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Relations between values at T-tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables

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Abstract. We give a new and very concise proof of the existence of a holomorphic continuation for a large class of twisted multivariable zeta functions. To do this, we use a simple method of "decalage" that avoids using an integral representation of the zeta function. This allows us to derive explicit recurrence relations between the values at T–tuples of negative integers. This also extends some earlier results of several authors where the underlying polynomials were products of linear forms.

1. Introduction.

Let $Q, P_1, \ldots, P_T \in \boldsymbol{R}[X_1, \ldots, X_N]$ and $\mu_1, \ldots, \mu_N \in \boldsymbol{C} - \{1\}$, each of modulus 1. To this data we can associate the following "twisted" multivariable zeta series:

$$Z(Q; P_1, \ldots, P_T; \mu_1, \ldots, \mu_N; s_1, \ldots, s_T) = \sum_{m_1 \geq 1, \ldots, m_N \geq 1} \frac{\left(\prod_{n=1}^{N} \mu_n^{m_n}\right) Q(m_1, \ldots, m_N)}{\prod_{t=1}^{T} P_t(m_1, \ldots, m_N)^{s_t}}$$

where $(s_1, \ldots, s_T) \in \boldsymbol{C}^T$.
In this article we will always assume that:

$$\forall t \in \{1, \ldots, T\},\ \forall \mathbf{x} \in [1, +\infty[^N,\ P_t(\mathbf{x}) > 0 \text{ and } \prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in J^N}]{} +\infty \qquad (\#)$$

It is not difficult to see that the condition (#) implies that $Z(Q; P_1, \ldots, P_T; \mu_1, \ldots, \mu_N; s_1, \ldots, s_T)$ is an absolutely convergent series when $\Re(s_1), \ldots, \Re(s_T)$ are sufficiently large.

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Cassou-Noguès ([**6**]) and Chen-Eie ([**7**]) proved in the case $T=1$, and $P=P_1$ a polynomial with positive coefficients, that the above series can be holomorphically continued to the whole complex plane and obtained very nice formulas for their values at negative integers. In [**8**] de Crisenoy extended these results by allowing $T>1$ and introducing the (HDF) hypothesis (see definition 2 in Section 2) that is much weaker than positivity of coefficients. The main result of [**8**] is the following:

THEOREM ([**8**]). *Let $Q, P_1, \ldots, P_T \in \boldsymbol{R}[X_1, \ldots, X_N]$ and $\boldsymbol{\mu} \in (\boldsymbol{T} \setminus \{1\})^N$.*
Assume that:
For each $t=1,\ldots,T$ P_t satisfies the (HDF) hypothesis and that $\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty \\ \mathbf{x}\in J^N}]{} +\infty.$
Then:

- *$Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; \cdot)$ can be holomorphically extended to $\boldsymbol{C}^T$*;
- *For all $k_1, \ldots, k_T \in \boldsymbol{N}$ if we set $Q \prod_{t=1}^{T} P_t^{k_t} = \sum_{\boldsymbol{\alpha}\in S} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$, we have*:

$$Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -k_1, \ldots, -k_T) = \sum_{\boldsymbol{\alpha}\in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \zeta_{\mu_n}(-\alpha_n).$$

where for all $\mu \in \boldsymbol{T}$, $\zeta_\mu(s) = \sum_{m=1}^{+\infty} \frac{\mu^m}{m^s}$.

To obtain this theorem he used an integral representation for the zeta series. The proof of the holomorphic continuation of the resulting integral is long and complicated.
By restricting to hypoelliptic polynomials (see definition 1 in Section 2) we can avoid the integrals:
The first result of this article gives a new proof of the holomorphic continuation of these series under the assumption that $P_1, \ldots, P_T$ are hypoelliptic. Our proof uses the "decalage" method of Essouabri ([**12**]). Particular interest for this paper is that this method does not use an integral representation for the zeta series. And the resulting proof is very concise and much simple.
The second result is relations between the values at T–tuples of negative integers of these series, relations that are true under the HDF hypothesis, and that give a mean to calculate by induction the values.
These relations are very simple in the case of linear forms, particularly interesting because of its link with the zeta functions of number fields ([**5**]). Similar results have been obtained by severals authors in particular cases of linear forms (see [**1**], [**3**] and [**4**]). Our method allows one also to obtain new relations even in the cases of linear forms.

2. Notations and preliminaries.

First, some notations:

1. Set $\boldsymbol{N} = \{0, 1, 2, \ldots\}, \boldsymbol{N}^* = \boldsymbol{N} - \{0\},\ J = [1, +\infty[$, and $\boldsymbol{T} = \{z \in \boldsymbol{C} \mid |z| = 1\}$.
2. The real part of $s \in \boldsymbol{C}$ will be denoted $\Re(s) = \sigma$ and its imaginary part $\Im(s) = \tau$.
3. Set $\mathbf{0} = (0, \ldots, 0) \in \boldsymbol{R}^N$ and $\mathbf{1} = (1, \ldots, 1) \in \boldsymbol{R}^N$.
4. For $\mathbf{x} = (x_1, \ldots, x_N) \in \boldsymbol{R}^N$ we set $|\mathbf{x}| = |x_1| + \ldots + |x_N|$.
5. For $\mathbf{z} = (z_1, \ldots, z_N) \in \boldsymbol{C}^N$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \boldsymbol{R}_+^N$ we set $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \ldots z_N^{\alpha_N}$.
6. The notation $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ (uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$) means that there exists $A = A(\mathbf{y}) > 0$, that depends neither on $\mathbf{x}$ nor λ, but could a priori depend on other parameters, and in particular on $\mathbf{y}$, such that: $\forall \mathbf{x} \in X\ \forall \lambda \in \Lambda, |f(\lambda, \mathbf{y}, \mathbf{x})| \leq A g(\mathbf{x})$.
   When there is no ambiguity, we will omit the word uniformly and the index $\mathbf{y}$.
7. The notation $f \asymp g$ means that we have both $f \ll g$ and $g \ll f$.
8. For $\mathbf{a} \in \boldsymbol{R}_+^N$ and $P \in \boldsymbol{R}[X_1, \ldots, X_N]$, we define $\Delta_{\mathbf{a}} P \in \boldsymbol{R}[X_1, \ldots, X_N]$ by $\Delta_{\mathbf{a}} P(\mathbf{X}) = P(\mathbf{X} + \mathbf{a}) - P(\mathbf{X})$. If no ambiguity, we will note simply $\Delta_{\mathbf{a}} P$ by ΔP.
9. Let $\mathbf{a} \in \boldsymbol{N}^N,\ N \in \boldsymbol{N}^*,\ I \subset \{1, \ldots, N\},\ q = \#I,\ I^c = \{1, \ldots, N\} \setminus I$ and $\mathbf{b} = (b_i)_{i \in I^c} \in \prod_{i \in I^c} \{0, \ldots, a_i\}$. Set $I = \{i_1, \ldots, i_q\}$ and $I^c = \{j_1, \ldots, j_{N-q}\}$. Then:
   (a) For all $\mathbf{d} = (d_1, \ldots, d_q) \in \boldsymbol{N}^q$ we define $\mathbf{m}^{\mathbf{a}, I, \mathbf{b}}(\mathbf{d}) = (m_1, \ldots, m_N) \in \boldsymbol{N}^N$ by $\forall k = 1, \ldots, q,\ m_{i_k} = a_{i_k} + d_k$ and $\forall k = 1, \ldots, N - q,\ m_{j_k} = b_{j_k}$;
   (b) For all $f : \boldsymbol{N}^N \to \boldsymbol{C}$ we define $f^{\mathbf{a}, I, \mathbf{b}} : \boldsymbol{N}^q \to \boldsymbol{C}$ by $f^{\mathbf{a}, I, \mathbf{b}}(\mathbf{d}) = f\big(\mathbf{m}^{\mathbf{a}, I, \mathbf{b}}(\mathbf{d})\big)$.

Convention. In this work we will say that a series defined by a sum over $N \geq 1$ variables is convergent when it is absolutely convergent.

Let us recall some definitions:

Definition 1. $P \in \boldsymbol{R}[X_1, \ldots, X_N]$ is said to be hypoelliptic if:

$$\forall \mathbf{x} \in J^N\ P(\mathbf{x}) > 0 \text{ and } \forall \boldsymbol{\alpha} \in \boldsymbol{N}^N \setminus \{\mathbf{0}\},\ \frac{\partial^{\boldsymbol{\alpha}} P}{P}(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in J^N}]{} 0.$$

Definition 2. Let $P \in \boldsymbol{R}[X_1, \ldots, X_N]$.
P is said to satisfy the weak decreasing hypothesis (denoted HDF in the rest of the article) if:

- $\forall \mathbf{x} \in J^N \ P(\mathbf{x}) > 0$,
- $\exists \epsilon_0 > 0$ such that for $\boldsymbol{\alpha} \in \boldsymbol{N}^N$ and $n \in \{1, \ldots, N\}$: $\alpha_n \geq 1 \Rightarrow \dfrac{\partial^{\boldsymbol{\alpha}} P}{P}(\mathbf{x}) \ll x_n^{-\epsilon_0}$ $(\mathbf{x} \in J^N)$.

REMARK 1. It follows from Hörmander ([**13**], p. 62) (see also Lichtin ([**14**], P. 342) that if $P \in \boldsymbol{R}[X_1, \ldots, X_N]$ is hypoelliptic then there exists $\epsilon > 0$ such that $\forall \boldsymbol{\alpha} \in \boldsymbol{N}^N \setminus \{\mathbf{0}\}$, $\dfrac{\partial^{\boldsymbol{\alpha}} P}{P}(\mathbf{x}) \ll \mathbf{x}^{-\epsilon \mathbf{1}}$ $(\mathbf{x} \in J^N)$. Therefore P satisfy also the HDF hypothesis.

3. Main results.

We first give a new and concise proof of the existence of a holomorphic continuation for our series when each polynomial is hypoelliptic. The procedure uses the "decalage" method, and does not require an integral representation.
A consequence of this method is found in corollaries 1 and 2. This gives simple recurrent relations between the values at T–tuples of negative integers when the polynomials P_i are products of linear polynomials or a class of quadratic polynomials. This procedure is different from that in [**8**] where the calculation of the special values was not an immediate consequence of the holomorphic continuation of the zeta series.

THEOREM 1. *Let* $\boldsymbol{\mu} \in (\boldsymbol{T} \setminus \{1\})^N$ *and* $Q, P_1, \ldots, P_T \in \boldsymbol{R}[X_1, \ldots, X_N]$.
We assume that $P_1, \ldots, P_T$ *are hypoelliptic and that at least one of them is not constant.*
Then $Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; \cdot)$ *can be holomorphically extended to* $\boldsymbol{C}^T$.
Let $\mathbf{a} \in \boldsymbol{N}^N$. *We set* $E(\mathbf{a}) = \{\mathbf{m} \in \boldsymbol{N}^{*N} \mid \mathbf{m} \not\geq \mathbf{a} + \mathbf{1}\} = \{\mathbf{m} \in \boldsymbol{N}^{*N} \mid \exists n \in \{1, \ldots, N\} \ m_n \leq a_n\}$.
Then for all $\mathbf{k} \in \boldsymbol{N}^T$ *we have the following relation*:

$$
\begin{aligned}
&(1 - \boldsymbol{\mu}^{\mathbf{a}}) Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}) \\
&\quad = \boldsymbol{\mu}^{\mathbf{a}} \sum_{\mathbf{0} \leq \mathbf{u} < \mathbf{k}} \binom{\mathbf{k}}{\mathbf{u}} Z\left(Q(\mathbf{X} + \mathbf{a}) \prod_{t=1}^{T} (\Delta_{\mathbf{a}} P_t)^{k_t - u_t}; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{u} \right) \\
&\qquad + \boldsymbol{\mu}^{\mathbf{a}} Z(\Delta Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}) + Z_{N-1}^{\mathbf{a}}(-\mathbf{k}).
\end{aligned}
$$

Where $Z_{N-1}^{\mathbf{a}}(\mathbf{s}) = \sum_{\mathbf{m} \in E(\mathbf{a})} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t}$. *By using notation* (9) *above it's easy to see that*:

$$Z_{N-1}^{\mathbf{a}}(\mathbf{s})$$
$$= \sum_{q=1}^{N-1} \sum_{\substack{I \subset \{1,\dots,N\} \\ \#I=q}} \sum_{\mathbf{b}=(b_i)\in\prod_{i\in I^c}\{0,\dots,a_i\}} \left(\prod_{i\in I^c} \mu_i^{b_i}\right) Z\Big(Q^{\mathbf{a},I,\mathbf{b}}; P_1^{\mathbf{a},I,\mathbf{b}}, \dots, P_T^{\mathbf{a},I,\mathbf{b}}; (\mu_i)_{i\in I}; \mathbf{s}\Big).$$

In particular, it is clear that $Z_{N-1}^{\mathbf{a}}$ is a finite linear combination of zeta series Z associated to hypoelliptic polynomials of at most $N-1$ variables.

Remark 2. When $\boldsymbol{\mu}^{\mathbf{a}} \neq 1$ (and we can of course choose $\mathbf{a}$ such that this is satisfied, this formula allows us to compute the values at T–tuples of negative integers of the series Z by recurrence because in each term on the right a integral value is strictly less than the corresponding value on the left: $|\mathbf{u}| < |\mathbf{k}|$, $\deg(\Delta Q) < \deg Q$, $N-1 < N$.

Now, using theorem A of [**8**](that gives the existence of the holomorphic continuation under the HDF hypothesis) and theorem B of [**8**] (that gives closed formula for the values, still under HDF) and the preeceding theorem, we show that the relations remain true under the HDF hypothesis:

Theorem 2. *Let $\boldsymbol{\mu} \in (\boldsymbol{T} \setminus \{1\})^N$ and $Q, P_1, \dots, P_T \in \boldsymbol{R}[X_1, \dots, X_N]$. We assume that $P_1, \dots, P_T$ satisfies the HDF hypothesis and that $\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty \\ \mathbf{x}\in J^N}]{} +\infty$.*
Then, the relations of theorem 1 are still true.

Now we deal with the particular case of linear forms. In this case the relations become particularly simple.

Corollary 1. *Let $\boldsymbol{\mu} \in (\boldsymbol{T} \setminus \{1\})^N$ and $L_1, \dots, L_T$ linear forms with positive coefficients.*
We assume that for each n there exists t such that L_t really depends on X_n.
We set $Z(\boldsymbol{\mu}; \cdot) = Z(1; L_1, \dots, L_T; \boldsymbol{\mu}; \cdot)$.
Let $\mathbf{a} \in \boldsymbol{N}^N$. We set $E(\mathbf{a}) = \{\mathbf{m} \in \boldsymbol{N}^{*N} \mid \mathbf{m} \not\geq \mathbf{a}+\mathbf{1}\} = \big\{\mathbf{m} \in \boldsymbol{N}^{*N} \mid \exists n \in \{1, \dots, N\},\ m_n \leq a_n\big\}$.
For all $t \in \{1, \dots, T\}$ we set: $\delta_t = L_t(\mathbf{X}+\mathbf{a}) - L_t(\mathbf{X})$. $\delta_t \in \boldsymbol{R}$.
Then, for all $\mathbf{k} \in \boldsymbol{N}^T$ we have the following relation:

$$(1-\boldsymbol{\mu}^{\mathbf{a}})Z(\boldsymbol{\mu};-\mathbf{k}) = \boldsymbol{\mu}^{\mathbf{a}} \sum_{\mathbf{0}\le\mathbf{u}<\mathbf{k}} \boldsymbol{\delta}^{\mathbf{k}-\mathbf{u}} \binom{\mathbf{k}}{\mathbf{u}} Z(\boldsymbol{\mu};-\mathbf{u}) + Z^{\mathbf{a}}_{N-1}(\boldsymbol{\mu},-\mathbf{k}).$$

Where $Z^{\mathbf{a}}_{N-1}(\mathbf{s}) = \sum_{\mathbf{m}\in E(\mathbf{a})} \boldsymbol{\mu}^{\mathbf{m}} \prod_{t=1}^{T} L_t(\mathbf{m})^{-s_t}$. *By using notation* (9) *above it's easy to see that*:

$$Z^{\mathbf{a}}_{N-1}(\mathbf{s}) = \sum_{q=1}^{N-1} \sum_{\substack{I\subset\{1,\dots,N\}\\ \#I=q}} \sum_{\mathbf{b}=(b_i)\in\prod_{i\in I^c}\{0,\dots,a_i\}} \left(\prod_{i\in I^c} \mu_i^{b_i}\right) Z\Big(1; L_1^{\mathbf{a},I,\mathbf{b}},\dots,L_T^{\mathbf{a},I,\mathbf{b}}; (\mu_i)_{i\in I}; \mathbf{s}\Big).$$

In particular, it is clear that $Z^{\mathbf{a}}_{N-1}$ *is a finite linear combination of zeta series* Z *associated to linear forms of at most* $N-1$ *variables.*

REMARK 3. Of course, here as well this formula allows a calculus by induction.

Since the works of Cassou-Noguès and Shintani, we know that the case of linear forms is particulaly interesting for algebraic number theory because of the link with the zeta functions of number fields. See also [**16**] for more motivations.

REMARK 4. Let us assume that $P \in \boldsymbol{R}[X_1,\dots,X_N]$ is a product of linear forms: $P = \prod_{t=1}^{T} L_t$ where $L_1,\dots,L_T$ have real positive coefficients, and that we want to evaluate the numbers $Z(1;P;\boldsymbol{\mu};-k)$ where $k \in \boldsymbol{N}$.
We could use Theorem 1, but it seems more interesting to note that
$\forall s \in \boldsymbol{C}\ \ Z(1;P;\boldsymbol{\mu};s) = Z(1;L_1,\dots,L_T;\boldsymbol{\mu};s,\dots,s)$ and then to consider the numbers $Z(1;P;\boldsymbol{\mu};-k) = Z(1;L_1,\dots,L_T;\boldsymbol{\mu};-k,\dots,-k)$ inside the family $Z(1;L_1,\dots,L_T;\boldsymbol{\mu};-k_1,\dots,-k_T)$ because of the simple relations between these numbers given by the preceeding proposition.

The particular case of linear forms is not the only case when relations of theorems 1 and 2 become particularly simple. The great flexibility in the assumptions of these theorems, allowed one to obtain also very simple relations in some other cases as in the following:

COROLLARY 2. *Let* $\boldsymbol{\mu} \in (\boldsymbol{T}\setminus\{1\})^N$ *and let* $P_1,\dots,P_T \in \boldsymbol{R}[X_1,\dots,X_N]$ *be polynomials of degree at most* 2. *Suppose that for all* $t = 1,\dots,T$:

$$P_t(\mathbf{X}) = P_t(X_1, \ldots, X_N) = \sum_{k=1}^{r_t} \left(\langle \alpha^{t,k}, \mathbf{X} \rangle\right)^2 + \sum_{n=1}^{N} c_{t,n} X_n + d_t$$

where $\alpha^{t,k} \in \boldsymbol{R}^N$, $c_{t,i} \in \boldsymbol{R}_+^*$ *and* $d_t \in \boldsymbol{R}_+$ $\forall t, k, i$.
Assume that there exists $\mathbf{a} \in \boldsymbol{N}^N \setminus \{\mathbf{0}\}$ *such that* $\langle \alpha^{t,k}, \mathbf{a} \rangle = 0$ *for all* t, k.
Then:

1. *for all* $t \in \{1, \ldots, T\}$, $\delta_t := P_t(\mathbf{X} + \mathbf{a}) - P_t(\mathbf{X}) \in \boldsymbol{R}$;
2. *the relations of Corollary* 1 *are still true.*

4. Proof of theorem 1.

Let $t \in \{1, \ldots, T\}$. It's clear that there exists $\boldsymbol{\alpha} \in \boldsymbol{N}^N$ such that $\partial^{\boldsymbol{\alpha}} P_t$ is a non-vanishing constant polynomial. So the hypoellipticity of P_t implies that $P_t(\mathbf{x}) \to +\infty$ when $|\mathbf{x}| \to +\infty$ ($\mathbf{x} \in [1, +\infty[^N$). By using Tarski-Seidenberg (see for example [**11**], Lemme 1), we know that there exists $\delta > 0$ such that $P_t(\mathbf{x}) \gg (x_1 \ldots x_N)^\delta$ uniformly in $\mathbf{x} \in [1, +\infty[^N$. Therefore, its clear that there exists σ_0 such that if $\sigma_1, \ldots, \sigma_T > \sigma_0$ then $Z(Q, P_1, \ldots, P_T, \boldsymbol{\mu}, \mathbf{s})$ converges.
By Remark 2.1 above, there exists also a fixed $\epsilon > 0$ such that $\forall t \in \{1, \ldots, T\}$ we have

$$\forall \boldsymbol{\alpha} \in \boldsymbol{N}^N \setminus \{\mathbf{0}\}, \quad \frac{\partial^{\boldsymbol{\alpha}} P}{P}(\mathbf{x}) \ll \mathbf{x}^{-\epsilon \mathbf{1}} \ (\mathbf{x} \in J^N).$$

Step 1. We establish a formula $(*)$.

Proof of step 1.
$P_1, \ldots, P_T$ are fixed in the whole proof so we will denote $Z(Q, \boldsymbol{\mu}, \cdot)$ instead of $Z(Q, P_1, \ldots, P_T, \boldsymbol{\mu}, \cdot)$. With this notation for all $\mathbf{s}$ such that $\sigma_1, \ldots, \sigma_T > \sigma_0$ we have:

$$\begin{aligned} Z(Q, \boldsymbol{\mu}, \mathbf{s}) &= \sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} \\ &= \sum_{\mathbf{m} \geq \mathbf{a}+\mathbf{1}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} + Z_{N-1}^{\mathbf{a}}(\mathbf{s}) \\ &= \boldsymbol{\mu}^{\mathbf{a}} \sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m} + \mathbf{a}) \prod_{t=1}^{T} P_t(\mathbf{m} + \mathbf{a})^{-s_t} + Z_{N-1}^{\mathbf{a}}(\mathbf{s}). \end{aligned}$$

$U \in \boldsymbol{N}$ is a set for the whole step.
We have $g_U\colon \boldsymbol{C} \times \boldsymbol{C}\backslash]-\infty,-1] \to \boldsymbol{C}$ holomorphic and satisfying:
$\forall s \in \boldsymbol{C},\ \forall z \in \boldsymbol{C}\backslash]-\infty,-1],\ (1+z)^s = \sum_{u=0}^{U} \binom{s}{u} z^u + z^{U+1} g_U(s,z).$
$\forall k \in \boldsymbol{N}$ verifying $k \le U$ and $\forall z \in \boldsymbol{C}\backslash]-\infty,-1]$, $g_U(k,z) = 0$.

For $t \in \{1,\ldots,T\}$ we define $\Delta_t = \Delta_{\mathbf{a}} P_t$.
For $t \in \{1,\ldots,T\}$ and $\mathbf{m} \in \boldsymbol{N}^{*N}$, we define $H_{t,\mathbf{m},U}\colon \boldsymbol{C} \to \boldsymbol{C}$ by:

$$H_{t,\mathbf{m},U}(s_t) = \sum_{u_t=0}^{U} \binom{-s_t}{u_t} \Delta_t(\mathbf{m})^{u_t} P_t(\mathbf{m})^{-u_t}.$$

$\forall t \in \{1,\ldots,T\}$ we have:

$$\begin{aligned} &P_t(\mathbf{m}+\mathbf{a})^{-s_t} \\ &\quad= [P_t(\mathbf{m}) + \Delta_t(\mathbf{m})]^{-s_t} \\ &\quad= P_t(\mathbf{m})^{-s_t}\Big[1 + \Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big]^{-s_t} \\ &\quad= P_t(\mathbf{m})^{-s_t}\Big[H_{t,\mathbf{m},U}(s_t) + \Delta_t(\mathbf{m})^{U+1} P_t(\mathbf{m})^{-(U+1)} g_U\Big(-s_t, \Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big)\Big]. \end{aligned}$$

For $x_1,\ldots,x_T,y_1,\ldots,y_T \in \boldsymbol{R}$, we have $\prod_{t=1}^{T}(x_t+y_t) = \sum_{\boldsymbol{\epsilon}\in\{0,1\}^T} \prod_{t=1}^{T} x_t^{1-\epsilon_t} y_t^{\epsilon_t}$, so:

$$\begin{aligned} &\prod_{t=1}^{T} P_t(\mathbf{m}+\mathbf{a})^{-s_t} \\ &\quad= \sum_{\boldsymbol{\epsilon}\in\{0,1\}^T} \prod_{t=1}^{T} H_{t,\mathbf{m},U}(s_t)^{1-\epsilon_t} \Delta_t(\mathbf{m})^{\epsilon_t(U+1)} P_t(\mathbf{m})^{-s_t-\epsilon_t(U+1)} g_U\Big(-s_t, \Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big)^{\epsilon_t}. \end{aligned}$$

For $\mathbf{m} \in \boldsymbol{N}^{*N}$ and $\boldsymbol{\epsilon} \in \{0,1\}^T$ we define $f_{\mathbf{m},U,\boldsymbol{\epsilon}}\colon \boldsymbol{C}^T \to \boldsymbol{C}$ thanks to the following formula:

$$f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s}) = \prod_{t=1}^{T} H_{t,\mathbf{m},U}(s_t)^{1-\epsilon_t} \Delta_t(\mathbf{m})^{\epsilon_t(U+1)} P_t(\mathbf{m})^{-s_t-\epsilon_t(U+1)} g_U\Big(-s_t, \Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big)^{\epsilon_t}.$$

So for all $\mathbf{m} \in \boldsymbol{N}^{*N}$ and $\mathbf{s} \in \boldsymbol{C}^T$ we have: $\prod_{t=1}^{T} P_t(\mathbf{m}+\mathbf{a})^{-s_t} = \sum_{\boldsymbol{\epsilon}\in\{0,1\}^T} f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s})$.
We define $Z_U(Q,\boldsymbol{\mu},\cdot)$ by: $Z_U(Q,\boldsymbol{\mu},\mathbf{s}) := \sum_{\boldsymbol{\epsilon}\in\{0,1\}^T\backslash\{\mathbf{0}\}} \sum_{\mathbf{m}\in\boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}+\mathbf{a}) f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s})$.
We will see in step 2 that for U large enough $Z_U(Q,\boldsymbol{\mu},\cdot)$ exists and is holomorphic

on

$$\left\{\mathbf{s} \in \boldsymbol{C}^T \mid \sigma_1, \ldots, \sigma_T > \sigma_0\right\}.$$

It's clear that $Z_U(Q, \boldsymbol{\mu}, \mathbf{s}) = \sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m} + \mathbf{a}) \sum_{\boldsymbol{\epsilon} \in \{0,1\}^T \setminus \{\mathbf{0}\}} f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s})$.
So $Z(Q, \boldsymbol{\mu}, \mathbf{s}) = \boldsymbol{\mu}^{\mathbf{a}} \sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m} + \mathbf{a}) f_{\mathbf{m},U,\mathbf{0}}(\mathbf{s}) + \boldsymbol{\mu}^{\mathbf{a}} Z_U(Q, \boldsymbol{\mu}, \mathbf{s}) + Z_{N-1}^{\mathbf{a}}(\mathbf{s})$.
By definition $f_{\mathbf{m},U,\mathbf{0}}(\mathbf{s}) = \prod_{t=1}^{T} H_{t,\mathbf{m},U}(s_t) P_t(\mathbf{m})^{-s_t}$ so:

$$\begin{aligned}
f_{\mathbf{m},U,\mathbf{0}}(\mathbf{s}) &= \prod_{t=1}^{T} \sum_{u_t=0}^{U} \binom{-s_t}{u_t} \Delta_t(\mathbf{m})^{u_t} P_t(\mathbf{m})^{-(s_t+u_t)} \\
&= \sum_{0 \le u_1, \ldots, u_T \le U} \prod_{t=1}^{T} \binom{-s_t}{u_t} \Delta_t(\mathbf{m})^{u_t} P_t(\mathbf{m})^{-(s_t+u_t)} \\
&= \sum_{\mathbf{u} \in \{0,\ldots,U\}^T} \binom{-\mathbf{s}}{\mathbf{u}} \prod_{t=1}^{T} \Delta_t(\mathbf{m})^{u_t} P_t(\mathbf{m})^{-(s_t+u_t)}
\end{aligned}$$

then, for all $\mathbf{s}$ such that $\sigma_1, \ldots, \sigma_T > \sigma_0$, we have:

$$\begin{aligned}
&\sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m} + \mathbf{a}) f_{\mathbf{m},U,\mathbf{0}}(\mathbf{s}) \\
&\quad = \sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m} + \mathbf{a}) \sum_{\mathbf{u} \in \{0,\ldots,U\}^T} \binom{-\mathbf{s}}{\mathbf{u}} \prod_{t=1}^{T} \Delta_t(\mathbf{m})^{u_t} P_t(\mathbf{m})^{-(s_t+u_t)} \\
&\quad = \sum_{\mathbf{u} \in \{0,\ldots,U\}^T} \binom{-\mathbf{s}}{\mathbf{u}} \sum_{\mathbf{m} \in \boldsymbol{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m} + \mathbf{a}) \prod_{t=1}^{T} \Delta_t(\mathbf{m})^{u_t} P_t(\mathbf{m})^{-(s_t+u_t)} \\
&\quad = \sum_{\mathbf{u} \in \{0,\ldots,U\}^T} \binom{-\mathbf{s}}{\mathbf{u}} Z\left(Q(\mathbf{X} + \mathbf{a}) \prod_{t=1}^{T} \Delta_t^{u_t}, \boldsymbol{\mu}, \mathbf{s} + \mathbf{u}\right).
\end{aligned}$$

The combination of the precedings results gives us:

$$Z(Q,\boldsymbol{\mu},\mathbf{s})$$
$$=\boldsymbol{\mu}^{\mathbf{a}}\sum_{\mathbf{u}\in\{0,\dots,U\}^T}\binom{-\mathbf{s}}{\mathbf{u}}Z\left(Q(\mathbf{X}+\mathbf{a})\prod_{t=1}^{T}\Delta_t^{u_t},\boldsymbol{\mu},\mathbf{s}+\mathbf{u}\right)+\boldsymbol{\mu}^{\mathbf{a}}Z_U(Q,\boldsymbol{\mu},\mathbf{s})+Z_{N-1}^{\mathbf{a}}(\mathbf{s}).$$

The following $(*)$ formula is now clear:

$$(1-\boldsymbol{\mu}^{\mathbf{a}})Z(Q,\boldsymbol{\mu},\mathbf{s})=\boldsymbol{\mu}^{\mathbf{a}}\sum_{\mathbf{u}\in\{0,\dots,U\}^T\setminus\{\mathbf{0}\}}\binom{-\mathbf{s}}{\mathbf{u}}Z\left(Q(\mathbf{X}+\mathbf{a})\prod_{t=1}^{T}\Delta_t^{u_t},\boldsymbol{\mu},\mathbf{s}+\mathbf{u}\right)$$
$$+\boldsymbol{\mu}^{\mathbf{a}}Z(\Delta Q,\boldsymbol{\mu},\mathbf{s})+Z_{N-1}^{\mathbf{a}}(\mathbf{s})+\boldsymbol{\mu}^{\mathbf{a}}Z_U(Q,\boldsymbol{\mu},\mathbf{s}).\qquad(*)$$

Step 2. For all $a\in\boldsymbol{R}_+$ there exists $U_0\in\boldsymbol{N}$ such that for all $U\geq U_0$, $Z_U(Q,\boldsymbol{\mu},\cdot)$ exists and is holomorphic on $\big\{\mathbf{s}\in\boldsymbol{C}^T\mid\forall t\in\{1,\dots,T\},\ \sigma_t>-a\big\}$.

Proof of step 2.
Let $a\in\boldsymbol{R}$. Let K be a compact of $\boldsymbol{C}^T$ included in $\{\mathbf{s}\in\boldsymbol{C}^T\mid\forall t\in\{1,\dots,T\},\ \sigma_t>-a\}$.
$\star$ Let $t\in\{1,\dots,T\}$.
We know that $P_t(\mathbf{x})\gg 1$ $(\mathbf{x}\in J^N)$ so it's easy to seen that: $P_t(\mathbf{x})^{-s_t}\ll P_t(\mathbf{x})^a$ $(\mathbf{x}\in J^N,\ \mathbf{s}\in K)$.
Let denote $p=\max\big\{\deg_{X_n}P_t\mid 1\leq n\leq N,\ 1\leq t\leq T\big\}$.
From now we assume $a>0$. So $P_t(\mathbf{x})^a\ll\mathbf{x}^{pa\mathbf{1}}$ $(\mathbf{x}\in J^N)$.
As a conclusion we have: $P_t(\mathbf{x})^{-s_t}\ll\mathbf{x}^{pa\mathbf{1}}$ $(\mathbf{x}\in J^N,\ \mathbf{s}\in K)$.
Let $U\in\boldsymbol{N}^*$ and $\boldsymbol{\epsilon}\in\{0,1\}^T$. By definition $H_{t,\mathbf{m},U}(\mathbf{s})=\sum_{u_t=0}^{U}\binom{-s_t}{u_t}\left(\frac{\Delta_t(\mathbf{m})}{P_t(\mathbf{m})}\right)^{u_t}$, so hypoellipticity of P_t implies that $H_{t,\mathbf{m},U}(\mathbf{s})\ll 1$ $(\mathbf{m}\in\boldsymbol{N}^{*N},\mathbf{s}\in K)$.
It implies also that there exists a compact of $]-1,+\infty[$ containing all the $\dfrac{\Delta_t(\mathbf{m})}{P_t(\mathbf{m})}$ where $\mathbf{m}$ is in $\boldsymbol{N}^{*N}$ so: $g_U\Big(-s_t,\Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big)^{\epsilon_t}\ll 1$ $(\mathbf{m}\in\boldsymbol{N}^{*N},\ \mathbf{s}\in K)$.
Thanks to the Taylor formula and to the choice of ϵ we have $\dfrac{\Delta_t(\mathbf{m})}{P_t(\mathbf{m})}\ll\mathbf{m}^{-\epsilon\mathbf{1}}$ $(\mathbf{m}\in\boldsymbol{N}^{*N})$.
From what proceeds we deduce that we have, uniformely in $\mathbf{m}\in\boldsymbol{N}^{*N}$ and $\mathbf{s}\in K$:
$$H_{t,\mathbf{m},U}(\mathbf{s})^{1-\epsilon_t}\left(\frac{\Delta_t(\mathbf{m})}{P_t(\mathbf{m})}\right)^{\epsilon_t(U+1)}P_t(\mathbf{m})^{-s_t}g_U\Big(-s_t,\Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big)^{\epsilon_t}\ll\mathbf{m}^{pa\mathbf{1}}\mathbf{m}^{-\epsilon\epsilon_t(U+1)\mathbf{1}}.$$
$\star$ By definition

$$f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s}) = \prod_{t=1}^{T} H_{t,\mathbf{m},U}(\mathbf{s})^{1-\epsilon_t}\left(\frac{\Delta_t(\mathbf{m})}{P_t(\mathbf{m})}\right)^{\epsilon_t(U+1)} P_t(\mathbf{m})^{-s_t} g_U\Big(-s_t, \Delta_t(\mathbf{m})P_t(\mathbf{m})^{-1}\Big)^{\epsilon_t}.$$

Since $\boldsymbol{\epsilon} \neq \mathbf{0}$, we have: $f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s}) \ll \mathbf{m}^{Tpa\mathbf{1}}\mathbf{m}^{-\epsilon(U+1)\mathbf{1}}$ $(\mathbf{m} \in \boldsymbol{N}^{*N})$.
We denote $q = \max\big\{\deg_{X_n} Q \mid 1 \leq n \leq N\big\}$ (obviously we can assume that $Q \neq 0$).
We see that $Q(\mathbf{m}+\mathbf{a})f_{\mathbf{m},U,\boldsymbol{\epsilon}}(\mathbf{s}) \ll \mathbf{m}^{(q+Tpa-\epsilon(U+1))\mathbf{1}}$ $(\mathbf{m} \in \boldsymbol{N}^{*N})$.
So it is enough to have $q + Tpa - \epsilon(U+1) \leq -2$, for U to fit.
Thus it is enough to choose $U \in \boldsymbol{N}$ such that $U \geq U_0 := \left[\dfrac{q+Tpa+2}{\epsilon}\right] + 1$.

Convention.

Let us take a convention, that we will use until the end of the proof. Let $a \in \boldsymbol{R}$. We will say that a function Y is an entire combination until a of the functions $Y_1, \ldots, Y_k$ if there exists:

- ⋆ entire functions $\lambda_1, \ldots, \lambda_k$: $\boldsymbol{C}^T \to \boldsymbol{C}$,
- ⋆ one function λ: $\big\{\mathbf{s} \in \boldsymbol{C}^T \mid \forall t \in \{1, \ldots, T\},\ \sigma_t > a\big\} \to \boldsymbol{C}$ holomorphic,

such that $Y = \lambda + \sum_{i=1}^{k} \lambda_i Y_i$.

A definition and a remark.

For $\mathbf{u} \in \boldsymbol{N}^T$ and $Q \in \boldsymbol{R}[X_1, \ldots, X_N]$, we denote $\mathscr{E}_{\mathbf{u}}(Q)$ the subspace of $\boldsymbol{R}[X_1, \ldots, X_N]$ generated by the polynomials of the following form: $\partial^{\boldsymbol{\beta}} Q \prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t$ where:

- ⋆ $\boldsymbol{\beta} \in \boldsymbol{N}^N$,
- ⋆ $F_1, \ldots, F_T$ are finite subset of $\boldsymbol{N}$ satisfying $|F_t| = u_t$,
- ⋆ $\forall t \in \{1, \ldots, T\}$ f_t is a function from F_t into $\boldsymbol{N}^N \setminus \{\mathbf{0}\}$.

We remark that $\mathscr{E}_{\mathbf{u}}(Q)$ is stable under partial derivations.

We are now beginning the proof of the existence of an holomorphic continuation.
The proof is by recurrence on N; it will be clear that the proof that rank $N-1$ implies rank $N-1$ gives the result at rank $N=1$.
Let $N \geq 1$. We assume that the result is true at rank $N-1$.

Step 3. Let $Q \in \boldsymbol{R}[X_1, \ldots, X_N]$ and $a \in \boldsymbol{R}_+$.
Then $Z(Q, \boldsymbol{\mu}, \mathbf{s})$ is an entire combination until $-a$ of functions of the type $Z(R, \boldsymbol{\mu}, \mathbf{s}+\mathbf{u})$ where $\mathbf{u} \in \boldsymbol{N}^N \setminus \{\mathbf{0}\}$ and $R \in \mathscr{E}_{\mathbf{u}}(Q)$.

PROOF OF STEP 3.

We are going to show by recurrence on $d \in \boldsymbol{N}$ that if $\deg Q < d$ then the result is true.
For $d = 0$ it is clear.
Let us assume the result for $d \geq 1$.
Let $Q \in \boldsymbol{R}[X_1, \ldots, X_N]$ such that $\deg Q < d + 1$.
Thanks to step 2 we set U such that $Z_U(Q, \boldsymbol{\mu}, \cdot)$ is holomorphic on $\{\mathbf{s} \in \boldsymbol{C}^T \mid \forall t \in \{1, \ldots, T\},\ \sigma_t > -a\}$.
We are now going to use the formula $(*)$ obtained in step 1. We look at each of the 4 term on the right.
• Let $\mathbf{u} \in \{0, \ldots, U\}^T \setminus \{\mathbf{0}\}$.
Thanks to the Taylor formula, it is easy to see that $Q(\mathbf{X} + \mathbf{a}) \prod_{t=1}^{T} \Delta_t^{u_t} \in \mathscr{E}_{\mathbf{u}}(Q)$.
• $\deg \Delta Q < d$ so the recurrence hypothesis on d implies that, $Z(\Delta Q, \boldsymbol{\mu}, \mathbf{s})$ is an entire combination until $-a$ of functions of the type $Z(R, \boldsymbol{\mu}, \mathbf{s} + \mathbf{u})$ where $\mathbf{u} \in \boldsymbol{N}^N \setminus \{\mathbf{0}\}$ and $R \in \mathscr{E}_{\mathbf{u}}(\Delta Q)$.
Furthermore, clearly, $\mathscr{E}_{\mathbf{u}}(\Delta Q) \subset \mathscr{E}_{\mathbf{u}}(Q)$.
• Thanks to the recurrence hypothesis on N, $Z_{N-1}^{\mathbf{a}}$ can be holomorphically extended to $\boldsymbol{C}^T$.
• We chose U so that $Z_U(Q, \boldsymbol{\mu}, \cdot)$ is holomorphic on $\{\mathbf{s} \in \boldsymbol{C}^T \mid \forall t \in \{1, \ldots, T\},\ \sigma_t > -a\}$.
So the formula $(*)$ gives the result.

STEP 4. $R \in \mathscr{E}_{\mathbf{u}}(Q)$ and $S \in \mathscr{E}_{\mathbf{v}}(R) \Rightarrow S \in \mathscr{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

PROOF OF STEP 4.

S is a linear combination of terms of the form $\partial^{\boldsymbol{\beta}} R \prod_{t=1}^{T} \prod_{k \in F'_t} \partial^{f'_t(k)} P_t$ where: $\boldsymbol{\beta} \in \boldsymbol{N}^N$, $F'_1, \ldots, F'_T$ are finite subsets of $\boldsymbol{N}$ such that $\forall t$, $|F'_t| = v_t$, and $f'_t \colon F'_t \to \boldsymbol{N}^N \setminus \{\mathbf{0}\}$.
$R \in \mathscr{E}_{\mathbf{u}}(Q)$ so $\partial^{\boldsymbol{\beta}} R \in \mathscr{E}_{\mathbf{u}}(Q)$ and then $\partial^{\boldsymbol{\beta}} R$ is a linear combination of terms of the form: $\partial^{\boldsymbol{\gamma}} Q \prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t$ where:
$\boldsymbol{\gamma} \in \boldsymbol{N}^N$, $F_1, \ldots, F_T$ are finite subsets of $\boldsymbol{N}$ such that $\forall t$, $|F_t| = u_t$, $f_t \colon F_t \to \boldsymbol{N}^N \setminus \{\mathbf{0}\}$.
We can assume that $\forall t, t' \in \{1, \ldots, T\}$, $F_t \cap F'_{t'} = \emptyset$.

To conclude it is enough to show that:
$U \overset{\text{def}}{=} \partial^{\boldsymbol{\gamma}} Q \Big(\prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t \Big) \Big(\prod_{t=1}^{T} \prod_{k \in F'_t} \partial^{f'_t(k)} P_t \Big)$ is in $\mathscr{E}_{\mathbf{u}+\mathbf{v}}(Q)$.
For $t \in \{1, \ldots, T\}$ we define $g_t \colon F_t \sqcup F'_t \to \boldsymbol{N}^N \setminus \{\mathbf{0}\}$ in the following way:
$g_t(k) = f_t(k)$ if $k \in F_t$, $g_t(k) = f'_t(k)$ if $k \in F'_t$.
Then $U = \partial^{\boldsymbol{\gamma}} Q \prod_{t=1}^{T} \prod_{k \in F_t \sqcup F'_t} \partial^{g_t(k)} P_t$ and $\forall t \in \{1, \ldots, T\}$, $|F_t \sqcup F'_t| = u_t + v_t$.

So it is now clear that $U \in \mathscr{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

STEP 5. Let $Q \in \boldsymbol{R}[X_1, \ldots, X_N]$, $a \in \boldsymbol{R}$ and $b \in \boldsymbol{N}^*$.
Then $Z(Q, \boldsymbol{\mu}, \mathbf{s})$ is an entire combination until $-a$ of functions of the type $Z(R, \boldsymbol{\mu}, \mathbf{s} + \mathbf{u})$ where $\mathbf{u} \in \boldsymbol{N}^N$ satisfies $|\mathbf{u}| \geq b$ and $R \in \mathscr{E}_{\mathbf{u}}(Q)$.

PROOF OF STEP 5.
The proof is by recurrence on $b \in \boldsymbol{N}^*$.
For $b = 1$, it comes from step 3.
The combination of step 3 and step 4 allows us to deduce the result at rank $b + 1$ from the result at rank b.

LAST STEP. Conclusion:
Let $Q \in \boldsymbol{R}[X_1, \ldots, X_N]$ and $a \in \boldsymbol{R}_+$.
We wish to show that $Z(Q, \boldsymbol{\mu}, \cdot)$ can be holomorphically extended until $-a$.
Let $b \in \boldsymbol{N}$. The value of b will be precised in the sequel.
By step 5 $Z(Q, \boldsymbol{\mu}, \mathbf{s})$ is an entire combination until $-a$ of functions of the type $Z(R, \boldsymbol{\mu}, \mathbf{s} + \mathbf{u})$ where $\mathbf{u} \in \boldsymbol{N}^N$ satisfies $|\mathbf{u}| \geq b$ and $R \in \mathscr{E}_{\mathbf{u}}(Q)$.
Let us consider $\mathbf{u} \in \boldsymbol{N}^N$ satisfying $|\mathbf{u}| \geq b$ and $R \in \mathscr{E}_{\mathbf{u}}(Q)$.
R is a linear combination of polynomials of the form $S = \partial^{\boldsymbol{\beta}} Q \prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t$ where:
$\boldsymbol{\beta} \in \boldsymbol{N}^N$, $F_1, \ldots, F_T$ are finite subsets of $\boldsymbol{N}$ satisfying $\forall t$, $|F_t| = u_t$ and $f_t\colon F_t \to \boldsymbol{N}^N \setminus \{\mathbf{0}\}$.

$$
\begin{aligned}
\frac{\prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t}{\prod_{t=1}^{T} P_t^{u_t}} (\mathbf{m}) &= \prod_{t=1}^{T} \prod_{k \in F_t} \frac{\partial^{f_t(k)} P_t}{P_t} (\mathbf{m}) \\
&\ll \prod_{t=1}^{T} \prod_{k \in F_t} \mathbf{m}^{-\epsilon \mathbf{1}} \ (\mathbf{m} \in \boldsymbol{N}^{*N}) \\
&\ll \prod_{t=1}^{T} \mathbf{m}^{-\epsilon u_t \mathbf{1}} \ (\mathbf{m} \in \boldsymbol{N}^{*N}) \\
&\ll \mathbf{m}^{-\epsilon |\mathbf{u}| \mathbf{1}} \ (\mathbf{m} \in \boldsymbol{N}^{*N}) \\
&\ll \mathbf{m}^{-\epsilon b \mathbf{1}} \ (\mathbf{m} \in \boldsymbol{N}^{*N}).
\end{aligned}
$$

Let K be a compact of $\boldsymbol{C}^T$ included in $\left\{\mathbf{s} \in \boldsymbol{C}^T \mid \forall t \in \{1, \ldots, T\},\ \sigma_t > -a\right\}$.
As in step 2: $\partial^{\boldsymbol{\beta}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} \ll \mathbf{m}^{\boldsymbol{q}\mathbf{1} + Tpa\mathbf{1}}$ $(\mathbf{m} \in \boldsymbol{N}^{*N},\ \mathbf{s} \in K)$.

$$
S(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-(s_t + u_t)} = \partial^{\boldsymbol{\beta}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} \frac{\prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t}{\prod_{t=1}^{T} P_t^{u_t}} (\mathbf{m}),
$$

so

$$S(\mathbf{m})\prod_{t=1}^{T}P_t(\mathbf{m})^{-(s_t+u_t)} \ll \mathbf{m}^{(q+Tpa-\epsilon b)\mathbf{1}} \quad (\mathbf{m}\in \boldsymbol{N}^{*N},\ \mathbf{s}\in K).$$

We choose $b\in \boldsymbol{N}$ satisfying $b\geq \dfrac{q+Tpa+2}{\epsilon}$ so that $q+Tpa-\epsilon b\leq -2$.
We see that $Z(R,\boldsymbol{\mu},\mathbf{s}+\mathbf{u})$ is holomorphic on $\{\mathbf{s}\in \boldsymbol{C}^T \mid \forall t\in\{1,\dots,T\},\sigma_t>-a\}$, so the proof is done. To obtain the formula of the theorem, it suffices to make $\mathbf{s}=-\mathbf{k}$ in the formula $(*)$ proved in step 1 and to remark that $Z_U(Q,\boldsymbol{\mu},-k)=0$ when we choose U such that $U>\max\{k_1,\dots,k_T\}$.
This ends the proof of theorem 1.

5. Proof of theorem 2.

Let $\boldsymbol{\mu}\in(\boldsymbol{T}\setminus\{1\})^N$ and $\mathbf{k}\in \boldsymbol{N}^N$ fixed. Let $d\in \boldsymbol{N}$ be fixed.
We denote $\boldsymbol{R}_d[X_1,\dots,X_N]$ the set of the real polynomials of N variables with degree at most d.
Let $D=card\{\boldsymbol{\alpha}\in \boldsymbol{N}^N \mid |\boldsymbol{\alpha}|\leq d\}$.
Let $\phi\colon \boldsymbol{R}^D\to \boldsymbol{R}_d[X_1,\dots,X_N]$ defined by: $A=(a_{\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|\leq d}\mapsto \phi(A)=\sum_{|\boldsymbol{\alpha}|\leq d}a_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$.
It is an isomorphism of real vector spaces.
Thanks to theorem B of [**8**], we know that there exists a polynomial $G\in \boldsymbol{R}[X_1,\dots,X_{D(T+1)}]$ such that for all $B,A_1,\dots,A_T\in \boldsymbol{R}^D$ such that $\phi(A_1),\dots,\phi(A_T)$ satisfy assumptions of Theorem 2, we have $Z(\phi(B);\phi(A_1),\dots,\phi(A_T);\boldsymbol{\mu};-\mathbf{k})=G(B,A_1,\dots,A_T)$.

Theorem B of [**8**] implies that if we restrain to polynomials of degree at most d with $P_1,\dots,P_T$ satisfying assumptions of Theorem 2, the following relation, that we want to establish:

$$\begin{aligned}&(1-\boldsymbol{\mu}^{\boldsymbol{\ell}})Z(Q;P_1,\dots,P_T;\boldsymbol{\mu};-\mathbf{k})\\ &\quad=\boldsymbol{\mu}^{\boldsymbol{\ell}}\sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{k}}\binom{\mathbf{k}}{\mathbf{u}}Z\left(Q(\mathbf{X}+\boldsymbol{\ell})\prod_{t=1}^{T}(\Delta P_t)^{u_t};P_1,\dots,P_T;\boldsymbol{\mu};-\mathbf{k}+\mathbf{u}\right)\\ &\qquad+\boldsymbol{\mu}^{\boldsymbol{\ell}}Z(\Delta Q;P_1,\dots,P_T;\boldsymbol{\mu};-\mathbf{k})+Z_{N-1}^{\boldsymbol{\ell}}(-\mathbf{k})\end{aligned}$$

is equivalent to $G_1(B,A_1,\dots,A_T)=G_2(B,A_1,\dots,A_T)$ (with $B=\phi^{-1}(Q)$ and $\forall t,\ A_t=\phi^{-1}(P_t)$) with $G_1,G_2\in \boldsymbol{R}[X_1,\dots,X_{D(T+1)}]$ depending only on $\mathbf{k},d,N,T$ and $\boldsymbol{\mu}$ and repectively associated to the right side and left side.
It is easy to see that if $A\in \boldsymbol{R}_+^{*D}$ then $\phi(A)$ is hypoelliptic and non constant.
Therefore Theorem 1 implies that for all $B\in \boldsymbol{R}^D$, and for all $A_1,\dots,A_T\in \boldsymbol{R}_+^{*D}$, $G_1(B,A_1,\dots,A_T)=G_2(B,A_1,\dots,A_T)$. Since G_1 and G_2 are polynomials, this

implies that $G_1 = G_2$.
This end the poof of theorem 2.

6. Proof of corollaries.

Corollary 1 is a direct consequence of theorem 1.
Point 1 of corollary 2 follows from assumption on $\mathbf{a}$ by easy computation.

PROOF OF THE POINT 2 OF COROLLARY 2. By using theorem 1, to finish the proof of corollary 2, it's enough to verify that each polynomial P_t is hypoelliptic. Let $t \in \{1, \ldots, T\}$ and $n \in \{1, \ldots, N\}$ fixed. We have uniformly in $[1, +\infty[^N$:

$$\begin{aligned} \frac{\partial P_t(\mathbf{x})}{\partial x_n} P_t^{-1}(\mathbf{x}) &\ll \frac{\sum_{k=1}^{r_t} \alpha_n^{t,k} |\langle \alpha^{t,k}, \mathbf{x}\rangle| + c_{t,n}}{\sum_{k=1}^{r_t} (\langle \alpha^{t,k}, \mathbf{x}\rangle)^2 + \sum_{j=1}^{N} c_{t,j} x_j + d_t} \\ &\ll \frac{1}{\sqrt{\sum_{k=1}^{r_t} (\langle \alpha^{t,k}, \mathbf{x}\rangle)^2 + \sum_{j=1}^{N} c_{t,j} x_j + d_t}} \\ &\ll \frac{1}{(x_1 + \ldots + x_N)^{1/2}}. \end{aligned}$$

But $\deg P_t \leq 2$. So the previous implies that P_t is an hypoelliptic polynomial. This completes the proof of corollary 2.

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