J. Math. Soc. Japan
Vol. 60, No. 2 (2008) pp. 341–361
doi: 10.2969/jmsj/06020341

Varieties of small degree with respect to codimension and ample vector bundles

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(Received Feb. 19, 2007)

Abstract. Let $\mathscr{E}$ be an ample vector bundle on a projective manifold X, with a section vanishing on a smooth subvariety Z of the expected dimension, and let H be an ample line bundle on X inducing a very ample ample line bundle H_Z on Z. Triplets $(X, \mathscr{E}, H)$ as above are classified assuming that Z, embedded by $|H_Z|$, is a variety of small degree with respect to codimension.

Introduction.

As is well known, the degree d of any non-degenerate projective variety $Z \subset \boldsymbol{P}^N$ satisfies the inequality $d \geq \operatorname{codim}_{\boldsymbol{P}^N} Z + 1$. Smooth non-degenerate projective varieties of degree

$$d \leq 2 \operatorname{codim}_{\boldsymbol{P}^N} Z + 1, \tag{1}$$

have been studied and classified by Ionescu ([**7**]). Recently, several classification problems for projective manifolds having a special variety among their hyperplane sections have been revisited in the setting of ample or very ample vector bundles ([**12**], [**16**], [**14**], [**18**]). In this paper we are interested in ample vector bundles admitting a section whose zero locus is a projective manifold of small degree with respect to codimension in the sense of (1). More precisely, the setting we consider is as follows.

0.1. Let X be a smooth complex projective variety of dimension n and let $\mathscr{E}$ be an ample vector bundle of rank $r \geq 2$ on X such that there exists a section $s \in \Gamma(\mathscr{E})$ whose zero locus $Z := (s)_0$ is a smooth subvariety of X of the expected dimension $n - r \geq 3$.

2000 *Mathematics Subject Classification.* Primary 14J60; Secondary 14F05, 14C20, 14J40, 14N30.

Key Words and Phrases. ample vector bundles, special varieties, Δ-genus, adjunction theory, Fano manifolds, classification.

Next, consider an ample line bundle H on X and suppose that its restriction H_Z to Z is very ample. Then, Z embedded in $\boldsymbol{P}^N$ by the complete linear system $|H_Z|$ is a projective manifold of degree $d(Z, H_Z)$. We are interested in triplets $(X, \mathscr{E}, H)$ as above such that $d := d(Z, H_Z)$ satisfies (1).

Comparing the pairs (Z, H_Z) in our setting with the projective manifolds studied in [**7**], the reader will note that we do not consider case $\dim Z = 2$. In fact we could describe the structure of pairs $(X, \mathscr{E})$ even in this case by using [**17**]; however a lack of information concerning H cannot be avoided (see Remark (3.5)). This is the technical reason why we assume $n - r \geq 3$ in (0.1).

The list of triplets $(X, \mathscr{E}, H)$ we are looking for is provided by Theorem (2.1), which is the main result of this paper. Essentially there are 17 types. Moreover, all of them do really occur (Remarks (3.1) and (3.3)). Here is a sketch of the proof. Most of the projective manifolds appearing in Ionescu's classification are special varieties in adjunction theory. So, the most appropriate tool to investigate triplets $(X, \mathscr{E}, H)$ compatible with them is a series of results of Maeda and the first author [**11**], [**12**], [**13**] on special varieties in adjunction theory occurring as zero loci of sections of ample vector bundles. In some cases they allow us to infer the structure of $(X, \mathscr{E}, H)$ from that of (Z, H_Z) rather quickly. In other cases, however, this requires some extra work. In particular, we have to pay attention to scrolls over a smooth surface. Indeed, to restrict the possible candidates for $(X, \mathscr{E}, H)$ in this case, we have to analyze whether the scroll structure of (Z, H_Z) can coexist with further possible structures inherited from (X, H). Another situation requiring special care is that of pairs (Z, H_Z) which admit a non-trivial reduction in the adjunction theoretic sense. To deal with them we first show that the reduction morphism of (Z, H_Z) extends to a birational morphism from X to a projective manifold X' contracting finitely many (-1)-hyperplanes. This gives rise to a new triplet $(X', \mathscr{E}', H')$ defining, in turn, a polarized manifold which is the minimal reduction of (Z, H_Z). Then results from [**13**] apply to $(X', \mathscr{E}', H')$. However, compatibility results for different adjunction theoretic structures on the same variety are needed once more. Some of them already exist in the literature; other ones are developed in Section 1.

Notice that condition (1) for our (Z, H_Z) is equivalent to

$$d(Z, H_Z) \geq 2\Delta(Z, H_Z) + 1, \tag{2}$$

where $\Delta(Z, H_Z)$ is the Δ-genus of the polarized manifold (Z, H_Z). So, with a little change of perspective, Theorem (2.1) provides also the classification of triplets $(X, \mathscr{E}, H)$ where $(X, \mathscr{E})$ is as in (0.1) and H is simply an ample line bundle on X such that H_Z admits a ladder and condition (2) is satisfied. Actually, under these assumptions, the very ampleness of H_Z, which is a hypothesis in Theorem (2.1), turns out to be a consequence of a result of Fujita [**5**, Theorem 3.5].

Finally we would like to note that this paper can be regarded as a continuation of our previous investigation of projective manifolds of small Δ-genera as zero loci of sections of ample vector bundles [**15**]. Actually in terms of Δ-genus, condition (1) can be rephrased also as $\Delta(Z, H_Z) \le \operatorname{codim}_{\boldsymbol{P}^N} Z$ (or $\Delta(Z, H_Z) < (1/2)d(Z, H_Z)$, equivalently). From this point of view our Theorem (2.1) improves [**15**, Corollary 5.3] considerably.

1. Background material.

We use the standard notation from algebraic geometry. By a little abuse we make no distinction between a line bundle and the corresponding invertible sheaf. Moreover, the tensor products of line bundles are denoted additively. The pullback $i^*\mathscr{E}$ of a vector bundle $\mathscr{E}$ on X by an embedding of projective varieties $i\colon Y \hookrightarrow X$ is denoted by $\mathscr{E}_Y$. We denote by K_X the canonical bundle of a smooth variety X. The blow-up of a variety X along a smooth subvariety Y is denoted by $\mathrm{Bl}_Y(X)$.

A polarized manifold is a pair $(X, \mathscr{L})$ consisting of a smooth complex projective variety X and an ample line bundle $\mathscr{L}$ on X. The degree and the Δ-genus of a polarized manifold $(X, \mathscr{L})$ are defined as $d(X, \mathscr{L}) = \mathscr{L}^{\dim X}$ and $\Delta(X, \mathscr{L}) = \dim X + d(X, \mathscr{L}) - h^0(X, \mathscr{L})$, respectively. A polarized manifold $(X, \mathscr{L})$ is said to be a scroll over a smooth variety W if there exists a surjective morphism $f\colon X \longrightarrow W$ such that $(F, \mathscr{L}_F) \cong (\boldsymbol{P}^r, \mathscr{O}_{\boldsymbol{P}^r}(1))$ with $r = \dim X - \dim W$ for any fiber F of f. This condition is equivalent to saying that $(X, \mathscr{L}) \cong (\boldsymbol{P}_W(\mathscr{F}), H(\mathscr{F}))$ for some ample vector bundle $\mathscr{F}$ on W, where $H(\mathscr{F})$ is the tautological line bundle on the projective space bundle $\boldsymbol{P}_W(\mathscr{F})$ associated to $\mathscr{F}$. A polarized manifold $(X, \mathscr{L})$ is said to be a quadric fibration over a smooth curve W if there exists a surjective morphism $f\colon X \longrightarrow W$ and any general fiber F of f is a smooth quadric hypersurface $\boldsymbol{Q}^{n-1}$ in $\boldsymbol{P}^n$ with $n = \dim X$ such that $\mathscr{L}_F \cong \mathscr{O}_{\boldsymbol{Q}^{n-1}}(1)$. A polarized manifold $(X, \mathscr{L})$ is said to be a Veronese bundle over a smooth curve W if there exists a $\boldsymbol{P}^2$-bundle $p\colon X \longrightarrow W$ such that $\mathscr{L}_F \cong \mathscr{O}_{\boldsymbol{P}^2}(2)$ for any fiber F of p. A polarized manifold $(X, \mathscr{L})$ is said to be a del Pezzo manifold if $K_X + (\dim X - 1)\mathscr{L} = \mathscr{O}_X$.

Let $(X, \mathscr{L})$ be a polarized manifold. An effective divisor $E \subset X$ is called a (-1)-hyperplane if $E \cong \boldsymbol{P}^{n-1}$, $\mathscr{O}_E(E) \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}(-1)$ and $\mathscr{L}_E \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}(1)$. Sometimes we write E_E for $\mathscr{O}_E(E)$ for shortness. Note that the set of (-1)-hyperplanes contained in X is finite and, in case $\dim X \ge 3$, any two (-1)-hyperplanes are disjoint. Let $\dim X \ge 3$. We will call a pair (Y, L) the reduction of $(X, \mathscr{L})$ if there exists a birational morphism $\sigma\colon X \longrightarrow Y$ which is the contraction of all (-1)-hyperplanes $E_1, \dots, E_s$ contained in X and L is the (unique) line bundle on Y such that $\mathscr{L} = \sigma^* L - E_1 - \cdots - E_s$; in this case L is ample on Y, see for instance [**3**, Lemma 5.7]. Let $y_i = \sigma(E_i)$. Recall that there is a bijection between the linear system $|\mathscr{L}|$ and the sublinear system $|L-y_1-\cdots-y_s|$ of $|L|$ consisting of elements passing through $y_1,\dots,y_s$. We will need the following

LEMMA 1.1. *Let $(X,\mathscr{L})$ be a polarized manifold containing a (-1)-hyperplane E and let $\sigma\colon X\longrightarrow X'$ be the birational morphism onto a smooth projective variety, contracting E. Let $\mathscr{E}$ be an ample vector bundle of rank $r\geq 2$ on X and suppose that $\mathscr{E}_E\cong\mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus r}$. Then there exist an ample line bundle $\mathscr{L}'$ and an ample vector bundle $\mathscr{E}'$ of rank r on X' such that*

$$\mathscr{L}=\sigma^*\mathscr{L}'-E \quad \text{and} \quad \mathscr{E}=\sigma^*\mathscr{E}'\otimes\mathscr{O}_X(-E).$$

PROOF. Since $\mathscr{L}_E=\mathscr{O}_E(1)$ and $E_E=\mathscr{O}_E(-1)$, we have that $(\mathscr{L}+E)_E=\mathscr{O}_E$. It follows that there exists $\mathscr{L}'\in\operatorname{Pic}(X')$ such that $\mathscr{L}+E=\sigma^*\mathscr{L}'$, i.e. $\mathscr{L}=\sigma^*\mathscr{L}'-E$. Moreover [**3**, Lemma 5.7] implies that $\mathscr{L}'$ is ample. The remaining part of the statement comes from [**12**, Lemma 5.1]. □

The following well-known facts will be used several times.

THEOREM 1.2 (Remmert–Van de Ven). *Let S be a smooth surface and let $f\colon\boldsymbol{P}^2\longrightarrow S$ be a surjective morphism. Then $S\cong\boldsymbol{P}^2$.*

Let X, $\mathscr{E}$ and Z be as in (0.1). We recall that $K_Z=(K_X+\det\mathscr{E})_Z$, by adjunction. Moreover, for any rational curve $C\subset X$ we have $\det\mathscr{E}\cdot C\geq r$.

THEOREM 1.3 (Lefschetz–Sommese). *Let $X,\mathscr{E}$ and Z be as in* (0.1). *Then the restriction homomorphism* $\operatorname{Pic}(X)\longrightarrow\operatorname{Pic}(Z)$ *is an isomorphism.*

We recall the following result [**13**, Theorem 3] (the final assertions are shown in the proof).

THEOREM 1.4. *Let X, $\mathscr{E}$ and Z be as in* (0.1). *Let H be an ample line bundle on X. Suppose that $K_Z+(\dim Z-1)H_Z$ is nef but $K_Z+(\dim Z-2)H_Z$ is not nef. If the Picard number of X is $\rho(X)>1$, then one of the following holds*:

(I) *there exists a (-1)-hyperplane E of (X,H) such that $\mathscr{E}_E\cong\mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus r}$;*
(II) *X is a $\boldsymbol{P}^{n-1}$-bundle over a smooth curve C and $(\mathscr{E}_F,H_F)\cong(\mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus(n-3)},\mathscr{O}_{\boldsymbol{P}^{n-1}}(2))$ for any fiber F of the projection $\psi\colon X\longrightarrow C$;*
(III) *(X,H) is a scroll over a smooth curve C and $\mathscr{E}_F\cong\mathscr{O}_{\boldsymbol{P}^{n-1}}(2)\oplus\mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus(r-1)}$ for any fiber F of the projection $\psi\colon X\longrightarrow C$;*
(IV) *(X,H) is a quadric fibration over a smooth curve C and $\mathscr{E}_F\cong\mathscr{O}_{\boldsymbol{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber F of the fibration $\psi\colon X\longrightarrow C$;*
(V) *(X,H) is a scroll over a smooth surface Σ and $\mathscr{E}_F\cong\mathscr{O}_{\boldsymbol{P}^{n-2}}(1)^{\oplus r}$ for any fiber F of the projection $\psi\colon X\longrightarrow\Sigma$.*

Moreover, in case (II) (Z, H_Z) *is a Veronese bundle over* C*, in cases* (III) *and* (IV) (Z, H_Z) *is a quadric fibration over* C*, in case* (V) (Z, H_Z) *is a scroll over* Σ*; all these structures are induced by* ψ*.*

Now we collect some facts that we will use in the proof of Theorem (2.1).

PROPOSITION 1.5. *Let* X*,* $\mathscr{E}$*,* Z*,* H*, and* E *be as in case* (I) *of Theorem* (1.4)*. Then* $E \cap Z$ *is a* (-1)*-hyperplane of* (Z, H_Z)*.*

PROOF. Set $e = E \cap Z$. As $\mathscr{E}_E \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus r}$, we have that e is a linear subspace of E of dimension $m \geq n-1-r$. Clearly, since e is contained in Z, its dimension is $\leq n-r$. Moreover, if equality holds, then $Z = e \cong \boldsymbol{P}^{n-r}$. So Z, hence X, would have Picard number 1, which is a contradiction. Therefore $m = n-r-1$, i.e. $e \cong \boldsymbol{P}^{n-r-1}$ is a divisor inside Z. Now we can compute $(K_Z)_e = ((K_X + \det\mathscr{E})_Z)_e = ((K_X + \det\mathscr{E})_E)_e$. Since $\mathscr{O}_{\boldsymbol{P}^{n-1}}(-n) \cong K_E = (K_X + E)_E \cong (K_X)_E + \mathscr{O}_{\boldsymbol{P}^{n-1}}(-1)$, we derive $(K_Z)_e = (\mathscr{O}_E(-n+1) + \mathscr{O}_E(r))_e = \mathscr{O}_e(-(n-r-1))$. On the other hand, as $e \cong \boldsymbol{P}^{n-r-1}$ is a divisor in Z, by adjunction we have $\mathscr{O}_e(-(n-r)) = K_e = (K_Z + e)_e = \mathscr{O}_e(-(n-r-1)) + \mathscr{O}_e(e)$. Hence we obtain $\mathscr{O}_e(e) = \mathscr{O}_e(-1)$, thus e is an exceptional divisor on Z. Moreover, $(H_Z)_e = (H_E)_e = (\mathscr{O}_E(1))_e = \mathscr{O}_e(1)$, therefore e is a (-1)-hyperplane of (Z, H_Z). □

Of course $(\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^1}(2,1))$ can be regarded both as a Veronese bundle over $\boldsymbol{P}^1$ and as a scroll over $\boldsymbol{P}^2$. The former structure, however, is not compatible with case (II) of Theorem (1.4). In fact we can prove the following.

LEMMA 1.6. *Let* X*,* $\mathscr{E}$*,* Z*,* H*, and* E *be as in case* (II) *of Theorem* (1.4) *and suppose that* $C \cong \boldsymbol{P}^1$*. Then* (Z, H_Z) *cannot be* $(\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^1}(2, a))$ *for* $a = 1$ *or* 2*.*

PROOF. We can write $X = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathscr{W})$, where $\mathscr{W}$ is a vector bundle of rank n on $\boldsymbol{P}^1$, normalized as in [**1**, Lemma 3.2.4], i.e. $\mathscr{W} = \oplus_{i=1}^n \mathscr{O}_{\boldsymbol{P}^1}(\alpha_i)$ with $\alpha_1 \geq \cdots \geq \alpha_n = 0$. Let ξ be the tautological line bundle on X and let F be a fiber of the bundle projection $\psi\colon X \longrightarrow \boldsymbol{P}^1$. Since $(\mathscr{E} \otimes \xi^{-1})_F \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}^{\oplus(n-3)}$, we get $\mathscr{E} \cong \xi \otimes \psi^*\mathscr{G}$, where $\mathscr{G}$ is a vector bundle of rank $n-3$ on $\boldsymbol{P}^1$. So $\mathscr{G} \cong \oplus_{j=1}^{n-3} \mathscr{O}_{\boldsymbol{P}^1}(b_j)$. Therefore $\mathscr{E} \cong \oplus_{j=1}^{n-3}[\xi + b_j F]$, and the ampleness of $\mathscr{E}$ combined with [**1**, Lemma 3.2.4] implies $b_j \geq 1$ for any $j = 1, \ldots, n-3$. In particular

$$\deg\mathscr{W} \geq 0 \quad \text{and} \quad \deg\mathscr{G} \geq n-3. \tag{1.6.1}$$

Now suppose by contradiction that $(Z, H_Z) \cong (\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^1}(2, a))$, $a = 1$ or 2. Recall that the structure of (Z, H_Z) as Veronese bundle is induced by ψ.

Note that $\xi_Z = \mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^1}(1,e)$ for some $e \geq 0$ because ξ is spanned. From the ampleness of H_Z we thus get $H_Z = \mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^1}(2,a) = 2\xi_Z + (a-2e)F_Z$. Thus $H = 2\xi + (a-2e)F$ by the Lefschetz–Sommese Theorem. Moreover

$$a - 2e > 0 \tag{1.6.2}$$

by ampleness, due to [**1**, Lemma 3.2.4]. On the other hand, $2K_Z + 3H_Z = (3a-4)F_Z$, which implies $2(K_X + \det\mathcal{E}) + 3H = (3a-4)F$. We can rewrite this relation as

$$2(-n\xi + (\deg\mathcal{W} - 2)F + (n-3)\xi + \deg\mathcal{G}\ F) + 3(2\xi + (a-2e)F) - (3a-4)F = 0,$$

from which we derive

$$\deg\mathcal{W} + \deg\mathcal{G} = 3e. \tag{1.6.3}$$

Now, (1.6.2) shows that $e = 0$ because $a \leq 2$; but then (1.6.3) contradicts (1.6.1). □

The result is no longer true if $a \geq 3$, as the following example shows.

Example 1.7. Consider the exact sequence of vector bundles on $\boldsymbol{P}^1$

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^1}(-1)^{\oplus 3} \longrightarrow \mathcal{O}_{\boldsymbol{P}^1}^{\oplus 6} \longrightarrow \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus 3} \longrightarrow 0$$

given by three copies of the Euler sequence twisted by $\mathcal{O}_{\boldsymbol{P}^1}(-1)$. Let $\mathcal{W} = \mathcal{O}_{\boldsymbol{P}^1}^{\oplus 6}$, $\mathcal{V} = \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus 3}$ and consider $X = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{W}) \cong \boldsymbol{P}^5 \times \boldsymbol{P}^1$, $Z = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{V}) \cong \boldsymbol{P}^2 \times \boldsymbol{P}^1$. Then Z is contained in X fiberwise and $\xi_Z \cong \mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^1}(1,1)$. Then arguing as in the proof of Lemma (1.6) we see that Z is the zero locus of a general section of the (very) ample vector bundle $\mathcal{E} = [\xi + F]^{\oplus 3}$. Moreover letting $H = 2\xi + F$ we get a very ample line bundle on X such that $(Z, H_Z) \cong (\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^1}(2,3))$.

Lemma 1.8. *Let $(Y, \mathcal{L})$ be a scroll over a smooth surface S and suppose that Y contains a (-1)-hyperplane with respect to $\mathcal{L}$, say e. Then $(Y, \mathcal{L})$ is a del Pezzo threefold of degree 7, namely $Y = \mathrm{Bl}_p(\boldsymbol{P}^3)$ is the blow-up of $\boldsymbol{P}^3$ at one point p, and $\mathcal{L} = \sigma^*\mathcal{O}_{\boldsymbol{P}^3}(2) - e$, where $\sigma\colon Y \longrightarrow \boldsymbol{P}^3$ is the blowing-up. Equivalently, $Y \cong \boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{O}_{\boldsymbol{P}^2}(2) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1))$, $\mathcal{L}$ being the tautological line bundle.*

Proof. Let $\pi\colon Y \longrightarrow S$ be the projection. Then $\pi(e)$ is either a point or the whole S. In the former situation the morphism π would have a divisorial fiber, which is a contradiction, so we are left with the latter case. Here the dimension of e has to be equal to 2, hence $\dim Y = 3$; moreover $S \cong \boldsymbol{P}^2$, by Theorem

(1.2). So $(Y, \mathscr{L})$ is a 3-dimensional scroll over $\boldsymbol{P}^2$. Set $\mathscr{W} = \pi_*\mathscr{L}$ and $M = \pi^*\ell$, where $\ell \subset \boldsymbol{P}^2$ is any line. Then $\mathscr{L}$ is the tautological line bundle of $\mathscr{W}$ on Y and $\operatorname{Pic}(Y) \cong \boldsymbol{Z}^2$ can be generated by $\mathscr{L}$ and M. Thus, since e is a divisor inside Y, we can write $e = a\mathscr{L} - bM$ for some integers a, b. Taking into account that $M^3 = 0$ we get

$$1 = (e_e)^2 = e^3 = a(a^2\mathscr{L}^3 - 3ab\mathscr{L}^2 \cdot M + 3b^2\mathscr{L} \cdot M^2).$$

Hence $a = 1$. So, for any fibre f of π, we have

$$e \cdot f = e \cdot M^2 = a\mathscr{L} \cdot M^2 = a = 1.$$

This shows that e is a section of π. In particular, $M_e = \mathscr{O}_e(1)$ due to the isomorphism $\pi|_e \colon e \cong \boldsymbol{P}^2$. Moreover, $\mathscr{L} = e + bM$. On the other hand, since

$$\mathscr{O}_e(1) = \mathscr{L}_e = e_e + bM_e = \mathscr{O}_e(-1 + b),$$

we conclude that $b = 2$, i.e. $\mathscr{L} = e + 2M$. Set $\mathscr{U} = \pi_*\mathscr{O}_Y(e)$. Then $Y = \boldsymbol{P}_{\boldsymbol{P}^2}(\mathscr{U})$, e being the tautological section. Moreover, $\mathscr{U} = \mathscr{W}(-2)$ and for any line $\ell \subset \boldsymbol{P}^2$ we have that $M = \pi^*\ell = \boldsymbol{P}_\ell(\mathscr{U}_\ell)$, e_M being the tautological section on M. Note that

$$(e_M)^2 = M_e \cdot e_e = \mathscr{O}_e(1) \cdot \mathscr{O}_e(-1) = -1.$$

It turns out from well-known properties of the Segre–Hirzebruch surfaces that $\mathscr{U}_\ell = \mathscr{O}_\ell \oplus \mathscr{O}_\ell(-1)$. Equivalently, $\mathscr{W}_\ell = \mathscr{O}_\ell(2) \oplus \mathscr{O}_\ell(1)$. This happens for any line ℓ, hence $\mathscr{W}$ is uniform, and then a theorem of Van de Ven [**19**, Chapter III, Theorem 2.2.2] implies that $\mathscr{W}$ is either $\mathscr{O}_{\boldsymbol{P}^2}(2) \oplus \mathscr{O}_{\boldsymbol{P}^2}(1)$ or the tangent bundle $T_{\boldsymbol{P}^2}$. However, the latter case is impossible because $h^0(\mathscr{W}(-2)) = h^0(\mathscr{U}) = h^0(e) > 0$, while the Euler sequence twisted by $\mathscr{O}_{\boldsymbol{P}^2}(-2)$ shows that $h^0(T_{\boldsymbol{P}^2}(-2)) = 0$. Therefore, $Y \cong \boldsymbol{P}_{\boldsymbol{P}^2}(\mathscr{O}_{\boldsymbol{P}^2} \oplus \mathscr{O}_{\boldsymbol{P}^2}(-1))$, i.e. Y is $\boldsymbol{P}^3$ blown-up at one point. Let $\sigma \colon Y \to \boldsymbol{P}^3$ be the blowing-up. Then $M + e = \sigma^*\mathscr{O}_{\boldsymbol{P}^3}(1)$, which gives $\mathscr{L} = e + 2M = \sigma^*\mathscr{O}_{\boldsymbol{P}^3}(2) - e$. In other words, $(Y, \mathscr{L})$ is the del Pezzo threefold of degree 7. □

2. Small degree with respect to codimension.

Theorem 2.1. *Let $X, \mathscr{E}$ and Z be as in* (0.1). *Let H be an ample line bundle on X such that H_Z is very ample and set* $\dim |H_Z| = N$. *Assume that*

$$d(Z, H_Z) \leq 2(N - \dim Z) + 1. \tag{2.1.1}$$

Then $(X, \mathscr{E}, H)$ *is one of the following*:

(1) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus r}, \mathscr{O}_{\boldsymbol{P}^n}(1))$;
(2) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-3)}, \mathscr{O}_{\boldsymbol{P}^n}(2))$;
(3) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(2) \oplus \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(r-1)}, \mathscr{O}_{\boldsymbol{P}^n}(1))$;
(4) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(2)^{\oplus 2} \oplus \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(r-2)}, \mathscr{O}_{\boldsymbol{P}^n}(1))$;
(5) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(3) \oplus \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(r-1)}, \mathscr{O}_{\boldsymbol{P}^n}(1))$;
(6) $(\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(1)^{\oplus r}, \mathscr{O}_{\boldsymbol{Q}^n}(1))$;
(7) $(\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(2) \oplus \mathscr{O}_{\boldsymbol{Q}^n}(1)^{\oplus(r-1)}, \mathscr{O}_{\boldsymbol{Q}^n}(1))$;
(8) (X, H) *is a del Pezzo manifold with* $\mathrm{Pic}(X) \cong \boldsymbol{Z}$ *generated by* H, $d(X, H) \geq 3$, *and* $\mathscr{E} = H^{\oplus r}$;
(9) (X, H) *is a scroll over a smooth curve* C *and* $\mathscr{E}_F \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus r}$ *for any fiber* F *of the scroll projection*;
(10) (X, H) *is a scroll over a smooth curve* C *and* $\mathscr{E}_F \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}(2) \oplus \mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus(r-1)}$ *for any fiber* F *of the scroll projection*;
(11) (X, H) *is a quadric fibration over a smooth curve* C *and* $\mathscr{E}_F \cong \mathscr{O}_{\boldsymbol{Q}^{n-1}}(1)^{\oplus r}$ *for any general fiber* F *of the fibration*;
(12) (X, H) *is a scroll over a (birationally) ruled surface* S *and* $\mathscr{E}_F \cong \mathscr{O}_{\boldsymbol{P}^{n-2}}(1)^{\oplus r}$ *for any fiber* F *of the scroll projection*;
(13) *there exist a birational morphism* $f\colon X \longrightarrow X'$ *expressing* X *as the blow-up of a projective manifold* X' *at a finite set of points* $y_1, \dots, y_s$ $(s \geq 0)$, *an ample line bundle* H' *and an ample vector bundle* $\mathscr{E}'$ *of rank* r *on* X' *such that*

$$H = f^*H' - (E_1 + \cdots + E_s) \quad \text{and} \quad \mathscr{E} = f^*\mathscr{E}' \otimes \mathscr{O}_X(-E_1 - \cdots - E_s),$$

where $E_i = f^{-1}(y_i)$, $i = 1, \dots, s$; *moreover,* $(X', \mathscr{E}', H')$ *is one of the following triplets*

(13-1) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-3)}, \mathscr{O}_{\boldsymbol{P}^n}(3))$;
(13-2) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-4)}, \mathscr{O}_{\boldsymbol{P}^n}(2))$;
(13-3) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(2) \oplus \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-4)}, \mathscr{O}_{\boldsymbol{P}^n}(2))$;
(13-4) $(\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(1)^{\oplus(n-3)}, \mathscr{O}_{\boldsymbol{Q}^n}(2))$;
(13-5) X' *is a* $\boldsymbol{P}^{n-1}$*-bundle over* $\boldsymbol{P}^1$ *and* $(\mathscr{E}'_{F'}, H'_{F'}) \cong (\mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus(n-3)}, \mathscr{O}_{\boldsymbol{P}^{n-1}}(2))$ *for any fiber* F' *of the bundle projection* $X' \longrightarrow \boldsymbol{P}^1$.

For more information concerning the points $y_1, \dots, y_s$ *see Remark* (3.3).

PROOF. The possible pairs (Z, H_Z) satisfying condition (2.1.1) are listed in [**7**, Theorem I]. Noting that the first possibility is ruled out by the assumption that $\dim Z \geq 3$, we are left with the following pairs:

(i) (Z, H_Z) is a scroll over a smooth curve C;

(ii) (Z, H_Z) is a scroll over a (birationally) ruled surface S;
(iii) (Z, H_Z) is a quadric fibration over a smooth curve C;
(iv) (Z, H_Z) is a del Pezzo manifold;
(v) $(Z, H_Z) \cong (\boldsymbol{P}^{n-r}, \mathscr{O}_{\boldsymbol{P}^{n-r}}(1))$;
(vi) $(Z, H_Z) \cong (\boldsymbol{Q}^{n-r}, \mathscr{O}_{\boldsymbol{Q}^{n-r}}(1))$;
(vii) (Z, H_Z) admits a reduction (Y, L) which is one of the following pairs:
(vii-a) $(\boldsymbol{P}^3, \mathscr{O}_{\boldsymbol{P}^3}(3))$;
(vii-b) $(\boldsymbol{Q}^3, \mathscr{O}_{\boldsymbol{Q}^3}(2))$;
(vii-c) $(\boldsymbol{P}^4, \mathscr{O}_{\boldsymbol{P}^4}(2))$;
(vii-d) a Veronese bundle over a smooth curve.

We proceed with a case-by-case analysis.

Case (i). Let $f \cong \boldsymbol{P}^{n-r-1}$ be any fiber of the scroll projection $Z \longrightarrow C$. Then, by adjunction, $K_f = (K_Z + f)_f = (K_Z)_f$. This implies that $(K_Z + (\dim Z - 1)H_Z)_f \cong \mathscr{O}_{\boldsymbol{P}^{n-r-1}}(-1)$, and so $K_Z + (\dim Z - 1)H_Z$ is not nef. On the other hand, $K_Z + (\dim Z)H_Z$ is nef, otherwise, by [**8**, Theorem 1.3] (see also [**4**, Theorem 1]), we would have the contradiction $(Z, H_Z) \cong (\boldsymbol{P}^{n-r}, \mathscr{O}_{\boldsymbol{P}^{n-r}}(1))$. Therefore we are in the assumption of [**13**, Theorem 2]. Note that cases (1)–(3) of that list are ruled out because our (Z, H_Z) has a scroll structure and $\dim Z \geq 3$. Hence we are left with the last case, which is case (9) of our statement.

Case (ii). Denote by $f \cong \boldsymbol{P}^{n-r-2}$ any fiber of the scroll projection $\pi\colon Z \longrightarrow S$. Then $(K_Z + (\dim Z - 2)H_Z)_f \cong \mathscr{O}_{\boldsymbol{P}^{n-r-2}}(-1)$, so $K_Z + (\dim Z - 2)H_Z$ is not nef. On the other hand we can assume that $K_Z + (\dim Z - 1)H_Z$ is nef, otherwise, recalling that the Picard number $\rho(Z) > 1$, we would fall in Case (i) by [**8**, Theorem 1.5] (see also [**4**, Theorems 1 and 2]). Therefore all the possible triplets $(X, \mathscr{E}, H)$ are as in Theorem (1.4). We proceed with a case-by-case analysis.

In case (I), set $e = E \cap Z$. By Proposition (1.5) and in view of Lemma (1.8), we get that (Z, H_Z) is a del Pezzo threefold of degree 7. But this fits into Case (iv), which will be discussed later.

Let now $(X, \mathscr{E}, H)$ be as in case (II). Then (Z, H_Z) is endowed with both a structure of scroll over the smooth surface S and a structure of Veronese bundle over the smooth curve C. Note that, since the scroll projection of Z has 1-dimensional fibers, by restricting π to any fiber of the Veronese bundle we get a surjection $\boldsymbol{P}^2 \longrightarrow S$. Therefore $S \cong \boldsymbol{P}^2$, by Theorem (1.2). We can thus apply [**2**, Theorem 2] to conclude that $(Z,\ H_Z) \cong (\boldsymbol{P}^2 \times \boldsymbol{P}^1,\ \mathscr{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^1}(2, 1))$. Hence, in view of Lemma (1.6), we get a contradiction.

Now we deal with cases (III) and (IV). In both cases (Z, H_Z) has a structure of scroll over the smooth surface S and a structure of quadric fibration over the smooth curve C. Then, according to [**10**, Theorem], $\dim Z = 3$ and either

$(Z, H_Z) \cong (\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,1,1))$, which fits into Case (iv), or $S \cong \boldsymbol{P}_C(\mathscr{F})$ for some vector bundle $\mathscr{F}$ of rank 2 on C. In this last situation, computing the Picard numbers, we have $\rho(X) = \rho(Z) = \rho(S) + 1 = 3$, which is a contradiction, since $\rho(X)$ turns out to be 2 in both cases (III) and (IV) (in case (IV) recall that $n \geq 5$).

Finally let $(X, \mathscr{E}, H)$ be as in case (V). Then the pair (Z, H_Z) inherits a scroll structure $\varphi\colon Z \longrightarrow \Sigma$ over the smooth surface Σ from ψ. Suppose that the scroll structures given by $\pi\colon Z \longrightarrow S$ and by φ are different. Then there exists a fiber $D \cong \boldsymbol{P}^{\dim Z-2}$ of φ such that $\pi(D)$ is not a point. Assume moreover that $\dim Z \geq 4$. In this case D dominates S via π, hence D has to be 2-dimensional and Theorem (1.2) implies that $S \cong \boldsymbol{P}^2$. Arguing symmetrically on π we deduce that $\Sigma \cong \boldsymbol{P}^2$, too. Therefore we have either

(A) $\pi = \varphi$, up to an automorphism of the base surface $S = \Sigma$,
(B) $\dim Z = 4$ and Z has two distinct $\boldsymbol{P}^2$-bundle structures over $\boldsymbol{P}^2$, or
(C) $\dim Z = 3$ and π and φ give different $\boldsymbol{P}^1$-bundle structures.

In case (A) we get case (12) of our statement, recalling that S is a (birationally) ruled surface.

In case (B), by using a result of Sato [**20**, Theorem A], we conclude that $(Z, H_Z) \cong (\boldsymbol{P}^2 \times \boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^2}(1,1))$, and then we can refer to the discussion of Case (iv).

It remains to discuss case (C). Due to the two scroll structures of (Z, H_Z) we can write $Z = \boldsymbol{P}_S(\mathscr{V})$ and $Z = \boldsymbol{P}_\Sigma(\mathscr{W})$ for two ample vector bundles $\mathscr{V}$ and $\mathscr{W}$ on S and Σ respectively, both of rank 2 and both having H_Z as tautological line bundle on Z. Let $T \in |H_Z|$ be a smooth element and look at the birational morphisms $\pi|_T\colon T \longrightarrow S$ and $\varphi|_T\colon T \longrightarrow \Sigma$. Note that $\pi|_T$ contracts $s = c_2(\mathscr{V})$ (-1)-lines $e_1, \dots, e_s$, while $\varphi|_T$ contracts $t = c_2(\mathscr{W})$ (-1)-lines $\varepsilon_1, \dots, \varepsilon_t$ of (T, H_T). In particular it turns out that S and Σ are birationally equivalent (via the birational map $\varphi|_T \circ \pi|_T^{-1}$). Since

$$\rho(S) + 1 = \rho(Z) = \rho(\Sigma) + 1$$

and

$$\rho(S) + s = \rho(T) = \rho(\Sigma) + t,$$

we see that

$$t = s. \tag{2.1.2}$$

By the canonical bundle formula, we have

$$K_T+H_T = (K_Z+2H_Z)_T = (-2H_Z+\pi^*(K_S+\det\mathscr{V})+2H_Z)_T = \pi|_T^*(K_S+\det\mathscr{V}).$$

Hence $H_T = \pi|_T^* \det\mathscr{V} - \sum_{i=1}^s e_i$ and $K_T = \pi|_T^* K_S + \sum_{i=1}^s e_i$. From these and the corresponding relations involving $\varphi|_T$ instead of $\pi|_T$ we get from (2.1.2):

$$c_1(\mathscr{V})^2 = c_1(\mathscr{W})^2, \quad K_S^2 = K_\Sigma^2, \quad \det\mathscr{V}\cdot K_S = \det\mathscr{W}\cdot K_\Sigma.$$

Now, consider a curve ε_j for some j. If $\pi(\varepsilon_j)$ is a point, then there exists an index i such that $\varepsilon_j = e_i$. By applying the rigidity lemma [**1**, Lemma 4.1.13] we conclude that there are a morphism $h\colon S\longrightarrow\Sigma$ such that $\varphi = h\circ\pi$ and a morphism $k\colon\Sigma\longrightarrow S$ such that $\pi = k\circ\varphi$. But then $h\colon S\longrightarrow\Sigma$ would be an isomorphism, with π and φ inducing the same $\boldsymbol{P}^1$-bundle structure on Z, a contradiction. Therefore $\varepsilon_j \neq e_i$ for any i and j. In particular $C_j := \pi(\varepsilon_j)$ is an irreducible curve of S. Recall that ε_j and e_i are two distinct lines inside (T, H_T). Hence $\varepsilon_j\cdot e_i = 0$ or 1. Since the multiplicity of C_j at the point $p_i = \pi(e_i)$ is $m_{j,i} = \varepsilon_j\cdot e_i$, we conclude that C_j has no singular points, hence, due to the birationality of $\pi|_T$, it is a smooth rational curve. We have $\varepsilon_j = \pi|_T^* C_j - \sum_{i=1}^s m_{j,i}\, e_i$. Let γ_j be the number of the $m_{j,i}$ different from zero. Then:

$$1 = H_T\cdot\varepsilon_j = \det\mathscr{V}\cdot C_j - \gamma_j, \tag{2.1.3}$$

and

$$-1 = K_T\cdot\varepsilon_j = K_S\cdot C_j + \gamma_j. \tag{2.1.4}$$

Recalling that $\mathscr{V}$ is an ample vector bundle and C_j a smooth $\boldsymbol{P}^1$, (2.1.3) gives

$$2 \le \deg\mathscr{V}_{C_j} = \det\mathscr{V}\cdot C_j = 1+\gamma_j.$$

Therefore $\gamma_j \ge 1$; in other words, there exist indexes j and i such that $\varepsilon_j\cdot e_i = 1$. Summing-up (2.1.3) and (2.1.4) for such a j we thus get

$$0 = (K_S+\det\mathscr{V})\cdot C_j,$$

which shows that the adjoint bundle $K_S+\det\mathscr{V}$ is not ample. Since $\mathscr{V}$ has rank 2, [**6**, Main Theorem] tells us that $(S,\mathscr{V})$ is one one the following pairs:

(C1) $(\boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^2}(1)^{\oplus 2})$;
(C2) $(\boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^2}(2)\oplus\mathscr{O}_{\boldsymbol{P}^2}(1))$;
(C3) $(\boldsymbol{P}^2, T_{\boldsymbol{P}^2})$;

(C4) $(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,1)^{\oplus 2})$;
(C5) S is a $\boldsymbol{P}^1$-bundle over a smooth curve B and $\mathscr{V}_f \cong \mathscr{O}_{\boldsymbol{P}^1}(1)^{\oplus 2}$ for every fibre f of the projection $S \longrightarrow B$.

Of course case (C4) fits into case (C5). In case (C1) the pair $(Z, H_Z) \cong (\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(1,1))$, but this clearly fits in Case (i) (already discussed). Note that in cases (C2)–(C4) the pair (Z, H_Z) is a del Pezzo manifold and then we can refer again to the discussion of Case (iv). Now suppose that (Z, H_Z) is as in case (C5). Then S is a $\boldsymbol{P}^1$-bundle over a smooth curve B. In particular, looking at the Picard numbers, this gives $\rho(Z) = \rho(S) + 1 = 3$. Therefore

$$\rho(X) = 3, \tag{2.1.5}$$

by the Lefschetz–Sommese theorem.

Now, by composing the ruling projection $p\colon S \longrightarrow B$ with the scroll projection $\pi\colon Z \longrightarrow S$ we get a morphism $p \circ \pi\colon Z \longrightarrow B$ making Z a $(\boldsymbol{P}^1 \times \boldsymbol{P}^1)$-bundle over B. To see this, recall that $Z = \boldsymbol{P}_S(\mathscr{V})$. Then, letting f_b and G_b denote the fibers of p and of $p \circ \pi$ over a point $b \in B$, respectively, we have

$$G_b = \pi^{-1}(f_b) = \boldsymbol{P}_{f_b}(\mathscr{V}_{f_b}) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1.$$

Moreover, (Z, H_Z) turns out to be a quadric fibration over B, since

$$H_{G_b} \cong \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,1)$$

under the isomorphism above. Indeed, let $\mathfrak{f}$ be a fiber of $\pi|_{G_b}\colon G_b \longrightarrow f_b$. Then $G_b = \mathfrak{f} \times f_b$ and $\mathfrak{f} \in |\mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(0,1)|$. Note that $H_{G_b} = (H_Z)_{G_b}$ is the tautological line bundle on G_b of $\mathscr{V}_{f_b} \cong \mathscr{O}_{f_b}(1) \otimes \mathscr{O}_{f_b}^{\oplus 2}$. Hence $H_{G_b} \cong \mathfrak{f} + \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,0)$, the latter summand being the tautological line bundle of $\mathscr{O}_{f_b}^{\oplus 2}$. Therefore $H_{G_b} \cong \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,1)$. It thus follows from [**14**, Theorem 0.4] that either

(a) (X, H) is a scroll over B, and $\mathscr{E}_D \cong \mathscr{O}_{\boldsymbol{P}^{n-1}}(2) \oplus \mathscr{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus(n-4)}$ for any fiber D of the projection $X \longrightarrow B$; or
(b) (X, H) is a quadric fibration over B and $\mathscr{E}_D \cong \mathscr{O}_{\boldsymbol{Q}^{n-1}}(1)^{\oplus(n-3)}$ for any general fiber D of the fibration $X \longrightarrow B$.

In both cases we would get $\rho(X) = \rho(B) + 1 = 2$ (in case (b) recall that $n \geq 5$). But this contradicts (2.1.5).

CASE (iii). We are in the assumption of [**14**, Theorem 0.4]; hence we obtain cases (10) and (11) of the statement.

Case (iv). We can apply [**13**, Theorem 4 and Remark], whose cases (2), (3), (6), (9) and (10) are ruled out by our assumption $n-r \geq 3$; the remaining cases (1), (4), (5), (7) and (8) are those listed as (2), (5), (4), (7) and (8) in our statement. In case (8) note that $d(X,H) = H^n = H^{n-r} \cdot c_r(\mathscr{E}) = H_Z^{n-r} = d(Z, H_Z) \geq 3$ because H_Z is very ample.

Case (v). As $Z \cong \boldsymbol{P}^{n-r}$ has dimension ≥ 3, it follows from [**11**, Theorem A] that $(X, \mathscr{E}) \cong (\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus r})$. Then $H \cong \mathscr{O}_{\boldsymbol{P}^n}(1)$ in view of Theorem (1.3). This gives case (1) in the statement.

Case (vi). As $Z \cong \boldsymbol{Q}^{n-r}$ has dimension ≥ 3, it follows from [**11**, Theorem B] that $(X, \mathscr{E})$ is either $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(2) \oplus \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(r-1)})$ or $(\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(1)^{\oplus r})$. By the Lefschetz–Sommese theorem this leads to (3) and (6) in the statement.

Case (vii). Let $\sigma\colon Z \longrightarrow Y$ be the reduction morphism leading to a reduction (Y, L) of (Z, H_Z) as in (vii-a)–(vii-d). Assume that σ is non-trivial. Then σ is the blowing-up of Y at a finite set $\{y_1, \ldots, y_s\}$ and $H_Z = \sigma^* L - \sum_{i=1}^s e_i$, where $e_i = \sigma^{-1}(y_i)$ for all $i = 1, \ldots, s$.

Recall that $K_Z = \sigma^* K_Y + (\dim Z - 1) \sum_{i=1}^s e_i$. Set $e = e_1$, for simplicity. Since $\mathscr{O}_e(e) = \mathscr{O}_e(-1)$ and $(H_Z)_e = \mathscr{O}_e(1)$, we get $(K_Z + (\dim Z - 2) H_Z)_e = \mathscr{O}_e(-1)$, which shows that $K_Z + (\dim Z - 2) H_Z$ is not nef. On the other hand $K_Z + (\dim Z - 1) H_Z$ is nef, otherwise, by [**8**, Theorem 1.5] (see also [**4**, Theorems 1 and 2]), (Z, H_Z) should be one of the following:

(α) $(\boldsymbol{P}^{n-r}, \mathscr{O}_{\boldsymbol{P}^{n-r}}(1))$;
(β) $(\boldsymbol{Q}^{n-r}, \mathscr{O}_{\boldsymbol{Q}^{n-r}}(1))$;
(γ) $(\boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^2}(2))$;
(δ) a scroll over a smooth curve.

But in all these cases (Z, H_Z) cannot have a non-trivial reduction. Therefore we are in the assumptions of Theorem (1.4), and we proceed with a case-by-case analysis. First we investigate cases (II)–(V), showing that they lead to a contradiction; then we will consider case (I).

So let $(X, \mathscr{E}, H)$ be as in case (II). Then Z is 3-dimensional and the pair (Z, H_Z) has a structure of Veronese bundle $\varphi\colon Z \longrightarrow C$ induced by ψ and, at the same time, Z contains an exceptional divisor e. Clearly $\varphi(e) = \varphi(\boldsymbol{P}^2)$ is a point of C, thus e is contained in, and hence is, a fiber of φ. Therefore $\mathscr{O}_e(e)$ has to be trivial, which is a contradiction, since e is an exceptional divisor.

Now we deal with cases (III) and (IV). In both cases the image of $e \cong \boldsymbol{P}^{n-r-1}$ via $\varphi := \psi|_Z$ must be a point, so e is contained in a fiber of φ. Since all these fibers are irreducible, e turns out to be a fiber of the quadric fibration φ, but this is a contradiction.

Assume now that $(X, \mathscr{E}, H)$ is as in case (V). Note that (Z, H_Z) satisfies the assumption of Lemma (1.8), hence the pair (Z, H_Z) is a del Pezzo threefold which has already been discussed. Therefore this situation is ruled out by the previous analysis of Case (iv).

So we are left with $(X, \mathscr{E}, H)$ as in case (I). Put $\varepsilon = E \cap Z$. Then ε is a (-1)-hyperplane of (Z, H_Z), by Proposition (1.5). Since σ contracts all (-1)-hyperplanes of (Z, H_Z), it follows that $\varepsilon = e_i$ for some i. Let $f' \colon X \longrightarrow X'$ be the contraction of E to a point of a smooth projective variety X'. Then $f'|_Z \colon Z \longrightarrow Z' := f'(Z)$ contracts ε, and σ factors through $f'|_Z$. Therefore, by Lemma (1.1), there exist an ample line bundle H' and an ample vector bundle $\mathscr{E}'$ of rank r on X' with a section vanishing on Z' such that

$$H = f'^{*}H' - E \quad \text{and} \quad \mathscr{E} = f'^{*}\mathscr{E}' \otimes \mathscr{O}_X(-E).$$

Now, as $\sigma \colon Z \longrightarrow Y$ factors through $f'|_Z \colon Z \longrightarrow Z'$, either $(Z', H'_{Z'}) \cong (Y, L)$, or, arguing as before, $K_{Z'} + (\dim Z' - 2)H'_{Z'}$ is not nef but $K_{Z'} + (\dim Z' - 1)H'_{Z'}$ is nef. Then, repeating the argument for $(X', \mathscr{E}', H', Z')$, we conclude that there exists a divisor $E' \cong \boldsymbol{P}^{n-1} \subseteq X'$ as in case (I) of Theorem (1.4). By Proposition (1.5) again this divisor defines a (-1)-hyperplane of $(Z', H'_{Z'})$ and we can contract E' by $f'' \colon X' \longrightarrow X''$. Moreover, there exist an ample line bundle H'' and an ample vector bundle $\mathscr{E}''$ of rank r on X'' such that $H' = f''^{*}H'' - E'$ and $\mathscr{E}' = f''^{*}\mathscr{E}'' \otimes \mathscr{O}_{X'}(-E')$.

Note that each step of this procedure decreases by 1 the Picard number of X and Z. So, after finitely many steps we get a birational morphism $f \colon X \longrightarrow X_0$ onto a projective manifold X_0 (consisting of finitely many contractions); moreover there exist both an ample line bundle H_0 and an ample vector bundle $\mathscr{E}_0$ of rank r on X_0 such that

$$H = f^{*}H_0 - E_1 - \cdots - E_s \quad \text{and} \quad \mathscr{E} = f^{*}\mathscr{E}_0 \otimes \mathscr{O}_X(-E_1 - \cdots - E_s).$$

Note that by construction $f|_Z = \sigma$, $\mathscr{E}_0$ has a section vanishing on Y and $(H_0)_Y = L$. Now consider $(X_0, \mathscr{E}_0, Y, H_0)$, with $(H_0)_Y = L$.

In case (vii-a) we can apply [**11**, Theorem A], hence $(X_0, \mathscr{E}_0) \cong (\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-3)})$. Moreover, as $\operatorname{Pic} X_0 \cong \operatorname{Pic} Y$ by Theorem (1.3), from $L = \mathscr{O}_Y(3)$ we conclude that $H_0 \cong \mathscr{O}_{\boldsymbol{P}^n}(3)$. Therefore we obtain case (13-1) of the statement.

As to case (vii-b), by [**11**, Theorem B], we have $(X_0, \mathscr{E}_0) \cong (\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(2) \oplus \mathscr{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-4)})$ or $(X_0, \mathscr{E}_0) \cong (\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(1)^{\oplus(n-3)})$. Moreover, H_0 is $\mathscr{O}_{\boldsymbol{P}^n}(2)$ in the former case and $\mathscr{O}_{\boldsymbol{Q}^n}(2)$ in the latter. Therefore we obtain cases (13-3) and (13-4) of the statement.

Arguing as in case (vii-a), in case (vii-c) we obtain (13-2) of the statement.

Now consider case (vii-d). As a first thing note that the pair (Z, H_Z) satisfies the hypothesis of [**9**, Section 4, p. 904, lines 8–6 from the bottom]; hence the bound on the degree provided by (2.1.1) implies that the base curve of the Veronese bundle structure of (Y, L) is $\boldsymbol{P}^1$. We have $(2K_Y + 3L)_F = \mathscr{O}_F$ for any fiber F of the projection $p\colon Y \longrightarrow \boldsymbol{P}^1$, hence $2(K_Y + L)_F = -L_F$, which says that $K_Y + (\dim Y - 2)L$ is not nef. On the other hand, we claim that $K_Y + (\dim Y - 1)L$ is nef. Otherwise the pair (Y, L) would be as in cases (α)–(δ) above, by [**8**, Theorem 1.5] (see also [**4**, Theorems 1 and 2]). Cases (α)–(γ) are ruled out noting that $\rho(Y) = 2$; as to case (δ), if $\pi\colon Y \longrightarrow B$ is the scroll projection, then any fiber G of π has to be also a fiber F of p. So $\mathscr{O}_{\boldsymbol{P}^2}(1) \cong H_G \cong H_F \cong \mathscr{O}_{\boldsymbol{P}^2}(2)$, which is a contradiction. This proves the claim. So we can apply Theorem (1.4). Checking the list, note that only case (II) can occur. Indeed, in case (I), in view of Proposition (1.5), Y would contain a (-1)-hyperplane with respect to L, which is a contradiction, since (Y, L) is a reduction. In cases (III) and (IV) the pair (Y, L) would be a quadric fibration over a smooth curve; but then any fiber $\cong \boldsymbol{P}^2$ of the Veronese bundle would be contained in a quadric surface, which is a contradiction. Finally, in case (V), we get a contradiction by [**2**, Theorem 2]. Hence we are left with case (II), which gives (13-5) of the statement. □

Two special cases deserve some attention.

Corollary 2.2. *Let* $(X, \mathscr{E}, H)$ *be as in Theorem* (2.1).

(a) *If* $H = \det \mathscr{E}$, *then only cases* (2) *and* (13) *with* $s = 0$ *occur; moreover,* $r = 2$ *except case* (13-1), *where* $r = 2, 3$.
(b) *If* $\mathscr{E} = H^{\oplus r}$, *then only cases* (1), (6), (8), (9), (11), (12) *occur.*

Proof. If $H = \det \mathscr{E}$ then for any rational curve $C \subset X$ we have $H \cdot C = \det \mathscr{E} \cdot C \geq r$. In particular (X, H) cannot contain lines, which implies that $s = 0$ in case (13). Thus a close inspection of the list in Theorem (2.1) gives assertion (a). Assertion (b) is immediate.

3. Examples and remarks.

All cases (1)–(13) in Theorem (2.1) are effective. This follows immediately in cases (1)–(8). For case (13) see Remark (3.3). We provide examples for the remaining cases.

Examples 3.1. Examples for cases (9)–(11) are the following.

Case (9): $X = \boldsymbol{P}_{\boldsymbol{P}^1}(\oplus_{i=1}^n \mathscr{O}_{\boldsymbol{P}^1}(a_i))$, with $a_1 \geq \cdots \geq a_n = 0$, $\mathscr{E} = \oplus_{j=1}^r (\xi + b_j F)$, $H = \xi + bF$, where ξ is the tautological line bundle on X, F is a fiber of the bundle projection $X \longrightarrow \boldsymbol{P}^1$ and $b > 0$, $b_j > 0$ for every j.

Case (10): $X = \boldsymbol{P}^{n-1} \times \boldsymbol{P}^1$, $\mathscr{E} = \mathscr{O}_{\boldsymbol{P}^{n-1}\times\boldsymbol{P}^1}(2,1) \oplus \mathscr{O}_{\boldsymbol{P}^{n-1}\times\boldsymbol{P}^1}(1,1)^{\oplus(r-1)}$, $H = \mathscr{O}_{\boldsymbol{P}^{n-1}\times\boldsymbol{P}^1}(1,1)$.

Case (11): $X = \boldsymbol{Q}^{n-1} \times \boldsymbol{P}^1$, $\mathscr{E} = \mathscr{O}_{\boldsymbol{Q}^{n-1}\times\boldsymbol{P}^1}(1,1)^{\oplus r}$, $H = \mathscr{O}_{\boldsymbol{Q}^{n-1}\times\boldsymbol{P}^1}(1,1)$.

For the above triplets we have $d(Z, H_Z) \geq n$ in the first example, while $d(Z, H_Z) = 2n-1$, $2n$, in the second and in the third one, respectively. On the other hand, $\Delta(Z, H_Z) = 0$, $n-2$, $n-2$ in the three examples, respectively. Hence (2.1.1) holds in the equivalent form provided by (2) in the Introduction. Here is an example as in case (12).

Case (12): $X = \boldsymbol{P}^3 \times \boldsymbol{P}^2$, $\mathscr{E} = \mathscr{O}_{\boldsymbol{P}^3\times\boldsymbol{P}^2}(1,1)^{\oplus 2}$, $H = \mathscr{O}_{\boldsymbol{P}^3\times\boldsymbol{P}^2}(1,1)$.

In this example $d(Z, H_Z) = 10$, while $\Delta(Z, H_Z) = 3$, hence (2) holds.

Concerning case (12), more generally, we can prove the following fact.

PROPOSITION 3.2. *Let S be a smooth regular surface (i.e. $h^1(\mathscr{O}_S) = 0$) polarized by a very ample line bundle $\mathscr{L}$ and let $X = \boldsymbol{P}^{n-2} \times S$, $\mathscr{E} = \mathscr{O}_{\boldsymbol{P}^{n-2}\times S}(1,1)^{\oplus r}$, $H = \mathscr{O}_{\boldsymbol{P}^{n-2}\times S}(1,1)$, where $\mathscr{O}_{\boldsymbol{P}^{n-2}\times S}(0,1)$ stands for the pull-back of $\mathscr{L}$ via the second projection. Then condition* (2.1.1) *holds if and only if $(S, \mathscr{L})$ is either $(\boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^2}(1))$ with $(n,r) = (5,2), (6,2), (6,3), (7,2), (7,3), (7,4)$, or $(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,1))$ and $(n,r) = (5,2)$.*

PROOF. We have

$$d(Z, H_Z) = \mathscr{O}_{\boldsymbol{P}^{n-2}\times S}(1,1)^n = \binom{n}{2}\mathscr{L}^2.$$

Consider a smooth ladder $X = X_0 \supset X_1 \supset \cdots \supset X_r = Z$, where $X_i \in |\mathscr{O}_{X_{i-1}}(1,1)|$ and the exact sequences $0 \longrightarrow \mathscr{O}_{X_{i-1}} \longrightarrow H_{X_{i-1}} \longrightarrow H_{X_i} \longrightarrow 0$ for $i = 1, \dots, r$. We have $h^1(\mathscr{O}_X) = h^1(\mathscr{O}_S) = 0$, hence $h^1(\mathscr{O}_{X_i}) = 0$ for $i = 1, \dots, r$ by the Lefschetz theorem. Set $h = h^0(\mathscr{L})$. Noting that $h^0(H) = h^0(\mathscr{L}^{\oplus(n-1)}) = (n-1)h$, by induction we get

$$h^0(H_Z) = h^0(H) - r = (n-1)h - r.$$

Thus condition (2.1.1) applied to (Z, H_Z) reads as

$$\binom{n}{2}\mathscr{L}^2 \leq 2(n-1)h - 2(n+1) + 1. \tag{3.2.1}$$

On the other hand, since $\Delta(S, \mathscr{L}) \geq 0$, we know that

$$\binom{n}{2}(h-2) \leq \binom{n}{2}\mathscr{L}^2. \tag{3.2.2}$$

Combining (3.2.1) with (3.2.2), and recalling that $n \geq 5$ and $h \geq 3$, we easily see that $5 \leq n \leq 7$, with $h \leq 4$ if $n = 5$, and $h = 3$ in the remaining cases. So, if $h = 3$, taking into account the range of r, we get $(S, \mathscr{L}) \cong (\boldsymbol{P}^2, \mathscr{O}_{\boldsymbol{P}^2}(1))$ with all pairs (n, r) listed in the statement. Let $h = 4$. Then $n = 5$, hence $r = 2$; moreover S, embedded by $|\mathscr{L}|$, is a non-degenerate smooth surface in $\boldsymbol{P}^3$. In this case (3.2.1) gives $10\mathscr{L}^2 \leq 21$. Therefore $(S, \mathscr{L}) \cong (\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1,1))$ and this concludes the proof. □

Remark 3.3. Note that the points $y_1, \dots, y_s$ appearing in case (13) of Theorem (2.1) are the same arising in case (vii) in the proof because $f|_Z = \sigma$, as already observed. Some upper bounds on s are mentioned in [**7**, Remark (1.6)] and fit into a more general set of restrictions concerning them. In fact we show that $s \leq 6, 5, 1, 5$ in cases (vii-a)–(vii-d), respectively. Moreover, if $y_1, \dots, y_s$ are general enough to insure the ampleness of H_Z, then H_Z is very ample and the conditions they impose to the linear system $|L|$ are linearly independent. To prove this facts, as a first thing, we note the following. In case (vii-a), the points $y_1, \dots, y_s$ can be neither three on a line, nor six on a conic; in case (vii-b), neither two on a line, nor four on a conic (contained in $\boldsymbol{Q}^3$). This is immediate. E.g., in the last case suppose that $y_1, \dots, y_4 \in \gamma$, a conic inside $\boldsymbol{Q}^3$; let $C := \sigma^{-1}(\gamma)$, where $\sigma\colon Z \longrightarrow Y$ is the reduction morphism. Then

$$H_Z \cdot C = (\sigma^* \mathscr{O}_{\boldsymbol{Q}^3}(2) - e_1 - \cdots - e_4 - \cdots - e_s) \cdot C = \mathscr{O}_{\boldsymbol{Q}^3}(2) \cdot \gamma - 4 = 0,$$

but then H_Z could not be ample, a contradiction. The remaining conditions follow arguing in the same way. In case (vii-c) the same argument shows that no two of the y_i's can be collinear, hence $s \leq 1$. Moreover, if $s = 1$, we can immediately observe that H_Z is very ample. Indeed, Z is isomorphic to $\boldsymbol{P}_{\boldsymbol{P}^3}(\mathscr{O}_{\boldsymbol{P}^3}(2) \oplus \mathscr{O}_{\boldsymbol{P}^3}(1))$ and H_Z is the tautological line bundle. Furthermore, an immediate check shows that condition (2.1.1) is satisfied for $s \leq 1$. Finally consider case (vii-d). Note that $y_1, \dots, y_s$ belong to distinct fibers of p. To see this suppose by contradiction that y_1 and y_2 are in the same fiber $F \cong \boldsymbol{P}^2$ of p, let ℓ be the line $\langle y_1, y_2 \rangle$ and let $C := \sigma^{-1}(\ell)$. Then

$$H_Z \cdot C = (\sigma^* L - e_1 - e_2 - \cdots - e_s) \cdot C = L_F \cdot \ell - 2 = 0,$$

which contradicts the ampleness of H_Z.

Now suppose that the position of $y_1, \dots, y_s$ is general enough to insure that the line bundle $H_Z = \sigma^* L - \sum_{i=1}^{s} e_i$ is ample. As we said in the Introduction, condition (2.1.1) can be rephrased in terms of Δ-genus as

$$d \geq 2\Delta(Z, H_Z) + 1. \tag{3.3.1}$$

We have

$$d = H_Z^{\dim Z} = L^{\dim Y} - s \quad \text{and} \quad h^0(H_Z) = h^0(L) - t, \tag{3.3.2}$$

where t is the number of linearly independent linear conditions imposed by the points $y_1, \dots, y_s$ to the linear system $|L|$. Let $g = g(Z, H_Z)$ be the sectional genus and recall that $g(Z, H_Z) = g(Y, L)$. Looking at cases (vii-a), (vii-b) and (vii-d) separately, we get $(d, \Delta, g) = (27 - s, 10 + t - s, 10)$ in case (vii-a) and $(16 - s, 5 + t - s, 5)$ in case (vii-b). In case (vii-d) we can write $Y = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathscr{V})$ where $\mathscr{V} = \oplus_{j=1}^3 \mathscr{O}_{\boldsymbol{P}^1}(b_j)$ is normalized as in [**1**, Lemma 3.2.4], i.e. $b_1 \geq b_2 \geq b_3 = 0$. Let ξ be the tautological line bundle of $\mathscr{V}$ and let $b = \sum_{j=1}^3 b_j$. Then we can write $L = 2\xi + \beta F$, where $F \cong \boldsymbol{P}^2$ is a fiber of the projection $p \colon Y \longrightarrow \boldsymbol{P}^1$. Note that $\beta > 0$, due to the ampleness of L. We can compute the degree

$$L^3 = (2\xi + \beta F)^3 = 8\xi^3 + 12\beta\xi^2 \cdot F = 8b + 12\beta = 2(4b + 6\beta).$$

On the other hand, we can compute $h^0(L) = h^0(S^2\mathscr{V} \otimes \mathscr{O}_{\boldsymbol{P}^1}(\beta))$. Now, $S^2\mathscr{V} = \oplus_{i \leq j} \mathscr{O}_{\boldsymbol{P}^1}(b_i + b_j)$, so $h^0(S^2\mathscr{V} \otimes \mathscr{O}_{\boldsymbol{P}^1}(\beta)) = \sum_{i \leq j} h^0(\mathscr{O}_{\boldsymbol{P}^1}(b_i + b_j + \beta))$. Therefore

$$h^0(L) = 4b + 6\beta + 6.$$

Moreover, by adjunction, the canonical bundle formula for $\boldsymbol{P}$-bundles gives

$$2g(L) - 2 = (K_Y + 2L) \cdot L^2 = (\xi + (b + 2\beta - 2)F) \cdot (4\xi^2 + 4\beta\xi \cdot F) = 8b + 12\beta - 8.$$

Then recalling (3.3.2) we obtain $(d, \Delta, g) = (8b+12\beta-s, 4b+6\beta-3+t-s, 4b+6\beta-3)$ in case (vii-d). Since $t \leq s$, we thus see that $\Delta \leq g$ in all cases, equality implying $t = s$. Therefore, in view of (3.3.1), [**5**, Theorem 3.5] applies to our polarized manifold (Z, H_Z) and tells us that

$$\Delta = g \quad \text{and} \quad H_Z \text{ is very ample.}$$

In particular, $t = s$. Thus the assumption on (Z, H_Z) in Theorem (2.1), rephrased by (3.3.1), says that $s \leq 6, 5, 5$ in cases (vii-a), (vii-b) and (vii-d), respectively. Conversely, if these bounds hold, then (3.3.1) is fulfilled and so H_Z is very ample.

Remark 3.4. A triplet $(X, \mathscr{E}, H)$ as in case (13-5) of Theorem (2.1) with $s = 0$ is provided by Example (1.7). Another example is the following. Putting together two copies of the Euler sequence twisted by $\mathscr{O}_{\boldsymbol{P}^1}(-1)$ with the identity of $\mathscr{O}_{\boldsymbol{P}^1}$ we can construct an exact sequence

$$0 \longrightarrow \mathscr{O}_{\boldsymbol{P}^1}(-1)^{\oplus 2} \longrightarrow \mathscr{O}_{\boldsymbol{P}^1}^{\oplus 5} \longrightarrow \mathscr{O}_{\boldsymbol{P}^1}(1)^{\oplus 2} \oplus \mathscr{O}_{\boldsymbol{P}^1} \longrightarrow 0.$$

Let $\mathscr{W} = \mathscr{O}_{\boldsymbol{P}^1}^{\oplus 5}$, $\mathscr{V} = \mathscr{O}_{\boldsymbol{P}^1}(1)^{\oplus 2} \oplus \mathscr{O}_{\boldsymbol{P}^1}$ and consider $X = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathscr{W})$, $Z = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathscr{V})$. Then Z is contained in X fiberwise. Note that Z is the blow-up of a quadric cone $Q_0 \subset \boldsymbol{P}^4$ of rank 4, the blowing-up being given by the morphism associated with $|\xi_Z|$ (it contracts the section of Z corresponding to the surjection $\mathscr{V} \longrightarrow \mathscr{O}_{\boldsymbol{P}^1}(1)$ to the vertex of Q_0). Then arguing as in the proof of Lemma (1.6) one sees that Z is the zero locus of a general section of the (very) ample vector bundle $\mathscr{E} = [\xi + F]^{\oplus 2}$ on X. Moreover $H = 2\xi + F$ is very ample and induces the structure of Veronese bundle on Z.

Remark 3.5. As observed in the Introduction, unlike in [**7**], our setting includes only projective manifolds Z with $\dim Z \geq 3$. To deal with case $\dim Z = 2$, we have to consider the further possibility ([**7**, Theorem I, case a)]) that

Z is a (birationally) ruled surface. $\quad (*)$

Of course cases (ii) and (vii) listed in the proof of Theorem (2.1) do not occur. Moreover the remaining cases (which could be settled for $\dim Z = 2$ in a similar way as we did for $\dim Z \geq 3$) fit into $(*)$. On the other hand, in case $(*)$ by adjunction we see that $K_X + \det \mathscr{E}$ is not nef, hence to lift this situation to the ample vector bundle setting we can use [**17**, Theorem]. This provides 14 possibilities for $(X, \mathscr{E})$; however, a lack of information concerning H cannot be avoided. More precisely, if $\dim Z = 2$ and (Z, H_Z) satisfies (2.1.1), we can make explicit more than what is said in [**7**, Theorem I, case a)]; namely, the hyperplane bundle H_C of a general curve section C of (Z, H_Z) has to be non-special. Actually, if H_C were special, then Clifford's theorem and the exact cohomology sequence of $0 \longrightarrow \mathscr{O}_Z \longrightarrow H_Z \longrightarrow H_C \longrightarrow 0$ would give $h^0(H_Z) \leq 1 + h^0(H_C) \leq (d/2) + 2$, where $d = d(Z, H_Z)$, but this contradicts the inequality

$$h^0(H_Z) \geq \frac{1}{2}(d+5) \tag{3.5.1}$$

coming from (2.1.1). On the other hand the non-speciality of H_C is not sufficient for a ruled surface (Z, H_Z) to satisfy (2.1.1). The Bordiga surface of degree 6 in $\boldsymbol{P}^4$ is a convincing example. In fact, confining to rational surfaces, condition (2.1.1) turns out to be equivalent to

$$d \geq 2g + 1, \tag{3.5.2}$$

where $g = g(C)$ is the sectional genus of (Z, H_Z). To see this, first note that Z

rational with H_C non-special implies

$$h^0(H_Z) = h^0(H_C) + 1 = d + 2 - g. \tag{3.5.3}$$

Now, if (2.1.1) holds, then, as we said, H_C is non-special, hence (3.5.2) follows from (3.5.3) and (3.5.1). Conversely, suppose that (3.5.2) holds. Then taking into account (3.5.3) we get

$$d \leq 2d - 2g - 1 = 2(d - 1 - g) + 1 = 2(h^0(H_Z) - 3) + 1,$$

which is (2.1.1). However, describing explicitly which rational surfaces satisfy (3.5.2) seems a hard problem. Notice that (3.5.2) is true for $\boldsymbol{P}^2$ and for all Segre–Hirzebruch surfaces for any polarization.

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