2 = 0, \tag{5.12}$$

$$\sigma_1^2 + 4G\rho_1^2 - \varepsilon(\sigma_2^2 + 4G\rho_2^2) = 0. \tag{5.13}$$

We prove (5.12) and (5.13) in the case where γ is not a helix. The proof of the case where γ is a helix is written in Section 6. It follows from (5.5) that

$$|H|^2 = (\rho_1^2 - \varepsilon\rho_2^2)r(t)^2 + \frac{\varepsilon\rho_2^2}{G}. \tag{5.14}$$

Also, since γ is not a helix, $|H|^2 = k(t)^2 + 4\nu^2a^2$ is not a constant function. Thus, $r(t)^2$ is not a constant function. Hence (5.10) yields (5.12). Similarly, (5.13) follows from (5.11). □

By Lemma 5.1 and the assumption of the signatures of σ_1, σ_2, ρ_1 and ρ_2, we obtain the following

PROPOSITION 5.2.

(1) $\rho_1 = \sqrt{(h+\sqrt{h^2-4Gf^2})/8}$, $\rho_2 = \sqrt{\varepsilon(h-\sqrt{h^2-4Gf^2})/8}$.
(2) *If* $\rho_1 = 0$ ($\Leftrightarrow h < 0$ *and* $f = 0$), *then* $\sigma_1 = \sqrt{Gh}$. *If* $\rho_1 \neq 0$, *then* $\sigma_1 = Gf/(2\rho_1)$.
(3) *If* $\rho_2 = 0$ ($\Leftrightarrow h > 0$ *and* $f = 0$), *then* $\sigma_2 = \sqrt{\varepsilon Gh}$. *If* $\rho_2 \neq 0$, *then* $\sigma_2 = \varepsilon Gf/(2\rho_2)$.

Now, we state the main theorem. We introduce the following notation:

$$A_1 = \frac{-\sigma_1}{4G}\left(\frac{\varepsilon\rho_2^2}{G} + \frac{\alpha S}{2w^2}\right) + 2\nu a\rho_1, \quad A_2 = \frac{-\sigma_2}{4G}\left(\frac{\rho_1^2}{G} + \frac{\alpha S}{2w^2}\right) + 2\nu a\rho_2,$$

$$C_1 = \frac{A_1}{y(\alpha + 4\nu^2a^2 - \varepsilon\rho_2^2/G)}, \qquad C_2 = \frac{A_2}{y(\alpha + 4\nu^2a^2 - \rho_1^2/G)},$$

$$B_1 = \frac{\alpha q^2}{\alpha + 4\nu^2a^2 - \varepsilon\rho_2^2/G}, \qquad B_2 = \frac{\alpha q^2}{\alpha + 4\nu^2a^2 - \rho_1^2/G}.$$

It should be noted that all of these are expressed by (α, η, p, w). We denote by

$$\Pi(x,c,p) = \int_0^x \frac{dx}{1 - c\,\mathrm{sn}^2(x,p)}$$

the incomplete elliptic integral of the third kind, where c, p are real numbers satisfying $c \leq 1$, $0 \leq p \leq 1$.

THEOREM 5.3. *Let* $r(t)$, $\theta(t)$, $\psi(t)$, $w_1(t)$, $w_2(t)$ *and* $w_3(t)$ *denote the* r, θ, ψ, w_1, w_2 *and* w_3 *components of* $\gamma(t)$.

CASE 1. *The case of* $h^2 - 4Gf^2 > 0$.

(1) *$r(t)$ is given by*

$$r(t) = \sqrt{\frac{(\alpha + 4\nu^2 a^2 - \varepsilon\rho_2^2/G) - \alpha q^2 \operatorname{sn}^2(yt, p)}{\rho_1^2 - \varepsilon\rho_2^2}}.$$

(a) *There exists $t \in \boldsymbol{R}$ satisfying $r(t) = 0$ if and only if*

$$p \neq 1, \ \ U_2 = 0 \ \ \text{and} \ \ L_2 \leq 0. \tag{5.15}$$

If (5.15) holds, then the set of all zeros of $r(t)$ is $\{(2m+1)K(p)/y\,;\, m \in \boldsymbol{Z}\}$.

(b) *There exists $t \in \boldsymbol{R}$ satisfying $\bar{r}(t) = 0$ if and only if*

$$\mathscr{M} = S^3, \ \ U_1 = 0 \ \ \text{and} \ \ L_1 \geq 0. \tag{5.16}$$

If (5.16) holds and $p \neq 1$, then the set of all zeros of $\bar{r}(t)$ is $\{2mK(p)/y\,;\, m \in \boldsymbol{Z}\}$. If (5.16) holds and $p = 1$, then the set of all zeros of $\bar{r}(t)$ is $\{0\}$.

(2) *$\theta(t)$ is given as follows:*
If (5.15) does not hold, then

$$\theta(t) = \frac{-\sigma_1}{4G}t + C_1 \varPi(yt, B_1, p) + \theta(0).$$

If (5.15) holds, then

$$\theta(t) = \frac{-\sigma_1}{4G}t + m\pi + \theta(0), \qquad \frac{(2m-1)K(p)}{y} < t < \frac{(2m+1)K(p)}{y},$$

where m is an arbitrary integer.

(3) *$\psi(t)$ is given as follows:*
If (5.16) does not hold, then

$$\psi(t) = \frac{-\sigma_2}{4G}t + C_2 \varPi(yt, B_2, p) + \psi(0).$$

If (5.16) holds and $p \neq 1$, then

$$\psi(t) = \frac{-\sigma_2}{4G}\left(t - \frac{K(p)}{y}\right) + m\pi + \psi\left(\frac{K(p)}{y}\right), \quad \frac{2mK(p)}{y} < t < \frac{(2m+2)K(p)}{y},$$

where m is an arbitrary integer.
If (5.16) holds and $p=1$, then

$$\psi(t)=\frac{-\sigma_2}{4G}\left(t-\frac{K(p)}{y}\right)+n(t)\pi+\psi\left(\frac{K(p)}{y}\right),$$

where

$$n(t)=\begin{cases}0 & \text{if } t>0,\\ -1 & \text{if } t<0.\end{cases}$$

Case 2. *The case of $\mathscr{M}=H^3$ and $h^2-4Gf^2=0$.*
$w_1(t)$, $w_2(t)$ and $w_3(t)$ are given as follows:

$$\begin{aligned}
w_1(t)&=\frac{1}{-4G}\left[t+\frac{S}{w\sqrt{\alpha}(1+4\nu^2\eta^2)}\varPi\left(yt,\frac{q^2}{1+4\nu^2\eta^2},p\right)\right]+w_1(0),\\
w_2(t)&=\frac{\pm 2\nu\eta w}{\sqrt{-G}\alpha(1+4\nu^2\eta^2)}\varPi\left(yt,\frac{q^2}{1+4\nu^2\eta^2},p\right)+w_2(0),\\
w_3(t)&=\frac{1}{-2G\sqrt{\alpha(1+4\nu^2\eta^2-q^2\operatorname{sn}^2(yt,p))}}.
\end{aligned}$$

Case 3. *The case of $\mathscr{M}=S^3$ and $h^2-4Gf^2=0$.*
In this case, $p=0$ and $\eta=\big(V/(2w)+\sqrt{G/\alpha}\big)/\nu$ or $\big(V/(2w)-\sqrt{G/\alpha}\big)/\nu$.
If $\eta=\big(V/(2w)+\sqrt{G/\alpha}\big)/\nu$, then $r(t)$, $\theta(t)$ and $\psi(t)$ are given as follows:

$$r(t)=\sqrt{\frac{1-(\xi_2^2/G)}{\xi_1^2-\xi_2^2}},\quad \theta(t)=\xi_1 t+\theta(0),\quad \psi(t)=\xi_2 t+\psi(0), \tag{5.17}$$

where

$$(\xi_1,\xi_2)=\left(\pm\frac{\sqrt{\alpha}-\sqrt{D_1}}{2w},\frac{\sqrt{\alpha}+\sqrt{D_1}}{2w}\right),\ D_1=\alpha w^2+(\sqrt{\alpha}V+2\sqrt{G}w)^2.$$

If $\eta=\big(V/(2w)-\sqrt{G/\alpha}\big)/\nu$, then $r(t)$, $\theta(t)$ and $\psi(t)$ are given by (5.17), where

$$(\xi_1,\xi_2)=\left(\mp\frac{\sqrt{\alpha}-\sqrt{D_2}}{2w},\frac{\sqrt{\alpha}+\sqrt{D_2}}{2w}\right),\ D_2=\alpha w^2+(\sqrt{\alpha}V-2\sqrt{G}w)^2.$$

Proof. First, we consider Case 1. By Lemma 5.1, $\rho_1^2 - \varepsilon\rho_2^2 = \sqrt{h^2 - 4Gf^2}/4 > 0$. Thus it follows from (5.14) that

$$r(t) = \sqrt{\frac{k(t)^2 + 4\nu^2 a^2 - \varepsilon\rho_2^2/G}{\rho_1^2 - \varepsilon\rho_2^2}}.$$

Substituting (2.4) yields the expression of $r(t)$ of (1).

We show (a). In the case of $p = 1$, it follows from $k^2 = \alpha\,\mathrm{sech}^2(yt)$ that r does not attain the minimum value and is positive everywhere. Next, let $p \neq 1$. Then r attains the maximum (resp. minimum) value $r_{\max}$ (resp. $r_{\min}$) precisely when $k(t)^2 = \alpha$ (resp. β), that is, when $t = 2mK(p)/y$ (resp. $(2m+1)K(p)/y$), where m is an arbitrary integer. By $|H|_\beta^2 = \beta + 4\nu^2 a^2$ and (4.6), we see

$$\begin{aligned}(r_{\min})^2 &= \frac{\beta + 4\nu^2 a^2 - \varepsilon\rho_2^2/G}{\rho_1^2 - \varepsilon\rho_2^2} \\ &= \frac{4G|H|_\beta^2 - |J|_\beta^2 + \sqrt{(4G|H|_\beta^2 - |J|_\beta^2)^2 + 16G|J \times H|_\beta^2}}{8G(\rho_1^2 - \varepsilon\rho_2^2)}.\end{aligned}$$

Thus, $r_{\min} = 0$ if and only if $|J \times H|_\beta = 0$ and $4G|H|_\beta^2 - |J|_\beta^2 \leq 0$. It follows from (5.4) that $r_{\min} = 0$ if and only if $U_2 = 0$ and $L_2 \leq 0$. Hence we complete the proof of (a).

Next, we show (b). In the case of $\mathscr{M} = H^3$, since $\bar{r}^2 = -1/G + r^2 \geq -1/G$, $\bar{r}(t)$ is positive everywhere. Let $\mathscr{M} = S^3$. Then $\bar{r}$ attains the minimum value $\bar{r}_{\min}$ precisely when $k(t)^2 = \alpha$, that is, when

$$t = \begin{cases} 2mK(p)/y \quad (m \in \boldsymbol{Z}) & \text{if} \quad p \neq 1, \\ 0 & \text{if} \quad p = 1, \end{cases}$$

and

$$\begin{aligned}(\bar{r}_{\min})^2 &= \frac{-(\alpha + 4\nu^2 a^2 - \rho_1^2/G)}{\rho_1^2 - \varepsilon\rho_2^2} \\ &= \frac{4G|H|_\alpha^2 - |J|_\alpha^2 - \sqrt{(4G|H|_\alpha^2 - |J|_\alpha^2)^2 + 16G|J \times H|_\alpha^2}}{-8G(\rho_1^2 - \varepsilon\rho_2^2)}.\end{aligned} \tag{5.18}$$

Thus $\bar{r}_{\min} = 0$ if and only if $|J \times H|_\alpha = 0$ and $4G|H|_\alpha^2 - |J|_\alpha^2 \geq 0$. By (5.3), $\bar{r}_{\min} = 0$ is equivalent to $U_1 = 0$ and $L_1 \geq 0$, which completes the proof of (b).

We seek for $\theta(t)$ and $\psi(t)$. By (5.5), $\tilde{J} \times \tilde{H} = (\sigma_1\rho_2 - \sigma_2\rho_1)(\partial/\partial\theta \times \partial/\partial\psi)$. Since Lemma 4.3 and the assumption $h^2 - 4Gf^2 > 0$ imply that $J \times H \neq 0$ for some t, we see $\sigma_1\rho_2 - \sigma_2\rho_1 \neq 0$, and so

$$\frac{\partial}{\partial\theta} = \frac{1}{\sigma_1\rho_2 - \sigma_2\rho_1}(\rho_2\tilde{J} - \sigma_2\tilde{H}), \quad \frac{\partial}{\partial\psi} = \frac{1}{\sigma_1\rho_2 - \sigma_2\rho_1}(-\rho_1\tilde{J} + \sigma_1\tilde{H}).$$

Thus, at a point where $r(t) > 0$, we have

$$\theta'(t) = \frac{\langle T, \partial/\partial\theta\rangle}{|\partial/\partial\theta|^2} = \frac{\rho_2\langle J, T\rangle - \sigma_2\langle H, T\rangle}{(\sigma_1\rho_2 - \sigma_2\rho_1)r(t)^2},$$

where we write $\gamma^*(\partial/\partial\theta)$ as $\partial/\partial\theta$, for short. Now, (5.6), (5.7), (5.8) and (5.9) yield $\sigma_1\rho_1 = \varepsilon\sigma_2\rho_2$ and $\sigma_1^2 - \varepsilon\sigma_2^2 = -4G(\rho_1^2 - \varepsilon\rho_2^2)$, from which we obtain

$$\frac{(\rho_1^2 - \varepsilon\rho_2^2)\rho_2}{\sigma_1\rho_2 - \sigma_2\rho_1} = \frac{-\sigma_1}{4G}, \quad \frac{(\rho_1^2 - \varepsilon\rho_2^2)\sigma_2}{\sigma_1\rho_2 - \sigma_2\rho_1} = -\rho_1.$$

By these expressions, (3.1) and (3.2),

$$\theta'(t) = \frac{-\sigma_1}{4G} + \frac{A_1}{(\rho_1^2 - \varepsilon\rho_2^2)r(t)^2} = \frac{-\sigma_1}{4G} + \frac{yC_1}{1 - B_1\,\mathrm{sn}^2(yt, p)}.$$

Hence, by integration in t, we obtain the former part of (2).

Similarly, at a point where $\bar{r}(t) > 0$, it follows that

$$\begin{aligned}\psi'(t) &= \frac{\langle T, \partial/\partial\psi\rangle}{|\partial/\partial\psi|^2} = \frac{-\sigma_2}{4G} + \frac{A_2}{-\varepsilon(\rho_1^2 - \varepsilon\rho_2^2)\bar{r}(t)^2} \\ &= \frac{-\sigma_2}{4G} + \frac{A_2}{(\alpha + 4\nu^2a^2 - \rho_1^2/G) - \alpha q^2\,\mathrm{sn}^2(yt, p)}.\end{aligned}$$

Suppose that $\bar{r}(t) > 0$ for all $t \in \boldsymbol{R}$. Then, in the case of $\mathscr{M} = S^3$, by using (5.18), we see $\alpha + 4\nu^2a^2 - \rho_1^2/G \neq 0$. In the case of $\mathscr{M} = H^3$, it is clear that $\alpha + 4\nu^2a^2 - \rho_1^2/G \neq 0$. Thus, B_2 is well-defined, and hence

$$\psi'(t) = \frac{-\sigma_2}{4G} + \frac{yC_2}{1 - B_2\,\mathrm{sn}^2(yt, p)},$$

from which the first part of (3) follows.

Next we consider the case where r (resp. $\bar{r}$) vanishes. Then, at a point where $r(t)=0$ (resp. $\bar{r}(t)=0$), $\theta(t)$ (resp. $\psi(t)$) jumps by π. More precisely, the following holds.

Lemma 5.4.

(1) *Suppose that* (5.15) *holds. Let* t_1 *satisfy* $r(t_1)=0$. *Then, both* $\lim_{t\to t_1+0}\theta(t)$ *and* $\lim_{t\to t_1-0}\theta(t)$ *exist, and* $\lim_{t\to t_1+0}\theta(t)-\lim_{t\to t_1-0}\theta(t)=\pi \pmod{2\pi}$.

(2) *Suppose that* (5.16) *holds. Let* t_2 *satisfy* $\bar{r}(t_2)=0$. *Then, both* $\lim_{t\to t_2+0}\psi(t)$ *and* $\lim_{t\to t_2-0}\psi(t)$ *exist, and* $\lim_{t\to t_2+0}\psi(t)-\lim_{t\to t_2-0}\psi(t)=\pi \pmod{2\pi}$.

Proof. We show (1). It is checked that (x_1,x_2,ψ) is a local coordinate system around $\gamma(t_1)$. Let $x_1(t)$, $x_2(t)$ and $\psi(t)$ be the x_1, x_2 and ψ components of $\gamma(t)$. To prove (1), it is sufficient to prove that $x_1'(t)(\partial/\partial x_1)+x_2'(t)(\partial/\partial x_2)$ does not vanish at $t=t_1$. Since

$$\left|x_1'\frac{\partial}{\partial x_1}+x_2'\frac{\partial}{\partial x_2}\right|=\left|r'\frac{\partial}{\partial r}+\theta'\frac{\partial}{\partial\theta}\right|\geq\left|r'\frac{\partial}{\partial r}\right|=\frac{|r'|}{\sqrt{\varepsilon G}\,\bar{r}}$$

except at $t=t_1$, it is sufficient to show that $|r'|$ approaches a positive number as $t\to t_1$. It follows from $r_{\min}=0$ that $\beta+4\nu^2a^2-\varepsilon\rho_2^2/G=0$, and hence

$$r(t)=\sqrt{\frac{\alpha q^2}{\rho_1^2-\varepsilon\rho_2^2}}|\mathrm{cn}(yt,p)|.$$

Thus,

$$|r'(t)|=y\sqrt{\frac{\alpha q^2}{\rho_1^2-\varepsilon\rho_2^2}}|\mathrm{sn}(yt,p)|\,\mathrm{dn}(yt,p)\to y\sqrt{\frac{\alpha q^2(1-p^2)}{\rho_1^2-\varepsilon\rho_2^2}}$$

as $t\to t_1$. Since $p\neq 1$, the above limit value is positive.

By using the local coordinate system (x_3,x_4,θ) around $\gamma(t_2)$, we can verify (2) in the same way as (1). □

We show the latter part of (2) of the theorem. Suppose that (5.15) holds. Since it follows from $r_{\min}=0$ that $\alpha+4\nu^2a^2-\varepsilon\rho_2^2/G=\alpha q^2$, we have

$$\theta(t)-\theta(0)=\int_0^t\theta'(t)dt=\frac{-\sigma_1 t}{4G}+\frac{A_1}{\alpha q^2}\int_0^t\frac{dt}{\mathrm{cn}^2(yt,p)}$$

for all $t \in (-K/y, K/y)$. If $A_1 \neq 0$, then the right hand side diverges as $t \to K/y - 0$, which contradicts the existence of $\lim_{t \to K/y-0} \theta(t)$. Consequently, $A_1 = 0$, and hence $\theta'(t) = -\sigma_1/(4G)$ except at the zeros of r. Combining this with Lemma 5.4, we obtain the latter part of (2). Similarly, we can check the second and third parts of (3). The proof of Case 1 is completed.

We consider Case 2, that is, the case where $\mathscr{M} = H^3$ and $\tilde{J}$ is parabolic. Proposition 4.5 implies $\tilde{J} = \partial/\partial w_1$ and $\tilde{H} = (1/(2\sqrt{-G}))\partial/\partial w_2$. Thus, it follows from $|\partial/\partial w_2|^2 = 1/(-Gw_3^2)$ that $|H|^2 = 1/(4G^2 w_3(t)^2)$. Hence $w_3(t) = 1/(-2G\sqrt{k(t)^2 + 4\nu^2 a^2})$. Also, by $|\partial/\partial w_1|^2 = 1/(-Gw_3^2)$,

$$w_1'(t) = \frac{\langle T, \partial/\partial w_1 \rangle}{|\partial/\partial w_1|^2} = -Gw_3(t)^2 \langle T, J \rangle = \frac{1}{-4G}\left(1 + \frac{-\mu + 2\nu a^2 - 4\nu^2 a^2}{k(t)^2 + 4\nu^2 a^2}\right).$$

Substituting (2.4) yields the expressions of $w_1(t)$ and $w_3(t)$. Similarly, we have the expression of $w_2(t)$.

We consider Case 3. Then γ is a helix and $T = \xi_1(\partial/\partial\theta) + \xi_2(\partial/\partial\psi)$. In the same way as in the proof of Lemma 4.2, we have (4.3). It follows from $|T| = 1$ that

$$r_0 = \sqrt{(1 - \xi_2^2/G)/(\xi_1^2 - \xi_2^2)}.$$

Let us express ξ_1 and ξ_2 by (α, η, p, w). By (4.4) and (4.5),

$$\begin{aligned} \alpha &= Gr_0^2\bar{r}_0^2(\xi_1^2 - \xi_2^2)^2 = \xi_1^2 + \xi_2^2 - (\xi_1\xi_2)^2/G - G, \\ \tau &= \langle \nabla_T N, B \rangle = -\xi_1\xi_2/\sqrt{G}. \end{aligned}$$

Thus, by the assumption $\xi_2 \geq |\xi_1|$, we have

$$\begin{aligned} \xi_1 &= \frac{1}{2}\Big(\sqrt{\alpha + (\tau - \sqrt{G})^2} - \sqrt{\alpha + (\tau + \sqrt{G})^2}\Big), \\ \xi_2 &= \frac{1}{2}\Big(\sqrt{\alpha + (\tau - \sqrt{G})^2} + \sqrt{\alpha + (\tau + \sqrt{G})^2}\Big). \end{aligned} \tag{5.19}$$

It follows from Lemma 4.3 that $\eta = \big(V/(2w) + \sqrt{G/\alpha}\big)/\nu$ or $\eta = \big(V/(2w) - \sqrt{G/\alpha}\big)/\nu$. If $\eta = \big(V/(2w) + \sqrt{G/\alpha}\big)/\nu$, then $\tau = \pm\sqrt{\alpha}(V/(2w) + \nu\eta) = \pm(\sqrt{\alpha}V/w + \sqrt{G})$. Substituting this to (5.19) yields the formula of (ξ_1, ξ_2) in terms of (α, η, p, w). Similarly, when $\eta = \big(V/(2w) - \sqrt{G/\alpha}\big)/\nu$ as well, we have the formula of (ξ_1, ξ_2). $\square$

6. Appendix.

In this appendix, we prove (5.12) and (5.13) of Lemma 5.1 in the case where γ is a helix. Note that $h^2 - 4Gf^2 > 0$ has been assumed. Since γ is a helix, we see $J = c_1T + c_3B$, $H = c_2T + c_4B$, where $c_1 = \alpha - \mu + 2\nu a^2 = 2\tau(\tau - 2\nu a) - 2G$, $c_2 = 2\nu a$, $c_3 = 2\sqrt{\alpha}(\tau - 2\nu a)$ and $c_4 = \sqrt{\alpha}$. It follows from Lemma 4.3 that $J \times H = (c_2c_3 - c_1c_4)N$ does not vanish, and hence T and B are expressed as linear combinations of J and H, respectively. Thus, T is expressed as $T = \xi_1\gamma^*(\partial/\partial\theta) + \xi_2\gamma^*(\partial/\partial\psi)$ for some $\xi_1, \xi_2 \in \boldsymbol{R}$. Hence we obtain $r(t) = r_0$, $\theta'(t) = \xi_1$ and $\psi'(t) = \xi_2$, where r_0 is a constant satisfying $r_0 \neq 0$ and $\bar{r}_0 \neq 0$.

Let us express σ_1, σ_2, ρ_1 and ρ_2 by ξ_1, ξ_2, c_1, c_2, c_3, c_4 and r_0. In the same way as in the proof of Lemma 4.2, we have (4.4), (4.5) and

$$\tau = \langle \nabla_T N, B \rangle = \pm \xi_1\xi_2/\sqrt{\varepsilon G}. \tag{6.1}$$

Thus,

$$\begin{pmatrix} \sigma_1 & \rho_1 \\ \sigma_2 & \rho_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \mp\delta\xi_2\bar{r}_0/r_0 \\ \xi_2 & \pm\delta\xi_1 r_0/\bar{r}_0 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}.$$

Hence we have

$$r_0\bar{r}_0(\sigma_1\rho_1 - \varepsilon\sigma_2\rho_2) = c_1c_2r_0\bar{r}_0(\xi_1^2 - \varepsilon\xi_2^2) + \frac{c_3c_4(\bar{r}_0^4\xi_2^2 - \varepsilon r_0^4\xi_1^2)}{r_0\bar{r}_0} \mp \frac{\delta(c_2c_3 + c_1c_4)\xi_1\xi_2}{\varepsilon G}.$$

Now, it follows from $|T| = 1$ that

$$r_0^2 = (1 - \varepsilon\xi_2^2/G)/(\xi_1^2 - \varepsilon\xi_2^2), \quad \bar{r}_0^2 = \varepsilon(\xi_1^2/G - 1)/(\xi_1^2 - \varepsilon\xi_2^2).$$

By using these two expressions together with (4.4) and (6.1), we have

$$r_0\bar{r}_0(\sigma_1\rho_1 - \varepsilon\sigma_2\rho_2) = \frac{\delta}{\sqrt{\varepsilon G}}\left[c_1c_2\sqrt{\alpha} + \frac{c_3c_4(\tau^2 - G)}{\sqrt{\alpha}} - (c_2c_3 + c_1c_4)\tau\right].$$

The definitions of $c_1, \ldots, c_4$ and a straightforward calculation yield that the right hand side is equal to zero. Since r_0, $\bar{r}_0 \neq 0$, we obtain (5.12). By a similar argument, we can verify $(\sigma_1^2 - \varepsilon\sigma_2^2) + 4G(\rho_1^2 - \varepsilon\rho_2^2) = 0$, and hence (5.13) holds.

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