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On factorization of the solutions of second order linear differential equations

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Abstract. In this paper, we discuss factorization of the solutions of some linear ordinary differential equations with transcendental entire coefficients. We give a condition which shows that the solutions for some differential equations are prime, for some are factorizable.

1. Introduction.

Let $f(z)$ be a meromorphic function. $f(z)$ is said to be factorizable, if there exist a transcendental mermorphic function $h(\zeta)$ and a transcendental entire function $g(z)$ such that $f(z) = h(g(z))$; $f(z)$ is said to be prime (pseudo-prime, left-prime), if every factorization $f(z) = h(g(z))$ implies that either $h(\zeta)$ is bilinear (a rational function, bilinear) or $g(z)$ is linear (a polynomial). N. Steinmetz ([**9**]) proved that each meromorphic solution of linear ordinary differential equation with rational coefficients is pseudo-prime. A natural question arises: how about the solutions of linear differential equations with transcendental coefficients? In this paper, we shall point out firstly that certain class of entire functions are prime, and then show that the solutions of some linear differential equations with transcendental coefficients are prime. We also discuss the factorization of the solution of some other differential equations and prove that the solutions are factorizable.

Throughout this paper, by $\rho(f)$ we denote the order of $f(z)$. We assume that the reader is familiar with Nevanlinna's theory of meromorphic functions and the meaning of the symbols $T(r, f), N(r, f)$ and etc. For other notation and terminology, the reader is referred to [**6**].

2. A class of prime entire functions.

In this section, we obtain a class of prime entire functions, which is used to discuss the factorization of the solutions of certain differential equation with the coefficients of periodic entire functions.

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THEOREM 2.1. *Let* $f(z) = \psi(e^z)\exp(\Phi(z) + dz)$, *where* d *is not a rational number,* $\psi(\zeta)$ *is a polynomial having at least a non-zero simple zero, and* $\Phi(z)$ *is a non-constant periodic entire function with period* τ *and such that* $\rho(\Phi(z)) < \infty$. *Then* $f(z)$ *is prime*.

Since $f(z)$ is not periodic, a result of Gross ([**5**]) shows that the primeness of $f(z)$ in general sense is equivalent to that in entire sense. Therefore we need only prove $f(z)$ is prime in entire sense, which follows immediately from the following Lemmas 2.2 and 2.4.

Let $f(z)$ in Theorem 2.1 have the factorization

$$f(z) = h(g(z)), \tag{2.1}$$

where $h(\zeta)$ and $g(z)$ are entire functions.

LEMMA 2.2. *Suppose that* $h(\zeta)$ *in* (2.1) *has an infinite number of zeros, then* $g(z)$ *is linear*.

To prove Lemma 2.2, the following lemma is needed:

LEMMA 2.3 ([**10**]). *Let* $F(z)$ *be an entire function. Assume that there exists an unbounded sequence* $\{w_n\}$ *such that all roots of the equations* $F(z) - w_n = 0$ $(n = 1, 2, \dots)$ *lie in the half plane* $\{z, |\arg z - \pi| < \pi/2\}$ *and*

$$\liminf_{r\to\infty} \frac{T(r,F)}{r} = 0,$$

then $F(z)$ *is a polynomial of degree not greater than two*.

Now we go back to the proof of Lemma 2.2. Let $\{w_n\}_{n=1}^{\infty}$ be the sequence of zeros of $h(\zeta)$. By Nevanlinna's second fundamental theorem, we know that for any integer $m \geq 2$

$$\begin{aligned} N\left(r, \frac{1}{\psi(e^z)}\right) &> \sum_{j=1}^{m} N\left(r, \frac{1}{g - w_j}\right) \\ &\geq (m - 1 - o(1))T(r, g), \quad \text{for } r \,\bar{\in}\, E, \end{aligned}$$

where E is a set of finite linear measure. On the other hand, we have

$$\begin{aligned} N\left(r, \frac{1}{\psi(e^z)}\right) &\leq T(r, \psi(e^z)) + O(1) \\ &= \frac{\deg_\zeta \psi(\zeta)}{\pi} r + O(1) \end{aligned}$$

Therefore

$$\frac{T(r,g)}{r} < \frac{\deg_\zeta \psi(\zeta)}{\pi(m-1-o(1))},$$

noting m is arbitrary we have

$$\liminf_{r\to\infty} \frac{T(r,g)}{r} \le \liminf_{\substack{r\to\infty \\ r\bar{\in} E}} \frac{T(r,g)}{r} = 0.$$

Obviously, the set of zeros of $\psi(e^z)$ is the same as the set of all the roots of $g(z) - w_n = 0$, $(n = 1, 2, \ldots)$. Suppose that $\zeta_1, \ldots, \zeta_k$ are the zeros of $\psi(\zeta)$, $\zeta_0 = \max_{1\le j\le k}\{\log|\zeta_j|\} > -\infty$. Then all the zeros of $\psi(e^z)$ must lie on the left hand side of the straight $\{z, \operatorname{Re} z = \zeta_0\}$, and so do all the roots of $g(z) - w_n = 0$. By Lemma 2.3, $g(z)$ is a polynomial of degree ≤ 2.

If $\deg g(z) = 2$ and $g(z) = (\alpha z + \beta)^2 + \gamma$, then set $Z = \alpha z + \beta$ or $z = aZ + b$, $g(z) = g(aZ + b) = Z^2 + \gamma$ is even in Z, and $f(aZ + b)$ is even, writing Z by z, we have

$$\begin{aligned}\psi(e^{az+b})&\exp(\Phi(az+b) + d(az+b)) \\ &= \psi(e^{-az+b})\exp(\Phi(-az+b) + d(-az+b)),\end{aligned}$$

therefore

$$\exp(2adz) = \frac{\psi(e^{-az+b})}{\psi(e^{az+b})}\exp(\Phi(-az+b) - \Phi(az+b)).$$

This implies that $\Phi(-az+b) - \Phi(az+b)$ is a constant, for it is periodic. Therefore $\exp(2adz)$ has period $2\pi i/a$, $2d$ must be an integer. This is clearly a contradiction. Therefore $g(z)$ must be linear. Lemma 2.2 is proved.

Lemma 2.4. *Suppose that $h(\zeta)$ in* (2.1) *has only finitely many zeros, then $h(\zeta)$ is linear*.

Proof. Since $h(\zeta)$ has only finitely many zeros, then $h(\zeta)$ must be of the form

$$h(\zeta) = C(\zeta)e^{B(\zeta)} \tag{2.2}$$

for a non-constant polynomial $C(\zeta)$ and an entire function $B(\zeta)$. Since $f(z)$ has infinitely many zeros, $g(z)$ must be transcendental. It follows from (2.1) that

$$C(g(z)) = \psi(e^z)e^{M(z)+dz} \tag{2.3}$$

and

$$B(g(z)) = \Phi(z) - M(z) + c. \tag{2.4}$$

We shall prove that $B(t)$ is a constant. To the end, suppose that $B(t) \not\equiv$ constant. From (2.3), we have

$$T(r, M) = o(T(r, g)), \quad r \notin F,$$

and from (2.4), we have

$$T(r, B(g)) \sim T(r, B(g) + M(z) - c) = T(r, \Phi(z)), \quad (r \to +\infty, r \notin F),$$

where F denotes the set of r with finite logarithmic measure, i.e., $\int_F dt/t < \infty$, and then $g(z)$ is of finite order, it follows from (2.3) that $M(z)$ is a polynomial.

Below we prove that $M(z) \equiv$ constant. To the end, suppose that $M(z) \not\equiv$ constant. By a result of Urabe's ([**10**]), for any non constant polynomial $p(z)$, $\Phi(z) + p(z)$ is left-prime. Therefore $B(g(z))$ is left-prime, i.e., $B(\zeta)$ is linear. We may assume that $B(z) = az$, $a \in \boldsymbol{C}\setminus\{0\}$. From (2.4) we have

$$g(z) = \Phi(z)/a - M(z)/a + c/a \tag{2.5}$$

Substituting (2.5) into (2.3), we get

$$C((\Phi(z) - M(z) + c)/a) = \psi(e^z)e^{M(z)+dz}.$$

Then

$$C((\Phi(0) - M(n\tau) + c)/a) = \psi(e^{n\tau})e^{M(n\tau)+n\,d\tau}.$$

Obviously, $C((\Phi(0) - M(n\tau) + c)/a) \sim \alpha n^k$, as $n \to \infty$, where α is a non-zero constant and $k = \deg C \deg M$. On the other hand, $\psi(e^{n\tau})e^{M(n\tau)+n\,d\tau}$ is either bound or equivalent to $\beta \exp(\gamma n^m)$, as $n \to +\infty$, where m is a positive integer. Thus we have derived a contradiction. Therefore $M(z) \equiv c_0$, and then

$$B(g(z)) = \Phi(z) + c_1.$$

Since $\rho(\Phi(z)) < +\infty$ and g is transcendental, we have $\rho(B) = 0$, and therefore from $B(g(z))$ being periodic it follows that $g(z)$ is periodic. On the other hand, since d is not a rational number, $C(g(z)) = \psi(e^z)e^{dz+c_0}$ is not periodic. We know, however, that $C(g(z))$ is periodic if and only if $g(z)$ is periodic. And then $g(z)$ is not periodic. The contradiction shows that $B(\zeta) \equiv const$, which implies $h(\zeta)$ is a polynomial.

Suppose that $h(\zeta)$ has at least two distinct zeros. By the similar method to the proof of Lemma 2.2, we have that $\rho(g) \leq 1$, and then $f(z) = h(g(z))$ is of order at most 1. This is impossible. Therefore $h(\zeta)$ has only one zero. But the given condition that $\psi(\zeta)$ has a non-zero simple zero implies that $h(\zeta)$ is linear. The proof of Lemma 2.4 is now completed. □

3. Solutions of $w'' + A(e^z)w = 0$.

With respect to representations of solutions of periodic second order linear differential equation

$$w'' + A(e^z)w = 0, \tag{3.1}$$

where $A(t) = \sum_{j=0}^{p} b_j t^j$, $b_p \neq 0$, $p \geq 2$, S. Bank and I. Laine [**2**] proved the following

THEOREM A. *Let $f(z)(\not\equiv 0)$ be a solution of* (3.1) *with the property that the exponent of convergence for the zero-sequence of $f(z)$ is finite, then the following is true*:

(I) *If $f(z)$ and $f(z + 2\pi i)$ are linearly dependent, then $f(z)$ can be represented in the form*

$$f(z) = \Psi(e^z)\exp\left(\sum_{j=q}^{m} d_j e^{jz} + dz\right). \tag{3.2}$$

(II) *If $f(z)$ and $f(z + 2\pi i)$ are linearly independent, then $f(z)$ can be represented in the form*

$$f(z) = \Psi(e^{z/2})\exp\left(\sum_{j=q}^{m} d_j e^{(j+1/2)z} + dz\right), \tag{3.3}$$

where $\Psi(t)$ is a polynomial all of whose roots are simple and nonzero, m and $q \in \boldsymbol{Z}$ with $m \geq q$ and $d, d_q, \dots, d_m \in C$ with $d_q \cdot d_m \neq 0$.

In this section, we shall point out that in (3.2) and (3.3), $m > 0$, $q \geq 0$ and $d = \sqrt{-b_0}$.

THEOREM 3.1. *Let $f(z)$ be defined as in Theorem A. If $f(z)$ and $f(z + 2\pi i)$ are linearly dependent, then in* (3.2), *$d = \sqrt{-b_0}$, $m = p/2$ and $q \geq 0$.*

PROOF. By Theorem A, we have $f(z) = t^d G(t)$, where $t = e^z$,

$$G(t) = \Psi(t)\exp\left(\sum_{j=q}^{m} d_j t^j\right).$$

Since

$$\frac{df(z)}{dz} = \frac{d(t^d G(t))}{dt} \cdot \frac{dt}{dz} = d \cdot t^d G(t) + t^{d+1} G'(t),$$

$$\frac{d^2 f(z)}{dz^2} = d^2 t^d G(t) + (2d + 1)t^{d+1} G'(t) + t^{d+2} G''(t),$$

from (3.1) we get

$$t^2G''(t) + (2d+1)tG'(t) + (A(t)+d^2)G(t) = 0$$

and then $\Psi(t) = a_0 + a_1 t + \cdots + a_n t^n$ $(a_0 \neq 0)$ satisfies

$$\begin{aligned} t^2\Psi'' &+ \left(2\sum_{j=q}^{m} jd_jt^j + 2d + 1\right)t\Psi' \\ &+ \left(\sum_{j=q}^{m} j(j-1)d_jt^j + \left(\sum_{j=q}^{m} jd_jt^j\right)^2 \right. \\ &\left. + (2d+1)\sum_{j=q}^{m} jd_jt^j + A(t) + d\right)\Psi(t) = 0. \end{aligned} \tag{3.4}$$

We shall point out here that $m > 0$. Indeed, if $m \leq 0$, then as $t \to \infty$, we have

$$\sum_{j=q}^{m} jd_jt^j \to 0, \quad \sum_{j=q}^{m} j(j-1)jd_jt^j \to 0$$

and

$$\frac{t^2\Psi''(t)}{\Psi(t)} \to n(n-1), \quad \frac{t\Psi'(t)}{\Psi(t)} \to n.$$

Since $A(t) \to \infty$ as $t \to \infty$, (3.4) implies a contradiction. Dividing (3.4) by $t^{2m}\Psi(t)$ and setting $t \to \infty$, we have

$$\frac{A(t)}{t^{2m}} \to -m^2d_m^2 \neq 0.$$

Thus it shows that $p = 2m$.

We set $H(s) = \Psi(s^{-1}) = a_0 + a_1s^{-1} + \cdots + a_ns^{-n}$. Similarly, from (3.4) we can show that $H(s)$ satisfies the following equation

$$\begin{aligned} s^2H'' &+ \left(1 - 2d - 2\sum_{j=q}^{m} jd_js^{-j}\right)sH' \\ &+ \left(\sum_{j=q}^{m} j(j-1)d_js^{-j} + \left(\sum_{j=q}^{m} jd_js^{-j}\right)^2 \right. \\ &\left. + (2d^2+1)\sum_{j=q}^{m} jd_js^{-j} + A(s^{-1}) + d^2\right)H = 0. \end{aligned} \tag{3.5}$$

We shall prove $q \geq 0$. In fact, if $q < 0$, then by using the same method as in the above, dividing both sides of (3.5) by s^{-2q} and setting $s \to \infty$, we get $q^2 d_q^2 a_0 = 0$. This is a contradiction. Therefore it must be $q \geq 0$. Since $s^2 H''$ and $sH'(s)$ tend to zero as $s \to \infty$, it follows from (3.5) that $b_0 + d^2 = 0$, and further $d = (-b_0)^{1/2}$. Theorem 3.1 follows. □

THEOREM 3.2. *Let $f(z)$ be defined as in Theorem A. If $f(z)$ and $f(z+2\pi i)$ are linearly independent, then in* (3.3), $d = \sqrt{-b_0}$, $m = (p-1)/2$ *and* $q \geq 0$.

PROOF. We set

$$f(z) = t^{2d} G_1(t),$$

where $G_1(t) = \Psi_1(t)\exp(\sum_{j=q}^{m} d_j t^{2j+1})$, $t = e^{z/2}$, where $\Psi_1(t) = a_0 + a_1 t + \cdots + a_n t^n$, $a_0 \neq 0$. From (3.1), it is easy to see that $\Psi_1(t)$ satisfies the following equation

$$\begin{aligned} &t^2\Psi_1'' + \left(2\sum_{j=q}^{m}(2j+1)d_j t^{2j+1} + 4d + 1\right)t\Psi_1' \\ &\quad + \left(\sum_{j=q}^{m} 2j(2j+1)d_j t^{2j+1} + \left(\sum_{j=q}^{m}(2j+1)d_j t^{2j+1}\right)^2\right. \\ &\quad \left. + (4d+1)\sum_{j=q}^{m}(2j+1)d_j t^{2j+1} + 4d^2 + 4A(t^2)\right)\Psi_1 = 0. \end{aligned} \tag{3.6}$$

Similar to Theorem 3.1, we can also point out that $m > 0$, and then dividing both sides of (3.6) by $\Psi_1(t)t^{4m+2}$ and setting $t \to \infty$, we obtain

$$\frac{A(t^2)f}{t^{4m+1}} \to (2m+1)^2 d_m^2 \neq 0,$$

so that $m = (p-1)/2$.

Set $H_1(s) = \Psi_1(s^{-1})$, we can show that $H_1(s)$ solves the following equation

$$\begin{aligned} &s^2 H_1'' + \left(1 - 2d - \sum_{j=q}^{m}(2j+1)d_j s^{-(2j+1)}\right)sH_1' \\ &\quad + \left(\sum_{j=q}^{m} 2j(2j+1)d_j s^{-(2j+1)} + \left(\sum_{j=q}^{m}(2j+1)d_j s^{-(2j+1)}\right)^2\right. \\ &\quad \left. + (4d+1)\sum_{j=q}^{m}(2j+1)d_j s^{-(2j+1)} + 4d^2 + 4A(s^{-2})\right)H_1 = 0. \end{aligned} \tag{3.7}$$

Similarly we can prove that $q \geq 0$. Since $s^2 H_1''(s)$ and $sH_1'(s)$ tend to zero as $s \to \infty$, from (3.7) we have that $4d^2 + 4b_0 = 0$, thus $d = (-b_0)^{1/2}$.

Combining Theorem A, 2.1, 3.1 and 3.2 we immediately have

THEOREM 3.3. *Let $f(z)$ be a solution of* (3.1) *with the properties that* 0 *is not a Picard exceptional value of $f(z)$ and the exponent of convergence for the zeros of $f(z)$ is finite. If $\gamma = \sqrt{-b_0}$ is not a rational number, then $f(z)$ is prime.*

REMARK. 1. If in Theorem 3.3, $\gamma = \sqrt{-b_0}$ is a rational number, then the solution of (3.1) is factorizable (see [**7**]).

2. The following example was provided by N. Yanagihara which shows that if the solution of (3.1) takes zero as Picard exceptional value, then it may be factorizable.

Suppose that

$$w'' + A(e^z)w = 0, \tag{3.8}$$

where $A(e^z) = -(\gamma^2 + (2\gamma+1)e^z + e^{2z})$. Then it is easy to show $f(z) = \exp(\gamma z + e^z)$ is a solution of (3.8), but $f(z)$ has the factorization

$$f(z) = h(g(z)), \quad h(\zeta) = e^{\zeta}, \quad g(z) = \gamma z + e^z.$$

Further if γ is a rational number, then $f(z)$ has another factorization. Indeed, we may write $\gamma = p/q$ (both p and q are integers),

$$f(z) = h_1(g_1(z)), \quad h_1(\zeta) = \zeta^p e^{\zeta^q} \quad \text{and} \quad g_1(z) = e^{z/q}.$$

4. Solution of $w''' - Kw' + A(e^z)w = 0$.

Suppose that $f(z)$ is a solution of the equation

$$w''' - Kw' + (b_0 + b_1 e^z + b_2 e^{2z})w = 0 \tag{4.1}$$

such that

$$\log^+ N\left(r, \frac{1}{f}\right) = o(r) \quad \text{as } r \to \infty. \tag{4.2}$$

Then by S. Bank and J. K. Langley ([**3**])

$$f(z) = \Psi(e^{z/q}) \exp\{dz + S(e^{z/q})\},$$

where $\Psi(\zeta) = \sum_{j=-r}^{s} c_j \zeta^j$, $S(\zeta) = \sum_{j=-n}^{m} d_j \zeta^j$, d is a constant, $1 \leq q \leq 3$. We have

THEOREM 4.1. *Suppose that* (4.1) *admits a non-trivial solution satisfying* (4.2). *Then it must be*

$$K + b_0 = -1 \tag{4.3}$$

Furthermore, $f(z)$ *admits the following representation*

$$f(z) = \Psi(e^{z/3}) \exp\{-z + S(e^{z/3})\},$$

$$\Psi(\zeta) = \sum_{j=0}^{s} c_j \zeta^j, \quad S(\zeta) = d_0 + d_2\zeta^2, \quad d_2 = -\frac{3}{2}\sqrt[3]{b_2}.$$

Corollary. *Let* $f(z)$ *be a solution of* (4.1) *with the condition* (4.2), *then* $f(z)$ *is factorizable. Moreover, one of the factorization is that* $f(z) = h(g(z))$ *where* $g(z) = e^{z/3}$, $h(\zeta) = \psi(\zeta)\zeta^{-3} \exp S(\zeta)$.

Proof of Theorem 4.1. Let $\hat{d} = d - \dfrac{r}{q}$, $\hat{s} = s + r$, we may rewrite $f(z)$ as

$$\begin{aligned} f(z) &= \left\{ \sum_{j=0}^{\hat{s}} \hat{c}_j e^{zj/q} \right\} \exp\left\{ \hat{d}z + \sum_{j=-n}^{m} d_j e^{zj/q} \right\} \\ &= \hat{\Psi}(e^{z/q}) \exp\{\hat{d}z + S(e^{z/q})\}. \end{aligned} \tag{4.4}$$

Set $\zeta = e^{z/q}$, then $f(z) = \zeta^{\hat{d}q} G(\zeta)$, $G(\zeta) = \hat{\Psi}(\zeta) \exp S(\zeta)$, where $\hat{\Psi}(\zeta) = \sum_{j=0}^{\hat{s}} \hat{c}_j \zeta^j$, $S(\zeta) = \sum_{j=-n}^{m} d_j \zeta^j$. A computation implies that

$$\begin{aligned} f'(z) &= \hat{d}\zeta^{\hat{d}q} G(\zeta) + \frac{1}{q}\zeta^{\hat{d}q+1} G'(\zeta), \\ f''(z) &= \hat{d}^2\zeta^{\hat{d}q} G(\zeta) + \frac{1}{q^2}(2\hat{d}q+1)\zeta^{\hat{d}q+1} G'(\zeta) + \frac{1}{q^2}\zeta^{\hat{d}q+2} G''(\zeta), \\ f'''(z) &= \hat{d}^3\zeta^{\hat{d}q} G(\zeta) + \frac{1}{q^3}(1 + 3\hat{d}q + 3\hat{d}^2q^2)\zeta^{\hat{d}q+1} G'(\zeta) + \frac{3}{q^3}(1+\hat{d}q)\zeta^{\hat{d}q+2} G''(\zeta) \\ &\quad + \frac{1}{q^3}\zeta^{\hat{d}q+3} G'''(\zeta). \end{aligned}$$

Substituting the above expressions into (4.1), we get

$$\begin{aligned} &\zeta^3 G'''(\zeta) + 3(\hat{d}q+1)\zeta^2 G''(\zeta) - \{Kq^2 - (1+\hat{d}q)^3 + \hat{d}^3q^3\}\zeta G'(\zeta) \\ &\quad + q^3(\hat{d}^3 - K\hat{d} + b_0 + b_1\zeta^2 + b_2\zeta^{2q}) G(\zeta) = 0. \end{aligned} \tag{4.5}$$

We may assume that $m \neq 0$, $d_m \neq 0$, then by substituting the representation of $G'(\zeta)/G(\zeta)$, $G''(\zeta)/G(\zeta)$ and $G'''(\zeta)/G(\zeta)$ with respect to ζ into (4.5), we have

$$\begin{aligned} &(md_m)^3\zeta^{3m} + 3(md_m)^2(m-1)d_{m-1}\zeta^{3m-1} + 3(\hat{d}q+1)(md_m)^2\zeta^{2m} \\ &\quad - \{Kq^2 - (1+\hat{d}q)^3 + \hat{d}^3q^3\} md_m \zeta^m \\ &\quad + q^3(\hat{d}^3 - K\hat{d} + b_0 + b_1\zeta^q + b_2\zeta^{2q}) + O(\zeta^{3m-2}) = 0 \end{aligned} \tag{4.6}$$

Collecting the coefficient of the highest term ζ^{3m} in (4.6), we get

$$3m = 2q \quad \text{and} \quad (md_m)^3 + q^3 b_2 = 0.$$

Noting $1 \le q \le 3$, it must be $q = 3$, $m = 2$ and then $d_2 = -(3/2)\sqrt[3]{b_2}$. Likewise, the coefficient of the term ζ^{3m-1} is $3(md_m)^2(m-1)d_{m-1} = 0$, it must be $d_{m-1} = d_1 = 0$. Therefore $S(\zeta) = d_2\zeta^2 + d_0 + d_{-1}\zeta^{-1} + \cdots + d_{-n}\zeta^{-n}$. We are going to show $n = 0$. To the end, set $t = 1/\zeta$, $G(\zeta) = G(1/t) = G_1(t)$ and then $G_1(t) = \psi_1(t)\exp S_1(t)$, where

$$\psi_1(t) = \sum_{j=0}^{\hat{s}} \hat{c}_j t^{-j}, \quad S_1(t) = d_2 t^{-2} + d_0 + d_{-1}t + \cdots + d_{-n}t^n.$$

A computation implies that $G_1(t)$ satisfies

$$t^3 G_1'''(t) + 3(1 - 3\hat{d})t^2 G_1''(t) + (1 - 9\hat{d} + 27\hat{d}^2 - 9K)tG_1'(t) \\ - 27(\hat{d}^3 - K\hat{d} + b_0 + b_1 t^{-3} + b_2 t^{-6})G_1(t) = 0 \tag{4.7}$$

On the other hand, it is easy to show that if $n \ne 0$, we may assume $nd_{-n} \ne 0$, then substituting the representation of $G_1'(t)/G_1(t)$, $G_1''(t)/G_1(t)$ and $G_1'''(t)/G_1(t)$ with respect to t into (4.7), and then dividing both sides of resulting equality by t^{3n} and letting $t \to \infty$, we get $(nd_{-n})^3 = 0$, which is a contradiction to $n \ne 0$. Therefore $S(\zeta) = d_2\zeta^2 + d_0$ and then $S'(\zeta) = 2d_2\zeta$, $S''(\zeta) = 2d_2$, $S'''(\zeta) = 0$, so that we have $G'(\zeta)/G(\zeta) = 2d_2\zeta + O(\zeta^{-1})$, $G''(\zeta)/G(\zeta) = 4d_2^2\zeta^2 + 6d_2 + O(\zeta^{-1})$ and $G'''(\zeta)/G(\zeta) = 8d_2^3\zeta^3 + 24d_2^2\zeta + O(1)$. From (4.5), it follows that

$$8d_2^3\zeta^6 + 24d_2^2\zeta^4 + 12d_2^2(q\hat{d}+1)\zeta^4 + 12d_2(q\hat{d}+1)\zeta^2 \\ - 2d_2[Kq^2 - (1+\hat{d}q)^3 + \hat{d}^3q^3]\zeta^2 + q^3(\hat{d}^3 - K\hat{d} + b_0 + b\zeta^3 + b_2\zeta^6) + O(\zeta^3) = 0.$$

Making use of $q = 3$ and $8d_2^3 + 27b_2 = 0$, collecting the coefficient of the term ζ^4, we get

$$24d_2^2 + 12d_2^2(3\hat{d} + 1) = 0$$

noting $d_2 \ne 0$, it follows that $\hat{d} = -1$.

Therefore $G(\zeta) = \hat{\Psi}(\zeta)\exp(d_0 + d_2\zeta)$, and then substituting the representation of $G'(\zeta)$, $G''(\zeta)$ and $G'''(\zeta)$ into (4.5) and making use of $q = 3$, we have

$$\zeta^3\hat{\Psi}''' + \{6d_2\zeta^3 + 3(3\hat{d}+1)\zeta^2\}\hat{\Psi}'' + \{12d_2^2\zeta^5 + 6d_2\zeta^3 + 12d_2(3\hat{d}+1)\zeta^3 \\ + (1 + 9\hat{d} + 27\hat{d}^2 - 9K)\zeta\}\hat{\Psi}' + \{[12d_2\zeta^4 + 3(3\hat{d}+1)(2d_2 + 4d_2\zeta^2)]\zeta^2 \\ + 2d_2(1 + 9\hat{d} + 27\hat{d}^2 - 9K)\zeta^2 + 27(b_0 + b_1\zeta^3 + \hat{d}^3 - K\hat{d})\}\hat{\Psi} = 0. \tag{4.8}$$

Since $\hat{\Psi}(\zeta) = \sum_{j=0}^{\hat{s}} \hat{c}_j \zeta^j$ and

$$\hat{\Psi}^{(k)}(\zeta) = \sum_{j=k}^{\hat{s}} j(j-1)\cdots(j-k+1)\hat{c}_j\zeta^{j-k}, \quad k = 1, 2, 3,$$

substituting these equalities into (4.8) and collecting the coefficient of constant term, we get

$$27\hat{c}_0(b_0 + \hat{d}^3 - K\hat{d}) = 0,$$

which shows that $\hat{d}$ is a root of the algebraic equation

$$x^3 - Kx + b_0 = 0.$$

Further we have $K + b_0 = -1$ which is desired (4.3). □

References

[1] S. Bank, On the growth of solutions of algebraic differential equation. Trans. Amer. Math. Soc., **240** (1978), 195–212.

[2] S. Bank and I. Laine, Representations of solutions of periodic second order linear differential equations, Journ. Reine Angew. Math., **344** (1983), 1–22.

[3] S. Bank and J. K. Langley, Oscillations of higher order differential equation with entire periodic coefficients, comment. Math. Univ. St. Paul, **41** (1992), 65–85.

[4] Y. M. Chiang, I. Laine and S. P. Wang, An oscillation result of a third order linear differential equation with entire periodic coefficients, Tech. Report. The Hong Kong Univ. of Sci. and Tech., 1995.

[5] F. Gross, On factorization of meromorphic functions, Trans. Amer. Math. Soc., **131** (1968), 215–222.

[6] F. Gross, Factorization of meromorphic functions, Gov. Print, Washington D. C. 1972.

[7] He Yuzan, Zou Xiulin and Lin Yong, Meromorphic solutions of ordinary differential equations, Acta Math. Sinica, **34** (1991), 289–298.

[8] T. Kobayashi, Distribution of values of entire functions of lower order less than one, Kodai Math. Sem. Rep., **28** (1976) 33–37.

[9] N. Steinmetz, Uber die faktorisierbaren Losungen gewohnlicher Differentialgleichungen, Math. Z., **170** (1980) 168–180.

[10] H. Urabe, On a factorization problem of certain entire functions and a theorem of Ostrovskii, Bulletin of Kyoto Univ. of Education, Ser., B. No. 53, 1978.

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