J. Math. Soc. Japan
Vol. 60, No. 2 (2008) pp. 511–549
doi: 10.2969/jmsj/06020511

Structure of locally convex quasi C^*-algebras

By Fabio Bagarello, Maria Fragoulopoulou, Atsushi Inoue and Camillo Trapani

(Received Aug. 25, 2006)
(Revised Jul. 30, 2007)

Abstract. The completion of a (normed) C^*-algebra $\mathscr{A}_0[\|\cdot\|_0]$ with respect to a locally convex topology τ on $\mathscr{A}_0$ that makes the multiplication of $\mathscr{A}_0$ separately continuous is, in general, a quasi $*$-algebra, and not a locally convex $*$-algebra [**10**], [**15**]. In this way, one is led to consideration of locally convex quasi C^*-algebras, which generalize C^*-algebras in the context of quasi $*$-algebras. Examples are given and the structure of these relatives of C^*-algebras is investigated.

1. Introduction.

The study of the structure and representation theory of the completion of a (normed) C^*-algebra $\mathscr{A}_0[\|\cdot\|_0]$ with respect to a locally convex topology τ on $\mathscr{A}_0$ "compatible" with the corresponding *-norm topology started in [**10**] and was continued in [**15**]. When the multiplication of $\mathscr{A}_0$ with respect to τ is jointly continuous, the completion $\widetilde{\mathscr{A}_0}[\tau]$ of $\mathscr{A}_0[\tau]$ is a GB^*-algebra over the unit ball $\mathscr{U}(\mathscr{A}_0) \equiv \{x \in \mathscr{A}_0 : \|x\|_0 \leq 1\}$ of $\mathscr{A}_0[\|\cdot\|_0]$ if and only if $\mathscr{U}(\mathscr{A}_0)$ is τ-closed [**15**, Corollary 2.2]. When the multiplication of $\mathscr{A}_0$ with respect to τ is just separately continuous, $\widetilde{\mathscr{A}_0}[\tau]$ may fail to be a locally convex $*$-algebra, but may well carry the structure of a quasi $*$-algebra. The properties and the $*$-representation theory of $\widetilde{\mathscr{A}_0}[\tau]$, in this case, have been studied in [**10**, Section 3] and [**15**, Section 3]. Continuing this project we are led to the introduction of locally convex quasi C^*-algebras in the present study (see Definition 3.3). In this way, the notion of a C^*-algebra is incorporated within the context of quasi $*$-algebras. Topological quasi $*$-algebras were first introduced by G. Lassner (see [**18**], [**19**]) for solving problems in quantum statistics and quantum dynamics that could not be resolved within the algebraic formulation of quantum theories developed by Haag and Kastler in [**16**]. However, the bimodule axiom (which is crucial for many considerations such as $*$-representation theory) was missing therein and also from many subsequent research

2000 *Mathematics Subject Classification.* Primary 47L60; Secondary 46K10, 46K70, 46L05.
Key Words and Phrases. quasi $*$-algebras, quasi-positivity, locally convex quasi C*-algebras, unbounded $*$-representations.

papers for about 20 years. The first correct definition was given in [**20**, p. 90], where also large classes of O^*-algebra examples have been derived. Furthermore, quasi $*$-algebras appeared later in [**21**], [**22**] and [**12**], [**13**]. These algebras constitute an interesting class of the so-called partial $*$-algebras, introduced by J.-P. Antoine and W. Karwowski in [**7**], [**8**] and studied extensively in [**3**], [**4**], [**5**], [**11**] and [**6**]. Partial $*$-algebras and quasi $*$-algebras play an important role in the theory of unbounded operators, which in its turn has numerous applications in mathematical physics (see, for instance, [**6**], [**17**], [**22**], [**9**]).

Our motivation for the present study is clear from the preceding discussion. The results that we shall exhibit are structured as follows: After the background material in Section 2, Section 3 defines two notions of positivity in the quasi $*$-algebra $\widetilde{\mathscr{A}_0}[\tau]$, called "quasi-positivity" and "commutatively quasi-positivity"; besides, it introduces locally convex quasi C^*-algebras (Definition 3.3) and gives examples from various classes of topological algebras. Since locally convex quasi C^*-algebras of operators are of particular interest (see, for instance, Remark 4.2 and Propositions 4.3 and 4.5), we study them separately in Section 4. In Section 5, the structure of commutative locally convex quasi C^*-algebras is investigated taking into account [**1**, Section 6] and [**10**], [**15**]. In Section 6 we apply the results of Sections 3 and 5 and also ideas developed in [**14**, Section 4] and [**15**] to present a functional calculus for the quasi-positive elements of a commutative locally convex quasi C^*-algebra. As a consequence the quasi nth-root of a quasi-positive element of such an algebra is, for instance, defined (Corollary 6.7). In Section 7, the structure of a noncommutative locally convex quasi C^*-algebra is studied. More precisely, if $\mathscr{A}[\tau]$ is a noncommutative locally convex quasi C^*-algebra, necessary and sufficient conditions are given (see Theorems 7.3 and 7.5) such that $\mathscr{A}[\tau]$ is continuously embedded in a locally convex quasi C^*-algebra of operators. Further, a functional calculus for commutatively quasi-positive elements in $\mathscr{A}[\tau]$ is investigated (Theorem 7.8).

2. Preliminaries.

All algebras that we deal with are complex and the topological spaces are supposed to be Hausdorff. If an algebra $\mathscr{A}$ has an identity, this will be denoted by 1. An algebra $\mathscr{A}$ with identity 1, will be called unital.

Let $\mathscr{A}_0[\|\cdot\|_0]$ be a C^*-algebra. We shall use the symbol $\|\cdot\|_0$ of the C^*-norm to denote the corresponding topology. Suppose that τ is a topology on $\mathscr{A}_0$ such that $\mathscr{A}_0[\tau]$ is a locally convex $*$-algebra. Then, the topologies τ, $\|\cdot\|_0$ on $\mathscr{A}_0$ are compatible whenever each Cauchy net in both topologies that converges with respect to one of them, also converges with respect to the other one. The symbol $\widetilde{\mathscr{A}_0}[\tau]$ denotes the completion of $\mathscr{A}_0[\tau]$.

A partial $*$-algebra is a vector space $\mathscr{A}$ equipped with a vector space involution $* : \mathscr{A} \to \mathscr{A} : x \mapsto x^*$ and a partial multiplication defined on a set $\Gamma \subset \mathscr{A} \times \mathscr{A}$ in such a way that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ and $\lambda, \mu \in \boldsymbol{C}$ imply $(x, \lambda y_1 + \mu y_2) \in \Gamma$;

(iii) for every $(x, y) \in \Gamma$, a product $xy \in \mathscr{A}$ is defined,

such that xy depends linearly on y and satisfies the equality $(xy)^* = y^* x^*$.

Whenever $(x, y) \in \Gamma$, we say that x is a left multiplier of y and y a right multiplier of x and we write $x \in L(y)$, respectively $y \in R(x)$.

Quasi $*$-algebras are important examples of partial $*$-algebras.

If $\mathscr{A}$ is a vector space and $\mathscr{A}_0$ is a subspace of $\mathscr{A}$ such that is also a $*$-algebra, then $\mathscr{A}$ is said to be a quasi $*$-algebra over $\mathscr{A}_0$ whenever the next properties are valid:

(i)$'$ The multiplication of $\mathscr{A}_0$ is extended on $\mathscr{A}$ as follows: The assignments

$\mathscr{A} \times \mathscr{A}_0 \to \mathscr{A} : (a, x) \mapsto ax$ (left multiplication of x by a) and
$\mathscr{A}_0 \times \mathscr{A} \to \mathscr{A} : (x, a) \mapsto xa$ (right multiplication of x by a)

are always defined and are bilinear;

(ii)$'$ $x_1(x_2 a) = (x_1 x_2)a, (ax_1)x_2 = a(x_1 x_2)$ and $x_1(ax_2) = (x_1 a)x_2$, for all $x_1, x_2 \in \mathscr{A}_0$ and $a \in \mathscr{A}$;

(iii)$'$ the involution $*$ of $\mathscr{A}_0$ is extended on $\mathscr{A}$, denoted also by $*$, such that $(ax)^* = x^* a^*$ and $(xa)^* = a^* x^*$, for all $x \in \mathscr{A}_0$ and $a \in \mathscr{A}$.

For further information see [**6**]. If $\mathscr{A}_0[\tau]$ is a locally convex $*$-algebra, with separately continuous multiplication, its completion $\widetilde{\mathscr{A}_0}[\tau]$ is a quasi $*$-algebra over $\mathscr{A}_0$ under the following operations: Given $x \in \mathscr{A}_0$ and $a \in \widetilde{\mathscr{A}_0}[\tau]$

- $ax := \lim\limits_{\alpha} x_\alpha x$ (left multiplication)
- $xa := \lim\limits_{\alpha} x x_\alpha$ (right multiplication)

with $\{x_\alpha\}_{\alpha \in \Delta}$ a net in $\mathscr{A}_0$ such that $a = \tau\text{-}\lim\limits_{\alpha} x_\alpha$.

- An involution on $\widetilde{\mathscr{A}_0}[\tau]$ like in (iii)$'$ is the continuous extension of the involution given on $\mathscr{A}_0$.

A $*$-invariant subspace $\mathscr{A}$ of $\widetilde{\mathscr{A}_0}[\tau]$ containing $\mathscr{A}_0$ is called a quasi $*$-subalgebra of $\widetilde{\mathscr{A}_0}[\tau]$ if ax, xa belong to $\mathscr{A}$ for any $x \in \mathscr{A}_0$, $a \in \mathscr{A}$. Then, one easily shows that $\mathscr{A}$ is a quasi $*$-algebra over $\mathscr{A}_0$. Moreover, $\mathscr{A}[\tau]$ is a locally convex space that contains $\mathscr{A}_0$ as a dense subspace and for every fixed $x \in \mathscr{A}_0$, the maps $\mathscr{A}[\tau] \to \mathscr{A}[\tau]$ with $a \mapsto ax$ and $a \mapsto xa$ are continuous. An algebra of this kind is called locally convex quasi $*$-algebra over $\mathscr{A}_0$.

Another concept we need is that of a GB^*-algebra introduced by G.R. Allan in 1967 [**2**] for generalizing C^*-algebras (also see [**14**]). Let $\mathscr{A}[\tau]$ be a unital locally convex $*$-algebra. Let $\mathscr{B}^*$ be the collection of all closed, bounded, absolutely

convex subsets B of $\mathscr{A}[\tau]$ with the properties: $1 \in B, B^* = B$ and $B^2 \subset B$. For every $B \in \mathscr{B}^*$, the linear span $A[B]$ of B is a normed $*$-algebra under the Minkowski functional $\|\cdot\|_B$ of B. If $A[B]$ is complete for every $B \in \mathscr{B}^*$, then $\mathscr{A}[\tau]$ is said to be pseudo-complete. Every sequentially complete locally convex $*$-algebra is pseudo-complete [**1**, Proposition (2.6)]. Now, a unital pseudo-complete locally convex $*$-algebra $\mathscr{A}[\tau]$, such that $\mathscr{B}^*$ has a greatest member, denoted by B_0, and $(1 + x^*x)^{-1}$ exists and belongs to $A[B_0]$ for every $x \in \mathscr{A}$, is called a GB^*-algebra over B_0. In this case $A[B_0]$ is a C^*-algebra.

3. Locally convex quasi C^*-algebras.

Throughout this Section $\mathscr{A}_0[\|\cdot\|_0]$ denotes a unital C^*-algebra and τ a locally convex topology on $\mathscr{A}_0$ compatible with the corresponding $\|\cdot\|_0$-topology. Under certain conditions on τ a quasi $*$-subalgebra $\mathscr{A}$ of the quasi $*$-algebra $\widetilde{\mathscr{A}_0}[\tau]$ over $\mathscr{A}_0$ is formed, which is named locally convex quasi C^*-algebra. Examples and basic properties of such algebras are presented. So, let $\mathscr{A}_0[\|\cdot\|_0]$ and τ be as above with $\{p_\lambda\}_{\lambda\in\Lambda}$ a defining family of seminorms for τ. Suppose that τ satisfies the properties:

(T_1) $\mathscr{A}_0[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication.

(T_2) $\tau \preceq \|\cdot\|_0$.

Then, the identity map $\mathscr{A}_0[\|\cdot\|_0] \to \mathscr{A}_0[\tau]$ extends to a continuous $*$-linear map $\mathscr{A}_0[\|\cdot\|_0] \to \widetilde{\mathscr{A}_0}[\tau]$ and since $\tau, \|\cdot\|_0$ are compatible, the C^*-algebra $\mathscr{A}_0[\|\cdot\|_0]$ can be regarded embedded into $\widetilde{\mathscr{A}_0}[\tau]$. It is easily shown that $\widetilde{\mathscr{A}_0}[\tau]$ is a quasi $*$-algebra over $\mathscr{A}_0$ (cf. [**15**, Section 3]).

The next Definition 3.1 provides concepts of positivity for elements of a quasi $*$-algebra $\widetilde{\mathscr{A}_0}[\tau]$.

Definition 3.1. An element a of $\widetilde{\mathscr{A}_0}[\tau]$ is called quasi-positive (resp. commutatively quasi-positive) if there is a net (resp. commuting net) $(x_\alpha)_{\alpha\in\Delta}$ of the positive cone $(\mathscr{A}_0)_+$ of the C^*-algebra $\mathscr{A}_0[\|\cdot\|_0]$, which converges to a with respect to the topology τ.

We have already used the symbol $(\mathscr{A}_0)_+$ for the set of all positive elements of the C^*-algebra $\mathscr{A}_0[\|\cdot\|_0]$. The set of all quasi-positive (resp. commutatively quasi-positive) elements of $\widetilde{\mathscr{A}_0}[\tau]$, we shall denote by $\widetilde{\mathscr{A}_0}[\tau]_{q+}$ (resp. $\widetilde{\mathscr{A}_0}[\tau]_{cq+}$). Then, $\widetilde{\mathscr{A}_0}[\tau]_{q+}$ is a wedge (that is, for any $a, b \in \widetilde{\mathscr{A}_0}[\tau]_{q+}$ and $\lambda \geq 0$, the elements $a + b$ and λa belong to $\widetilde{\mathscr{A}_0}[\tau]_{q+}$), but it is not necessarily a positive cone (i.e. $\widetilde{\mathscr{A}_0}[\tau]_{q+} \cap (-\widetilde{\mathscr{A}_0}[\tau]_{q+}) \neq \{0\}$). The set $\widetilde{\mathscr{A}_0}[\tau]_{cq+}$ is not even, in general, a wedge. But, if $\mathscr{A}_0$ is commutative, then of course, $\widetilde{\mathscr{A}_0}[\tau]_{q+} = \widetilde{\mathscr{A}_0}[\tau]_{cq+}$.

Further, we employ the following two extra conditions (T_3), (T_4) for the locally convex topology τ on $\mathscr{A}_0$ and examine the effect on $\widetilde{\mathscr{A}_0}[\tau]_{cq+}$:

(T_3) For each $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda$ such that

$$p_\lambda(xy) \leq \|x\|_0 p_{\lambda'}(y), \forall\, x, y \in \mathscr{A}_0 \text{ with } xy = yx;$$

(T_4) The set $\mathscr{U}(\mathscr{A}_0)_+ := \{x \in (\mathscr{A}_0)_+ : \|x\|_0 \leq 1\}$ is τ-closed, and $\widetilde{\mathscr{A}_0}[\|\cdot\|]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+$.

PROPOSITION 3.2. *Let $\mathscr{A}_0[\|\cdot\|_0]$ be a unital C^*-algebra and τ a locally convex topology on $\mathscr{A}_0$. Suppose that τ fulfils the conditions (T_1)–(T_4). Then, $\widetilde{\mathscr{A}_0}[\tau]$ is a locally convex quasi $*$-algebra over $\mathscr{A}_0$ with the properties*:

(1) *For every $a \in \widetilde{\mathscr{A}_0}[\tau]_{cq+}$, the element $1 + a$ is invertible and its inverse $(1 + a)^{-1}$ belongs to $\mathscr{U}(\mathscr{A}_0)_+$.*

(2) *For a given $a \in \widetilde{\mathscr{A}_0}[\tau]_{cq+}$ and any $\varepsilon > 0$, let*

$$a_\varepsilon = a(1 + \varepsilon a)^{-1}.$$

Then, $\{a_\varepsilon\}_{\varepsilon>0}$ is a commuting net in $(\mathscr{A}_0)_+$ such that $a - a_\varepsilon \in \widetilde{\mathscr{A}_0}[\tau]_{cq+}$ and $a = \tau\text{-}\lim_{\varepsilon\to 0} a_\varepsilon$.

(3) $\widetilde{\mathscr{A}_0}[\tau]_{cq+} \cap (-\widetilde{\mathscr{A}_0}[\tau]_{cq+}) = \{0\}$.

(4) *If $a \in \widetilde{\mathscr{A}_0}[\tau]_{cq+}$ and $b \in (\mathscr{A}_0)_+$ such that $b - a \in \widetilde{\mathscr{A}_0}[\tau]_{q+}$, then $a \in (\mathscr{A}_0)_+$.*

PROOF. (1) Let $a \in \widetilde{\mathscr{A}_0}[\tau]_{cq+}$. Then, there is a net $\{x_\alpha\}_{\alpha\in\Delta}$ in $(\mathscr{A}_0)_+$, such that $x_\alpha x_\beta = x_\beta x_\alpha$, for all $\alpha, \beta \in \Delta$, and $x_\alpha \underset{\tau}{\to} a$. Using properties of the positive elements of a C^*-algebra and the condition (T_3), we have that for every $\lambda \in \Lambda$, there is $\lambda' \in \Lambda$ with

$$\begin{aligned} p_\lambda((1 + x_\alpha)^{-1} - (1 + x_\beta)^{-1}) &= p_\lambda((1 + x_\alpha)^{-1}(x_\alpha - x_\beta)(1 + x_\beta)^{-1}) \\ &\leq \|(1 + x_\alpha)^{-1}\|_0 \|(1 + x_\beta)^{-1}\|_0 p_{\lambda'}(x_\alpha - x_\beta) \\ &\leq p_{\lambda'}(x_\alpha - x_\beta) \to 0. \end{aligned}$$

So, $\{(1 + x_\alpha)^{-1}\}_{\alpha\in\Delta}$ is a Cauchy net in $\mathscr{A}_0[\tau]$ consisting of elements of $\mathscr{U}(\mathscr{A}_0)_+$, which by (T_4) is τ-closed. Hence,

$$(1 + x_\alpha)^{-1} \underset{\tau}{\to} y \in \mathscr{U}(\mathscr{A}_0)_+. \tag{3.1}$$

We shall show that $(1 + a)^{-1}$ exists and equals y. Indeed: Using again condition

(T_3), for each $\lambda \in \Lambda$, there is $\lambda' \in \Lambda$ with

$$p_\lambda(1-(1+a)(1+x_\alpha)^{-1}) = p_\lambda((x_\alpha - a)(1+x_\alpha)^{-1}) \\ \leq \|(1+x_a)^{-1}\|_0 p_{\lambda'}(x_\alpha - a) \leq p_{\lambda'}(x_\alpha - a) \to 0.$$

Therefore,

$$(1+a)(1+x_\alpha)^{-1} \underset{\tau}{\to} 1. \tag{3.2}$$

On the other hand, since

$$x_\beta y = \tau - \lim_\alpha x_\beta(1+x_\alpha)^{-1} = \tau - \lim_\alpha (1+x_\alpha)^{-1}x_\beta = yx_\beta, \ \forall\ \beta \in \Delta,$$

we have $ay = ya$. Further, we can show that

$$(1+a)(1+x_\alpha)^{-1} \underset{\tau}{\to} (1+a)y. \tag{3.3}$$

Indeed, since $x_\alpha \underset{\tau}{\to} a$, for any $\varepsilon > 0$ there exists $\alpha_0 \in \Delta$ such that for all $\alpha \geq \alpha_0$ and all $\lambda \in \Lambda$ one has $p_\lambda(x_\alpha - a) < \varepsilon$. Now, by (T_3) we have that for any $\alpha \in \Delta$

$$\begin{aligned} &p_\lambda((1+a)(1+x_\alpha)^{-1} - (1+a)y) \\ &\quad \leq p_\lambda((1+a)(1+x_\alpha)^{-1} - (1+x_{\alpha_0})(1+x_\alpha)^{-1}) \\ &\quad + p_\lambda((1+x_{\alpha_0})(1+x_\alpha)^{-1} - (1+x_{\alpha_0})y) + p_\lambda((1+x_{\alpha_0})y - (1+a)y) \\ &\quad \leq p_{\lambda'}(a - x_{\alpha_0}) + \|1+x_{\alpha_0}\|_0 p'_\lambda((1+x_\alpha)^{-1} - y) + p_\lambda(x_{\alpha_0} - a) \\ &\quad < 2\varepsilon + \|1+x_{\alpha_0}\|_0 p'_\lambda((1+x_\alpha)^{-1} - y), \end{aligned}$$

which by (3.1) implies that $\lim_\alpha p_\lambda((1+a)(1+x_\alpha)^{-1} - (1+a)y) = 0$. Thus, by (3.2) and (3.3) we have $(1+a)y = 1 = y(1+a)$. Hence, $(1+a)^{-1}$ exists and belongs to $\mathscr{U}(\mathscr{A}_0)_+$ (since y does).

(2) It is clear from (1) that for every $\varepsilon > 0$ the element $(1+\varepsilon a)^{-1}$ exists in $\widetilde{\mathscr{A}_0}[\tau]$ and belongs to $\mathscr{U}(\mathscr{A}_0)_+$. In particular, applying (T_3) we get that for each $\lambda \in \Lambda$, there is $\lambda' \in \Lambda$ with

$$p_\lambda(1-(1+\varepsilon a)^{-1}) = \varepsilon p_\lambda(a(1+\varepsilon a)^{-1}) \leq \varepsilon \|(1+\varepsilon a)^{-1}\|_0 p_{\lambda'}(a) \leq \varepsilon p_{\lambda'}(a),$$

so that

$$\tau\text{-}\lim_{\varepsilon\to 0}(1+\varepsilon a)^{-1} = 1. \tag{3.4}$$

On the other hand, from the very definitions one has

$$a_\varepsilon = a(1+\varepsilon a)^{-1} = (1+\varepsilon a)^{-1}a = \frac{1}{\varepsilon}(1-(1+\varepsilon a)^{-1}), \quad \forall\, \varepsilon > 0, \quad \text{and}$$
$$a - a_\varepsilon = a(1-(1+\varepsilon a)^{-1}) = (1-(1+\varepsilon a)^{-1})a \in \widetilde{\mathscr{A}_0}[\tau]_{cq+}. \tag{3.5}$$

Now, by the same way as in (3.3), we conclude from (3.4) and (3.5) that $\tau\text{-}\lim_{\varepsilon\to 0} a_\varepsilon = a$.

(3) Let $a \in \widetilde{\mathscr{A}_0}[\tau]_{cq+} \cap (-\widetilde{\mathscr{A}_0}[\tau]_{cq+})$. For any $\varepsilon > 0$, we have by (2) that

$$(\mathscr{A}_0)_+ \ni a(1+\varepsilon a)^{-1} \underset{\tau}{\to} a \text{ and } (\mathscr{A}_0)_+ \ni (-a)(1-\varepsilon a)^{-1} \underset{\tau}{\to} -a.$$

Thus, if

$$x_\varepsilon := a(1+\varepsilon a)^{-1} - (-a)(1-\varepsilon a)^{-1}, \tag{3.6}$$

we get

$$\begin{aligned} x_\varepsilon = a((1+\varepsilon a)^{-1} + (1-\varepsilon a)^{-1}) &= a(1+\varepsilon a)^{-1}(1-\varepsilon a+1+\varepsilon a)(1-\varepsilon a)^{-1} \\ &= 2a(1+\varepsilon a)^{-1}(1-\varepsilon a)^{-1}, \end{aligned}$$

where by (1) and (2) we conclude that $(1-\varepsilon a)^{-1} \in (\mathscr{A}_0)_+$ and $a(1+\varepsilon a)^{-1} \in (\mathscr{A}_0)_+$ respectively. Therefore, $x_\varepsilon \in (\mathscr{A}_0)_+$ according to the functional calculus in commutative C^*-algebras. Similarly, we have that

$$-x_\varepsilon = 2(-a)(1-\varepsilon a)^{-1}(1+\varepsilon a)^{-1} \in (\mathscr{A}_0)_+$$

since $(-a)(1-\varepsilon a)^{-1}$ and $(1+\varepsilon a)^{-1}$ belong to $(\mathscr{A}_0)_+$. Thus,

$$x_\varepsilon \in (\mathscr{A}_0)_+ \cap (-(\mathscr{A}_0)_+) = \{0\}$$

and so (see (3.6))

$$a(1+\varepsilon a)^{-1} = -a(1-\varepsilon a)^{-1}.$$

Taking τ-limits with $\varepsilon \to 0$, we get $a = -a$, i.e., $a = 0$.

(4) By (2) and the assumptions in (4), $b-a$ and $a-a_\varepsilon$ are contained in $\widetilde{\mathscr{A}_0}[\tau]_{q+}$. Since, $\widetilde{\mathscr{A}_0}[\tau]_{q+}$ is a wedge, $b - a_\varepsilon = (b-a) + (a-a_\varepsilon) \in \widetilde{\mathscr{A}_0}[\tau]_{q+}$. Furthermore, by (T_4)

$$b - a_\varepsilon \in \widetilde{\mathscr{A}_0}[\tau]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+, \quad \forall\, \varepsilon > 0.$$

Hence,

$$\|a_\varepsilon\|_0 \leq \|b\|_0, \quad \forall\, \varepsilon > 0,$$

so that if $b = 0$, then $a = 0 \in (\mathscr{A}_0)_+$ since $a = \tau\text{-}\lim_{\varepsilon \to 0} a_\varepsilon$. If $b \neq 0$ then $\left\{a_\varepsilon/\|b\|_0 : \varepsilon > 0\right\} \subset \mathscr{U}(\mathscr{A}_0)_+$ and by (T_4) $\mathscr{U}(\mathscr{A}_0)_+$ is τ-closed; so again we get that $a \in (\mathscr{A}_0)_+$. □

The above lead to the following

DEFINITION 3.3. A quasi $*$-subalgebra $\mathscr{A}$ of the locally convex quasi $*$-algebra $\widetilde{\mathscr{A}_0}[\tau]$ over $\mathscr{A}_0$, where $\mathscr{A}_0[\|\cdot\|_0]$ is a unital C^*-algebra and τ a locally convex topology on $\mathscr{A}_0$ satisfying the conditions (T_1)–(T_4), is said to be a locally convex quasi C^*-algebra over $\mathscr{A}_0$.

We present now some examples of locally convex quasi C^*-algebras.

EXAMPLE 3.4 (GB^*-algebras). Let $\mathscr{A}[\tau]$ be a GB^*-algebra over B_0 (see Section 2). Then, $\mathscr{A}_0[\|\cdot\|_0] \equiv A[B_0]$ is a C^*-algebra under the C^*-norm $\|\cdot\|_0 \equiv \|\cdot\|_{B_0}$ given by the Minkowski functional of B_0. Assume that the locally convex topology τ fulfils the condition (T_3). Then, it is easily checked that $\mathscr{A}[\tau]$ is a locally convex quasi C^*-algebra over $\mathscr{A}_0$.

EXAMPLE 3.5 (Banach quasi C^*-algebras). Let $\mathscr{A}_0[\|\cdot\|_0]$ be a unital C^*-algebra and $\tau = \|\cdot\|$ a norm topology on $\mathscr{A}_0$ with the properties (T_1)–(T_4). That is,

(T_1) $\mathscr{A}_0[\|\cdot\|]$ is a locally convex $*$-algebra;
(T_2) $\|\cdot\| \preceq \|\cdot\|_0$;
(T_3) $\|xy\| \leq \|x\|_0\|y\|$, $\forall\, x, y \in \mathscr{A}_0$ with $xy = yx$;
(T_4) $\mathscr{U}(\mathscr{A}_0)_+$ is $\|\cdot\|$-closed, and $\widetilde{\mathscr{A}_0}[\|\cdot\|]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+$.

Then, a locally convex quasi C^*-algebra over $\mathscr{A}_0$ is called a normed quasi C^*-algebra over $\mathscr{A}_0$. In particular, the completion $\widetilde{\mathscr{A}_0}[\|\cdot\|]$ of $\mathscr{A}_0[\|\cdot\|]$ is said to be a Banach quasi C^*-algebra over $\mathscr{A}_0$.

Notice that the Banach space $L^p[0,1]$, $1 \leq p < \infty$, is a Banach quasi C^*-

algebra over the C^*-algebra $L^\infty[0,1]$.

EXAMPLE 3.6 (proper CQ^*-algebras). A quasi $*$-algebra $(\mathscr{X}, \mathscr{A}_0)$ is said to be a Banach quasi $*$-algebra over $\mathscr{A}_0$ (see [**12**]), if a norm $\|\cdot\|$ is defined on $\mathscr{X}$ with the properties:

(i) $\mathscr{X}[\|\cdot\|]$ is a Banach space;
(ii) $\|x^*\| = \|x\|, \ \forall\ x \in \mathscr{X}$;
(iii) $\mathscr{A}_0$ is dense in $\mathscr{X}[\|\cdot\|]$;
(iv) for each $a \in \mathscr{A}_0$, the map $L_a : \mathscr{X} \to \mathscr{X} : x \mapsto ax$, is continuous.

The continuity of the involution implies that for each $a \in \mathscr{A}_0$, the map $R_a : \mathscr{X} \to \mathscr{X} : x \mapsto xa$, is continuous.

The identity of $(\mathscr{X}, \mathscr{A}_0)$ is an element $1 \in \mathscr{A}_0$ such that $1x = x1 = x$, for each $x \in \mathscr{X}$. Let $(\mathscr{X}, \mathscr{A}_0)$ be a unital Banach quasi $*$-algebra. Then, $\mathscr{A}_0$ is a normed $*$-algebra under the norm

$$\|a\|_{op} := \max\{\|L_a\|, \|R_a\|\}, \quad \forall\ a \in \mathscr{A}_0, \text{ and}$$

$$\|a\| \le \|a\|_{op}, \quad \forall\ a \in \mathscr{A}_0, \tag{3.7}$$

$$\|ab\| \le \|a\|\|b\|_{op}, \quad \forall\ a, b \in \mathscr{A}_0. \tag{3.8}$$

An element x of $\mathscr{X}$ is said to be bounded if the map $R_x : \mathscr{A}_0 \to \mathscr{X} : a \mapsto ax$ is continuous, equivalently the map $L_x : \mathscr{A}_0 \to \mathscr{X} : a \mapsto xa$ is continuous. Then, R_x respectively L_x extend to bounded linear operators $\overline{R}_x$ resp. $\overline{L}_x$. Denote by $\mathscr{X}_b$ the set of all bounded elements of $\mathscr{X}$. Then $\mathscr{X}$ is said to be normal [**23**] if $\overline{L}_x y = \overline{R}_y x$ for every $x, y \in \mathscr{X}_b$. In this case, $\mathscr{X}_b$ is a Banach $*$-algebra equipped with the multiplication

$$x \circ y = \overline{L}_x y, \quad \forall\ x, y \in \mathscr{X}_b$$

and the norm $\|x\|_b := \max\{\|\overline{L}_x\|, \|\overline{R}_x\|\}, x \in \mathscr{X}_b$ (see [**23**, Corollary 2.14]). Furthermore, we have

$$\overline{\mathscr{U}(\mathscr{A}_0[\|\cdot\|_{op}])}^{\|\cdot\|} \subset \mathscr{U}(\mathscr{X}_b). \tag{3.9}$$

Indeed, take an arbitrary $x \in \overline{\mathscr{U}(\mathscr{A}_0[\|\cdot\|_{op}])}^{\|\cdot\|}$. Then, there is a sequence $\{a_n\}$ in $\mathscr{U}(\mathscr{A}_0[\|\cdot\|_{op}])$ such that $\lim\limits_{n\to\infty} \|a_n - x\| = 0$. On the other hand, using (3.8), we have that for each $b \in \mathscr{A}_0$

$$\|xb\| = \lim_n \|a_n b\| \le \overline{\lim_{n\to\infty}} \|a_n\|_{op}\|b\| \le \|b\|$$

and similarly $\|bx\| \leq \|b\|$. Hence, $x \in \mathscr{U}(\mathscr{X}_b)$.

If $\mathscr{A}_0 = \mathscr{X}_b$, then the Banach quasi $*$-algebra $(\mathscr{X}, \mathscr{A}_0)$ is said to be full. If $\mathscr{A}_0[\|\cdot\|_{op}]$ is a C^*-algebra, then $(\mathscr{X}, \mathscr{A}_0)$ is called a proper CQ^*-algebra [**12**].

Let $(\mathscr{X}, \mathscr{A}_0)$ be a full proper CQ^*-algebra. Suppose $\widetilde{\mathscr{A}_0}[\|\cdot\|]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+$. Then, $\mathscr{U}(\mathscr{A}_0)_+$ is $\|\cdot\|$-closed. Indeed, take an arbitrary $x \in \overline{\mathscr{U}(\mathscr{A}_0)_+}^{\|\cdot\|}$. Then, there is a sequence $\{a_n\}$ in $\mathscr{U}(\mathscr{A}_0)_+$ such that $\lim_{n\to\infty} \|a_n - x\| = 0$. Since $(\mathscr{X}, \mathscr{A}_0)$ is full, it follows from (3.9) that $x \in \mathscr{U}(\mathscr{A}_0)$, which implies $x \in \widetilde{\mathscr{A}_0}[\|\cdot\|]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+$. Thus, $\mathscr{U}(\mathscr{A}_0)_+$ is $\|\cdot\|$-closed.

Banach quasi C^*-algebras are related to proper CQ^*-algebras in the following way:

1. If $(\mathscr{X}, \mathscr{A}_0)$ is a full proper CQ^*-algebra with $\widetilde{\mathscr{A}_0}[\|\cdot\|]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+$, then $\mathscr{X}$ is a Banach quasi C^*-algebra over the C^*-algebra $\mathscr{A}_0[\|\cdot\|_{op}]$.

This follows by the very definitions (in this respect, see also Example 3.5) and (3.7), (3.8), (3.9).

2. Conversely, suppose that $\mathscr{A}$ is a Banach quasi C^*-algebra over the C^*-algebra $\mathscr{A}_0[\|\cdot\|_0]$. Then, $(\mathscr{A}, \mathscr{A}_0)$ is a proper CQ^*-algebra if and only if $\|a\|_{op} = \|a\|_0$, for all $a \in \mathscr{A}_0$.

We consider the following realization of this situation. Let I be a compact interval of $\boldsymbol{R}$. Then, it is shown that the proper CQ^*-algebra $(L^p(I), L^\infty(I))$ is a Banach quasi C^*-algebra over $L^\infty(I)$, but the proper CQ^*-algebra $(L^p(I), C(I))$ is not a Banach quasi C^*-algebra over $C(I)$.

A noncommutative example of a proper CQ^*-algebra, which is also a Banach quasi C^*-algebra, can be constructed as follows. Let S be a (possibly unbounded) selfadjoint operator in a Hilbert space $\mathscr{H}$, with $S \geq I$. Let $\mathscr{C}(S)$ be the von Neumann algebra

$$\mathscr{C}(S) = \{X \in \mathscr{B}(\mathscr{H}) : XS^{-1} = S^{-1}X\},$$

where $\mathscr{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators on $\mathscr{H}$. We denote with $\|\cdot\|_0$ the operator norm in $\mathscr{B}(\mathscr{H})$. Let us define on $\mathscr{C}(S)$ the norm

$$\|X\| = \|S^{-1}XS^{-1}\|_0, \quad X \in \mathscr{C}(S).$$

Let $\widetilde{\mathscr{C}(S)}$ denote the $\|\cdot\|$-completion of $\mathscr{C}(S)$. Then, it is easily seen that $(\widetilde{\mathscr{C}(S)}, \mathscr{C}(S))$ is a proper CQ^*-algebra. Making use of the weak topology of $\mathscr{B}(\mathscr{H})$, one can prove that (T_4) also holds on $\mathscr{C}(S)$. The proof will be given in the next Section in a more general context. Then, $\widetilde{\mathscr{C}(S)}$ is a locally convex quasi C^*-algebra.

EXAMPLE 3.7. In this example we will shortly discuss the so-called physical topologies on a noncommutative C^*-algebra, first introduced by Lassner [**18**], [**19**] in the early 1980's. Thereafter these topologies revealed to be very useful for the description of many quantum physical models with an infinite number of degrees of freedom (for reviews see [**22**], [**9**] and [**6**, Ch. 11]). In view of these applications, it seems interesting to consider the question under which conditions they can be cast in the framework developed in this paper.

Let $\mathscr{A}_0$ be a C^*-algebra and $\Sigma = \{\pi_\alpha; \alpha \in I\}$ a system of $*$-representations of $\mathscr{A}_0$ on a dense subspace $\mathscr{D}_\alpha$ of a Hilbert space $\mathscr{H}_\alpha$, i.e. each π_α is a $*$-homomorphism of $\mathscr{A}_0$ into the O^*-algebra $\mathscr{L}^\dagger(\mathscr{D}_\alpha)$ (see Section 4). Since $\mathscr{A}_0$ is a C^*-algebra, each π_α is a bounded $*$-representation, i.e. $\overline{\pi_\alpha(x)} \in \mathscr{B}(\mathscr{H}_\alpha)$, for every $x \in \mathscr{A}_0$. The system Σ is supposed to be faithful, in the sense that if $x \in \mathscr{A}_0$, $x \neq 0$, then there exists $\alpha \in \Sigma$ such that $\pi_\alpha(x) \neq 0$. The physical topology τ_Σ is the coarsest locally convex topology on $\mathscr{A}_0$ such that every $\pi_\alpha \in \Sigma$ is continuous from $\mathscr{A}[\tau_\Sigma]$ into $\mathscr{L}^\dagger(\mathscr{D}_\alpha)[\tau_u(\mathscr{L}^\dagger(\mathscr{D}_\alpha))]$, where $\tau_u(\mathscr{L}^\dagger(\mathscr{D}_\alpha))$ is the $\mathscr{L}^\dagger(\mathscr{D}_\alpha)$-uniform topology of $\mathscr{L}^\dagger(\mathscr{D}_\alpha)$ (see Section 4). This topology depends, of course, on the choice of an appropriate system Σ of $*$-representations of $\mathscr{A}_0$; these $*$-representations are, in general nothing but the GNS representations constructed starting from a family ω_α of states which are relevant (and they are usually called in this way) for the physical model under consideration. Every physical topology satisfies the conditions (T_1), (T_2) and (T_4), but it does not necessarily satisfy (T_3). Here we show that $\widetilde{\mathscr{A}_0[\tau_\Sigma]}$ is a locally convex quasi C^*-algebra over $\mathscr{A}_0$ for some special choice of the system Σ of *-representations of $\mathscr{A}_0$. Suppose that $\mathscr{D}_\alpha = \mathscr{D}^\infty(M_\alpha) = \bigcap_{n\in\boldsymbol{N}} \mathscr{D}(M_\alpha^n)$, where M_α is a selfadjoint unbounded operator. Without loss of generality we may assume that $M_\alpha \geq I_\alpha$, with I_α the identity operator in $\mathscr{B}(\mathscr{H}_\alpha)$. Let Σ be a system of representations π_α of $\mathscr{A}_0$ on $\mathscr{D}_\alpha$ such that $\pi_\alpha(x)M_\alpha\xi = M_\alpha\pi_\alpha(x)\xi$, for every $x \in \mathscr{A}_0$ and for every $\xi \in \mathscr{D}_\alpha$. Then $\widetilde{\mathscr{A}_0[\tau_\Sigma]}$ is a locally convex quasi C^*-algebra over $\mathscr{A}_0$. This follows from the fact that, in this case, the physical topology τ_Σ is defined by the family of seminorms

$$p_\alpha^f(x) := \|f(M_\alpha)\overline{\pi_\alpha(x)}\|_0 \text{ (operator } C^*\text{-norm)}, \ \forall\, x \in \mathscr{A}_0,$$

where $\pi_\alpha \in \Sigma$ and f runs over the set $\mathscr{F}$ of all positive, bounded and continuous functions on $\boldsymbol{R}^+$ such that $\sup_{x\in\boldsymbol{R}^+} x^k f(x) < \infty$, for every $k = 0, 1, 2, \ldots$ [**19**, Lemma 2.8], and from the inequality

$$p_\alpha^f(xy) = \|f(M_\alpha)\overline{\pi_\alpha(x)\pi_\alpha(y)}\|_0 \leq \|\overline{\pi_\alpha(x)}\|_0 p_\alpha^f(y), \ \forall\, x, y \in \mathscr{A}_0.$$

4. Locally convex quasi C^*-algebras of operators.

Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathscr{H}$. Let $\mathscr{L}(\mathscr{D})$ be the algebra (under usual algebraic operations) of all linear operators from $\mathscr{D}$ to $\mathscr{D}$ and $\mathscr{L}^\dagger(\mathscr{D}) := \{X \in \mathscr{L}(\mathscr{D}) : \mathscr{D}(X^*) \supset \mathscr{D} \text{ and } X^*\mathscr{D} \subset \mathscr{D}\}$, where $\mathscr{D}(X^*)$ stands for the domain of the adjoint X^* of X. Then $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra under the involution $X^\dagger := X^*\lceil\mathscr{D}$ (see [**17**, p. 8]). Furthermore, let $\mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$ denote all linear operators X from $\mathscr{D}$ to $\mathscr{H}$ such that $\mathscr{D}(X^*) \supset \mathscr{D}$. Then, $\mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$ is a $*$-preserving vector space endowed with the usual linear operations and the involution $X^\dagger := X^*\lceil\mathscr{D}$ (ibid., p. 23). In particular, $\mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$ is a partial $*$-algebra [**6**, Proposition 2.1.11] under the (weak) partial multiplication $X\Box Y = X^{\dagger *}Y$, defined whenever $Y\mathscr{D} \subset \mathscr{D}(X^{\dagger *})$ and $X^\dagger\mathscr{D} \subset \mathscr{D}(Y^*), X, Y \in \mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$.

Let now $\mathscr{M}_0$ be a unital C^*-algebra over $\mathscr{H}$ that leaves $\mathscr{D}$ invariant, i.e., $\mathscr{M}_0\mathscr{D} \subset \mathscr{D}$. Then, the restriction $\mathscr{M}_0\lceil\mathscr{D}$ of $\mathscr{M}_0$ to $\mathscr{D}$ is an O^*-algebra on $\mathscr{D}$, therefore an element X of $\mathscr{M}_0$ is regarded as an element $X\lceil\mathscr{D}$ of $\mathscr{M}_0\lceil\mathscr{D}$. Moreover, let

$$\mathscr{M}_0 \subset \mathscr{M} \subset \mathscr{L}^\dagger(\mathscr{D},\mathscr{H}),$$

where $\mathscr{M}$ is an O^*-vector space on $\mathscr{D}$, that is, a $*$-invariant subspace of $\mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$. Denote by $\mathscr{B}(\mathscr{M})$ the set of all bounded subsets of $\mathscr{D}[t_{\mathscr{M}}]$, where $t_{\mathscr{M}}$ is the graph topology on $\mathscr{M}$ (see [**17**, p. 9]). Further, denote by $\mathscr{B}_f(\mathscr{D})$ the set of all finite subsets of $\mathscr{D}$. Then $\mathscr{B}_f(\mathscr{D}) \subset \mathscr{B}(\mathscr{M})$. A subset $\mathscr{B}$ of $\mathscr{B}(\mathscr{M})$ is called admissible if the following hold:

(i) $\mathscr{B}_f(\mathscr{D}) \subset \mathscr{B}$,
(ii) $\forall\ \mathfrak{M}_1, \mathfrak{M}_2 \in \mathscr{B},\ \exists\ \mathfrak{M}_3 \in \mathscr{B} : \mathfrak{M}_1 \cup \mathfrak{M}_2 \subset \mathfrak{M}_3$,
(iii) $A\mathfrak{M} \in \mathscr{B}$, $\forall\ A \in \mathscr{M}_0$ and $\forall\ \mathfrak{M} \in \mathscr{B}$.

It is clear that $\mathscr{B}_f(\mathscr{D})$ and $\mathscr{B}(\mathscr{M})$ are admissible. Consider now an arbitrary admissible subset $\mathscr{B}$ of $\mathscr{B}(\mathscr{M})$. Then, for any $\mathfrak{M} \in \mathscr{B}$ define the following seminorms on $\mathscr{M}$:

$$p_{\mathfrak{M}}(X) := \sup_{\xi,\eta\in\mathfrak{M}} |(X\xi|\eta)|, \quad X \in \mathscr{M} \tag{4.1}$$

$$p^{\mathfrak{M}}(X) := \sup_{\xi\in\mathfrak{M}} \|X\xi\|, \quad X \in \mathscr{M} \tag{4.2}$$

$$p^{\mathfrak{M}}_\dagger(X) := \sup_{\xi\in\mathfrak{M}} \{\|X\xi\| + \|X^\dagger\xi\|\}, \quad X \in \mathscr{M}. \tag{4.3}$$

We call the corresponding locally convex topologies on $\mathscr{M}$ defined by the families (4.1), (4.2) and (4.3) of seminorms, $\mathscr{B}$-uniform topology, strongly $\mathscr{B}$-uniform topology, resp. strongly* $\mathscr{B}$-uniform topology on $\mathscr{M}$ and denote them by $\tau_u(\mathscr{B})$,

$\tau^u(\mathscr{B})$, resp. $\tau^u_*(\mathscr{B})$. In particular, the $\mathscr{B}(\mathscr{M})$-uniform topology, the strongly $\mathscr{B}(\mathscr{M})$-uniform topology, resp. the strongly* $\mathscr{B}(\mathscr{M})$-uniform topology will be simply called $\mathscr{M}$-uniform topology, strongly $\mathscr{M}$-uniform topology, resp. strongly* $\mathscr{M}$-uniform topology and will be denoted by $\tau_u(\mathscr{M})$, $\tau^u(\mathscr{M})$, resp. $\tau^u_*(\mathscr{M})$. In the book of Schmüdgen [**20**], these topologies are called bounded topologies and $\tau_u(\mathscr{B})$, $\tau^u(\mathscr{B})$ are denoted by $\tau_{\mathscr{B}}$, $\tau^{\mathscr{B}}$, while $\tau_u(\mathscr{M})$, $\tau^u(\mathscr{M})$ are denoted by $\tau_{\mathscr{D}}$, $\tau^{\mathscr{D}}$, respectively. The $\mathscr{B}_f(\mathscr{D})$-uniform topology, the strongly $\mathscr{B}_f(\mathscr{D})$-uniform topology, resp. the strongly* $\mathscr{B}_f(\mathscr{D})$-uniform topology is called weak topology, strong topology, resp. strong*-topology on $\mathscr{M}$, denoted resp. by τ_w, τ_s and τ_{s^*}. All these topologies are related in the following way:

$$\begin{array}{ccccc} \tau_w & \preceq & \tau_u(\mathscr{B}) & \preceq & \tau_u(\mathscr{M}) \\ \wedge| & & \wedge| & & \wedge| \\ \tau_s & \preceq & \tau^u(\mathscr{B}) & \preceq & \tau^u(\mathscr{M}) \\ \wedge| & & \wedge| & & \wedge| \\ \tau_{s^*} & \preceq & \tau^u_*(\mathscr{B}) & \preceq & \tau^u_*(\mathscr{M}). \end{array} \tag{4.4}$$

We investigate now whether $\widetilde{\mathscr{M}_0}[\tau_u(\mathscr{B})]$ and $\widetilde{\mathscr{M}_0}[\tau^u_*(\mathscr{B})]$ are locally convex quasi C^*-algebras over $\mathscr{M}_0$. So, we must check the properties (T_1)–(T_4) (stated before and after Definition 3.1) for the locally convex topologies $\tau_u(\mathscr{B}), \tau^u_*(\mathscr{B})$ and the operator C^*-norm $\|\cdot\|_0$ on $\mathscr{M}_0$.

(T_1) This follows easily for both topologies, since $\mathscr{B}$ is admissible and $\mathscr{M}_0\mathscr{D} \subset \mathscr{D}$.

(T_2) Notice that for all $X \in \mathscr{M}_0$ and $\mathfrak{M} \in \mathscr{B}$ we have:

$$p^{\mathfrak{M}}_\dagger(X) = \sup_{\xi\in\mathfrak{M}} \{\|X\xi\| + \|X^\dagger\xi\|\} \leq \Big(2 \sup_{\xi\in\mathfrak{M}} \|\xi\|\Big)\|X\|_0,$$

so by (4.4) we conclude that $\tau_u(\mathscr{B}) \preceq \tau^u_*(\mathscr{B}) \preceq \|\cdot\|_0$.

(T_3) Concerning $\tau^u_*(\mathscr{B})$, the property (T_3) follows easily from the very definitions. Now, notice the following: For any $X, Y \in \mathscr{M}_0$ with $XY = YX$ and $Y^* = Y$, one concludes that

$$p_{\mathfrak{M}}(XY) \leq \|X\|_0 \sup_{\xi\in\mathfrak{M}} (|Y|\xi \mid \xi), \quad \forall\ \mathfrak{M} \in \mathscr{B}, \tag{4.5}$$

where $|Y| := (Y^2)^{1/2}$. Then, it follows that for any $X, Y \in \mathscr{M}_0$ with $XY = YX$ and $Y \geq 0$, one has

$$p_{\mathfrak{M}}(XY) \leq \|X\|_0 \sup_{\xi\in\mathfrak{M}}(Y\xi \mid \xi), \quad \forall\, \mathfrak{M} \in \mathscr{B}.$$

We prove (4.5). From the polar decomposition of Y, there is a unique partial isometry V from $\mathscr{H}$ to $\mathscr{H}$ such that

$$Y = V|Y| = |Y|V, \quad \ker(V) = \ker(Y) \text{ and } VY = |Y|.$$

By continuous functional calculus it follows that: X commutes with both $|Y|$ and $|Y|^{1/2}$, but also $V|Y|^{1/2} = |Y|^{1/2}V$. Thus,

$$\begin{aligned} p_{\mathfrak{M}}(XY) &= \sup_{\xi,\eta\in\mathfrak{M}} \big|\big(XY\xi \mid \eta\big)\big| = \sup_{\xi,\eta\in\mathfrak{M}} \big|\big(V|Y|X\xi \mid \eta\big)\big| \\ &= \sup_{\xi,\eta\in\mathfrak{M}} \big|\big(X|Y|^{1/2}\xi \mid |Y|^{1/2}V\eta\big)\big| \leq \sup_{\xi,\eta\in\mathfrak{M}} \|X\|_0 \big\||Y|^{1/2}\xi\big\| \big\||Y|^{1/2}\eta\big\| \\ &\leq \frac{1}{2}\|X\|_0 \sup_{\xi,\eta\in\mathfrak{M}} \big(\big\||Y|^{1/2}\xi\big\|^2 + \big\||Y|^{1/2}\eta\big\|^2\big) \\ &\leq \|X\|_0 \sup_{\xi\in\mathfrak{M}} \big(|Y|\xi \mid \xi\big), \quad \forall\, \mathfrak{M} \in \mathscr{B}. \end{aligned}$$

But, we can not say whether (T_3) holds for $\tau_u(\mathscr{B})$. In the case when $\mathscr{M}_0$ is a von Neumann algebra we have the following:

- If $\mathscr{M}_0$ is commutative, then (T_3) holds for the topology τ_w.
- If $\mathscr{M}$ is a commutative O^*-algebra (see [**17**, p. 8]) on $\mathscr{D}$ in $\mathscr{H}$, containing $\mathscr{M}_0$, then (T_3) holds for the topology $\tau_u(\mathscr{M})$.

Indeed: Suppose that $\mathscr{M}$ is commutative with $\mathscr{M}_0 \subset \mathscr{M}$. For each $\mathfrak{M} \in \mathscr{B}(\mathscr{M})$ consider the set

$$\mathfrak{M}' := \cup\{V\mathfrak{M} : V \text{ partial isometry in } \mathscr{M}_0\}.$$

Commutativity of $\mathscr{M}$ implies that $\mathfrak{M}' \in \mathscr{B}(\mathscr{M})$. Moreover, $\mathfrak{M} \subset \mathfrak{M}'$. Let now $X, Y \in \mathscr{M}_0$. Let $Y = V|Y|$ be the polar decomposition of Y. Since $\mathscr{M}_0$ is a von Neumann algebra, we have $V \in \mathscr{M}_0$, which implies that

$$\begin{aligned} p_{\mathfrak{M}}(XY) &= \sup_{\xi,\eta\in\mathfrak{M}} \big|(XY\xi \mid \eta)\big| = \sup_{\xi,\eta\in\mathfrak{M}} \big|\big(VX|Y|^{1/2}\xi \mid |Y|^{1/2}\eta\big)\big| \\ &\leq \|VX\|_0 \sup_{\xi,\eta\in\mathfrak{M}} \big\||Y|^{1/2}\xi\big\| \big\||Y|^{1/2}\eta\big\| = \|X\|_0 \sup_{\xi\in\mathfrak{M}} \big(|Y|\xi \mid \xi\big) \\ &= \|X\|_0 \sup_{\xi\in\mathfrak{M}} \big(Y\xi \mid V^*\xi\big) \leq \|X\|_0 \sup_{\xi,\eta\in\mathfrak{M}'} \big|\big(Y\xi \mid \eta\big)\big| = \|X\|_0 p_{\mathfrak{M}'}(Y). \end{aligned}$$

Hence, (T_3) holds for $\tau_u(\mathscr{M})$.

(T_4) This property holds for all topologies in (4.4). It suffices to prove (T_4) for the topology τ_w. So, let $X \in \overline{\mathscr{U}(\mathscr{M}_0)}^{\tau_w}$ be arbitrary. Then, there is a net $\{X_\alpha\}$ in $\mathscr{U}(\mathscr{M}_0)$ with $X_\alpha \underset{\tau_w}{\longrightarrow} X$. Notice that the sesquilinear form defined on $\mathscr{D} \times \mathscr{D}$ by

$$\mathscr{D} \times \mathscr{D} \ni (\xi, \eta) \mapsto \lim_\alpha \big(X_\alpha \xi \mid \eta\big) \in \boldsymbol{C},$$

is bounded. Hence, X can be regarded as a bounded linear operator on $\mathscr{H}$ such that

$$\|X\|_0 = 1 \text{ and } \big(X\xi \mid \eta\big) = \lim_\alpha \big(X_\alpha \xi \mid \eta\big), \quad \forall\, \xi, \eta \in \mathscr{D}.$$

Since $\mathscr{D}$ is dense in $\mathscr{H}$, an easy computation shows that

$$\big(Xx \mid y\big) = \lim_\alpha \big(X_\alpha x \mid y\big), \quad \forall\, x, y \in \mathscr{H}. \tag{4.6}$$

This proves that $X \in \mathscr{M}_0 \cap \mathscr{B}(\mathscr{H})_1 = \mathscr{U}(\mathscr{M}_0)$, which means that $\mathscr{U}(\mathscr{M}_0)$ is τ_w-closed. A consequence of (4.6) is now that $\mathscr{U}(\mathscr{M}_0)_+$ is weakly closed. Similarly we can show that $\widetilde{\mathscr{M}_0}[\tau_w]_{q+} \cap \mathscr{M}_0 = (\mathscr{M}_0)_+$, therefore (T_4) holds for the topology τ_w on $\mathscr{M}_0$. From (4.4), (T_4) also holds for the topologies $\tau_u(\mathscr{B})$ and $\tau^u_*(\mathscr{B})$.

From the preceding discussion we conclude the following

Proposition 4.1. *Let $\mathscr{B}$ be an admissible subset of $\mathscr{B}(\mathscr{M})$. Then, $\widetilde{\mathscr{M}_0}[\tau^u_*(\mathscr{B})]$ and $\widetilde{\mathscr{M}_0}[\tau_{s^*}]$ are locally convex quasi C^*-algebras over $\mathscr{M}_0$. If $\mathscr{M}_0$ is a von Neumann algebra and there is a commutative O^*-algebra $\mathscr{M}$ on $\mathscr{D}$ in $\mathscr{H}$, containing $\mathscr{M}_0$, then $\widetilde{\mathscr{M}_0}[\tau_w]$ and $\widetilde{\mathscr{M}_0}[\tau_u(\mathscr{M})]$ are commutative locally convex quasi C^*-algebras over $\mathscr{M}_0$.*

Remark 4.2. (1) In general, we do not know whether $\widetilde{\mathscr{M}_0}[\tau_u(\mathscr{B})]$ and $\widetilde{\mathscr{M}_0}[\tau_w]$ are locally convex quasi C^*-algebras.

(2) The locally convex quasi C^*-algebra $\widetilde{\mathscr{M}_0}[\tau_{s^*}]$ over $\mathscr{M}_0$, equals to the completion $\widetilde{\mathscr{M}_0''}[\tau_{s^*}]$ of the von Neumann algebra $\mathscr{M}_0''$ with respect to the topology τ_{s^*}, but $\widetilde{\mathscr{M}_0''}[\tau_{s^*}]$ is not necessarily a locally convex quasi C^*-algebra over $\mathscr{M}_0''$, since in general, $\mathscr{M}_0''\mathscr{D} \not\subset \mathscr{D}$. In the case when $\mathscr{M}_0''\mathscr{D} \subset \mathscr{D}$, one has the equality

$$\widetilde{\mathscr{M}_0''}[\tau_{s^*}] = \widetilde{\mathscr{M}_0}[\tau_{s^*}],$$

set-theoretically; but, the corresponding locally convex quasi C^*-algebras over $\mathscr{M}_0$

do not coincide. In particular, one has that

$$\widetilde{\mathscr{M}_0}[\tau_{s^*}]_{cq+} \subsetneq \widetilde{\mathscr{M}_0''}[\tau_{s^*}]_{cq+}.$$

We present now some properties of the locally convex quasi C^*-algebra $\widetilde{\mathscr{M}_0}[\tau_{s^*}]$.

PROPOSITION 4.3. *Let $A \in \widetilde{\mathscr{M}_0}[\tau_{s^*}]_{q+}$. Consider the following*:

(i) $A \in \widetilde{\mathscr{M}_0}[\tau_{s^*}]_{cq+}$.

(ii) $(I+A)^{-1}$ *exists and belongs to* $\mathscr{U}(\mathscr{M}_0)_+$.

(iii) *The closure* $\overline{A}$ *of* A *is a positive self-adjoint operator.*

Then, one has that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF. (i) $\Rightarrow$ (ii) It follows from Proposition 3.2, (1).

(ii) $\Rightarrow$ (iii) Since $(I+A)^{-1}$ is a bounded self-adjoint operator and $(I+A)^{-1}\mathscr{D} \subset \mathscr{D}$, it follows that

$$\big((I+A)^{-1}(I+A^*)\xi \mid \eta\big) = \big((I+A^*)\xi \mid (I+A)^{-1}\eta\big) = (\xi \mid \eta),$$

for all $\xi \in \mathscr{D}(A^*)$ and $\eta \in \mathscr{D}$, which implies

$$\begin{aligned}(A^*\xi \mid \zeta) &= \big((I+A^*)\xi \mid (I+A)^{-1}(I+A^*)\zeta\big) - (\xi \mid \zeta)\\ &= \big(\xi \mid (I+A^*)\zeta\big) - (\xi \mid \zeta) = (\xi \mid A^*\zeta),\ \forall\, \xi, \zeta \in \mathscr{D}(A^*).\end{aligned}$$

Hence, $\xi \in \mathscr{D}(\overline{A})$ and $\overline{A}\xi = A^*\xi$. It is now easily seen that $\overline{A}$ is a positive self-adjoint operator. □

COROLLARY 4.4. *Suppose that $A \in \widetilde{\mathscr{M}_0''}[\tau_{s^*}]$ and $\mathscr{M}_0''\mathscr{D} \subset \mathscr{D}$. Then, the following statements are equivalent*:

(i) $A \in \widetilde{\mathscr{M}_0''}[\tau_{s^*}]_{cq+}$.

(ii) $(I+A)^{-1} \in \mathscr{U}(\mathscr{M}_0'')_+$.

(iii) $\overline{A}$ *is a positive self-adjoint operator.*

PROOF. From Proposition 4.3 we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) This follows easily by considering the spectral decomposition of $\overline{A}$. □

It is natural now to ask whether there exists an extended C^*-algebra (abbreviated to EC^*-algebra) $\mathscr{M}$ on $\mathscr{D}$ such that

$$\mathscr{M}_0 \subset \mathscr{M} \subset \widetilde{\mathscr{M}_0}[\tau_{s^*}].$$

If $\mathscr{M}$ is a closed O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$, let $\mathscr{M}_b := \{X \in \mathscr{M} : \overline{X} \in \mathscr{B}(\mathscr{H})\}$ be the bounded part of $\mathscr{M}$, where $\mathscr{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators on $\mathscr{H}$. Then, when $\overline{\mathscr{M}_b} \equiv \{\overline{X} : X \in \mathscr{M}_b\}$ is a C^*-algebra on $\mathscr{H}$ and $(I + X^*X)^{-1} \in \mathscr{M}_b$, for each $X \in \mathscr{M}$, $\mathscr{M}$ is said to be an EC^*-algebra on $\mathscr{D}$.

In this regard, we have the following, which gives a characterization of certain EC^*-algebras on $\mathscr{D}$, through the set of commutatively quasi-positive elements of $\widetilde{\mathscr{M}_0}[\tau_{s^*}]$.

PROPOSITION 4.5. *Let $\mathscr{M}$ be a closed O^*-algebra on $\mathscr{D}$ such that $\mathscr{M}_0 \subset \mathscr{M} \subset \widetilde{\mathscr{M}_0}[\tau_{s^*}]$ and $\mathscr{M}_b = \mathscr{M}_0$. Then, $\mathscr{M}$ is an EC^*-algebra on $\mathscr{D}$ if and only if $\mathscr{M}_+ \subset \widetilde{\mathscr{M}_0}[\tau_{s^*}]_{cq+}$.*

PROOF. Suppose that $\mathscr{M}$ is an EC^*-algebra on $\mathscr{D}$ and let $A \in \mathscr{M}_+$ be arbitrary. Then, since $\mathscr{M}_b = \mathscr{M}_0$, $\overline{A}$ is a bounded positive self-adjoint operator with $(I + \overline{A})^{-1} \in \mathscr{U}(\mathscr{M}_0)_+$. But, $\widetilde{\mathscr{M}_0}[\tau_{s^*}]$ is a locally convex quasi C^*-algebra (Proposition 4.1), therefore $\mathscr{U}(\mathscr{M}_0)_+$ is τ_{s^*}-closed. Note that for each $n \in \boldsymbol{N}$, the elements $X_n := \overline{A}(I + \frac{1}{n}\overline{A})^{-1}$ belong to $(\mathscr{M}_0)_+$, are commuting and $X_n \underset{\tau_{s^*}}{\longrightarrow} A$, so Definition 3.1 implies that $A \in \widetilde{\mathscr{M}_0}[\tau_{s^*}]_{cq+}$.

Conversely, suppose that $\mathscr{M}_+ \subset \widetilde{\mathscr{M}_0}[\tau_{s^*}]_{cq+}$. So, $A \in \mathscr{M}$ implies $A^\dagger A \in \widetilde{\mathscr{M}_0}[\tau_{s^*}]_{cq+}$, therefore $(I + A^\dagger A)^{-1} \in \mathscr{U}(\mathscr{M}_0)_+$ from Proposition 3.2, (1). Now, since $\mathscr{M}_b = \mathscr{M}_0$ we finally get that $\mathscr{M}$ is an EC^*-algebra on $\mathscr{D}$. □

5. Structure of commutative locally convex quasi C^*-algebras.

Throughout this Section, $\mathscr{A}[\tau]$ is a commutative locally convex quasi C^*-algebra over a unital C^*-algebra $\mathscr{A}_0$. If the multiplication of $\mathscr{A}_0$ with respect to the topology τ is jointly continuous, then $\mathscr{A}[\tau]$ is a commutative GB^*-algebra [**15**, Theorem 2.1], and so $\mathscr{A}[\tau]$ is isomorphic to a $*$-algebra of $\boldsymbol{C}^*$-valued continuous functions on a compact space, which take the value ∞ on at most a nowhere dense subset [**2**, Theorem 3.9], where $\boldsymbol{C}^*$ is the extended complex plane in its usual topology as the one-point compactification of $\boldsymbol{C}$. The purpose of this Section is to consider a generalization of the above result in the case when the multiplication of $\mathscr{A}[\tau]$ is not jointly continuous. As a^*a is not necessarily defined for $a \in \mathscr{A}[\tau]$, it is impossible to extend any nonzero multiplicative linear functional φ on $\mathscr{A}_0$ to $\mathscr{A}[\tau]$, like in the case of [**1**, Proposition 6.8]. Here we show that φ is extendable to a $\boldsymbol{C}^*$-valued partial multiplicative linear functional φ' on $\mathscr{A}[\tau]_{q+}$, and that $\mathscr{A}[\tau]_{q+}$ is isomorphic to a wedge of $\boldsymbol{C}^*$-valued positive functions on a compact space, which take the value ∞ on at most a nowhere dense subset. This result will be applied

in Section 6 for studying a functional calculus for quasi-positive elements. Using the notation given after Definition 3.1, define now a wedge of $\mathscr{A}[\tau]$ as follows:

$$\mathscr{A}[\tau]_{q+} := \mathscr{A}[\tau] \cap \widetilde{\mathscr{A}_0}[\tau]_{q+} = \mathscr{A}[\tau] \cap \overline{(\mathscr{A}_0)_+}^{\tau}.$$

Then, let

$$\mathfrak{M}(\mathscr{A}_0, \mathscr{A}[\tau]_{q+}) := \{ax + y : a \in \mathscr{A}[\tau]_{q+},\ x, y \in \mathscr{A}_0\},$$

and denote by $\mathscr{M}(\mathscr{A}_0)$ the Gel'fand space of $\mathscr{A}_0$, i.e. the set of all nonzero multiplicative linear functionals on $\mathscr{A}_0$, endowed with the weak*-topology $\sigma(\mathscr{M}(\mathscr{A}_0), \mathscr{A}_0)$. Now, let $a \in \mathscr{A}[\tau]_{q+}$ and $x, y \in \mathscr{A}_0$. Suppose x is hermitian. Then, by continuous functional calculus, x is uniquely decomposed in the following way:

$$x = x_+ - x_-, \quad x_+, \quad x_- \in (\mathscr{A}_0)_+, \quad x_+ x_- = 0$$
$$|x| \equiv (x^* x)^{1/2} = x_+ + x_- \in (\mathscr{A}_0)_+.$$

Hence, $a|x|$, ax_+, $ax_- \in \mathscr{A}[\tau]_{q+}$, and by (1) and (2) of Proposition 3.2, $(1 + a|x|)^{-1}$, $a|x|(1 + a|x|)^{-1} \in (\mathscr{A}_0)_+$. Furthermore, since

$$a|x|(1 + a|x|)^{-1} - ax_+(1 + a|x|)^{-1} = ax_-(1 + a|x|)^{-1} \in \widetilde{\mathscr{A}_0}[\tau]_{q+},$$

Proposition 3.2, (4) implies that $ax_+(1 + a|x|)^{-1} \in (\mathscr{A}_0)_+$. Similarly, $ax_-(1 + a|x|)^{-1} \in (\mathscr{A}_0)_+$. Hence, we have

$$\begin{aligned}&(ax + y)(1 + a|x|)^{-1} \\ &\quad = ax_+(1 + a|x|)^{-1} - ax_-(1 + a|x|)^{-1} + y(1 + a|x|)^{-1} \in \mathscr{A}_0.\end{aligned}$$

Since a general element x of $\mathscr{A}_0$ is a linear combination of two hermitian elements of $\mathscr{A}_0$, we finally obtain that

$$(ax + y)(1 + a|x|)^{-1} \in \mathscr{A}_0,\ \forall\ a \in \mathscr{A}[\tau]_{q+} \text{ and } x, y \in \mathscr{A}_0.$$

Indeed: Let x be arbitrary in $\mathscr{A}_0$. Then, $x = x_1 + ix_2$, with x_1 and x_2 hermitian. An easy computation shows that

$$|x| \le |x_1| + |x_2|,\ |x_j| \le |x|,\ (1 + a|x_j|)(1 + a|x|)^{-1} \in \widetilde{\mathscr{A}_0}[\tau]_{q+},$$
$$\text{and}\ \ 1 - (1 + a|x_j|)(1 + a|x|)^{-1} \in \widetilde{\mathscr{A}_0}[\tau]_{q+},\ j = 1, 2.$$

The latter together with Proposition 3.2, (4) gives $(1+a|x_j|)(1+a|x|)^{-1} \in (\mathscr{A}_0)_+$; moreover, from the above $(ax_j + y)(1 + a|x_j|)^{-1} \in \mathscr{A}_0$. Thus, for $j = 1, 2$, we get

$$(ax_j + y)(1 + a|x|)^{-1} = ((ax_j + y)(1 + a|x_j|)^{-1})((1 + a|x_j|)(1 + a|x|)^{-1}) \in \mathscr{A}_0,$$

which implies

$$(ax + y)(1 + a|x|)^{-1} = (ax_1 + y)(1 + a|x|)^{-1} + ix_2(1 + a|x|)^{-1} \in \mathscr{A}_0.$$

Hence, the elements $\varphi((1 + a|x|)^{-1})$ and $\varphi((ax + y)(1 + a|x|)^{-1})$ are complex numbers for each $\varphi \in \mathscr{M}(\mathscr{A}_0)$, so that we can consider the correspondence

$$\varphi' : \mathfrak{M}(\mathscr{A}_0, \mathscr{A}[\tau]_{q+}) \longrightarrow \boldsymbol{C}^* \equiv \boldsymbol{C} \cup \{\infty\},\ \text{with}$$
$$ax + y \mapsto \varphi'(ax + y) = \begin{cases} \frac{\varphi((ax+y)(1+a|x|)^{-1})}{\varphi((1+a|x|)^{-1})} & \text{if } \varphi((1 + a|x|)^{-1}) \neq 0 \\ \infty & \text{if } \varphi((1 + a|x|)^{-1}) = 0. \end{cases}$$

Then, we have

LEMMA 5.1. *The following statements hold*:

(1) *For every $\varphi \in \mathscr{M}(\mathscr{A}_0)$ the correspondence φ', given above, is well-defined.*

(2) *Let $a \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$. Then, $(1 + a)^{-1}$ exists in $\mathscr{A}_0$* (from Proposition 3.2, (1)) *and we have*:

(i) $\varphi((1 + a|x|)^{-1}) = 0$ *implies* $\varphi((1 + a)^{-1}) = 0, \varphi \in \mathscr{M}(\mathscr{A}_0)$.

(ii) $\varphi((1 + a)^{-1}) = 0$ *and* $\varphi(x) \neq 0$ *imply* $\varphi((1 + a|x|)^{-1}) = 0, \varphi \in \mathscr{M}(\mathscr{A}_0)$.

PROOF. (1) Let $a, b \in \mathscr{A}[\tau]_{q+}$ and $x, y, z, w \in \mathscr{A}_0$ such that $ax+y = bz+w$. Then, for every $\varphi \in \mathscr{M}(\mathscr{A}_0)$ one has that

$$\varphi((1 + a|x|)^{-1}) = 0 \Leftrightarrow \varphi((1 + b|z|)^{-1}) = 0. \tag{5.1}$$

Indeed, we first show (5.1) in case x and z are hermitian. Since $ax + y = bz + w$, we have

$$(1 + a|x|) - 2ax_- + y = (1 + b|z|) - 2bz_- + w.$$

We multiply the last equality by $(1+a|x|)^{-1}(1+b|z|)^{-1}$ and get

$$\begin{aligned}&(1+b|z|)^{-1}-2ax_-(1+a|x|)^{-1}(1+b|z|)^{-1}+y(1+a|x|)^{-1}(1+b|z|)^{-1}\\&\quad=(1+a|x|)^{-1}-2bz_-(1+b|z|)^{-1}(1+a|x|)^{-1}+w(1+a|x|)^{-1}(1+b|z|)^{-1}.\end{aligned}$$

This implies that for every $\varphi\in\mathscr{M}(\mathscr{A}_0)$

$$\varphi((1+a|x|)^{-1})=0\Leftrightarrow\varphi((1+b|z|)^{-1})=0. \tag{5.2}$$

We next prove (5.1) in the case when x and z are arbitrary elements of $\mathscr{A}_0$. Then, the elements x,y,z and w are decomposed into

$$x=x_1+ix_2,\quad y=y_1+iy_2,\quad z=z_1+iz_2,\quad w=w_1+iw_2,$$

where x_j,y_j,z_j,w_j $(j=1,2)$ are hermitian elements in $\mathscr{A}_0$ that satisfy the equations:

$$ax_1+y_1=bz_1+w_1,\quad ax_2+y_2=bz_2+w_2. \tag{5.3}$$

We show now that

$$\begin{aligned}\varphi((1+a|x|)^{-1})=0\ \Leftrightarrow\ &\text{either}\ \ \varphi((1+a|x_1|)^{-1})=0\\&\text{or}\ \ \varphi((1+a|x_2|)^{-1})=0.\end{aligned} \tag{5.4}$$

Suppose that $\varphi((1+a|x_1|)^{-1})\neq0$ and $\varphi((1+a|x_2|)^{-1})\neq0$. Then,

$$\begin{aligned}&(1+a(|x_1|+|x_2|))^{-1}-(1+a|x_1|)^{-1}(1+a|x_2|)^{-1}\\&\quad=(1+a(|x_1|+|x_2|))^{-1}(a|x_1|(1+a|x_1|)^{-1})(a|x_2|(1+a|x_2|)^{-1})\in(\mathscr{A}_0)_+,\end{aligned}$$

whence

$$\varphi((1+a(|x_1|+|x_2|))^{-1})\geq\varphi((1+a|x_1|)^{-1}(1+a|x_2|)^{-1})>0.$$

Furthermore, since $|x|\leq|x_1|+|x_2|$, we have

$$0<\varphi((1+a(|x_1|+|x_2|))^{-1})\leq\varphi((1+a|x|)^{-1}).$$

Hence, $\varphi((1+a|x|)^{-1}) \neq 0$. Conversely, suppose $\varphi((1+a|x_1|)^{-1}) = 0$ or $\varphi((1+a|x_2|)^{-1}) = 0$. Then, since $(1+a|x_j|)^{-1} \geq (1+a|x|)^{-1}$, $j = 1, 2$, we have that $\varphi((1+a|x|)^{-1}) = 0$.

Now from (5.2), (5.3) and (5.4) we get

$$\begin{aligned}\varphi((1+a|x|)^{-1}) = 0 &\Leftrightarrow \varphi((1+a|x_1|)^{-1}) = 0 \text{ or } \varphi((1+a|x_2|)^{-1}) = 0\\ &\Leftrightarrow \varphi((1+b|z_1|)^{-1}) = 0 \text{ or } \varphi((1+b|z_2|)^{-1}) = 0\\ &\Leftrightarrow \varphi((1+b|z|)^{-1}) = 0.\end{aligned}$$

Thus, (5.1) has been shown. Now, by assumption $ax + y = bz + w$, consequently

$$\varphi'(ax+y) = \infty \Leftrightarrow \varphi'(bz+w) = \infty.$$

On the other hand, from (5.1) it follows that

$$\varphi'(ax+y) < \infty \Leftrightarrow \varphi'(bz+w) < \infty.$$

In this case,

$$\varphi'(ax+y) = \frac{\varphi((ax+y)(1+a|x|)^{-1}(1+b|z|)^{-1})}{\varphi((1+a|x|)^{-1})\varphi((1+b|z|)^{-1})} = \varphi'(bz+w)$$

and this completes the proof of (1).

(2) (i) Suppose $\varphi((1+a|x|)^{-1}) = 0, \varphi \in \mathscr{M}(\mathscr{A}_0)$, Then,

$$\begin{aligned}(1+a)^{-1} &= (1+a|x|)^{-1}(1+a|x|)(1+a)^{-1}\\ &= (1+a|x|)^{-1}((1+a)^{-1} + |x|a(1+a)^{-1})\\ &= (1+a|x|)^{-1}((1+a)^{-1} + |x| - |x|(1+a)^{-1})\\ &= (1+a|x|)^{-1}((1-|x|)(1+a)^{-1} + |x|),\end{aligned}$$

where $(1-|x|)(1+a)^{-1} + |x| \in \mathscr{A}_0$. So applying φ we have $\varphi((1+a)^{-1}) = 0$.

(ii) Suppose that $\varphi((1+a)^{-1}) = 0$ and $\varphi(x) \neq 0$, $\varphi \in \mathscr{M}(\mathscr{A}_0)$. Then, we apply φ to the final result of the preceding calculation in (i) and we take

$$\varphi((1+a|x|)^{-1})\varphi(|x|) = 0.$$

Since $\varphi(x) \neq 0$ if and only if $\varphi(|x|) \neq 0$, clearly we have $\varphi((1+a|x|)^{-1}) = 0$. $\square$

PROPOSITION 5.2. *For $\varphi \in \mathscr{M}(\mathscr{A}_0)$, the well defined map φ' has the following properties*:

(1) $\varphi' \supset \varphi$ (i.e., *φ' is an extension of φ*);

(2) $\varphi'(ax+y) = \varphi'(a)\varphi(x) + \varphi(y)$ *and* $\varphi'(ax) = \varphi'(a)\varphi(x)$, *whenever* $a \in \mathscr{A}[\tau]_{q+}$ *and* $x, y \in \mathscr{A}_0$ *such that* $\varphi'(a)\varphi(x) \neq \infty \cdot 0$;

(3) $\varphi'(a+b) = \varphi'(a) + \varphi'(b)$, *for all* $a, b \in \mathscr{A}[\tau]_{q+}$;

(4) $\varphi'(\lambda a) = \lambda\varphi'(a)$, *for all* $\lambda \in \boldsymbol{C}$ *and* $a \in \mathscr{A}[\tau]_{q+}$, *where* $0 \cdot \infty = 0$.

PROOF.

(1) It is trivial.

(2) Suppose that $\varphi'(a)\varphi(x) \neq \infty \cdot 0$, $\varphi \in \mathscr{M}(\mathscr{A}_0)$. Then, from the definition of φ' and Lemma 5.1, (2), we have the following implications (considering separately the cases where $\varphi'(a)$ is infinite or not):

- $$\begin{array}{ccc} \varphi'(ax+y) = \infty & \Leftrightarrow & \varphi'(a) = \infty \\ \Updownarrow & & \Updownarrow \\ \varphi((1+a|x|)^{-1}) = 0 & & \varphi((1+a)^{-1}) = 0 \\ & & \Downarrow \\ & & \varphi'(a)\varphi(x) + \varphi(y) = \infty. \end{array}$$
- $$\begin{array}{ccc} \varphi'(ax+y) < \infty & \Leftrightarrow & \varphi'(a) < \infty \\ \Updownarrow & & \Updownarrow \\ \varphi((1+a|x|)^{-1}) \neq 0 & & \varphi((1+a)^{-1}) \neq 0. \end{array}$$

So, in this case we also get

$$\begin{aligned} \varphi'(ax+y) &= \frac{\varphi(ax(1+a|x|)^{-1})}{\varphi((1+a|x|)^{-1})} + \varphi(y) \\ &= \frac{\varphi(a(1+a)^{-1})\varphi(x)}{\varphi((1+a)^{-1})} + \varphi(y) = \varphi'(a)\varphi(x) + \varphi(y), \end{aligned}$$

and this completes the proof of (2).

(3) Observe that for any $a, b \in \mathscr{A}[\tau]_{q+}$, one has

$$\begin{aligned} (1+a)^{-1}(1+b)^{-1} = {} & (1+a+b)^{-1}((1+a)^{-1}(1+b)^{-1} \\ & + a(1+a)^{-1}(1+b)^{-1} + (1+a)^{-1}b(1+b)^{-1}), \end{aligned}$$

where $(1+a)^{-1}(1+b)^{-1} + a(1+a)^{-1}(1+b)^{-1} + (1+a)^{-1}b(1+b)^{-1} \in \mathscr{A}_0$ (see Proposition 3.2). Thus, applying any $\varphi \in \mathscr{M}(\mathscr{A}_0)$ to the last equality we conclude

that

$$\begin{aligned}&\varphi((1+a+b)^{-1})=0 \text{ implies either}\\ &\varphi((1+a)^{-1})=0 \text{ or } \varphi((1+b)^{-1})=0.\end{aligned}\tag{5.5}$$

Conversely, observe that

$$(1+a)^{-1}=(1+a+b)^{-1}+b(1+a+b)^{-1}(1+a)^{-1},$$

where $b(1+a+b)^{-1}\in(\mathscr{A}_0)_+$ by Proposition 3.2, (4), since $(a+b)(1+a+b)^{-1}-a(1+a+b)^{-1}=b(1+a+b)^{-1}\in\widetilde{\mathscr{A}_0}[\tau]_{q+}$ with $(a+b)(1+a+b)^{-1}\in(\mathscr{A}_0)_+$. So, taking also into account an analogous equality for $(1+b)^{-1}$, as well as (5.5) we have that

$$\begin{aligned}\varphi((1+a+b)^{-1})=0 \Leftrightarrow \text{ either } &\varphi((1+a)^{-1})=0 \text{ or}\\ &\varphi((1+b)^{-1})=0, \quad \forall\ \varphi\in\mathscr{M}(\mathscr{A}_0).\end{aligned}$$

Using now the preceding equivalence, clearly we conclude that:

- $\varphi'(a+b)=\infty \Leftrightarrow$ either $\varphi'(a)=\infty$ or $\varphi'(b)=\infty$; thus,

$$\varphi'(a+b)=\varphi'(a)+\varphi'(b)=\infty;\ \text{or}$$

- $\varphi'(a+b)<\infty \Leftrightarrow \varphi'(a)<\infty$ and $\varphi'(b)<\infty$.

In this case,

$$\begin{aligned}&\varphi'(a+b)\\ &=\frac{\varphi(a(1+a)^{-1}(1+b)^{-1}(1+a+b)^{-1}+b(1+a)^{-1}(1+b)^{-1}(1+a+b)^{-1})}{\varphi((1+a)^{-1})\varphi((1+b)^{-1})\varphi((1+a+b)^{-1})}\\ &=\frac{\varphi(a(1+a)^{-1})}{\varphi((1+a)^{-1})}+\frac{\varphi(b(1+b)^{-1})}{\varphi((1+b)^{-1})}\\ &=\varphi'(a)+\varphi'(b).\end{aligned}$$

(4) It follows from (2) by replacing x with $\lambda 1, \lambda\in\boldsymbol{C}$, and y with 0. □

REMARK 5.3. In order to have all the values of φ' fully determined, we need to define the following:

- $\varphi'(a)\varphi(x)$, $\varphi'(ax)+\varphi'(bx)$ and $\varphi'(a)\varphi(x_1)+\varphi'(a)\varphi(x_2)$, for any $a,b\in$

$\mathscr{A}[\tau]_{q+}$ and $x_1, x_2 \in \mathscr{A}_0$.

From Proposition 5.2 we conclude that:

(i) $\varphi'(a)\varphi(x) = \varphi'(ax)$, for any $a \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$ with $\varphi'(a)\varphi(x) \neq \infty \cdot 0$.

(ii) $\varphi'(ax) + \varphi'(bx) = \varphi'((a+b)x)$, for any $a,\ b \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$ with either $\varphi'(a)\varphi(x) \neq \infty \cdot 0$ or $\varphi'(b)\varphi(x) \neq \infty \cdot 0$.

(iii) $\varphi'(a)\varphi(x_1 + x_2) = \varphi'(a(x_1 + x_2))$, for any $a \in \mathscr{A}[\tau]_{q+}$ and $x_1, x_2 \in \mathscr{A}_0$ with $\varphi'(a(x_1 + x_2)) \neq \infty \cdot 0$.

Furthermore, the definition of φ' and Proposition 5.2 imply that:

(1) When $\varphi'(a) = \infty$ and $\varphi(x) = 0$, the value $\varphi'(ax)$ of φ' depends upon a and x. For instance,

- $x = 0 \Rightarrow \varphi'(ax) = \varphi'(0) = \varphi(0) = 0$;
- $x = (1+a)^{-1} \Rightarrow \varphi'(a(1+a)^{-1}) = \varphi(a(1+a)^{-1}) = \varphi(1 - (1+a)^{-1}) = 1$.

(2) For $a, b \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$ such that either $\varphi'(a)\varphi(x) = \infty \cdot 0$ or $\varphi'(b)\varphi(x) = \infty \cdot 0$, the value $\varphi'((a+b)x)$ clearly depends upon a, b and x.

(3) For $a \in \mathscr{A}[\tau]_{q+}$ and $x_1, x_2 \in \mathscr{A}_0$ such that either $\varphi'(a)\varphi(x_1) = \infty \cdot 0$ or $\varphi'(a)\varphi(x_2) = \infty \cdot 0$, then again the value $\varphi'(a(x_1 + x_2))$ depends upon a, x_1 and x_2.

CONCLUSION. We define the requested values of φ' by (i), (ii) and (iii), for any $a,\ b \in \mathscr{A}[\tau]_{q+}$ and $x_1, x_2 \in \mathscr{A}_0$.

REMARK 5.4. We do not know whether φ' is defined or not on the linear span of $\mathfrak{M}(\mathscr{A}_0, \mathscr{A}[\tau]_{q+})$.

Now, for any $a \in \mathscr{A}[\tau]_{q+}$ and $x, y \in A_0$, we fix the notation:

$$\widehat{ax+y}(\varphi) \equiv \varphi'(ax + y), \quad \varphi \in \mathscr{M}(\mathscr{A}_0).$$

Then, we have the following

PROPOSITION 5.5. *$\widehat{ax+y}$ is a $\boldsymbol{C}^*$-valued continuous function on the compact Hausdorff space $\mathscr{M}(\mathscr{A}_0)$, which takes the value ∞ on at most a nowhere dense subset of $\mathscr{M}(\mathscr{A}_0)$.*

PROOF. We shall show that the set

$$N_{\widehat{ax+y}} \equiv \big\{\varphi \in \mathscr{M}(\mathscr{A}_0) : \widehat{ax+y}(\varphi) = \infty\big\},$$

is a nowhere dense closed subset of $\mathscr{M}(\mathscr{A}_0)$. Notice that

$$N_{\widehat{ax+y}} = \big\{\varphi \in \mathscr{M}(\mathscr{A}_0) : \varphi(1 + a|x|)^{-1} = 0\big\}, \tag{5.6}$$

from which it follows that $N_{\widehat{ax+y}}$ is closed. Now, suppose that

$$\exists\ \mathscr{U} \text{ non-empty open subset of } \mathscr{M}(\mathscr{A}_0) \text{ with } \mathscr{U} \subset N_{\widehat{ax+y}}.$$

From the commutative Gel'fand-Naimark theorem, $\mathscr{A}_0 \simeq \mathscr{C}(\mathscr{M}(\mathscr{A}_0))$, up to an isometric $*$-isomorphism. Thus, using Urysohn's lemma for $\mathscr{M}(\mathscr{A}_0)$ we get that

$$\exists\ b \in \mathscr{A}_0 : \|b\|_0 = 1 \text{ and } \hat{b}(\varphi) = \varphi(b) = 0, \quad \forall\ \varphi \notin \mathscr{U}.$$

But this together with (5.6) and the fact that $\mathscr{U} \subset N_{\widehat{ax+y}}$, implies

$$\varphi(b(1 + a|x|)^{-1}) = 0, \quad \forall\ \varphi \in \mathscr{M}(\mathscr{A}_0).$$

The afore-mentioned identification $\mathscr{A}_0 \simeq \mathscr{C}(\mathscr{M}(\mathscr{A}_0))$ gives now $b(1 + a|x|)^{-1} = 0$, which clearly yields $b = 0$, a contradiction to $\|b\|_0 = 1$. Hence, $N_{\widehat{ax+y}}$ is a nowhere dense closed subset of $\mathscr{M}(\mathscr{A}_0)$.

Next we show that $\widehat{ax + y}$ is continuous on $\mathscr{M}(\mathscr{A}_0)$. Put

$$z \equiv (1 + a|x|)^{-1} \text{ and } w \equiv ax(1 + a|x|)^{-1}.$$

Take an arbitrary $\varphi_0 \in \mathscr{M}(\mathscr{A}_0)$ and consider the cases:

- $\widehat{ax + y}(\varphi_0) \neq \infty$, i.e., $\hat{z}(\varphi_0) \neq 0$.

From the continuity of $\hat{z}$ there is a neighborhood $\mathscr{U}_{\varphi_0}$ of φ_0 with $\hat{z}(\varphi) \neq 0$, for all $\varphi \in \mathscr{U}_{\varphi_0}$. Thus, we get

$$\widehat{ax + y}(\varphi) = \frac{\hat{w}(\varphi)}{\hat{z}(\varphi)} + \hat{y}(\varphi), \quad \forall\ \varphi \in \mathscr{U}_{\varphi_0},$$

where all functions $\hat{w}, \hat{z}, \hat{y}$ are continuous at φ_0, so that the same is true for $\widehat{ax + y}$.

- $\widehat{ax + y}(\varphi_0) = \infty$, i.e., $\hat{z}(\varphi_0) = 0$.

Take an arbitrary net $\{\varphi_\alpha\}$ in $\mathscr{M}(\mathscr{A}_0)$ such that $\varphi_\alpha \to \varphi_0$, with respect to the weak*-topology $\sigma(\mathscr{M}(\mathscr{A}_0), \mathscr{A}_0)$. Then,

$$\hat{z}(\varphi_\alpha) \to \hat{z}(\varphi_0) = 0,$$

where $\hat{z}(\varphi_\alpha) \neq 0$, since $N_{\widehat{ax+y}}$ is a nowhere dense subset of $\mathscr{M}(\mathscr{A}_0)$. Since

$$
\begin{aligned}
|\widehat{ax}(\varphi_\alpha)| &= \frac{\varphi_\alpha((ax^*((1+a|x|)^{-1})(ax(1+a|x|)^{-1}))^{1/2}}{\varphi_\alpha((1+a|x|)^{-1})} \\
&= \frac{\varphi_\alpha((a(1+a|x|)^{-1})(x^*xa(1+a|x|)^{-1}))^{1/2}}{\varphi_\alpha((1+a|x|)^{-1})} \\
&= \frac{\varphi_\alpha((a|x|(1+a|x|)^{-1})^2)^{1/2}}{\varphi_\alpha((1+a|x|)^{-1})} \\
&= \frac{\varphi_\alpha(a|x|(1+a|x|)^{-1})}{\varphi_\alpha((1+a|x|)^{-1})} \\
&= \frac{\varphi_\alpha(1-(1+a|x|)^{-1})}{\varphi_\alpha((1+a|x|)^{-1})} \\
&= \frac{1}{\widehat{z}(\varphi_\alpha)} - 1,
\end{aligned}
$$

it follows that $\lim_\alpha \widehat{ax}(\varphi_\alpha) = \infty$, which implies

$$\lim_\alpha \widehat{ax+y}(\varphi_\alpha) = \infty = \widehat{ax+y}(\varphi_0).$$

This completes the proof of the continuity of $\widehat{ax+y}$ at φ_0; so the proof of Proposition 5.5 is finished. □

All the above lead to the following

Definition 5.6. Let W be a completely regular topological space and $\mathscr{F}(W)_+$ the set of all $\boldsymbol{C}^*$-valued positive continuous functions on W, which take the value ∞ on at most a nowhere dense subset W_0 of W. Then, $\mathscr{F}(W)_+$ is said to be a wedge on W, if for any $f, g \in \mathscr{F}(W)_+$ and $\lambda \geq 0$, the functions $f+g$ and λf defined pointwise on W_0 on which f and g are both finite, are extendible to $\boldsymbol{C}^*$-valued positive continuous functions on W that also belong to $\mathscr{F}(W)_+$. We keep the same symbols $f+g$ and λf for the respective extensions.

Consider now the set

$$\mathscr{F}(W) \equiv \{fg_0 + h_0 : f \in \mathscr{F}(W)_+, g_0, h_0 \in \mathscr{C}(W)\},$$

where $\mathscr{C}(W)$ is the $*$-algebra of all continuous $\boldsymbol{C}$-valued functions on W. Then, the set $\mathscr{F}(W)$ fulfils the following conditions:

- $(f_1 + f_2)g_0 = f_1g_0 + f_2g_0$,

- $(\lambda f)g_0 = \lambda(fg_0)$,
- $f(g_0 + h_0) = fg_0 + fh_0$,

for all $f, f_1, f_2 \in \mathscr{F}(W)_+$, $g_0, h_0 \in \mathscr{C}(W)$ and $\lambda \geq 0$.

DEFINITION 5.7. We call $\mathscr{F}(W)$ the set of $\boldsymbol{C}^*$-valued positive continuous functions on W generated by the wedge $\mathscr{F}(W)_+$ and the $*$-algebra $\mathscr{C}(W)$.

In this regard (see also Remark 5.3), we have the following

THEOREM 5.8. *Let $\mathscr{F}(\mathscr{M}(\mathscr{A}_0))_+ \equiv \{\hat{a} : a \in \mathscr{A}[\tau]_{q+}\}$. Then,*

(1) *$\mathscr{F}(\mathscr{M}(\mathscr{A}_0))_+$ is a wedge on $\mathscr{M}(\mathscr{A}_0)$.*

(2) *The map $\Phi : \mathfrak{M}(\mathscr{A}_0, \mathscr{A}[\tau]_{q+}) \to \mathscr{F}(\mathscr{M}(\mathscr{A}_0)) : ax + y \mapsto \widehat{ax + y}$, is a bijection satisfying the properties*:

(i) *$\Phi(\mathscr{A}[\tau]_{q+}) = \mathscr{F}(\mathscr{M}(\mathscr{A}_0))_+$, with $\Phi(a + b) = \Phi(a) + \Phi(b)$ and $\Phi(\lambda a) = \lambda\Phi(a)$, for all $a, b \in \mathscr{A}[\tau]_{q+}$ and $\lambda \geq 0$.*

(ii) *$\Phi(\mathscr{A}_0) = \mathscr{C}(\mathscr{M}(\mathscr{A}_0))$, Φ being an isometric $*$-isomorphism from $\mathscr{A}_0$ onto $\mathscr{C}(\mathscr{M}(\mathscr{A}_0))$.*

(iii) *$\Phi(ax) = \Phi(a)\Phi(x)$, for all $a \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$. $\Phi((a + b)x) = (\Phi(a) + \Phi(b))\Phi(x)$, for all $a, b \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$. $\Phi(\lambda ax) = \lambda\Phi(a)\Phi(x)$, for all $a \in \mathscr{A}[\tau]_{q+}, x \in \mathscr{A}_0$ and $\lambda \geq 0$. $\Phi(a(x_1 + x_2)) = \Phi(a)(\Phi(x_1) + \Phi(x_2))$, for all $a \in \mathscr{A}[\tau]_{q+}$ and $x_1, x_2 \in \mathscr{A}_0$.*

PROOF. The statements (1), (2)(i) and (2)(ii) follow from Propositions 5.2 and 5.5. We show the statement (2)(iii). Let $a \in \mathscr{A}[\tau]_{q+}$ and $x \in \mathscr{A}_0$. From Proposition 5.5, $\hat{a}$ and $\widehat{ax}$ are $\boldsymbol{C}^*$-valued continuous functions on $\mathscr{M}(\mathscr{A}_0)$ that take the value ∞ on at most a nowhere dense subset of $\mathscr{M}(\mathscr{A}_0)$. Hence, the set

$$\mathscr{K} \equiv \left\{\varphi \in \mathscr{M}(\mathscr{A}_0) : \hat{a}(\varphi) < \infty \text{ and } \widehat{ax}(\varphi) < \infty\right\}$$

is dense in $\mathscr{M}(\mathscr{A}_0)$ and

$$\widehat{ax}(\varphi) = \hat{a}(\varphi)\hat{x}(\varphi), \forall\ \varphi \in \mathscr{K},$$

therefore by the continuity of $\hat{a}$ and $\widehat{ax}$ we conclude that $\widehat{ax} = \hat{a}\hat{x}$, from which it follows that $\Phi(ax) = \Phi(a)\Phi(x)$. The rest of the properties in (2)(iii) are similarly proved. □

6. Functional calculus for quasi-positive elements.

Throughout this Section $\mathscr{A}[\tau]$ is a commutative locally convex quasi C^*-algebra over a C^*-algebra $\mathscr{A}_0$. Here we shall consider a functional calculus for

the quasi-positive elements of $\mathscr{A}[\tau]$, resulting, for instance, to consideration of the quasi nth-root of an element $a \in \mathscr{A}[\tau]_{q+}$ (see Corollary 6.7). For this purpose, we first need to extend the multiplication of $\mathscr{A}[\tau]$.

DEFINITION 6.1. Let $a, b \in \mathscr{A}[\tau]_{q+}$; a is called left-multiplier of b, and we write $a \in L(b)$, if there exist nets $\{x_\alpha\}, \{y_\beta\}$ in $(\mathscr{A}_0)_+$ such that $x_\alpha \underset{\tau}{\to} a$, $y_\beta \underset{\tau}{\to} b$ and $x_\alpha y_\beta \underset{\tau}{\to} c$ (in the sense that the double indexed net $\{x_\alpha y_\beta\}$ converges to c). The product of a, b denoted by ab is given as follows

$$ab := c = \tau\text{-}\lim_{\alpha,\beta} x_\alpha y_\beta.$$

LEMMA 6.2. *The product ab is well-defined, in the sense that it is independent of the selection of the nets $\{x_\alpha\}, \{y_\beta\}$.*

PROOF. Let $\{x_\alpha\}, \{y_\beta\}$ be two nets in $(\mathscr{A}_0)_+$ such that

$$x_\alpha \underset{\tau}{\to} a,\ y_\beta \underset{\tau}{\to} b \text{ and } x_\alpha y_\beta \underset{\tau}{\to} c.$$

Then (also see Proposition 3.2)

$$\begin{aligned}
&(1+x_\alpha)^{-1}x_\alpha y_\beta(1+y_\beta)^{-1}(1+c)^{-1} - (1+a)^{-1}c(1+c)^{-1}(1+b)^{-1}\\
&\quad = ((1+x_\alpha)^{-1}x_\alpha y_\beta(1+y_\beta)^{-1}(1+c)^{-1} - (1+x_\alpha)^{-1}c(1+c)^{-1}(1+y_\beta)^{-1})\\
&\qquad + ((1+x_\alpha)^{-1}c(1+c)^{-1}(1+y_\beta)^{-1} - (1+a)^{-1}c(1+c)^{-1}(1+y_\beta)^{-1})\\
&\qquad + ((1+a)^{-1}c(1+c)^{-1}(1+y_\beta)^{-1} - (1+a)^{-1}c(1+c)^{-1}(1+b)^{-1}).
\end{aligned}$$

As we have seen in the proof of Proposition 3.2,(1) $(1+x_\alpha)^{-1} \underset{\tau}{\to} a$, so taking τ-limits in the preceding equality, we conclude that

$$(1+x_\alpha)^{-1}x_\alpha y_\beta(1+y_\beta)^{-1}(1+c)^{-1} \underset{\tau}{\to} (1+a)^{-1}c(1+c)^{-1}(1+b)^{-1}.$$

On the other hand,

$$\begin{aligned}
&(1+x_\alpha)^{-1}x_\alpha y_\beta(1+y_\beta)^{-1}(1+c)^{-1} - ((1+a)^{-1}a)(b(1+b)^{-1})(1+c)^{-1}\\
&\quad = ((1+x_\alpha)^{-1}x_\alpha - (1+a)^{-1}a)y_\beta(1+y_\beta)^{-1}(1+c)^{-1}\\
&\qquad + (1+a)^{-1}a(y_\beta(1+y_\beta)^{-1} - b(1+b)^{-1})(1+c)^{-1},
\end{aligned}$$

from which, as before, we take that

$$(1+x_\alpha)^{-1}x_\alpha y_\beta(1+y_\beta)^{-1}(1+c)^{-1} \underset{\tau}{\to} ((1+a)^{-1}a)(b(1+b)^{-1})(1+c)^{-1}.$$

Hence, we finally obtain

$$(1+a)^{-1}c(1+b)^{-1} = ((1+a)^{-1}a)(b(1+b)^{-1}). \tag{6.1}$$

Suppose now that two other nets $\{x'_\lambda\}, \{y'_\mu\}$ exist in $(\mathscr{A}_0)_+$ such that

$$x'_\lambda \underset{\tau}{\to} a,\ y'_\mu \underset{\tau}{\to} b \text{ and } x'_\lambda y'_\mu \underset{\tau}{\to} c'.$$

Working exactly as before we come to the equality

$$(1+a)^{-1}c'(1+b)^{-1} = ((1+a)^{-1}a)(b(1+b)^{-1}),$$

which together with (6.1) gives

$$(1+a)^{-1}c(1+b)^{-1} = (1+a)^{-1}c'(1+b)^{-1} \Leftrightarrow c = c'. \qquad \square$$

We may now set the following

DEFINITION 6.3. Let $a, b \in \mathscr{A}[\tau]_{q+}$ with $a \in L(b)$ and $x, y \in \mathscr{A}_0$. The product of the elements ax, by is defined as follows:

$$(ax)(by) := (ab)xy.$$

Further, we consider the spectrum of an element $a \in \mathscr{A}[\tau]_{q+}$.

DEFINITION 6.4. Let $a \in \mathscr{A}[\tau]_{q+}$. The spectrum of a denoted by $\sigma_{\mathscr{A}_0}(a)$, is that subset of $\boldsymbol{C}^*$, defined in the following way:

- Let $\lambda \in \boldsymbol{C}$. Then $\lambda \in \sigma_{\mathscr{A}_0}(a) \Leftrightarrow \lambda 1 - a$ has no inverse in $\mathscr{A}_0$;
- $\infty \in \sigma_{\mathscr{A}_0}(a) \Leftrightarrow a \notin \mathscr{A}_0$.

LEMMA 6.5. *Let $a \in \mathscr{A}[\tau]_{q+}$. Then,*

$$\sigma_{\mathscr{A}_0}(a) = \{\hat{a}(\varphi) : \varphi \in \mathscr{M}(\mathscr{A}_0)\} \subset \boldsymbol{R}_+ \cup \{\infty\}.$$

In particular, $\sigma_{\mathscr{A}_0}(a)$ is a locally compact subset of $\boldsymbol{C}^*$.

PROOF. Let $\lambda \in \boldsymbol{C}$. Then (also see Theorem 5.8),

$$\lambda \notin \sigma_{\mathscr{A}_0}(a) \Leftrightarrow (\lambda 1 - a)^{-1} \in \mathscr{A}_0 \Leftrightarrow \lambda \neq \hat{a}(\varphi), \forall\ \varphi \in \mathscr{M}(\mathscr{A}_0).$$

Let now $\lambda = \infty$. Then,

$$\begin{aligned}\lambda \in \sigma_{\mathscr{A}_0}(a) \Leftrightarrow a \notin \mathscr{A}_0 &\Leftrightarrow \hat{a} \notin \mathscr{C}(\mathscr{M}(\mathscr{A}_0)) \\ &\Leftrightarrow \exists\ \varphi_0 \in \mathscr{M}(\mathscr{A}_0) : \hat{a}(\varphi_0) = \infty.\end{aligned}$$

The rest is clear. □

If $a \in \mathscr{A}[\tau]_{q+}$, denote by $\mathscr{C}_b(\sigma_{\mathscr{A}_0}(a))$, the C^*-algebra of all bounded continuous functions on $\sigma_{\mathscr{A}_0}(a)$. For $n \in \boldsymbol{N}$ and $f \in \mathscr{C}(\sigma_{\mathscr{A}_0}(a))$, define the function

$$g_n(\lambda) := \frac{f(\lambda)}{(1+\lambda)^n}, \quad \lambda \in \sigma_{\mathscr{A}_0}(a). \tag{6.2}$$

In this regard, set

$$C_n(\sigma_{\mathscr{A}_0}(a)) := \{f \in C(\sigma_{\mathscr{A}_0}(a) \cap \boldsymbol{R}) : g_n \in C_b(\sigma_{\mathscr{A}_0}(a))\}. \tag{6.3}$$

Then,

$$C_b(\sigma_{\mathscr{A}_0}(a)) \subset C_1(\sigma_{\mathscr{A}_0}(a)) \subset C_2(\sigma_{\mathscr{A}_0}(a)) \subset \cdots.$$

Now, the promised functional calculus for quasi-positive elements in $\mathscr{A}[\tau]$ is given by the following

THEOREM 6.6. *Let $a \in \mathscr{A}[\tau]_{q+}$. Suppose that the element a^n is well-defined for some $n \in \boldsymbol{N}$. Then, there is a unique $*$-isomorphism $f \mapsto f(a)$ from $\bigcup_{k=1}^n C_k(\sigma_{\mathscr{A}_0}(a))$ into $\mathscr{A}[\tau]$, in such a way that*:

(i) *If $u_0 \in \bigcup_{k=1}^n C_k(\sigma_{\mathscr{A}_0}(a))$, with $u_0(\lambda) = 1$, for each $\lambda \in \sigma_{\mathscr{A}_0}(a)$, then $u_0(a) = 1 \in \mathscr{A}_0 \hookrightarrow \mathscr{A}[\tau]$.*

(ii) *If $u_1 \in \bigcup_{k=1}^n C_k(\sigma_{\mathscr{A}_0}(a))$, with $u_1(\lambda) = \lambda$, for each $\lambda \in \sigma_{\mathscr{A}_0}(a)$, then $u_1(a) = a \in \mathscr{A}[\tau]$.*

(iii) *$\widehat{f(a)}(\varphi) = f(\hat{a}(\varphi))$, for any $f \in \bigcup_{k=1}^n C_k(\sigma_{\mathscr{A}_0}(a))$ and $\varphi \in \mathscr{M}(\mathscr{A}_0)$;*

(iv) *$(f_1 + f_2)(a) = f_1(a) + f_2(a)$, for any $f_1, f_2 \in \bigcup_{k=1}^n C_k(\sigma_{\mathscr{A}_0}(a))$, $(\lambda f)(a) = \lambda f(a)$, for any $f \in \bigcup_{k=1}^n C_k(\sigma_{\mathscr{A}_0}(a))$ and $\lambda \in \boldsymbol{C}$, $(f_1 f_2)(a) = f_1(a) f_2(a)$, for any $f_j \in C_{k_j}(\sigma_{\mathscr{A}_0}(a))$, $j = 1, 2$, with $k_1 + k_2 \leq n$.*

(v) *Restricted to $C_b(\sigma_{\mathscr{A}_0}(a))$ the map $f \mapsto f(a)$ is an isometric $*$-isomorphism*

of the C^*-algebra $C_b(\sigma_{\mathscr{A}_0}(a))$ onto the closed $*$-subalgebra of the C^*-algebra $\mathscr{A}_0$ generated by 1 and $(1+a)^{-1}$.

PROOF. Let $f \in \bigcup_{k=1}^{n} C_k(\sigma_{\mathscr{A}_0}(a))$. Then, $f \in C_k(\sigma_{\mathscr{A}_0}(a))$, for some k with $1 \leq k \leq n$, and $g_k \in C_b(\sigma_{\mathscr{A}_0}(a))$ with $g_k(\lambda) := f(\lambda)/(1+\lambda)^k, \lambda \in \sigma_{\mathscr{A}_0}(a)$. From Lemma 6.5 we have that $g_k \circ \hat{a} \in C(\mathscr{M}(\mathscr{A}_0))$, therefore (Gel'fand-Naimark theorem) there is a unique element $g_k(a) \in \mathscr{A}_0$ such that

$$\widehat{g_k(a)}(\varphi) = g_k(\hat{a}(\varphi)), \quad \forall\, \varphi \in \mathscr{M}(\mathscr{A}_0). \tag{6.4}$$

Now let

$$f(a) := g_k(a)(1+a)^k \in \mathscr{A}[\tau]. \tag{6.5}$$

We shall show that $f(a)$ does not depend on k, $1 \leq k \leq n$. Indeed, let $f \in C_j(\sigma_{\mathscr{A}_0}(a))$ with:

- $j \leq k$; then for each $\lambda \in \sigma_{\mathscr{A}_0}(a)$,

$$g_k(\lambda) = \frac{f(\lambda)}{(1+\lambda)^k} = \frac{f(\lambda)}{(1+\lambda)^j}\frac{1}{(1+\lambda)^{k-j}} = g_j(\lambda)\frac{1}{(1+\lambda)^{k-j}}.$$

Hence, $g_k(a) = g_j(a)(1+a)^{-(k-j)} \in \mathscr{A}_0$ and

$$g_k(a)(1+a)^k = g_j(a)(1+a)^j; \tag{6.6}$$

- $j > k$; in this case too, one takes (6.6) in a similar way. So, the element $f(a) \in \mathscr{A}[\tau]$ is well-defined by (6.5). Now, it is easily seen that the map

$$f \mapsto f(a) \text{ from } \bigcup_{k=1}^{n} C_k(\sigma_{\mathscr{A}_0}(a)) \text{ into } \mathscr{A}[\tau]$$

is a $*$-isomorphism with the properties (i), (ii), (iii).

(iv) Consider the functions $f_1 \in C_{k_1}(\sigma_{\mathscr{A}_0}(a))$, $f_2 \in C_{k_2}(\sigma_{\mathscr{A}_0}(a))$ with $k_1+k_2 \leq n$. Then (see (6.3) and discussion before (6.4)), $g_{k_i} \in C_b(\sigma_{\mathscr{A}_0}(a))$ with $g_{k_i}(a)$ unique in $\mathscr{A}_0$, $i = 1, 2$. Define the function $f(\lambda) := f_1(\lambda)f_2(\lambda)$, $\lambda \in \sigma_{\mathscr{A}_0}(a)$. Then, $f \in C_{k_1+k_2}(\sigma_{\mathscr{A}_0}(a))$ and

$$g_{k_1+k_2}(\lambda) = \frac{f(\lambda)}{(1+\lambda)^{k_1+k_2}} = g_{k_1}(\lambda)g_{k_2}(\lambda), \quad \lambda \in \sigma_{\mathscr{A}_0}(a),$$

that is $g_{k_1+k_2} \in C_b(\sigma_{\mathscr{A}_0}(a))$. Thus, $g_{k_1+k_2}(a) = g_{k_1}(a)g_{k_2}(a) \in \mathscr{A}_0$. Moreover (see also Definition 6.3 and (6.5))

$$\begin{aligned}(f_1 f_2)(a) = f(a) &= g_{k_1+k_2}(a)(1+a)^{k_1+k_2} \\ &= (g_{k_1}(a)(1+a)^{k_1})(g_{k_2}(a)(1+a)^{k_2}) \\ &= f_1(a)f_2(a).\end{aligned}$$

The first two equalities in (iv) are similarly shown.

(v) Arguing as in (6.4) and taking into account Lemma 6.5, we easily reach at the conclusion. □

COROLLARY 6.7. *Let $a \in \mathscr{A}[\tau]_{q+}$ and $n \in \boldsymbol{N}$. Then, there is a unique $b \in \mathscr{A}[\tau]_{q+}$ such that $a = b^n$. The element b is called quasi nth-root of a and is denoted by $a^{\frac{1}{n}}$. If, in particular, $n = 2$, the element $a^{\frac{1}{2}}$ is called quasi square-root of a.*

PROOF. Consider the functions $f_1(\lambda) := \lambda^{\frac{1}{n}}$ and $f_2(\lambda) := \lambda^{1-\frac{1}{n}}$, $\lambda \geq 0$, which clearly belong to $C_1(\sigma_{\mathscr{A}_0}(a))$. Then (see (6.2), (6.3)), $g_1, g_2 \in C_b(\sigma_{\mathscr{A}_0}(a))$ with $g_1(\lambda) = f_1(\lambda)(1+\lambda)^{-1}$, $g_2(\lambda) = f_2(\lambda)(1+\lambda)^{-1}$, $\lambda \geq 0$. Theorem 6.6 gives that the elements $f_1(a), f_2(a)$ are uniquely defined in $\mathscr{A}[\tau]$ with

$$f_1(a) = g_1(a)(1+a), \quad f_2(a) = g_2(a)(1+a),$$

where $g_i(a) \in (\mathscr{A}_0)_+$, $i = 1, 2$ (see, e.g., (6.4)). Moreover (also see Proposition 3.2, (1) and (2)), for each $\varepsilon > 0$

$$(\mathscr{A}_0)_+ \ni g_1(a)(1+a)(1+\varepsilon a)^{-1} \xrightarrow[\tau]{\varepsilon \to 0} f_1(a), \text{ resp.}$$

$$(\mathscr{A}_0)_+ \ni g_2(a)(1+a)(1+\varepsilon a)^{-1} \xrightarrow[\tau]{\varepsilon \to 0} f_2(a).$$

On the other hand, since $(f_1 f_2)(\lambda) = \lambda$, from Theorem 6.6, (ii) we have that $(f_1 f_2)(a) = a$, therefore (also see Proposition 3.2, (2))

$$(g_1(a)(1+a)(1+\varepsilon a)^{-1})(g_2(a)(1+a)(1+\varepsilon a)^{-1}) = a(1+\varepsilon a)^{-1} \xrightarrow[\tau]{\varepsilon \to 0} a.$$

So, from Definition 6.1, we conclude that

$$f_1(a) \in L(f_2(a)) \text{ and } a = f_1(a)f_2(a).$$

Now, since $f_2(a) \in \mathscr{A}[\tau]_{q+}$, we repeat the previous procedure with $f_2(a)$ in the place of a, so that continuing in this way we finally obtain

$$a = f_1(a)f_1(a)\cdots f_1(a) \quad (n\text{-times}).$$

The proof is completed by taking $b = f_1(a)$. □

7. Structure of noncommutative locally convex quasi C^*-algebras.

In this Section we consider a noncommutative locally convex quasi C^*-algebra $\mathscr{A}[\tau]$ over a unital C^*-algebra $\mathscr{A}_0$ and we investigate the following: (a) Conditions under which such an algebra is continuously embedded in a locally convex quasi C^*-algebra of operators (Theorems 7.3, 7.5); (b) a functional calculus for the commutatively quasi-positive elements in $\mathscr{A}[\tau]$ (Theorem 7.8).

Definition 7.1. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathscr{H}$. A $*$-representation π of $\mathscr{A}[\tau]$ is a linear map from $\mathscr{A}$ into $\mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$ (see beginning of Section 4) with the following properties:

(i) π is a $*$-representation of $\mathscr{A}_0$;

(ii) $\pi(a)^\dagger = \pi(a^*), \forall\ a \in \mathscr{A}$;

(iii) $\pi(ax) = \pi(a)\Box\pi(x)$ and $\pi(xa) = \pi(x)\Box\pi(a), \forall\ a \in \mathscr{A}$ and $x \in \mathscr{A}_0$, where $\Box$ is the (weak) partial multiplication in $\mathscr{L}^\dagger(\mathscr{D},\mathscr{H})$ (ibid.). Having a $*$-representation π as before, we write $\mathscr{D}(\pi)$ in the place of $\mathscr{D}$ and $\mathscr{H}_\pi$ in the place of $\mathscr{H}$. By a (τ,τ_{s^*})-continuous $*$-representation π of $\mathscr{A}[\tau]$, we clearly mean continuity of π, when $\mathscr{L}^\dagger(\mathscr{D}(\pi),\mathscr{H}_\pi)$ carries the locally convex topology τ_{s^*} (see Section 4).

Lemma 7.2. *Let π be a $*$-representation of $\mathscr{A}[\tau]$ with domain $\mathscr{D}(\pi)$ dense in $\mathscr{H}_\pi$. Let also $\mathscr{B}$ be an admissible subset of $\mathscr{B}(\pi(\mathscr{A}))$. The following holds*:

(1) *If π is (τ,τ_{s^*})-continuous, then $\pi(\mathscr{A})[\tau_{s^*}]$ is a locally convex quasi C^*-algebra over the C^*-algebra $\pi(\mathscr{A}_0)$.*

(2) *If π is $(\tau,\tau^u_*(\mathscr{B}))$-continuous* (in the spirit of Definition 7.1)*, then $\pi(\mathscr{A})[\tau^u_*(\mathscr{B})]$ is a locally convex quasi C^*-algebra over $\pi(\mathscr{A}_0)$.*

Proof. Clearly $\pi(\mathscr{A}_0)$ is a C^*-algebra and

$$\pi : \mathscr{A}[\tau] \to \pi(\mathscr{A})[\tau_{s^*}] \subset \widetilde{\pi(\mathscr{A}_0)}[\tau_{s^*}]$$

is a (τ,τ_{s^*})-continuous $*$-representation of $\mathscr{A}[\tau]$, with $\pi(\mathscr{A})$ a quasi $*$-algebra over $\pi(\mathscr{A}_0)$ and $\widetilde{\pi(\mathscr{A}_0)}[\tau_{s^*}]$ (similarly $\widetilde{\pi(\mathscr{A}_0)}[\tau^u_*(\mathscr{B})]$) a locally convex quasi C^*-algebra over $\pi(\mathscr{A}_0)$. So, (1) and (2) follow from Definition 3.3. □

Now, a sesquilinear form φ on $\mathscr{A} \times \mathscr{A}$ is called positive, resp. invariant, if and only if $\varphi(a,a) \geq 0$, for each $a \in \mathscr{A}$, resp. $\varphi(ax,y) = \varphi(x,a^*y)$, for all $a \in \mathscr{A}$ and $x, y \in \mathscr{A}_0$. Moreover, φ is called τ-continuous, if $|\varphi(a,b)| \leq p(a)p(b)$ for some τ-continuous seminorm p on $\mathscr{A}$ and all $a, b \in \mathscr{A}$.

Further, let φ be a τ-continuous positive invariant sesquilinear form on $\mathscr{A}_0 \times \mathscr{A}_0$. Then, $\tilde{\varphi}$ denotes the extension of φ to a τ-continuous positive invariant sesquilinear form on $\mathscr{A} \times \mathscr{A}$. Moreover, let $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ be the GNS-construction for φ (see, for instance, [**6**, Section 9.1]). Then, π_φ is extended on $\mathscr{A}$, as follows:

$$\pi_\varphi(a)\lambda_\varphi(x) := \lim_\alpha \pi_\varphi(x_\alpha)\lambda_\varphi(x), \quad \forall\, x \in \mathscr{A}_0, \tag{7.1}$$

where $\{x_\alpha\}$ is a net in $\mathscr{A}[\tau]$ with $a = \tau\text{-}\lim_\alpha x_\alpha$. By the very definitions and the τ-continuity of φ, it follows that π_φ is a (τ, τ_{s^*})-continuous $*$-representation of $\mathscr{A}$. Now, put

$$\mathscr{S}(\mathscr{A}_0) := \{\tau\text{-continuous positive invariant sesquilinear forms } \varphi \text{ on } \mathscr{A}_0 \times \mathscr{A}_0\}.$$

We shall say that the set $\mathscr{S}(\mathscr{A}_0)$ is sufficient, whenever

$$a \in \mathscr{A} \text{ with } \tilde{\varphi}(a,a) = 0, \forall\, \varphi \in \mathscr{S}(\mathscr{A}_0), \text{ implies } a = 0.$$

From the results that follow, Theorems 7.3, 7.5 (and, of course, Corollary 7.4) give answers to the question (a) stated at the beginning of this Section. These results can be viewed as analogues of the Gel'fand-Naimark theorem, in the case of locally convex quasi C^*-algebras.

THEOREM 7.3. *Let $\mathscr{A}[\tau]$ be a locally convex quasi C^*-algebra over a unital C^*-algebra $\mathscr{A}_0$. The following statements are equivalent*:

(1) *There exists a faithful (τ, τ_{s^*})-continuous $*$-representation π of $\mathscr{A}$.*

(2) *The set $\mathscr{S}(\mathscr{A}_0)$ is sufficient.*

PROOF. (1) $\Rightarrow$ (2) For every $\xi \in \mathscr{D}(\pi)$ define

$$\varphi_\xi(x,y) := (\pi(x)\xi | \pi(y)\xi), \quad \forall\, x, y \in \mathscr{A}_0.$$

Then, $\{\varphi_\xi : \xi \in \mathscr{D}(\pi)\} \subset \mathscr{S}(\mathscr{A}_0)$, so that from the preceding discussion it follows easily that $\mathscr{S}(\mathscr{A}_0)$ is sufficient.

(2) $\Rightarrow$ (1) Let $\varphi \in \mathscr{S}(\mathscr{A}_0)$ and $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ the GNS-construction for φ. Then, as we noticed before (see (7.1)), π_φ extends to a (τ, τ_{s^*})-continuous $*$-representation of $\mathscr{A}$ with $\mathscr{D}(\pi_\varphi) = \lambda_\varphi(\mathscr{A}_0)$. Now, take

$$\mathscr{D}(\pi) := \Big\{(\lambda_\varphi(x_\varphi))_{\varphi\in\mathscr{S}(\mathscr{A}_0)} \in \bigoplus_{\varphi\in\mathscr{S}(\mathscr{A}_0)} \mathscr{H}_\varphi : x_\varphi \in \mathscr{A}_0 \text{ and}$$

$$\lambda_\varphi(x_\varphi) = 0, \text{ except for a finite number of } \varphi\text{'s from } \mathscr{S}(\mathscr{A}_0)\Big\}$$

and define

$$\pi(a)(\lambda_\varphi(x_\varphi)) := (\lambda_\varphi(ax_\varphi)), \forall\ a \in \mathscr{A} \text{ and } (\lambda_\varphi(x_\varphi)) \in \mathscr{D}(\pi).$$

Then, it is easily seen that π is a faithful (τ, τ_{s^*})-continuous $*$-representation of $\mathscr{A}$. □

Results for quasi $*$-algebras over a unital C^*-algebra $\mathscr{A}_0$ related to Theorem 7.3, have been considered in [**10**, Theorem 3.3] and [**15**, Theorem 3.2].

Now an application of Theorem 7.3 and Lemma 7.2, gives the following

COROLLARY 7.4. *Let $\mathscr{A}[\tau]$, $\mathscr{A}_0$ be as in Theorem 7.3. Suppose that the set $\mathscr{S}(\mathscr{A}_0)$ is sufficient. Then, the locally convex quasi C^*-algebra $\mathscr{A}[\tau]$ over $\mathscr{A}_0$ is continuously embedded in a locally convex quasi C^*-algebra of operators.*

The next theorem gives further conditions under which a locally convex quasi C^*-algebra $\mathscr{A}[\tau]$ can be continuously embedded in a locally convex quasi C^*-algebra of operators.

THEOREM 7.5. *Let $\mathscr{A}[\tau]$ be a locally convex quasi C^*-algebra over $\mathscr{A}_0$. Suppose the multiplication of $\mathscr{A}_0$ satisfies the following condition*:

For every τ-bounded subset B of $\mathscr{A}_0$ and every $\lambda \in \Lambda$, there exist $\lambda' \in \Lambda$ and a positive constant c_B such that

$$\sup_{y\in B} p_\lambda(xy) \le c_B p_{\lambda'}(x), \quad \forall\ x \in \mathscr{A}_0.$$

Then, the next statements are equivalent:

(i) *There is a faithful $(\tau, \tau_*^u(\mathscr{B}))$-continuous $*$-representation π of $\mathscr{A}$, where $\mathscr{B}$ is an admissible subset of $\mathscr{B}(\pi(\mathscr{A}))$.*

(ii) *There is a faithful (τ, τ_{s^*})-continuous $*$-representation of $\mathscr{A}$.*

(iii) *The set $\mathscr{S}(\mathscr{A}_0)$ is sufficient.*

PROOF. (i) ⇒ (ii) It is trivial (see (4.3)).

(ii) ⇒ (iii) It follows from Theorem 7.3.

(iii) ⇒ (i) Let $\varphi \in \mathscr{S}(\mathscr{A}_0)$ and $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ the GNS-construction for φ (see

discussion before Theorem 7.3). Set

$$\mathscr{B}_\varphi := \{\lambda_\varphi(B) : B \text{ a } \tau\text{-bounded subset of } \mathscr{A}_0\}.$$

Then, for each τ-bounded subset B of $\mathscr{A}_0$, we have

$$\sup_{y\in B}\|\pi_\varphi(a)\lambda_\varphi(y)\| = \sup_{y\in B}\varphi(ay,ay)^{1/2} \le \sup_{y\in B} p_\lambda(ay) \le c_B p_{\lambda'}(a),$$

for all $a \in \mathscr{A}$ and some $\lambda, \lambda' \in \Lambda$. It is clear now that $\lambda_\varphi(B) \in \mathscr{B}(\pi_\varphi(\mathscr{A}))$ and that (see (4.2)) π_φ is $(\tau, \tau^u_*(\mathscr{B}_\varphi))$-continuous. Let now π be as in the proof of Theorem 7.3. Put

$$\mathscr{B}_\pi := \left\{ \bigoplus_{\varphi\in\mathscr{S}(\mathscr{A}_0)}^{\text{finite}} \lambda_\varphi(B_\varphi) : B_\varphi \text{ a } \tau\text{-bounded subset of } \mathscr{A}_0 \right\}.$$

Then, it is easily seen that $\mathscr{B}_\pi$ is an admissible subset of $\mathscr{B}(\pi(\mathscr{A}))$ and π a faithful $(\tau, \tau^u_*(\mathscr{B}_\pi))$-continuous $*$-representation of $\mathscr{A}$. □

An analogue of Corollary 7.4 is stated in the case of Theorem 7.5, too.

Taking again $\mathscr{A}[\tau], \mathscr{A}_0$ as in Theorem 7.3, we proceed to the study of a functional calculus for the commutatively quasi-positive elements of $\mathscr{A}[\tau]$ (see (b) at the beginning of this Section). So, let $a \in \mathscr{A}[\tau]_{cq+}$. Then, from Proposition 3.2,(1), the element $(1+a)^{-1}$ exists and belongs to $\mathscr{U}(\mathscr{A}_0)_+$. Consider the maximal commutative C^*-subalgebra $C^*(a)$ of $\mathscr{A}_0$ containing the elements $1, (1+a)^{-1}$. Then,

- $C^*(a)[\tau]$ satisfies the properties (T_1)–(T_4) of Section 3. The properties (T_1)–(T_3) are trivially checked. We must check (T_4).

First we prove that $\mathscr{U}(C^*(a))_+$ is τ-closed. Let $\{x_\alpha\}$ be a net in $\mathscr{U}(C^*(a))_+$ such that $x_\alpha \underset{\tau}{\to} x$. But, $\mathscr{U}(C^*(a))_+ \subset \mathscr{U}(\mathscr{A}_0)_+$ and since $\mathscr{U}(\mathscr{A}_0)_+$ is τ-closed we have that $x \in \mathscr{U}(\mathscr{A}_0)_+$. On the other hand,

$$xy \underset{\tau}{\leftarrow} x_\alpha y = y x_\alpha \underset{\tau}{\to} yx, \quad \forall\, y \in C^*(a).$$

Hence, $xy = yx$, which by the maximality of $C^*(a)$ means that $x \in C^*(a)$ and finally $x \in \mathscr{U}(C^*(a))_+$. Thus, $\mathscr{U}(C^*(a))+$ is τ-closed. Now, take an arbitrary $x \in \widetilde{C^*(a)}[\tau]_{q+} \cap C^*(a)$. Then, $x \in \mathscr{A}[\tau]_{q+} \cap \mathscr{A}_0 = (\mathscr{A}_0)_+$, and so $x \in C^*(a) \cap (\mathscr{A}_0)_+ = C^*(a)_+$. This completes the proof of (T_4). Thus, the following is proved:

PROPOSITION 7.6. *Let $\mathscr{A}[\tau]$ be a locally convex quasi C^*-algebra over a unital C^*-algebra $\mathscr{A}_0$. Let $a \in \mathscr{A}[\tau]_{cq+}$ and $C^*(a)$ the maximal commutative C^*-subalgebra of $\mathscr{A}_0$ containing $\{1, (1+a)^{-1}\}$. Then, $\widetilde{C^*(a)}[\tau]$ is a commutative locally convex quasi C^*-algebra over $C^*(a)$.*

COROLLARY 7.7. *The element a belongs to $\widetilde{C^*(a)}[\tau]_{q+}$.*

PROOF. Since $a \in \mathscr{A}[\tau]_{cq+}$, Proposition 3.2, (2) implies that for every $\varepsilon > 0$, $a(1+\varepsilon a)^{-1} = 1/\varepsilon(1-(1+\varepsilon a)^{-1}) \in (\mathscr{A}_0)_+$. Now, since $(1+a)^{-1}$ commutes with every element $\omega \in C^*(a)$, it follows that ω also commutes with $1+a$, hence with a, therefore with $(1+\varepsilon a)^{-1}$ too. Thus, $a(1+\varepsilon a)^{-1} \in C^*(a)$, for each $\varepsilon > 0$. Since moreover, $a = \tau\text{-}\lim_{\varepsilon \to 0} a(1+\varepsilon a)^{-1}$ (ibid.), Definition 3.1 gives that $a \in \widetilde{C^*(a)}[\tau]_{q+}$. □

It is now clear from Corollary 7.7 that making use of Theorem 6.6 for $\widetilde{C^*(a)}[\tau]_{q+}$, we can obtain the promised functional calculus for the commutatively quasi-positive elements of the noncommutative locally convex quasi C^*-algebra $\mathscr{A}[\tau]$. That is, we have the following

THEOREM 7.8. *Let $\mathscr{A}[\tau]$ be a noncommutative locally convex quasi C^*-algebra over a unital C^*-algebra $\mathscr{A}_0$. Let $a \in \mathscr{A}[\tau]_{cq+}$ such that a^n is well defined for some $n \in \boldsymbol{N}$. Then, there is a unique $*$-isomorphism $f \mapsto f(a)$ from $\bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ into $\mathscr{A}[\tau]$ such that:*

(1) If $u_0 \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ with $u_0(\lambda) = 1$, for each $\lambda \in \sigma_{C^*(a)}(a)$, then $u_0(a) = 1 \in C^*(a) \hookrightarrow \mathscr{A}[\tau]$.

(2) If $u_1 \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ with $u_1(\lambda) = \lambda$, for each $\lambda \in \sigma_{C^*(a)}(a)$, then $u_1(a) = a \in \mathscr{A}[\tau]$.

(3) $\widehat{f(a)}(\varphi) = f(\hat{a}(\varphi))$, for any $f \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ and $\varphi \in \mathscr{M}(C^*(a))$.

(4) $(f_1+f_2)(a) = f_1(a) + f_2(a)$, for any $f_1, f_2 \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$, $(\lambda f)(a) = \lambda f(a)$, for any $f \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ and $\lambda \in \boldsymbol{C}$, $(f_1 f_2)(a) = f_1(a) f_2(a)$, for any $f_j \in C_{k_j}(\sigma_{C^*(a)}(a))$, $j = 1, 2$, with $k_1 + k_2 \leq n$.

(5) Restricted to $C_b(\sigma_{C^*(a)}(a))$ the map $f \mapsto f(a)$ is an isometric $*$-isomorphism of the C^*-algebra $C_b(\sigma_{C^*(a)}(a))$ onto the closed $*$-subalgebra of the C^*-algebra $C^*(a)$ generated by 1 and $(1+a)^{-1}$.

Now, an application of Corollary 6.7 for the commutative locally convex quasi C^*-algebra $\widetilde{C^*(a)}[\tau]$ and Theorem 7.8 give the following

COROLLARY 7.9. *Let $\mathscr{A}[\tau]$, $\mathscr{A}_0$ be as in Theorem 7.8. Let $a \in \mathscr{A}[\tau]_{cq+}$ and $n \in \boldsymbol{N}$. Then, there is a unique element $b \in \mathscr{A}[\tau]_{cq+}$ such that $a = b^n$. The element b is called commutatively quasi nth-root of a and is denoted by $a^{\frac{1}{n}}$. If*

$n = 2$, the element $a^{\frac{1}{2}}$ is called commutatively quasi square root of a.

Acknowledgments. This work begun during the stay of two of us (F.B and C.T.) at the Department of Applied Mathematics, Fukuoka University, in the summer of 2005 and it was concluded during the 3-month stay of the second author (M.F.) at the same Department in the spring of 2006. All three of us gratefully acknowledge the financial support of this Department and the warm hospitality of its staff.

References

[1] G. R. Allan, A spectral theory for locally convex algebras, Proc. London Math. Soc. (3), **15** (1965), 399–421.

[2] G. R. Allan, On a class of locally convex algebras, Proc. London Math. Soc. (3), **17** (1967), 91–114.

[3] J.-P. Antoine, F. Bagarello and C. Trapani, Topological partial $*$-algebras: Basic properties and examples, Rev. Math. Phys., **11** (1999), 267–302.

[4] J.-P. Antoine, A. Inoue and C. Trapani, Partial $*$-algebras of closable operators, I, The basic theory and the abelian case, Publ. RIMS, Kyoto Univ., **26** (1990), 359–395; II, States and $*$-representations of partial $*$-algebras, ibid., **27** (1991), 399–340.

[5] J.-P. Antoine, A. Inoue and C. Trapani, Partial $*$-algebras of closable operators, Rev. Math. Phys., **8** (1996), 1–42.

[6] J.-P. Antoine, A. Inoue and C. Trapani, Partial $*$-Algebras and their Operator Realizations, Math. Appl., **553**, Kluwer Academic, Dordrecht, 2003.

[7] J.-P. Antoine and W. Karwowski, Partial $*$-algebras of closed operators in Quantum Theory of Particles and Fields, (eds. B. Jancewitz and J. Lukerski), World Scientific, Singapore, 1983, pp. 13–30.

[8] J.-P. Antoine and W. Karwowski, Partial $*$-algebras of closed linear operators in Hilbert space, Publ. RIMS, Kyoto Univ., **21** (1985), 205–236; Add./Err. ibid., **22** (1986), 507–511.

[9] F. Bagarello, Algebras of unbounded operators and physical applications, Rev. Math. Phys., **19** (2007), 231–272.

[10] F. Bagarello, M. Fragoulopoulou, A. Inoue and C. Trapani, The completion of a C^*-algebra with a locally convex topology, J. Operator Theory, **56** (2006), 357–376.

[11] F. Bagarello, A. Inoue and C. Trapani, Unbounded C^*-seminorms and $*$-representations of partial $*$-algebras, Z. Anal. Anwend., **20** (2001), 1–20.

[12] F. Bagarello and C. Trapani, States and representations of CQ^*-algebras, Ann. Inst. H. Poincaré, **61** (1994), 103–133.

[13] F. Bagarello and C. Trapani, CQ^*-algebras: Structure properties, Publ. RIMS, Kyoto Univ., **32** (1996), 85–116.

[14] P. G. Dixon, Generalized B^*-algebras, Proc. London Math. Soc., **21** (1970), 693–715.

[15] M. Fragoulopoulou, A. Inoue and K.-D. Kürsten, On the completion of a C^*-normed algebra under a locally convex algebra topology, Contemporary Math., **427** (2007), 89–95.

[16] R. Haag and D. Kastler, An algebraic approach to quantum field theory, J. Math. Phys., **5** (1964), 848–861.

[17] A. Inoue, Tomita-Takesaki Theory in Algebras of Unbounded Operators, Lecture Notes in Math., **1699**, Springer-Verlag, 1998.

[18] G. Lassner, Topological algebras and their applications in quantum statistics, Wiss. Z. KMU-Leipzig, Math. Naturwiss. R., **30** (1981), 572–595.

[19] G. Lassner, Algebras of unbounded operators and quantum dynamics, Phys. A, **124** (1984), 471–480.

[20] K. Schmüdgen, Unbounded Operator Algebras and Representation Theory, Birkhäuser-Verlag, Basel, 1990.

[21] C. Trapani, States and derivations on quasi $*$-algebras, J. Math. Phys., **29** (1988), 1885–1890.

[22] C. Trapani, Quasi $*$-algebras of operators and their applications, Rev. Math. Phys., **7** (1995), 1303–1332.

[23] C. Trapani, Bounded elements and spectrum in Banach quasi $*$-algebras, Studia Math., **172** (2006), 249–273.

Fabio Bagarello

Dipartimento di Matematica ed Applicazioni
Fac Ingegneria
Universita di Palermo
I-90128 Palermo, Italy
E-mail: bagarell@unipa.it

Atsushi Inoue

Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180, Japan
E-mail: a-inoue@fukuoka-u.ac.jp

Maria Fragoulopoulou

Department of Mathematics
University of Athens
Panepistimiopolis
Athens 15784, Greece
E-mail: fragoulop@math.uoa.gr

Camillo Trapani

Dipartimento di Matematica ed Applicazioni
Universita di Palermo
I-90123 Palermo, Italy
E-mail: trapani@unipa.it