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Strong unique continuation property for time harmonic Maxwell equations

Dedicated to Professor Kiyoshi Mochizuki on the occasion of his sixtieth birthday

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Abstract. We give a strong unique continuation theorem for time harmonic Maxwell's equations in inhomogeneous anisotropic media of three dimensional space. The components of the matrices in the constitutive relations are supposed to be Hölder continuous. Furthermore, they are supposed to be differentiable except at one point where they may have critical singularities of Coulomb type. These assumptions prevent us to use the usual second order approach, so that we have to utilize a certain nice structure of Maxwell's equations as the first order system.

1. Introduction.

There is a long history about the strong unique continuation property. It goes back to the works due to T. Carleman, C. Müller, E. Heinz, N. Aronszajn and H. O. Cordes. After their works many advances have been obtained. Among them the differential inequalities with critical singularities as well as subcritical ones are intensively investigated in connection with the absence of positive eigenvalues in the continuous spectrum [**1**], [**7**] and [**4**]. Contrary to this, only few attempts have so far been made at studying about unique continuation property for systems [**5**], [**9**], [**10**], [**3**] and [**6**].

In this paper we are concerned with Maxwell's equations in continuous medium $U \subset \boldsymbol{R}^3$:

$$\tag{1.1} \begin{cases} \partial_t B + \operatorname{curl} E = 0, & \operatorname{div} B = 0, \\ -\partial_t D + \operatorname{curl} H = J, & \operatorname{div} D = \rho. \end{cases}$$

In this system the vector-valued functions $E(x,t)$, $B(x,t)$, $D(x,t)$ and $H(x,t)$ are unknown. They are called respectively the electric field, the magnetic induction,

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the electric induction and the magnetic field. $\rho(x,t)$ is supposed known and called the charge density. On the other hand, the vector-valued function $J(x,t)$ is often supposed to be unknown, which are called the current density.

The constitutive relations are given by

$$D = \varepsilon E, \quad H = \mu^{-1} B, \quad J = \sigma E. \tag{1.2}$$

In isotropic and homogeneous media we assume that ε and μ are positive constants characteristic of the medium considered, called respectively the permittivity (or the dielectric constant) and magnetic permeability. Furthermore, in this case, σ is a nonnegative constant called the constant of conductivity. The third relation of (1.2) is called Ohm's law.

In inhomogeneous anisotropic media such as crystals, ε and μ are symmetric matrices depending only on the position x. Moreover, σ is also a nonnegative symmetric matrix, called conductivity tensor (cf. [**2**]).

The problem in which we are interested here is the stationary one. Let U be a connected open subset of $\boldsymbol{R}^3$ containing the origin and

$$E(x,t) = E_0(x)e^{i\lambda t}, \quad B(x,t) = B_0(x)e^{i\lambda t}, \tag{1.3}$$

where λ is a non-zero real constant. Substituting these functions into the equations (1.1), we obtain the time harmonic Maxwell equations in an anisotropic inhomogeneous medium U:

$$\begin{cases} -\operatorname{curl} E = i\lambda\mu(x)H, \\ \operatorname{curl} H = i\lambda\varepsilon(x)E + \sigma(x)E. \end{cases} \tag{1.4}$$

Here, $\mu(x)$, $\varepsilon(x)$ and $\sigma(x)$ denote 3×3 nonnegative real symmetric matrix-valued functions in U, and $\lambda \in \boldsymbol{R} \backslash \{0\}$. In what follows, we call them *constitutive matrices*. (1.4) is a 6×6 system with a weakly coupling lower order terms. Its principal part is not elliptic. In fact, the characteristic polynomial of the symbol of

$$\begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}$$

is equal to $\tau^2(\tau - i|\xi|)^2(\tau + i|\xi|)^2$.

The usual approach for studying (1.4) is to transform them into second order systems. Temporarily we assume that the three constitutive matrices are scalar. We shall use the following notation when there is no confusion:

$$\varepsilon(x) = \varepsilon(x)I_3, \quad \mu(x) = \mu(x)I_3, \quad \sigma(x) = \sigma(x)I_3.$$

As a direct consequence of (1.4), we have

$$\operatorname{div}(\mu(x)H) = \operatorname{div}((i\lambda\varepsilon(x) + \sigma(x))E) = 0. \tag{1.5}$$

Using the identity

$$\text{curl}\,\text{curl}\,H = -\triangle H + \text{grad}\,\text{div}\,H \tag{1.6}$$

and

$$\text{div}\,H = -\sum_{j=1}^{3} \mu^{-1}(\partial_j \mu) H_j,$$

we have the second order system:

$$-\triangle H + \text{grad} \sum_{j=1}^{3} \mu^{-1}(\partial_j \mu) H_j = \text{curl}(i\lambda\varepsilon + \sigma)E.$$

Likewise,

$$-\triangle E + \text{grad} \sum_{j=1}^{3} (\lambda\varepsilon + \sigma\tilde{\varepsilon})^{-1}(\partial_j(\lambda\varepsilon + \sigma))E_j = -\text{curl}(i\lambda\mu H).$$

However, this method requires the differentiability of μ, ε and σ up to the second order.

Instead of this second order approach, we will take advantage of a nice structure as the first order system (1.4).

Denote the smallest eigenvalue of $\mu(0)$ and $\varepsilon(0)$ by $\mu_{\min}$ and $\varepsilon_{\min}$, respectively. We shall assume that $\sigma(x) \equiv 0$ in the nonisotropic case. In addition, we require that all the component of $\mu_{jk}(x)$, $\varepsilon_{jk}(x)$ and $\sigma_{jk}(x)$ belong to $C^{\delta}(U) \cap C^1(U \backslash \{0\})$ with $\delta < 1$, and that there exist positive constants C_1 and C_2 such that for any j and $k = 1, 2, 3$,

$$\lim_{r \to 0} \sup_{0<|x|<r} |x|^{\kappa} |\nabla \mu_{jk}(x)| \le C_1 \mu_{\min}, \quad \lim_{r \to 0} \sup_{0<|x|<r} |x|^{\kappa} |\nabla \varepsilon_{jk}(x)| \le C_2 \varepsilon_{\min} \tag{1.7}$$

with $\kappa \le 1$. Here, for any nonnegative number $\delta \le 1$, $C^{\delta}(U)$ denotes the class of all Hölder continuous functions f defined in U such that

$$\exists C > 0, \quad \forall x \text{ and } y \in U, \quad |f(x) - f(y)| \le C|x - y|^{\delta}.$$

Needless to say, these assumptions prevent us to employ the usual second order approach.

Subcritical cases ($\kappa < 1$) has been studied, under the additional condition $\varepsilon(0) = \mu(0) = I$, by V. Vogelsang [**10**] where he used a skillful technique based on the symbol calculus. On the other hand, the critical cases ($\kappa = 1$) require more precise techniques based on the operational calculus. This often causes some difficulties in studying general system of inequalities. Fortunately the Maxwell system as well as the Dirac operator has a nice structure that we shall see later. For the sake of this fact, we can show that under a certain condition the Maxwell system with the critical singularities has the strong unique continuation property if the constants C_1 and C_2 are zero (Corollary 2.2). In addition, we can relax the condition for the isotropic case (Theorem 2.3). Finally, we remark that the Maxwell equations in a vacuum can be written in the form analogous to the Dirac equation (§8). Therefore, a part of our results for isotropic cases can be covered by the previous work [**3**]. However, we shall give an independent proof for that result.

2. Statement of results.

2.1. Main result.

Let U be a connected open subset of $\boldsymbol{R}^3$ containing the origin and $\dot{U}$ be $U \setminus \{0\}$. First of all, for a $\boldsymbol{C}^3$-valued function u, we investigate the following differential inequalities, called the Maxwell type.

$$\begin{cases} |\operatorname{curl} u| \leq A_1 |u|/|x| \\ |\operatorname{div}(\alpha u)| \leq A_2 |u|/|x|, \quad x \in \dot{U} \end{cases} \tag{2.1}$$

where A_j, $j = 1, 2$ are positive constants and $\alpha(x)$ is a 3×3 real positive symmetric matrix defined in U.

In what follows, $C^1_*(\dot{U})$ denotes the subset of $C^1(\dot{U})$ consisting of all functions f satisfying

$$\lim_{\rho \to 0} \sup_{0 < |x| < \rho} \{|x| \, |\nabla_x f(x)|\} = 0. \tag{2.2}$$

We assume that for some $0 < \delta < 1$,

$$\alpha(x) \in C^\delta(U) \cap C^1_*(\dot{U}). \tag{2.3}$$

In what follows, $\alpha_{\min}$ and $\alpha_{\max}$ respectively stand for the smallest and the largest eigenvalue of $\alpha(0)$, which are positive by our assumption. We say that a function $u \in L^2_{\mathrm{loc}}(U)$ vanishes of infinite order at the origin if

$$\lim_{r \to 0} r^{-N} \int_{|x| \leq r} |u|^2 \, dx = 0, \quad \forall N \in \boldsymbol{N}. \tag{2.4}$$

Theorem 2.1. *Let α satisfy* (2.3). *Suppose $u \in \{H^1_{\mathrm{loc}}(U)\}^3$ satisfies* (2.1) *with*

$$A_1 \alpha_{\max} + A_2 < \frac{\alpha_{\min}}{2}. \tag{2.5}$$

If u vanishes of infinite order at the origin, then u is identically equal to zero in U.

When the last statement in Theorem 2.1 holds, we say that the system has the strong unique continuation property in U.

As an application of Theorem 2.1, we can see that Maxwell's equations have this property. Keep the same notation as in the previous section.

Corollary 2.2. *Let* $\sigma(x) \equiv 0$. *Suppose that* $\mu(x)$ *and* $\varepsilon(x)$ *are real, strictly positive and symmetric in* U *such that*

$$\mu(0) = \lambda_0 \varepsilon(0), \quad \exists \lambda_0 > 0. \tag{2.6}$$

In addition, we assume that every component of these two matrices belongs to $C^{\delta}(U) \cap C_*^1(\dot{U})$ *with* $0 < \delta < 1$. *Then, the time harmonic Maxwell equations* (1.4) *have the strong unique continuation property in a neighborhood of the origin.*

Theorem 2.3. *Suppose that* μ, ε *and* σ *are scalar functions belonging to* $C^0(U) \cap C^1(\dot{U})$. *Let* $\tilde{\varepsilon}(x) = \lambda\varepsilon(x) - i\sigma(x)$. *In addition, we assume that there exist positive constants* M_j, $j = 1, 2$ *such that*

$$\begin{cases} \lim\limits_{r \to 0} \sup\limits_{0<|x|<r} |x| \, |\mu(x)|^{-1} |\nabla\mu(x)| \leq M_1, \\ \lim\limits_{r \to 0} \sup\limits_{0<|x|<r} |x| \, |\tilde{\varepsilon}(x)|^{-1} |\nabla\tilde{\varepsilon}(x)| \leq M_2. \end{cases} \tag{2.7}$$

If $\max(M_1, M_2) < 1/2$, *then the time harmonic Maxwell equations* (1.4) *have the strong unique continuation property in* U.

The proof for the nonisotropic case is much more difficult than for the isotropic case. We, at first, consider the isotropic cases in the section 4. Our strategy for proving the isotropic cases (Theorems 2.3) is as follows. First, we shall reduce the system (1.4) into two 3×3 elliptic systems of the Maxwell type (2.1) with a scalar matrix $\alpha(x)$. Introducing the polar coordinates, we shall transform the new system into a system having a nice radial and an angular part (Proposition 4.4). Thanks to their nice structure, we can obtain a precise information on the spectrum of its angular part to derive a so-called Carleman inequality by using a basic principle, which is described in the section 3.

The proof for the nonisotropic cases (Theorem 2.1 and Corollary 2.2) is given in the section 5. It is strongly connected to the theory of elliptic systems with variable coefficients. We will approximate it by the isotropic system and utilize two type of weight functions so that Carleman estimates hold.

3. Carleman estimates for the model equations.

We are now going to describe a basic principle which yields Carleman estimates for the equations with critical singularities. Let d be a positive integer. Suppose that G is a selfadjoint operator in $\{L^2(\boldsymbol{S}^2)\}^d$ with $D(G) \supset \{C^{\infty}(\boldsymbol{S}^2)\}^d$

and there exists in $\{L^2(\boldsymbol{S}^2)\}^d$ a complete orthonormal basis $\{\phi_j\}_{j=-\infty}^{+\infty}$ consisting of eigenvectors of G:

$$G\phi_j = \lambda_j \phi_j, \quad j \in \boldsymbol{Z},$$

where λ_j, $j \in \boldsymbol{Z}$, are numbered in nondecreasing order such that

$$\cdots \le \lambda_{-2} \le \lambda_{-1} < 0 \le \lambda_0 \le \lambda_1 \le \cdots.$$

Define $\Sigma = \{\lambda_j; j = 1, 2, \dots\}$, which is a subset of positive real numbers.

Proposition 3.1. *In addition to the above conditions, suppose*

$$\lim_{R\to\infty} \inf\{|x-y| : x \ne y,\ x, y \in \Sigma \cap [R, \infty)\} = \delta > 0.$$

Then we can find a sequence $\{\gamma_j\}_{j=1}^{\infty}$ of positive numbers which tends to ∞ as $j \to \infty$ such that

$$\int |x|^{-2\gamma_j - 3} |-r\partial_r u + Gu|^2\, dx \ge \left(\frac{\delta}{2} - \frac{1}{j}\right)^2 \int |x|^{-2\gamma_j - 3} |u|^2\, dx \tag{3.1}$$

for any $u \in \{C_0^\infty(0 < |x| < 1)\}^d$ and any sufficiently large $j \in \boldsymbol{N}$. In particular, if there exists a positive number R_0 such that

$$\inf\{|x-y| : x \ne y,\ x, y \in \Sigma \cap [R, \infty)\} = \delta, \quad \forall R > R_0, \tag{3.2}$$

then the similar estimate to (3.1) *without its error term* $-1/j$ *holds.*

Proof. From the hypothesis, it follows that

$$u(x) = \sum_{j\in\boldsymbol{Z}} u_j(r)\phi_j(\omega) \quad \text{in } \{L^2(\boldsymbol{S})^2\}^3. \tag{3.3}$$

Set $y = \log r$ and $I = (-\infty, 0)$. The next lemma gives a crucial step for estimating each coefficients $u_j(r)$ of the eigenfunction expansion (3.3).

Lemma 3.2. *Let v, $\gamma \in \boldsymbol{R}$.*

$$\int_I e^{-2\gamma y} \left| -\frac{d}{dy} f + vf \right|^2 dy = \int_I \left\{ (-\gamma + v)^2 |e^{-\gamma y} f(y)|^2 + \left| \frac{d}{dy} (e^{-\gamma y} f(y)) \right|^2 \right\} dy$$

for any $f \in C_0^\infty(I)$.

Proof. This identity can be easily verified by use of an integration by parts. ☐

We now continue the proof of Proposition 3.1. Applying this lemma with $v = \lambda_j \in \boldsymbol{Z}$ and $f(y) = u_j(e^y)$, we obtain

$$\int r^{-2\gamma-3}|-r\partial_r u_j + \lambda_j u_j|^2 r^2\,dr \geq (-\gamma + \lambda_j)^2 \int r^{-2\gamma-3}|u_j|^2 r^2\,dr \tag{3.4}$$

for any positive γ.

We define a sequence $\{v_k\}_{k \in \boldsymbol{N}}$ of positive numbers as $(\lambda_k + \lambda_{k+1})/2$ for $k \in \boldsymbol{N}$. Let N_0 be a positive integer satisfying $\delta - 2/N_0 > 0$. From the definition of δ, it follows that there exists a subsequence $\{v_k^1\}$ of $\{v_k\}$ such that for any $k \in \boldsymbol{N}$,

$$\min_{j \in \boldsymbol{Z}} |-v_k^1 + \lambda_j| \geq \frac{\delta}{2} - \frac{1}{N_0 + 1}.$$

Likewise, we can find a subsequence $\{v_k^2\}$ of $\{v_k^1\}$ such that for any $k \in \boldsymbol{N}$,

$$\min_{j \in \boldsymbol{Z}} |-v_k^2 + \lambda_j| \geq \frac{\delta}{2} - \frac{1}{N_0 + 2}.$$

Proceeding in the same manner, we can find a subsequence $\{v_k^N\}$ of $\{v_k^{N-1}\}$ such that for any $k \in \boldsymbol{N}$,

$$\min_{j \in \boldsymbol{Z}} |-v_k^N + \lambda_j| \geq \frac{\delta}{2} - \frac{1}{N_0 + N}.$$

Define a sequence $\{\gamma_k\}_{k \in \boldsymbol{N}}$ as $\{v_k^k\}$. Then,

$$\int r^{-2\gamma_k-3}|-r\partial_r u_j + G u_j|^2 r^2\,dr \geq \left(\frac{\delta}{2} - \frac{1}{N_0 + k}\right)^2 \int r^{-2\gamma_j-3}|u_j|^2 r^2\,dr \tag{3.5}$$

for any $k \in \boldsymbol{N}$ and $j \in \boldsymbol{Z}$. Let $B_{r'} = \{x \in \boldsymbol{R}^3 : |x| \leq r'\}$. In view of the identity

$$\|u(x)\|_{L^2(B_{r'})}^2 = \sum_{j \in \boldsymbol{Z}} \int_0^{r'} |u_j(r)|^2 r^2\,dr,$$

(3.1) follows from (3.5). If (3.2) holds, the proof becomes much simpler. So we omit it. This completes the proof of Proposition 3.1. □

In the subsequent sections we shall use the simple case in Proposition 3.1. In order to apply it, we have to reduce our system to the canonical form and investigate the spectrum of G. There is, however, a large difference in their treatment between the isotropic cases and the nonisotropic ones. First, we will consider the isotropic ones in the section 4 where we will give a beautiful structure of Maxwell's equations. In the section 5 we will treat the nonisotropic cases by use of perturbation method which is developed in the theory of single elliptic equations with variable coefficients (cf. [**8**]).

4. Isotropic cases.

In this section we consider the case where all the matrices ε and μ are scalar ones, which is called the isotropic case. The Carleman inequality we would like to show is the followings.

THEOREM 4.1. *Let Ω be any nonempty open ball with center at the origin of $\boldsymbol{R}^3$. Then,*

$$\frac{1}{2}\left\{\int |x|^{-2\gamma-2}|u|^2\,dx\right\}^{1/2} \le \left\{\int |x|^{-2\gamma}|\mathrm{curl}\,u|^2\,dx\right\}^{1/2} + \left\{\int |x|^{-2\gamma}|\mathrm{div}\,u|^2\,dx\right\}^{1/2}$$

for any $u \in \{C_0^\infty(\dot{\Omega})\}^3$ and any large positive $\gamma \in \boldsymbol{Z}$.

The proof of this result is given in Section 4.1 and 4.2.

4.1. The polar coordinates and the canonical form.

We use the polar coordinates: $x = r\omega$, where

$$\omega(\theta,\varphi) = \frac{x}{|x|} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{4.1}$$

The gradient operator is represented by

$$\nabla = \omega\partial_r + \frac{1}{r}\Omega, \quad \Omega = (\partial_\theta\omega)\partial_\theta + (\partial_\varphi\omega)\frac{1}{\sin^2\theta}\partial_\varphi = \omega \wedge \boldsymbol{L}.$$

Here, $i\boldsymbol{L} = (iL_1, iL_2, iL_3)$ are the angular momentum operators:

$$L_1 = x_2\partial_3 - x_3\partial_2, \quad L_2 = x_3\partial_1 - x_1\partial_3, \quad L_3 = x_1\partial_2 - x_2\partial_1.$$

They are written in the polar coordinates as follows.

$$\boldsymbol{L} = \frac{1}{\sin\theta}\{(\partial_\theta\omega)\partial_\varphi - (\partial_\varphi\omega)\partial_\theta\}.$$

It holds that

$$-L^2 = -\left\{\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi^2\right\}, \tag{4.2}$$

which is the Laplace-Beltrami operator on $\boldsymbol{S}^2$. We introduce an important matrix J defined as $Ja = \omega \wedge a$:

$$J = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

At the final stage of our proof, we shall use the inequality

$$|Ju| \leq |u|, \tag{4.3}$$

which follows from the fact that the eigenvalues of J are 0, $\pm i$. Thus,

LEMMA 4.2.

$$\begin{aligned} \operatorname{curl} a = \nabla \wedge a &= \omega \wedge \partial_r a + \frac{1}{r}\Omega \wedge a \\ &= J\partial_r a + \frac{1}{r}\Omega \wedge a, \\ \operatorname{div} a &= \Sigma_{j=1}^3 \omega_j \partial_r a_j + \Sigma_{j=1}^3 \frac{1}{r}\Omega_j a_j. \end{aligned} \tag{4.4}$$

In addition,

$$\omega \cdot \Omega = 0, \quad \Omega \cdot \omega = 2.$$

A direct calculation implies the following relation.

LEMMA 4.3.

$$J^2 = -I + \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \end{pmatrix}.$$

Taking the nice relation between Ω and ω into consideration, we obtain the beautiful identity.

PROPOSITION 4.4.

$$J \operatorname{curl} a = -\partial_r a + \frac{1}{r} Ga + F(\omega, a),$$

where

$$G = \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix} \quad \textit{and} \quad F(\omega, a) = \operatorname{div} a \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

PROOF. From Lemmas 4.2 and 4.3, it follows that

$$\begin{aligned} J \operatorname{curl} a = -\partial_r a + \frac{1}{r} \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} \begin{pmatrix} -\Sigma\Omega_j a_j \\ -\Sigma\Omega_j a_j \\ -\Sigma\Omega_j a_j \end{pmatrix} + \frac{1}{r} J(\Omega \wedge a) \\ + \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} \begin{pmatrix} \operatorname{div} a \\ \operatorname{div} a \\ \operatorname{div} a \end{pmatrix}. \end{aligned} \tag{4.5}$$

A simple calculation gives us

$$\begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} \begin{pmatrix} -\Sigma\Omega_j a_j \\ -\Sigma\Omega_j a_j \\ -\Sigma\Omega_j a_j \end{pmatrix} = \begin{pmatrix} -\omega_1\Omega_1 & -\omega_1\Omega_2 & -\omega_1\Omega_3 \\ -\omega_2\Omega_1 & -\omega_2\Omega_2 & -\omega_2\Omega_3 \\ -\omega_3\Omega_1 & -\omega_3\Omega_2 & -\omega_3\Omega_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{4.6}$$

and

(4.7)

$$J(\Omega \wedge a) = \begin{pmatrix} -\omega_2\Omega_2 - \omega_3\Omega_3 & \omega_2\Omega_1 & \omega_3\Omega_1 \\ \omega_1\Omega_2 & -\omega_1\Omega_1 - \omega_3\Omega_3 & \omega_3\Omega_2 \\ \omega_1\Omega_3 & \omega_2\Omega_3 & -\omega_1\Omega_1 - \omega_2\Omega_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Using the relations

$$\omega_2\Omega_1 - \omega_1\Omega_2 = -L_3, \quad \omega_3\Omega_2 - \omega_2\Omega_3 = -L_1, \quad \omega_1\Omega_3 - \omega_3\Omega_1 = -L_2,$$

and

$$\sum_{j=1}^{3} \omega_j\Omega_j = 0,$$

we see that the sum of the right hand sides in (4.6) and (4.7) is equal to

$$\begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix}.$$

Thus, Proposition 4.4 has been proved. $\square$

Since $G^* = G$ in $(L^2(\boldsymbol{S}^2))^3$, we can apply Proposition 3.1 if we check the spectral property of G. This will be done in the next subsection.

4.2. Spectrum of the angular part.

To prove the Carleman inequality, we look into the spectrum of the operator G. To investigate this, we use the spherical harmonics expansion of the square integrable function on $\boldsymbol{S}^2$. Let $\{Y_\ell^m : m \in \boldsymbol{Z}, |m| \le \ell, \ell = 0, 1, 2, \dots\}$ be the sequence of spherical harmonics. This is an orthonormal basis of $L^2(\boldsymbol{S}^2)$. Any $u \in L^2(\boldsymbol{R}^3)$ can be expanded as follows.

$$u(x) = \sum_{\ell=0}^{\infty} \sum_{m=\ell}^{\ell} u_{\ell,m}(r) Y_\ell^m(\theta, \varphi), \quad r = |x|,$$

where

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,e^{im\varphi}P_\ell^m(\cos\theta), \quad \text{for } \ell \geq m \geq 0,$$

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{m+\ell}}{dx^{m+\ell}}(x^2-1)^\ell \quad \text{and}$$

$$Y_\ell^{-m} = (-1)^m\,\overline{Y}_\ell^m \quad \text{for } -\ell \leq m < 0.$$

Lemma 4.5.

$$\left\{\begin{array}{l} (L_1 + iL_2)\,Y_\ell^m = -i\sqrt{(\ell+m+1)(\ell-m)}\,Y_\ell^{m+1} \\ (L_1 - iL_2)\,Y_\ell^m = -i\sqrt{(\ell-m+1)(\ell+m)}\,Y_\ell^{m-1} \\ L_3\,Y_\ell^m = -im\,Y_\ell^m \\ (L_1^2 + L_2^2 + L_3^2)\,Y_\ell^m = -\ell(\ell+1)\,Y_\ell^m, \quad |m| \leq \ell. \end{array}\right. \tag{4.8}$$

Define the subspace E_ℓ of $\{L^2(\boldsymbol{S}^2)\}^3$ whose components are spanned by $\{Y_\ell^m\}_{|m|\leq\ell}$. From Lemma 4.5, it follows that

Proposition 4.6. *For each n, E_n is invariant under G: $GE_n \subset E_n$.*

Thus, our problem is reduced to the linear transformation $G|_{E_n}$ on the finite dimensional vector space.

The following relations are very useful in the subsequent argument.

Lemma 4.7.

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2.$$

We introduce an auxiliary operator:

$$\mathrm{div}_L\, u = \sum_{j=1}^{3} L_j u_j.$$

In view of Lemma 4.7, a direct calculation gives a remarkable property of the eigenfunctions of G.

Proposition 4.8. *Let $u \in C^2(\boldsymbol{S}^2)$. Then,*

$$\mathrm{div}_L\, Gu = \mathrm{div}_L\, u.$$

Hence, if $Gu = \lambda u$ with $\lambda \neq 1$, then

$$\mathrm{div}_L\, u = 0.$$

Furthermore, this nice property leads to the following conclusion.

Proposition 4.9. *Suppose that $u \in C^\infty(\boldsymbol{S}^2)$ satisfies $Gu = \lambda u$ with $\lambda \neq 1$. Then,*

$$G^2 u = -L^2 u + Gu, \quad G(G-I)u = -L^2 u.$$

Proof. By a direct calculation we see that

$$G^2 = -L^2 I + \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix} \begin{pmatrix} L_1 & L_2 & L_3 \\ L_1 & L_2 & L_3 \\ L_1 & L_2 & L_3 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & [L_2, L_1] & [L_3, L_1] \\ [L_1, L_2] & 0 & [L_3, L_2] \\ [L_1, L_3] & [L_2, L_3] & 0 \end{pmatrix}.$$

In the last expression we use Lemma 4.7 and Proposition 4.8 to arrive at the conclusion. □

As a simple consequence of Propositions 4.6 and 4.9, we obtain the following result.

Proposition 4.10. *The spectrum of the operator G is a subset of $\boldsymbol{Z}$.*

Remark 4.1. In the subsequent argument we do not require a complete information on the spectrum of G. The important thing is an estimate on the gaps in its spectrum.

Since Proposition 4.4 and (4.3) imply that

$$|(-r\partial_r + G)u| \leq r|J \operatorname{curl} u| + r|\operatorname{div} u| \leq r|\operatorname{curl} u| + r|\operatorname{div} u|,$$

Theorem 4.1 is a direct consequence of Proposition 3.1 and Proposition 4.10 because the width of two adjacent spectrum of G is equal to one.

4.3. Proof of Theorem 2.3.

We are now in a position to prove the unique continuation property for the isotropic case. Theorem 2.3 follows directly from Theorem 4.1. The space $H^1_{0,\infty}(\Omega)$ denotes the set of all functions belonging to $H^1_0(\Omega)$ and vanishing of infinite order at the origin. Then, we note that Theorem 4.1 is valid for $u \in H^1_{0,\infty}(\Omega)$. By taking the limit, the statement of Theorem 4.1 is true for $u \in H^1_0(U)$ if $U \not\ni 0$. Let $\Phi \in C_0^\infty(B(1))$, $0 \leq \chi \leq 1$ be a cut-off function such that $\Phi(x) = 1$ if $|x| \leq 1/3$ and $\Phi = 0$ if $|x| \geq 1/2$. Suppose that $u \in H^1_{0,\infty}(\Omega)$. Let $\Phi_n(x) = \Phi(nx)$. We apply Theorem 4.1 to $\tilde{u} = (1 - \Phi_n(x))u$ and let $n \to \infty$. In view of (2.4), we see that

$$\lim_{n\to\infty} \left\{ \int |x|^{-2\gamma} |[\operatorname{curl}, \Phi_n]u|^2 \, dx + \int |x|^{-2\gamma} |[\operatorname{div}, \Phi_n]u|^2 \, dx \right\} = 0.$$

From this fact, we can easily verify that the inequality in Theorem 4.1 holds for u.

Let E and $H \in \{H^1_{\mathrm{loc}}(\dot{U})\}^3$ be a solution to (1.4) vanishing of infinite order at the origin.

Let ρ be a positive number satisfying

$$3\rho < \frac{1}{2} - \max(M_1, M_2). \tag{4.9}$$

We can see that

$$|\mathrm{curl}\, E| \leq C_1 |H|/|x|, \quad |\mathrm{curl}\, H| \leq C_2 |E|/|x|, \quad x \in U,$$

where the constants C_1 and C_2 are given by

$$C_1 = \sup_U |x| \, \|\mu(x)\|, \quad C_2 = \sup_U |x| \, \|\tilde{\varepsilon}(x)\|.$$

In view of the boundedness of μ and $\tilde{\varepsilon}$, the constants C_1 and C_2 can be taken to be an arbitrarily small positive number if the domain U is shrunk enough. We choose U so that $U \subset B_1(0)$ and

$$C_1 + C_2 \leq \rho.$$

Since μ and $\tilde{\varepsilon}$ are scalar functions, (1.5) implies that

$$\mathrm{div}\, E = -\sum_{j=1}^{3} \tilde{\varepsilon}^{-1}(x)(\partial_j \tilde{\varepsilon})(x) E_j \tag{4.10}$$

and

$$\mathrm{div}\, H = -\sum_{j=1}^{3} \mu^{-1}(x)(\partial_j \mu)(x) H_j. \tag{4.11}$$

From the assumption (2.7), it follows that there exists a positive number r_0 such that $B_{r_0}(0) \subset U$ and

$$|\mathrm{div}\, H| \leq (M_1 + \rho)|H|/|x|, \quad |\mathrm{div}\, E| \leq (M_2 + \rho)|E|/|x|$$

if $|x| \leq r_0$.

Let $\chi \in C_0^\infty(U)$, $0 \leq \chi \leq 1$ be a cut-off function such that $\chi(x) = 1$ if $|x| \leq r_0/3$ and $\chi = 0$ if $|x| \geq r_0/2$. It holds that for any $u \in \{H^1_{\mathrm{loc}}(B_1(0))\}^3$,

$$\int_U |x|^{-2\gamma_j} |[\mathrm{curl}, \chi]u|^2 \, dx \leq 4 \max |\nabla \chi|^2 C^2 \int_{|x| \geq r_0/3} |x|^{-2\gamma_j} |u|^2 \, dx$$

and

$$\int_U |x|^{-2\gamma_j} |[\mathrm{div}, \chi]u|^2 \, dx \leq \max |\nabla \chi|^2 \int_{|x| \geq r_0/3} |x|^{-2\gamma_j} |u|^2 \, dx.$$

If we apply Theorem 4.1 to χE and χH, we see that there exists a positive constant C independent of γ such that

$$
\begin{aligned}
(4.12)\qquad \frac{1}{2}\left\{\int_U |x|^{-2\gamma_j-2}|\chi E|^2\,dx\right\}^{1/2} &\le C_1\left\{\int_U |x|^{-2\gamma_j-2}|\chi H|^2\,dx\right\}^{1/2}\\
&\quad + (M_2+\rho)\left\{\int_U |x|^{-2\gamma_j-2}|\chi E|^2\,dx\right\}^{1/2}\\
&\quad + C\left\{\int_{|x|\ge r_0/3} |x|^{-2\gamma_j}|E|^2\,dx\right\}^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.13)\qquad \frac{1}{2}\left\{\int_U |x|^{-2\gamma_j-2}|\chi H|^2\,dx\right\}^{1/2} &\le C_2\left\{\int_U |x|^{-2\gamma_j-2}|\chi E|^2\,dx\right\}^{1/2}\\
&\quad + (M_1+\rho)\left\{\int_U |x|^{-2\gamma_j-2}|\chi H|^2\,dx\right\}^{1/2}\\
&\quad + C\left\{\int_{|x|\ge r_0/3} |x|^{-2\gamma_j}|H|^2\,dx\right\}^{1/2}.
\end{aligned}
$$

Summing up these two inequalities, we obtain

$$
\begin{aligned}
(4.14)\qquad &\left(\frac{1}{2} - M_1 - \rho - C_1\right)\left\{\int_U |x|^{-2\gamma_j-2}|H|^2\,dx\right\}^{1/2}\\
&\quad + \left(\frac{1}{2} - M_2 - \rho - C_2\right)\left\{\int_U |x|^{-2\gamma_j-2}|E|^2\,dx\right\}^{1/2}\\
&\quad \le 2C\left\{\int_{|x|\ge r_0/3} |x|^{-2\gamma_j}|E|^2\,dx\right\}^{1/2} + 2C\left\{\int_{|x|\ge r_0/3} |x|^{-2\gamma_j}|H|^2\,dx\right\}^{1/2}.
\end{aligned}
$$

Finally, we conclude that

$$
\begin{aligned}
(4.15)\qquad &\delta_0\left\{\int_{|x|\le r_0/4} |x|^{-2\gamma_j-2}|H|^2\,dx\right\}^{1/2} + \delta_0\left\{\int_{|x|\le r_0/4} |x|^{-2\gamma_j-2}|E|^2\,dx\right\}^{1/2}\\
&\quad \le 2C\left\{\int_{|x|\ge r_0/3} |x|^{-2\gamma_j}|E|^2\,dx\right\}^{1/2} + 2C\left\{\int_{|x|\ge r_0/3} |x|^{-2\gamma_j}|H|^2\,dx\right\}^{1/2}
\end{aligned}
$$

for any positive integer γ. Since $|x|^{-2\gamma_j-2}$ is strictly decreasing function, we see that

$$\|H\|_{L^2(|x|\le r_0/4)} + \|E\|_{L^2(|x|\le r_0/4)} \le (2\delta_0)^{-1} C r_0 (4/3)^{-\gamma_j}.$$

Letting $\gamma_j \to \infty$ in the last inequality, we can conclude that E and H are identically equal to zero in $B_{r_0/4}$. This leads to the conclusion of Theorem 2.3. $\square$

5. Non-isotropic cases.

When $\alpha(x)$ is a positive symmetric matrix-valued function, we shall use a perturbation method that is effective in the treatment of elliptic operators with variable coefficients (cf. [**8**]). Suppose that there exist a positive constant A_0 such that

$$\sup_{x\in\dot{U}} \left\{ \sum_{k=1}^{3} |x|^2 \left| \sum_{j=1}^{3} \partial_{x_j} \alpha_{jk}(x) \right|^2 \right\}^{1/2} \le A_0, \tag{5.1}$$

and

$$\sup_{x\in\dot{U}} \|(x,\nabla)\alpha(x)\| \le B_0. \tag{5.2}$$

Here and in what follows, the symbol $\|\cdot\|$ denotes the norm of matrices as a multiplication on $\boldsymbol{C}^3$.

First of all, we shall transform $\alpha(0)$ into the identity matrix. It is well known that the operator curl is invariant under any orthogonal matrix on $\boldsymbol{R}^3$ whose determinant is equal to one. Namely, let $\{e_j\}_{j=1}^3$ and $\{f_j\}_{j=1}^3$ be two orthonormal basis of $\boldsymbol{R}^3$ such that the determinant of the orthogonal matrix $A = (a_{jk})$ defined by the relation

$$f_j = \sum_{k=1}^{3} a_{jk} e_k, \quad j = 1, 2, 3$$

is equal to one. The point P of $\boldsymbol{R}^3$ can be written in these basis as follows.

$$P = \sum_{j=1}^{3} x_j e_j = \sum_{j=1}^{3} y_j f_j.$$

It follows that $y = Ax$ and

$$A\,\mathrm{curl}_x\, {}^t\!A = \mathrm{curl}_y.$$

In a similar manner we see that

$$\mathrm{div}_x(\alpha u) = \nabla_x \cdot (\alpha u) = {}^t\!A\nabla_y \cdot (\alpha\, {}^t\!AAu) = \nabla_y \cdot A\alpha\, {}^t\!A(Au). \tag{5.3}$$

Therefore, we may assume that $A\alpha(0){}^tA$ is a diagonal matrix. Actually, it has the positive entries by our assumption:

$$A\alpha(0){}^tA = \begin{pmatrix} \alpha_1(0) & 0 & 0 \\ 0 & \alpha_2(0) & 0 \\ 0 & 0 & \alpha_3(0) \end{pmatrix}, \quad \alpha_j(0) > 0, \quad j = 1, 2, 3.$$

Furthermore, we can transform our system into a new one for which $\alpha(0)$ is the identity matrix. Indeed, making the following transformations

$$\text{(5.4)} \qquad y_j = \sqrt{\alpha_j(0)}\, z_j, \quad v_j(x) = \sqrt{\alpha_j(0)}\, \tilde{u}_j, \quad j = 1, 2, 3 \text{ with } \tilde{u} = Au,$$

we see that

$$\operatorname{div}_x(\alpha(x)u) = \operatorname{div}_z(\tilde{\alpha}(z)v), \quad \tilde{\alpha}(z) = \alpha_0^{-1/2} A\alpha(A^{-1}\alpha_0^{1/2} z){}^tA\alpha_0^{-1/2},$$

where

$$\alpha_0^{1/2} = \begin{pmatrix} \alpha_1^{1/2}(0) & 0 & 0 \\ 0 & \alpha_2^{1/2}(0) & 0 \\ 0 & 0 & \alpha_3^{1/2}(0) \end{pmatrix}.$$

Needless to say, it holds that $\tilde{\alpha}(0) = I_3$. At the same time, the transformations (5.4) implies

$$\operatorname{curl}_x u = \frac{1}{\sqrt{\alpha_1(0)\alpha_2(0)\alpha_3(0)}} \begin{pmatrix} \sqrt{\alpha_1(0)} & 0 & 0 \\ 0 & \sqrt{\alpha_2(0)} & 0 \\ 0 & 0 & \sqrt{\alpha_3(0)} \end{pmatrix} \operatorname{curl}_z v.$$

Let $\tilde{U}$ be the image of U under the map $z = \alpha_0^{-1/2} Ax$. As a consequence, we see that (2.1) is transformed into

$$\text{(5.5)} \qquad |\operatorname{curl} v| \le D_1|v|/|z|, \quad |\operatorname{div}(\tilde{\alpha}v)| \le D_2|v|/|z|, \quad z \in \dot{\tilde{U}},$$

with

$$D_1 = A_1\alpha_{\max}/\alpha_{\min}, \quad D_2 = A_2/\alpha_{\min}.$$

The hypothesis (5.2) implies

$$\text{(5.6)} \qquad \sup_{x \in \dot{\tilde{U}}} \|(z, \nabla)\tilde{\alpha}(z)\| \le D_3$$

with $D_3 = B_0/\alpha_{\min}$.

In what follows, the bracket $(\cdot, \cdot)$ stands for the standard inner product of

$\boldsymbol{C}^3$. We denote the k-th row of the matrix a by $a_{(k)}$ and the j-th column by $a^{(j)}$. It holds that

$$\operatorname{div}(\tilde{\alpha}\xi) = \sum_{k=1}^{3}(\partial_{z_k}(\alpha_0^{-1/2}A\alpha)_{(k)}, {}^tA\alpha_0^{-1/2}\xi).$$

In view of

$$(5.7)\qquad \sum_{k=1}^{3}\partial_{z_k}(\alpha_0^{-1/2}A\alpha)_{(k)} = (\operatorname{div}(\alpha_0^{-1/2}A\alpha)^{(1)}, \operatorname{div}(\alpha_0^{-1/2}A\alpha)^{(2)}, \operatorname{div}(\alpha_0^{-1/2}A\alpha)^{(3)})$$
$$= (\operatorname{div}_x \alpha^{(1)}, \operatorname{div}_x \alpha^{(2)}, \operatorname{div}_x \alpha^{(3)}),$$

the assumption (5.1) implies

$$(5.8)\qquad \left|\sum_{j=1}^{3}\partial_{z_j}(\alpha_0^{-1/2}A\alpha)_{(j)}(z)\right| \leq \left\{\sum_{k=1}^{3}\left|\sum_{j=1}^{3}(\partial_{x_j}\alpha_{jk})(\alpha_0^{1/2}z)\right|^2\right\}^{1/2} \leq A_0/|\alpha_0^{1/2}z|.$$

Therefore, we can conclude that

$$(5.9)\qquad |\operatorname{div}(\tilde{\alpha}\xi)| = \left|\sum_{k=1}^{3}(\partial_{z_k}(\alpha_0^{-1/2}\alpha)_{(k)}, {}^tA\alpha_0^{-1/2}\xi)\right| \leq \alpha_{\min}^{-1}A_0|\xi|/|z|.$$

Hereafter, we denote $\alpha_{\min}^{-1}A_0$ by D_0.

Henceforth, we drop the symbol ˜ for $\tilde{\alpha}$ and use the symbols x and u for z and v, respectively. Then, we have the following intermediate estimate.

Theorem 5.1. *Let U be any nonempty open ball with center at the origin of $\boldsymbol{R}^3$. Suppose that $\alpha \in C^\delta(U) \cap C_*^1(\dot{U})$ satisfies*

$$|\operatorname{div}(\alpha\xi)| \leq D_0|\xi|/|x|, \quad \forall x \in \dot{U}, \quad \forall \xi \in \boldsymbol{C}^3.$$

Then, there exists a positive constant C such that

$$(5.10)\qquad \frac{1}{2}\left\{\int |x|^{-2\gamma-2}|u|^2\,dx\right\}^{1/2}$$
$$\leq \left\{\int |x|^{-2\gamma}|J\operatorname{curl} u|^2\,dx\right\}^{1/2} + \left\{\int |x|^{-2\gamma-2}|u|^2\,dx\right\}^{1/2}$$
$$+ \left\{\int |x|^{-2\gamma}|\operatorname{div}(\alpha u)|^2\,dx\right\}^{1/2} + C\left\{\int |x|^{-2\gamma+2\delta}(|\nabla_x u|^2 + |u|^2)\,dx\right\}^{1/2}$$

for any $u \in \{C_0^\infty(\dot{U})\}^3$ and any large positive $\gamma \in \boldsymbol{Z}$.

Proof. We introduce the polar coordinates:

$$x = (r, \omega), \quad r = |x|, \quad \omega = x/|x|.$$

For a scalar function $h(x)$ and a vector-valued function $g(x)$ defined in a neighborhood U of the origin, $\mathcal{O}(h(x))g(x)$ denotes a vector-valued function $R(x)$ satisfying that

$$|R(x)| \le C|h(x)|\,|g(x)|, \quad \forall x \in \dot{U}$$

with some positive constant C.

Then, the same procedure as in the isotropic case yields that

$$J \operatorname{curl}_x u = -\partial_r u + \frac{1}{r} Gu + \operatorname{div}(\alpha u)\omega + \operatorname{div}((1-\alpha)u)\omega. \tag{5.11}$$

Since $\alpha(0) = 1$ and $\alpha(z)$ is Hölder continuous, Leibniz's rule implies that

$$\operatorname{div}((1-\alpha)u) = -\operatorname{div}(\alpha\xi)|_{\xi=u} + \mathcal{O}(|x|^{\delta})|\nabla u|.$$

The first term of the above equation satisfies

$$|\operatorname{div}(\tilde{\alpha}\xi)|_{\xi=u}| \le D_0|u|/|x|.$$

Since the principal part in (5.11) is the same as in the isotropic case, the argument for proving Theorem 4.1 implies the assertion of Theorem 5.1. □

In order to estimate the last term of the right hand side of (5.10), we shall require an inequality of elliptic type.

Lemma 5.2. *Suppose that the same hypothesis as in Theorem* 5.1. *If U is sufficiently small, there exists a positive constant C such that*

$$\int |\nabla f|^2\,dx \le C\left\{\int |\operatorname{curl} f|^2\,dx + \int |\operatorname{div} \alpha f|^2\,dx + D_0^2 \int |x|^{-2}|f|^2\,dx\right\} \tag{5.12}$$

holds for all $f \in C_0^1(\dot{U})$.

Proof. It is easily verified that

$$\sum_{j=1}^{3} \int |\partial_{x_j} f|^2 = \int |\operatorname{curl} f|^2 + \int |\operatorname{div} f|^2, \quad \forall f \in \{C_0^1(U)\}^3 \tag{5.13}$$

and

$$|\nabla f|^2 = |\partial_r f|^2 + \frac{1}{r^2}|Lf|^2, \quad \forall f \in C^1(U).$$

Since

$$\sum_{j=1}^{3} \int |L_j f|^2\,d\omega = \sum_{j=1}^{3} \int |\Omega_j f|^2\,d\omega, \quad \forall f \in C^1(\boldsymbol{S}^2),$$

it follows that for any $f \in \{C_0^1(\dot{U})\}^3$

$$\int |\partial_y f|^2 e^y \, dyd\omega + \sum_{j=1}^{3} \int |\Omega_j f|^2 e^y \, dyd\omega \tag{5.14}$$
$$\leq C \int e^{3y} \{|\widetilde{\mathrm{curl}} f|^2 + |\widetilde{\mathrm{div}} f|^2\} \, dyd\omega$$
$$\leq C \int e^{3y} \{|\widetilde{\mathrm{curl}} f|^2 + |\widetilde{\mathrm{div}}(\alpha f)|^2\} \, dyd\omega$$
$$+ C \int e^y e^{2\delta y} \sum_{j=1}^{3} |\Omega_j f|^2 \, dyd\omega + C \int e^y \{e^{2\delta y} |\partial_y f|^2 + e^{-2y} |f|^2\} \, dyd\omega.$$

Here, $\widetilde{\mathrm{div}}$ and $\widetilde{\mathrm{curl}}$ denote the representation of the operators div and curl with respect to the coordinates (y, ω), respectively. If the support of f is sufficiently small, we arrive at the conclusion. □

Note that $[\nabla, |x|^{-\gamma}]u = \mathcal{O}(\gamma |x|^{-\gamma-1})u$,

$$[\mathrm{curl}, |x|^{-\gamma}]u = \mathcal{O}(\gamma |x|^{-\gamma-1})u \quad \text{and} \quad [\mathrm{div}, |x|^{-\gamma}]u = \mathcal{O}(\gamma |x|^{-\gamma-1})u.$$

Shrinking $\dot{U}$ and applying (5.12) to $f = |x|^{-\gamma} u$, we obtain the following weighted estimate of elliptic type.

Lemma 5.3. *Suppose that $\alpha(0) = I$. Then, there exists a positive constant K such that for any $\gamma > 1$ and any $u \in \{C_0^1(\dot{U})\}^3$*

$$\gamma^{-1} \left\{ \int |x|^{-2\gamma} |\nabla u|^2 \, dx \right\}^{1/2} \leq K \gamma^{-1} \left\{ \int |x|^{-2\gamma} |\mathrm{curl}\, u|^2 \, dx \right\}^{1/2} \tag{5.15}$$
$$+ K \left\{ \gamma^{-1} \left\{ \int |x|^{-2\gamma} |\mathrm{div}(\alpha u)|^2 \, dx \right\}^{1/2} \right.$$
$$\left. + \left\{ \int |x|^{-2\gamma-2} |u|^2 \, dx \right\}^{1/2} \right\}$$

if $\dot{U}$ is sufficiently small.

From Theorem 5.1 and Lemma 5.3, we shall show that the function $u \in \{C^1(U)\}^3$ satisfying the system of differential inequalities (5.5) vanishes exponentially at the origin if it vanishes of infinite order at the origin.

Proposition 5.4. *Let $u \in \{H^1_{\mathrm{loc}}(U)\}^3$ satisfy the system (5.5) and vanish of* *infinite order at the origin. Suppose that $D_0 + D_1 + D_2 < 1/2$. Then, there exist positive constants c_0 and C such that*

$$\int_{|x|\le r} (|u|^2 + |\nabla u|^2)\, dx \le C \exp(-c_0 r^{-\delta})$$

for any small positive r.

Proof. Let $\chi(r)$ be a nonnegative function belonging to $C_0^1((-\infty, 2))$ such that $\chi(r) = 1$ when $r < 1$. Set $\tilde{u}(x) = \chi(M\gamma^{1/\delta}|x|)u(x)$. Here, M is a large positive parameter, which will be defined later. Note that

$$[\mathrm{curl}, \chi(M\gamma^{1/\delta}|x|)]u = \mathcal{O}(M\gamma^{1/\delta})u \quad \text{and} \quad [\mathrm{div}, \chi(M\gamma^{1/\delta}|x|)]u = \mathcal{O}(M\gamma^{1/\delta})u.$$

We can apply Theorem 5.1 and Lemma 5.3 for $u \in H^1_{0,\infty}(U)$. To verify this, we apply them to $(1 - \chi(jx))\tilde{u}$, which belongs to $H_0^1(K)$ with a compact subset of $\dot{U}$. In view of (2.4) and (5.5), if we let $j \to \infty$ in the obtained inequality, we arrive at the conclusion because for each γ,

$$\lim_{j\to\infty} \int |x|^{-2\gamma} |[\nabla, (1 - \chi(jx))]\tilde{u}|^2\, dx = 0$$

and

$$\lim_{j\to\infty} \int |x|^{-2\gamma} \{|\mathrm{curl}(1 - \chi(jx))\tilde{u}|^2 + |\mathrm{div}(1 - \chi(jx))\tilde{u}|^2\}\, dx = \int |x|^{-2\gamma} \{|\mathrm{curl}\, \tilde{u}|^2 + |\mathrm{div}\, \tilde{u}|^2\}\, dx. \tag{5.16}$$

Since $\|J\| \le 1$, Theorem 5.1 and Lemma 5.3 imply that for any positive N,

$$\beta\left\{\int |x|^{-2\gamma-2}|\tilde{u}|^2\, dx\right\}^{1/2} + \gamma^{-1}N^{-1}\left\{\int |x|^{-2\gamma}|\nabla\tilde{u}|^2\, dx\right\}^{1/2} \le C\left\{\int |x|^{-2\gamma+2\delta}(|\nabla\tilde{u}|^2 + |\tilde{u}|^2)\, dx\right\}^{1/2} + C'M\gamma^{1/\delta}\left\{\int_{2\ge M\gamma^\delta|x|\ge 1} |x|^{-2\gamma}|u|^2\, dx\right\}^{1/2}, \tag{5.17}$$

where

$$\beta = \frac{1}{2} - (D_1 + D_2)(1 + K(N\gamma)^{-1}) - D_0 - \frac{K}{N}.$$

From our assumptions, it holds that β is positive if we take N and γ to be large enough.

Now, we choose M large enough such that $CM^{-2\delta} < 1/(4N)$, which implies that

$$C|x|^{2\delta} \le \gamma^{-2}/(2N), \quad \text{if } x \in \operatorname{supp} \chi(M\gamma^{1/\delta}|x|).$$

If we take γ to be large so that

$$\sup_{|x| \le 2/(M\gamma^{\delta})} (C|x|^{\delta+1}) \le \beta/2,$$

then (5.17) implies

$$(5.18) \qquad \frac{\beta}{2}\int_{B(2/(M\gamma^{1/\delta}))} |x|^{-2\gamma-2}|u|^2\,dx + (2\gamma^2 N)^{-1}\int_{B(2/(M\gamma^{1/\delta}))} |x|^{-2\gamma}|\nabla u|^2\,dx$$
$$\le C'M\gamma^{1/\delta}\int_{U\setminus B(1/(M\gamma^{1/\delta}))} |x|^{-2\gamma-2}|u|^2\,dx.$$

From this inequality, we find that

$$(5.19) \qquad (2M\gamma^{1/\delta})^{2\gamma}\int_{B(1/(2M\gamma^{1/\delta}))} \left\{\frac{\beta}{2}|u|^2 + (2\gamma^2 N)^{-1}|\nabla u|^2\right\}dx$$
$$\le C'M(M\gamma^{1/\delta})^{2\gamma+2}\int_{U\setminus B(1/(M\gamma^{1/\delta}))} |u|^2\,dx.$$

As a result, we obtain

$$\int_{B(1/(2M\gamma^{1/\delta}))} \{|u|^2 + (2\gamma^2 N)^{-1}|\nabla u|^2\}\,dx \le C'M^3\gamma^{3/\delta}2^{-2\gamma}\int_U |u|^2\,dx$$

for any large positive $\gamma \in \boldsymbol{N}$. This leads to the desired estimate in Proposition 5.4. $\square$

Proposition 5.4 enables us to choose a more singular weight with which a Carleman inequality holds.

Theorem 5.5. *Suppose that* $\alpha(0) = I$ *and* (2.2). *There exist a positive constant* C *and a sufficiently small neighborhood* U *of the origin such that*

$$(5.20) \qquad \gamma\int |x|^{-2}|\log|x||e^{\gamma(\log|x|)^2}|u|^2|x|^{-2}\,dx$$
$$\le C\int e^{\gamma(\log|x|)^2}|\operatorname{curl} u|^2|x|^{-2}\,dx + \int e^{\gamma(\log|x|)^2}|\operatorname{div}(\alpha u)|^2|x|^{-2}\,dx$$

for any $u \in \{C_0^\infty(\dot{U})\}^3$ *and any large* $\gamma > 0$.

To prove this theorem, we require several preparations.

Instead of the differential operator ∂_r in the radial direction, we shall use a scalar operator Q defined as

$$Qu = (\alpha\omega, \omega)^{-1}(\omega, \alpha\partial_x)u.$$

We decompose $\operatorname{curl} u$ into its radial and angular parts:

$$\operatorname{curl} u = \omega \wedge Qu + \tilde{L}u, \quad \tilde{L}u = \nabla \wedge u - \omega \wedge Qu,$$

which has been firstly used in [10]. By a direct calculation, it is easily verified that for any $f \in C^1([0, \infty))$, $u \in C^1(U)$ and $\psi \in C^1(\boldsymbol{S}^2)$, we have

$$Qf(r) = f'(r), \quad \tilde{L}(f(r)u) = f(r)\tilde{L}u \tag{5.21}$$

and

$$Q(\psi(\omega)u) = \mathcal{O}(r^{-1+\delta})u \tag{5.22}$$

because

$$Qu - \partial_r = \mathcal{O}(|x|^{\delta})|\nabla u|.$$

To make the subsequent equations simpler, we shall prepare several notations. For each $\xi \in \boldsymbol{R}^3$, we define the matrix J_ξ as

$$J_\xi u = \xi \wedge u \quad \text{and} \quad J_\nabla u = \nabla \wedge u.$$

$R_{-1}[u, v]$ denotes a function satisfying that for every small $\rho > 0$, there exists a small neighborhood U of the origin such that

$$|R_{-1}[u, v](x)| \leq \frac{\rho}{|x|}|u|\,|v| \quad x \in \dot{U}.$$

Furthermore, if a vector-valued function $R(x)$ satisfies that for every small $\eta > 0$, there exists a neighborhood V of the origin such that

$$|R(x)| \leq \eta|h(x)|\,|g(x)|, \quad \forall x \in \dot{V},$$

then, we write

$$R(x) = o(h(x))g(x).$$

Finally, we denote the inner product and the norm of $L^2(\boldsymbol{S}^2)^3$ by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The next lemma is an analogue to Lemma 3.2 and Lemma 3.3 in [**10**].

Lemma 5.6. *Suppose that $\alpha(0) = I$, (2.3) holds. For any $u, v \in C_0^1(U)$, $f, g \in \{C_0^1(U)\}^3$ and $h \in \{C_0^2(U)\}^3$, the following three statements hold.*

i)

$$\int \langle Qu, v\rangle\, dr = -\int \langle u, Qv\rangle\, dr - 2\int \langle u, r^{-1}v\rangle\, dr + \int R_{-1}[u, v]\, dx. \tag{5.23}$$

ii)

$$\int \langle \tilde{L}f, g\rangle\, dr = \int \langle f, \tilde{L}g\rangle\, dr - \int \langle f, 2r^{-1}\omega \wedge g\rangle\, dr + \int R_{-1}[f, g]\, dx. \tag{5.24}$$

iii)

$$Q\tilde{L}h = \tilde{L}Qh - r^{-1}\tilde{L}h + \mathcal{O}(|x|^{\delta-1})Qh + o(r^{-1})|\nabla h| + o(|x|^{-2})h. \tag{5.25}$$

Proof. Let $\kappa(x) = (\alpha(x)\omega, \omega)$. Since

$$\kappa^{-1}(x) - 1 = \mathcal{O}(|x|^{\delta}),$$

it holds that

$$\int \langle Qu, v\rangle\, dr = \int \langle u, Q^{*}v\rangle\, dr,$$

where

$$\begin{aligned} Q^{*}v = {} & -Qv + \{(\omega, \nabla)\kappa\}v - \sum_{j=1}^{3} \partial_j(\alpha\omega)_{(j)}v \\ & + \mathcal{O}(|x|^{\delta})\{(\omega, \nabla)\kappa^{-1}\}v + \mathcal{O}(|x|^{\delta}) \sum_{j=1}^{3} \partial_j(\alpha\omega)_{(j)}v. \end{aligned} \tag{5.26}$$

In view of

$$-\operatorname{div}\omega = -2r^{-1},$$

the identity (5.26) implies the statement i). The statement ii) follows from the definition of $\tilde{L}$ and i). To prove iii), we see that

$$\begin{aligned} Q(\omega \wedge h) &= \omega \wedge Qh + (Q\omega) \wedge h \\ Q^{2}(\omega \wedge h) &= Q(\omega \wedge Qh) + (Q\omega) \wedge Qh + (Q^{2}\omega) \wedge h. \end{aligned} \tag{5.27}$$

We note that

$$Q\omega = \mathcal{O}(|x|^{\delta-1}), \quad Q^{2}\omega = o(|x|^{-2}).$$

In addition,

$$\begin{aligned} \partial_{x_k} Qh_{\ell} &= Q\partial_{x_k} h_{\ell} - r^{-1}\omega_k Qh_{\ell} + r^{-1}\kappa^{-1} \sum_{j=1}^{3} \alpha_{k,j}\partial_{x_j} h_{\ell} + o(r^{-1})|\nabla h| \\ &= -r^{-1}\omega_k Qh_{\ell} + r^{-1}\partial_{x_k} h_{\ell} + o|\nabla h|. \end{aligned} \tag{5.28}$$

It follows from the last identity that

$$\nabla \wedge Qh = Q(\nabla \wedge h) + r^{-1}(\nabla \wedge h - Q(\omega \wedge h)) + o(r^{-1})|\nabla h| + o(r^{-2})|h|. \tag{5.29}$$

Subtracting (5.27) from (5.29), we obtain iii). □

Corollary 5.7. *Under the same assumption as in Lemma 5.6, it holds that*

$$\int (Qu, v)|x|^{-2}\,dx = -\int (u, Qv)|x|^{-2}\,dx + \int R_{-1}[u, v]|x|^{-2}\,dx. \tag{5.30}$$

The next is essentially obtained in [**10**] (cf. Lemma 3.4). We modify it in our setting and give its proof because this identity is crucial in proving Theorem 5.5.

Lemma 5.8. *Suppose the same assumption as in the previous lemma holds. For $g \in \{C^1(\dot{U})\}^3$, we have*

$$\tilde{L}((\alpha\omega) \wedge g) = (\operatorname{div} g)\alpha\omega - (Q(\omega, g))\alpha\omega - r^{-1}(\omega, g)\alpha\omega - r^{-1}g + o(r^{-1})g.$$

Proof. For proving Lemma 5.8, we shall use the vector rule

$$f \wedge (g \wedge h) = (f, h)g - (f, g)h. \tag{5.31}$$

We observe that

$$(g, \partial_x)(\alpha\omega) = r^{-1}\alpha g - r^{-1}(\omega, g)\alpha\omega + o(r^{-1})g, \quad \omega = x/|x|$$

and

$$\operatorname{div}(\alpha\omega) = 2r^{-1} + o(r^{-1}), \quad Q(\alpha\omega) = o(r^{-1}).$$

For the sake of these equations, the relation (5.31) implies that

$$\nabla \wedge (\alpha\omega \wedge g) = (\operatorname{div} g, \alpha\omega) - (\alpha\omega, \omega)Qg - r^{-1}g - r^{-1}(\omega, g)\alpha\omega + o(r^{-1})g \tag{5.32}$$

and

$$Q(\omega \wedge (\alpha\omega) \wedge g) = (Q(\omega, g))\alpha\omega - (\alpha\omega, \omega)Qg + o(r^{-1})g. \tag{5.33}$$

Subtracting (5.33) from (5.32), we arrive at the conclusion of Lemma 5.8. □

From Lemma 5.8, we can derive an analogue to the canonical form of Proposition 4.4.

Proposition 5.9.

$$\begin{aligned} \alpha J_{\alpha\omega} \operatorname{curl} u = {} & -Q(\alpha u) + \{\alpha J_{\alpha\omega}\tilde{L} - \tilde{L}J_{\alpha\omega}\alpha\}u + (\operatorname{div} \alpha u)\alpha\omega \\ & + (\alpha\omega, Qu)(\alpha - 1)\alpha\omega - r^{-1}u + J_{\alpha\omega}J_{(1-\alpha)\omega}Q\alpha u \\ & - r^{-1}(\omega, u)\omega + o(r^{-1})u. \end{aligned} \tag{5.34}$$

PROOF. Since

$$\mathrm{curl} = Q(J_\omega u) - \tilde{L}u = J_\omega Qu - \tilde{L}u + \mathcal{O}(r^{\delta-1})u,$$

and

$$J_\xi^2 u = -|\xi|^2 u + (\xi, u)\xi, \quad \forall \xi \in \boldsymbol{R}^3,$$

it follows that

$$\begin{aligned} (5.35)\qquad J_{\alpha\omega}\,\mathrm{curl}\,u &= J_{\alpha\omega}^2 Qu + J_{\alpha\omega}\tilde{L}u + J_{\alpha\omega}J_{(1-\alpha)\omega}Qu + o(r^{-1})u \\ &= -|\alpha\omega|^2 Qu + (\alpha\omega, Qu)(\alpha\omega) + J_{\alpha\omega}\tilde{L}u + J_{\alpha\omega}J_{(1-\alpha)\omega}Qu + o(r^{-1})u. \end{aligned}$$

On the other hand, Lemma 5.8 and ii) of Lemma 5.6 imply

$$(5.36)\qquad -\tilde{L}J_{\alpha\omega}\alpha u = -(\mathrm{div}\,\alpha u)\alpha\omega + (Q(\omega, \alpha u))\alpha\omega - r^{-1}u - (\omega, \alpha u)\alpha\omega + o(r^{-1})u.$$

Combining (5.35) with (5.36), we arrive at the conclusion. □

Now, we are in a position to prove Theorem 5.5. Define

$$\tilde{G} = \alpha J_{\alpha\omega}(\tilde{L} - r^{-1}J_\omega) - (\tilde{L} - r^{-1}J_\omega)J_{\alpha\omega}\alpha$$

and

$$R_Q u = (\alpha - 1)Qu + (\alpha\omega, Qu)(\alpha - 1)\alpha\omega + J_{\alpha\omega}J_{(1-\alpha)\omega}Q\alpha u.$$

Then, it holds that

$$(5.37)\qquad \alpha J_{\alpha\omega}\,\mathrm{curl}\,u = -Qu + \tilde{G}u + (\mathrm{div}\,\alpha u)\alpha\omega + R_Q u + \mathcal{O}(r^{-1})u,$$

$$\int \langle \tilde{G}u, v\rangle r^{-2}\,dr = \int \langle u, \tilde{G}v\rangle r^{-2}\,dr + \int o(r^{-1})\|u\|\,\|v\|r^{-2}\,dr$$

and

$$\int \langle Qu, v\rangle r^{-2}\,dr = -\int \langle u, Qv\rangle r^{-2}\,dr + \int o(r^{-1})\|u\|\,\|v\|r^{-2}\,dr.$$

For $v = e^{\varphi(r)}u$ and $\varphi' = (d/dr)\varphi(r)$, we see that

$$e^{\varphi(r)}Qu = (Q - \varphi')v, \quad e^{\varphi(r)}\tilde{L}u = \tilde{L}v$$

and

$$(5.38)\qquad e^{\varphi(r)}R_Q u = R_Q v - \varphi'\{(\alpha - 1)v + (\alpha\omega, v)(\alpha - 1)\alpha\omega + J_{\alpha\omega}J_{(1-\alpha)\omega}\alpha v\}.$$

Defining

$$Sv = (\alpha - 1)v + (\alpha\omega, v)(\alpha - 1)\alpha\omega + J_{\alpha\omega}J_{(1-\alpha)\omega}\alpha v,$$

we obtain

$$e^{\varphi(r)}\alpha J_{\alpha\omega}\,\mathrm{curl} = -(Q - \varphi')v + \tilde{G}v + e^{\varphi(r)}(\mathrm{div}\,\alpha u)\alpha\omega + R_Q v + \varphi' Sv + \mathcal{O}(r^{-1})v.$$

We write the right hand side of the last identity as the sum of the following two terms:

$$P_\varphi^{(1)}v = -(Q - \varphi')v + \tilde{G}v + \varphi' Sv$$

and

$$P_\varphi^{(2)}v = e^{\varphi(r)}(\operatorname{div}\alpha u)\alpha\omega + R_Q v + \mathcal{O}(|x|^{-1})v.$$

It follows that

$$\int |e^{\varphi(r)}\alpha J_{\alpha\omega}\operatorname{curl} u|^2|x|^{-2}\,dx \geq \frac{1}{2}\int |P_\varphi^{(1)}v|^2|x|^{-2}\,dx - \int |P_\varphi^{(2)}v|^2|x|^{-2}\,dx, \tag{5.39}$$

$$\begin{aligned}\int |P_\varphi^{(1)}v|^2|x|^{-2}\,dx = \int |Qv|^2|x|^{-2}\,dx + \int |\tilde{G}v - \varphi' v + \varphi' Sv|^2|x|^{-2}\,dx \\ - 2\operatorname{Re}\int (Qv, \tilde{G}v + \varphi' v + \varphi' Sv)|x|^{-2}\,dx\end{aligned} \tag{5.40}$$

and

$$\begin{aligned}\int |P_\varphi^{(2)}v|^2|x|^{-2}\,dx \leq 2\int e^{2\varphi(r)}|\operatorname{div}\alpha u|^2|x|^{-2}\,dx \\ + C\Big\{\int (\varphi')^2|x|^{2\delta}|v|^2|x|^{-2}\,dx + \int |x|^{2\delta}|Qv|^2\,dx \\ + \int |x|^{-2}|v|^2|x|^{-2}\,dx\Big\}.\end{aligned} \tag{5.41}$$

In order to estimate the third term in the last expression, we shall utilize Lemma 5.6.

Lemma 5.10.

$$\begin{aligned}2\operatorname{Re}\int (Qv, \tilde{G}v)|x|^{-2}\,dx = \int o(|x|^{-1})|\nabla v|\,|v|\,|x|^{-2}\,dx \\ + \int o(|x|^{-1})|Qv|\,|v|\,dx + \int \mathcal{O}(|x|^{-2})|v|^2|x|^{-2}\,dx.\end{aligned} \tag{5.42}$$

Proof. Let $\hat{L} = \tilde{L} - r^{-1}J_{\alpha\omega}$. Then, we note that the statement iii) of Lemma 5.6 is still valid for $\hat{L}$. By Lemma 5.6 and Corollary 5.7, it holds that

$$\begin{aligned}&\int (Qv, \alpha J_{\alpha\omega}\hat{L}v)|x|^{-2}\,dx \\ &= -\int (v, o(|x|^{-1})\hat{L}v)|x|^{-2}\,dx - \int (v, \alpha J_{\alpha\omega}Q\hat{L}v)|x|^{-2}\,dx \\ &= -\int (v, \alpha J_{\alpha\omega}\hat{L}Qv)|x|^{-2}\,dx + \int (|x|^{-1}v, \alpha J_{\alpha\omega}\hat{L}v)|x|^{-2}\,dx \\ &\quad - \int \mathcal{O}(|x|^{\delta-1})|v|\,|Qv|\,|x|^{-2}\,dx - \int o(|x|^{-1})|v|\,|\nabla v|\,|x|^{-2}\,dx \\ &\quad - \int o(|x|^{-2})|v|^2|x|^{-2}\,dx\end{aligned} \tag{5.43}$$

and

$$-\int (v, \alpha J_{\alpha\omega}\hat{L}Qv)|x|^{-2}\,dx = \int (\hat{L}J_{\alpha\omega}\alpha v, Qv)|x|^{-2}\,dx + \int o(|x|^{-1})|v|\,|Qv|\,|x|^{-2}\,dx.$$

Likewise,

$$\begin{aligned}(5.44)\qquad & \int (Qv, -\hat{L}J_{\alpha\omega}v\alpha)|x|^{-2}\,dx \\ &= -\int (\hat{L}Qv, J_{\alpha\omega}v\alpha)|x|^{-2}\,dx - \int (o(|x|^{-1})Qv, v)|x|^{-2}\,dx \\ &= -\int (Q\hat{L}v, J_{\alpha\omega}\alpha v)|x|^{-2}\,dx + \int (|x|^{-1}\hat{L}v, J_{\alpha\omega}\alpha v)|x|^{-2}\,dx \\ &\quad - \int \mathcal{O}(|x|^{\delta-1})|v|\,|Qv|\,|x|^{-2}\,dx - \int o(|x|^{-1})|v|\,|\nabla v|\,|x|^{-2}\,dx \\ &\quad - \int \mathcal{O}(|x|^{-2})|v|^2|x|^{-2}\,dx \\ &= \int (\alpha J_{\alpha\omega}\hat{L}v, Qv)|x|^{-2}\,dx + \int (v, |x|^{-1}\hat{L}J_{\alpha\omega}\alpha v)|x|^{-2}\,dx \\ &\quad - \int \mathcal{O}(|x|^{\delta-1})|v|\,|Qv|\,|x|^{-2}\,dx - \int o(|x|^{-1})|v|\,|\nabla v|\,|x|^{-2}\,dx \\ &\quad - \int o(|x|^{-2})|v|^2|x|^{-2}\,dx.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}(5.45)\qquad & 2\,\mathrm{Re}\int (Qv, \tilde{G}v)|x|^{-2}\,dx \\ &= \int o(|x|^{-1})|\nabla v|\,|v|\,|x|^{-2}\,dx + \int (\tilde{G}v, Qv)|x|^{-2}\,dx - \int (v, Q\tilde{G}v)|x|^{-2}\,dx \\ &= \int (|x|^{-1}v, \alpha J_{\alpha\omega}\hat{L}v)|x|^{-2}\,dx - \int (|x|^{-1}v, \hat{L}J_{\alpha\omega}\alpha v)|x|^{-2}\,dx \\ &\quad + \int o(|x|^{-1})|\nabla v|\,|v|\,|x|^{-2}\,dx + \int o(|x|^{-1})|Qv|\,|v|\,|x|^{-2}\,dx \\ &\quad + \int o(|x|^{-2})|v|^2|x|^{-2}\,dx.\end{aligned}$$

Since $\alpha(0) = 1$ and the Hölder continuity of α imply that

$$\alpha J_{\alpha\omega} - J_{\alpha\omega}\alpha = \mathcal{O}(|x|^{\delta}),$$

we note that

$$(5.46)\qquad -\int (|x|^{-1}v, \hat{L}J_{\alpha\omega}\alpha v)|x|^{-2}\,dx + \int (|x|^{-1}v, \alpha J_{\alpha\omega}\hat{L}v)|x|^{-2}\,dx$$
$$= \int \mathcal{O}(|x|^{\delta-1})|v|\,|\nabla v|\,|x|^{-2}\,dx + \int o(|x|^{-2})|v|^2|x|^{-2}\,dx.$$

As a result, we arrive at the assertion of Lemma 5.10. □

Similarly,

Lemma 5.11.

$$(5.47)\qquad 2\,\mathrm{Re}\int (Qv, -\varphi' v - \varphi' Sv)|x|^{-2}\,dx \geq \int (1 - \mathcal{O}(|x|^{\delta}))\varphi''|v|^2|x|^{-2}\,dx$$
$$- \int o(|x|^{-1})\varphi'|v|^2|x|^{-2}\,dx - \int o(|x|^{-1})|v|^2|x|^{-2}\,dx.$$

Proof. We see that

$$(5.48)\qquad -2\,\mathrm{Re}\int (Qv, \varphi' v)|x|^{-2}\,dx \geq \int \varphi''|v|^2|x|^{-2}\,dx - \int o(|x|^{-1})|\varphi'|\,|v|^2|x|^{-2}\,dx.$$

Since $Sv = \mathcal{O}(|x|^{\delta})v$ and $Qv - \partial_r v = \mathcal{O}(|x|^{\delta})|\nabla v|$, it holds that

$$(5.49)\qquad 2\,\mathrm{Re}\int (Qv, \varphi' Sv)|x|^{-2}\,dx = -\int (v, \varphi'' Sv)|x|^{-2}\,dx - \int (v, \varphi' QSv)|x|^{-2}\,dx$$
$$+ \int \mathcal{O}(|x|^{\delta-1})|v|\,|\varphi' v|\,|x|^{-2}\,dx$$

and

$$(5.50)\qquad QSv = SQv + (\partial_r S)v + \mathcal{O}(|x|^{\delta-1})v.$$

From the definition, it follows that

$$(5.51)\qquad (\partial_r S)v = (\partial_r \alpha)v + (\alpha\omega, v)(\partial_r \alpha)\alpha\omega - J_{\alpha\omega}J_{(\partial_r\alpha)\omega}\alpha v + \mathcal{O}(|x|^{\delta-1})|v|$$
$$= (\partial_r \alpha)v + (\omega, v)(\partial_r \alpha)\omega - J_{\omega}J_{(\partial_r\alpha)\omega}v + \mathcal{O}(|x|^{\delta-1})|v|.$$

Thus, we obtain

$$(5.52)\qquad |(\partial_r S)v| \leq 3\|\partial_r \alpha\|\,|v| + \mathcal{O}(|x|^{\delta-1})|v|.$$

From (5.49)–(5.52), it holds that

$$(5.53)\quad 2\,\mathrm{Re}\int (Qv, \varphi' Sv)|x|^{-2}\,dx \geq -\int o(r^{-1})|\varphi'|\,|v|^2|x|^{-2}\,dx \\ -\int \mathcal{O}(|x|^{\delta})\varphi''|v|^2|x|^{-2}\,dx - \int \mathcal{O}(|x|^{\delta-1})|v|^2|x|^{-2}\,dx.$$

From (5.47) and (5.53), we arrive at the conclusion of Lemma 5.11. □

Combining Lemma 5.10 with Lemma 5.11, we can conclude that

$$(5.54)\quad -2\,\mathrm{Re}\int (Qv, \tilde{G}v + \varphi' v + \varphi' Sv)|x|^{-2}\,dx \\ \geq \int \{\varphi''(1-\mathcal{O}(|x|^{\delta})) - o(r^{-1})\varphi'\}|v|^2|x|^{-2}\,dx \\ -\int o(|x|^{-1})|\nabla v|\,|v|\,|x|^{-2}\,dx - \int o(|x|^{-2})|v|^2|x|^{-2}\,dx.$$

We choose $\varphi(r) = \gamma(\log r)^2/2$. Then,

$$\varphi'' = \gamma r^{-2}|\log r|(1+|\log r|^{-1}), \quad r^{-1}\varphi' = \gamma r^{-2}\log r.$$

If we use the inequality $ab \leq \{\gamma a^2 + \gamma^{-1}b^2\}/2$, it holds that

$$r^{-1}|v|\,|\nabla u| \leq \frac{1}{2}\{\gamma r^{-1}|\log r|\,|v|^2 + \gamma^{-1}|\log r|^{-1}|\nabla v|^2\}, \\ r^{-1}|v|\,|Qv| \leq r^{-2}|\log r|\,|v|^2 + |\log r|^{-1}|Qv|^2.$$

In view of (5.39), (5.40), (5.41) and (5.54), we see that for every small $\theta > 0$, there exist a small neighborhood U of the origin and a positive constant C such that

$$(5.55)\quad \frac{\gamma}{2}(1 - D_0 - 4D_3)\int |x|^{-2}\big|\log|x|\big|\,|v|^2|x|^{-2}\,dx + \frac{1}{3}\int |Qv|^2|x|^{-2}\,dx \\ \leq C\left\{\int e^{2\varphi}|\alpha\,\mathrm{curl}\,u|^2|x|^{-2}\,dx + \int e^{2\varphi}|\mathrm{div}(\alpha u)|^2|x|^{-2}\,dx\right\} \\ + \theta\gamma^{-1}\sum_{j=1}^{3}\int \big|\log|x|\big|^{-1}|\partial_j v|^2|x|^{-2}\,dx$$

for all $v \in C_0^{\infty}(U)$. The terms with the first order derivatives in the right hand side of the last estimate are harmless. To see this, we shall use Lemma 5.2 and the fact that

$$(5.56)\qquad \left|\log|x|\right|^{-1/2}\nabla(e^{\varphi}u) = \nabla(\left|\log|x|\right|^{-1/2}v) + [\nabla, \left|\log|x|\right|^{-1/2}]e^{\varphi}u,$$
$$|[\nabla, \left|\log|x|\right|^{-1/2}]e^{\varphi}u| \le C|x|^{-1}\left|\log|x|\right|^{-3/2}|v|,$$

$$(5.57)\qquad \operatorname{curl}(\left|\log|x|\right|^{-1/2}v)$$
$$= \left|\log|x|\right|^{-1/2}\operatorname{curl} v + \mathcal{O}(|x|^{-1}\left|\log|x|\right|^{-3/2})|v|$$
$$= \left|\log|x|\right|^{-1/2}e^{\varphi}\operatorname{curl} u + \gamma\mathcal{O}(|x|^{-1}\left|\log|x|\right|^{1/2})v + \mathcal{O}(|x|^{-1}\left|\log|x|\right|^{-3/2})v$$

and

$$(5.58)\qquad \operatorname{div}(\left|\log|x|\right|^{-1/2}\alpha v)$$
$$= \left|\log|x|\right|^{-1/2}\operatorname{div}\alpha v + \mathcal{O}(|x|^{-1}\left|\log|x|\right|^{-3/2})|v|$$
$$= \left|\log|x|\right|^{-1/2}e^{\varphi}\operatorname{div}\alpha u + \gamma\mathcal{O}(|x|^{-1}\left|\log|x|\right|^{1/2})v + \mathcal{O}(|x|^{-1}\left|\log|x|\right|^{-3/2})v.$$

Therefore, applying the identity (5.13) to $f = \left|\log|x|\right|^{-1/2}e^{\varphi}u$, we see that for any $u \in \{C_0^1(\dot{U})\}^3$,

$$(5.59)\qquad \gamma^{-1}\sum_{j=1}^{3}\int \left|\log|x|\right|^{-1}|\partial_j v|^2|x|^{-2}\,dx$$
$$\le C\left\{\gamma^{-1}\int \left|\log|x|\right|^{-1}|e^{2\varphi}(|\operatorname{div}(\alpha u)|^2 + |\operatorname{curl} u|^2)|x|^{-2}\,dx\right\}$$
$$+ C\gamma\int \left|\log|x|\right|\,|x|^{-2}|v|^2|x|^{-2}\,dx.$$

Combining (5.55) with (5.59), we finally conclude that there exists a positive constant C such that if U is small enough,

$$\int e^{2\varphi}(|\operatorname{div}(\alpha u)|^2 + |\operatorname{curl} u|^2)|x|^{-2}\,dx \ge C\gamma\int \left|\log|x|\right|\,|x|^{-2}e^{2\varphi}|u|^2|x|^{-2}\,dx$$

for any $u \in \{C_0^1(\dot{U})\}^3$. This completes the proof of Theorem 5.5. □

6. Proof of Theorem 2.1.

With aid of Proposition 5.4, Theorem 2.1 follows from Theorem 5.5 by use of the standard procedure. Indeed, let $u \in \{H_{\mathrm{loc}}^1(U)\}^3$ satisfy (2.1) and vanish of infinite order at the origin. and χ be the same cut-off function as in Section 4.3.

In view of Proposition 5.4, we can apply Theorem 5.5 to $\tilde{u} = \chi(x)u(x)$. It follows that

$$\gamma \int_{|x| \le r_0/4} |x|^{-2} |\log|x|| e^{2\varphi} |u|^2 |x|^{-2}\, dx \le C \int_{|x| \ge r_0/3} e^{2\varphi} |\tilde{u}|^2 |x|^{-2}\, dx.$$

Since $(\log r)^2$ is a strictly decreasing function, we have

$$\gamma (r_0/4)^{-4} |\log r_0/4| \int |u|^2\, dx \le C' e^{\gamma\{(\log(r_0/3))^2 - (\log(r_0/4))^2\}}.$$

Thus, letting $\gamma \to \infty$, we conclude that $u = 0$ in $B_{r_0/4}(0)$. This completes the proof of Theorem 2.1. □

7. Proof of Corollary 2.2.

In this section we shall prove Corollary 2.2 and keep the same notation as in Section 4.3 except for the cut-off function. Since $\lambda_0 \varepsilon(0) = \mu(0)$, by using the mapping

$$\tilde{E} = \lambda_0^{-1/2} E, \quad \tilde{H} = H,$$

we see that

$$-\operatorname{curl} \tilde{E} = i\lambda \lambda_0^{-1/2} \mu \tilde{H}, \quad \operatorname{curl} \tilde{H} = i\lambda \lambda_0^{1/2} \varepsilon \tilde{E}.$$

Therefore, without loss of generality, we may assume that

$$\mu(0) = \varepsilon(0).$$

As in the beginning of Section 5, we can transform them into $\varepsilon(0) = \mu(0) = I$ simultaneously. Let $\chi(r)$ be a nonnegative function belonging to $C_0^1((-\infty, 2))$ such that $\chi(r) = 1$ when $r < 1$. Set $\chi_\gamma = \chi(M\gamma^{1/\delta}|x|)$. Here, M is a large positive parameter, which will play the same role in the proof of Proposition 5.4. Applying Theorem 5.1 to $\chi_\gamma E$, we see that for every small positive number ρ, if we shrink U enough, there exists a positive constant C independent of γ such that

$$\begin{aligned}(7.1) \qquad & \left(\frac{1}{2} - M_2 - \rho\right) \left\{ \int_U |x|^{-2\gamma-2} |\chi_\gamma E|^2\, dx \right\}^{1/2} \\ & \le \rho \left\{ \int_U |x|^{-2\gamma-2} |\chi_\gamma H|^2\, dx \right\}^{1/2} + CM\gamma^{1/\delta} \left\{ \int_{U \setminus B(1/M\gamma^\delta)} |x|^{-2\gamma} |E|^2\, dx \right\}^{1/2} \\ & \quad + C \left\{ \int |x|^{-2\gamma+2\delta} (|\nabla_x \chi_\gamma E|^2 + |\chi_\gamma E|^2)\, dx \right\}^{1/2}.\end{aligned}$$

Similarly, an application of Theorem 5.1 to $\chi_\gamma H$ implies

$$(7.2)\qquad \left(\frac{1}{2} - M_1 - \rho\right)\left\{\int_U |x|^{-2\gamma-2}|\chi_\gamma H|^2\,dx\right\}^{1/2}$$

$$\le \rho\left\{\int_U |x|^{-2\gamma-2}|\chi_\gamma E|^2\,dx\right\}^{1/2} + CM\gamma^{1/\delta}\left\{\int_{U\setminus B(1/M\gamma^\delta)} |x|^{-2\gamma}|H|^2\,dx\right\}^{1/2}$$

$$+ C\left\{\int |x|^{-2\gamma+2\delta}(|\nabla_x\chi_\gamma H|^2 + |\chi_\gamma H|^2)\,dx\right\}^{1/2}.$$

If we take the sum of (7.1) and (7.2), it follows that

$$(7.3)\qquad \rho_1\left\{\int_U |x|^{-2\gamma-2}|\chi_\gamma E|^2\,dx\right\}^{1/2} + \rho_2\left\{\int_U |x|^{-2\gamma-2}|\chi_\gamma H|^2\,dx\right\}^{1/2}$$

$$\le CM\gamma^{1/\delta}\left\{\int_{U\setminus B(1/M\gamma^\delta)} |x|^{-2\gamma}|E|^2\,dx\right\}^{1/2}$$

$$+ CM\gamma^{1/\delta}\left\{\int_{U\setminus B(1/M\gamma^\delta)} |x|^{-2\gamma}|H|^2\,dx\right\}^{1/2}$$

$$+ C\left\{\int |x|^{-2\gamma+2\delta}(|\nabla_x\chi_\gamma E|^2 + |\chi_\gamma E|^2)\,dx\right\}^{1/2}$$

$$+ C\left\{\int |x|^{-2\gamma+2\delta}(|\nabla_x\chi_\gamma H|^2 + |\chi_\gamma H|^2)\,dx\right\}^{1/2},$$

where

$$\rho_1 = \left(\frac{1}{2} - M_2 - \rho - \rho K_1\right) > 0, \quad \rho_2 = \left(\frac{1}{2} - M_1 - \rho - \rho K_2\right) > 0.$$

Deducing in the same manner as in the proof of Proposition 5.1, we conclude

Proposition 7.1. *If* $\max(M_1, M_2) < 1/2$, *then there exist positive constants* c_0 *and* C *such that*

$$\int_{B(r)} (|E|^2 + |H|^2)\,dx \le C\exp(-c_0 r^{-\delta})$$

for all small positive number r.

Now, we are in the final stage for proving Theorem 2.1. Let $\Phi \in C_0^\infty(U)$,

$0 \le \Phi \le 1$ be a cut-off function such that $\Phi(x) = 1$ if $|x| \le r_0/3$ and $\Phi = 0$ if $|x| \ge r_0/2$. If we apply Theorem 5.5 to ΦE and ΦH, it follows that

$$\gamma \int |x|^{-2} |\log|x|| e^{\gamma(\log|x|)^2} |\Phi E|^2 |x|^{-2}\, dx \le C \int e^{\gamma(\log|x|)^2} |\Phi \mu H|^2 |x|^{-2}\, dx + \int_{U \setminus B(r_0/3)} e^{\gamma(\log|x|)^2} |E|^2 |x|^{-2}\, dx \tag{7.4}$$

and

$$\gamma \int |x|^{-2} |\log|x|| e^{\gamma(\log|x|)^2} |\Phi H|^2 |x|^{-2}\, dx \le C \int e^{\gamma(\log|x|)^2} |\Phi \varepsilon E|^2 |x|^{-2}\, dx + \int_{U \setminus B(r_0/3)} e^{\gamma(\log|x|)^2} |H|^2 |x|^{-2}\, dx. \tag{7.5}$$

Adding (7.4) to (7.5), we see that

$$\frac{1}{2}\gamma \int_{|x| \le r_0/4} |x|^{-2} |\log|x|| e^{\gamma(\log|x|)^2} (|\Phi E|^2 + |\Phi H|^2) |x|^{-2}\, dx \le C \left\{ \int_{U \setminus B(r_0/3)} e^{\gamma(\log|x|)^2} (|E|^2 + |H|^2) |x|^{-2}\, dx \right\} \tag{7.6}$$

if r_0 is small enough and γ is sufficiently large. Since $\phi(r) = (\log r)^2$ is strictly decreasing when $0 < r < 1$, there exists a positive constant C_0 independent of γ such that

$$\int_{|x| \le r_0/4} (|E|^2 + |H|^2)\, dx \le C_0 e^{\gamma(\phi(r_0/3) - \phi(r_0/4))}.$$

Letting $\gamma \to \infty$, we conclude that E and H are equal to zero in $B(r_0/3)$. $\square$

8. An appendix.

Finally, we would like to give a remark on a similarity between the Dirac equation and the Maxwell equation in a vacuum:

$$\begin{cases} \operatorname{curl} E + (1/c)\partial_t H = 0, & \operatorname{div} H = 0, \\ \operatorname{curl} H - (1/c)\partial_t E = (4\pi/c) J, & \operatorname{div} E = 4\pi\rho. \end{cases}$$

This system can be written in the form analogous to the Dirac equation

$$-\frac{1}{i} \sum_{k=0}^{3} \alpha_k \partial_{x_k} \psi = -\frac{4\pi}{c} \Phi.$$

Indeed, we define the components of ψ as

$$\psi_0 = 0, \quad \psi_k = H_k - iE_k, \quad \Phi_0 = c\rho, \quad \Phi_k = j_k, \quad k = 1, 2, 3$$

and $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$. Furthermore, we have

$$\alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then, it is easily seen that if $k,\ \ell = 1, 2, 3$,

$$\alpha_k \alpha_\ell + \alpha_\ell \alpha_k = 2\delta_{k\ell} I \tag{8.1}$$

and

$$\alpha_1 \alpha_2 = i\alpha_3, \quad \alpha_2 \alpha_3 = i\alpha_1, \quad \alpha_3 \alpha_1 = i\alpha_2. \tag{8.2}$$

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