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The Stokes and Navier-Stokes equations in an aperture domain

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Abstract. We consider the nonstationary Stokes and Navier-Stokes equations in an aperture domain $\Omega \subset \boldsymbol{R}^n$, $n \geq 2$. For this purpose, we prove L^p-L^q type estimate of the Stokes semigroup in the aperture domain. Our proof is based on the local energy decay estimate obtained by investigation of the asymptotic behavior of the resolvent of the Stokes operator near the origin. We apply them to the Navier-Stokes initial value problem in the aperture domain. As a result, we can prove the global existence of a unique solution to the Navier-Stokes problem with the vanishing flux condition and some decay properties as $t \to \infty$, when the initial velocity is sufficiently small in the L^n space. Moreover we can prove the time-local existence of a unique solution to the Navier-Stokes problem with the non-trivial flux condition.

1. Introduction.

An aperture domain $\Omega \subset \boldsymbol{R}^n$ $(n \geq 2)$ is an unbounded domain with noncompact boundary $\partial\Omega$. Roughly speaking, Ω consists of two disjoint half-spaces separated by a wall and connected by a hole (aperture) through this wall (see Section 2 for detail).

We assume that $\partial\Omega$ is smooth enough, $\partial\Omega \in C^1$ for the Helmholtz decomposition, $\partial\Omega \in C^{2,\mu}(0 < \mu < 1)$ for the Stokes resolvent system and that Ω is divided into some upper domain Ω_+, some lower domain Ω_- and some smooth $(n-1)$-dimensional manifold M in the hole such that $\Omega = \Omega_+ \cup M \cup \Omega_-$.

In $\Omega \times (0,\infty)$, we consider the nonstationary Navier-Stokes initial boundary value problem:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla\pi = 0 & \text{in } \ \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \ \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on } \ \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in } \ \Omega \end{cases} \tag{NS}$$

for the unknown velocity field $u = (u_1, \ldots, u_n) \in W^{2,p}(\Omega)^n$ and the unknown scalar pressure term $\nabla\pi \in L^p(\Omega)^n$, where $1 < p < \infty$.

The aperture domain is a particularly interesting class of domains with noncompact boundaries. In 1976, Heywood [**23**] pointed out that the solution may not be uniquely

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determined by usual boundary conditions in this domain and therefore in order to get a unique solution u we may have to prescribe either the pressure drop $[\pi]$ at infinity between the upper and lower subdomains $\Omega_{\pm}$:

$$[\pi] = \lim_{|x|\to\infty,\ x\in\Omega_+} \pi(x) - \lim_{|x|\to\infty,\ x\in\Omega_-} \pi(x)$$

or the flux $\phi(u)$ through the aperture M:

$$\phi(u) = \int_M N \cdot u \, d\sigma,$$

where N denotes the normal vector on M directed to Ω_-, as an additional boundary condition. When $n = 2$, for $1 < p \le 2$ the solution is unique and the flux vanishes, whereas for $p > 2$ the flux has to be given. When $n \ge 3$, for $1 < p \le \frac{n}{n-1} (=: n')$ the solution is unique, without claiming any additional boundary condition. If $n' < p < n$, either the flux or the pressure drop can be prescribed, whereas for $p \ge n$ only the flux can be given (see Farwig [**15**]).

We shall introduce the known results concerning the aperture domain Ω. The results of Farwig and Sohr [**17**] and Miyakawa [**33**] are the first step to discuss the nonstationary problem (NS) in the L^p-space. They showed the Helmholtz decomposition of the L^p-space of vector fields $L^p(\Omega)^n = J^p(\Omega) \oplus G^p(\Omega)$ for $n \ge 2$ and $1 < p < \infty$, where $J^p(\Omega)$ and $G^p(\Omega)$ denote as follows:

$$J^p(\Omega) = \overline{\{u \in C_0^\infty(\Omega)^n \mid \nabla \cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)^n}},$$
$$G^p(\Omega) = \{\nabla \pi \in L^p(\Omega)^n \mid \pi \in L^p_{loc}(\overline{\Omega})\}.$$

The space $J^p(\Omega)$ is characterized as

$$J^p(\Omega) = \{u \in L^p(\Omega) \mid \nabla \cdot u = 0,\ \nu \cdot u|_{\partial\Omega} = 0,\ \phi(u) = 0\},$$

where ν is the unit outer normal vector on $\partial\Omega$ (see [**17**, Lemma 3.1]). Here the condition $\phi(u) = 0$ is automatically satisfied and may be omitted if $1 < q \le n'$ but otherwise the elements of $J^p(\Omega)$ have to possess this condition $\phi(u) = 0$.

Let P be a continuous projection from $L^p(\Omega)^n$ to $J^p(\Omega)$ associated with the Helmholtz decomposition. The Stokes operator A is defined by $A = -P\Delta$ with a domain which is introduced in Section 2. It is proved by Farwig and Sohr [**17**] that $-A$ generates a bounded analytic semigroup $T(t)$ on $J^p(\Omega)$.

The main purpose of this paper is to prove the L^p-L^q estimates of the Stokes semigroup:

$$\|T(t)a\|_{L^q(\Omega)^n} \le C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L^p(\Omega)^n}, \tag{1.1}$$

$$\|\nabla T(t)a\|_{L^q(\Omega)^{n^2}} \le C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L^p(\Omega)^n} \tag{1.2}$$

for $a \in J^p(\Omega)$ and $t > 0$, where $1 \le p \le q \le \infty$ $(p \neq \infty, q \neq 1)$ for (1.1) and $1 \le p \le q < \infty$ $(q \neq 1)$ for (1.2). In particular, the gradient estimate (1.2) without any restriction on (p, q) is our important contribution and also our result covers the case $n = 2$. Up to now, Abels [**2**] proved (1.1) for $1 < p \le q < \infty$ and (1.2) for $1 < p \le q < n$ when $n \ge 3$; and Hishida [**22**] proved (1.1) for $1 \le p \le q \le \infty$ $(p \neq \infty, q \neq 1)$ and (1.2) for $1 \le p \le q \le n$ $(q \neq 1)$ and $1 \le p < n < q < \infty$ when $n \ge 3$. Although the result of [**22**] is sufficient for the proof of the global existence of the Navier-Stokes flow with small L^n data $(n \ge 3)$, the improvement above of the gradient estimate is of own interest and also implies optimal decay rates of the gradient of the global solution of [**22**] in L^r with $r > n$; see Theorem 2.3. Recently in [**30**] the author and Shibata proved the L^p-L^q estimates of the Stokes semigroup for the same (p, q) as above and $n \ge 2$ in the case of a perturbed half-space by using a precise analysis of the resolvent for the half-space problem due to ourselves [**29**]. Since the aperture domain is obtained from upper and lower half-spaces by a perturbation within a bounded region, one can exactly follow the argument of [**30**] in the proof of (1.1) and (1.2). In this paper, we give the outline of the proof in our context of the aperture domain. As explained above, the aperture domain is physically more interesting than the perturbed half-space; for instance, one can discuss the fluid motion when a non-trivial flux $\phi(u)$ through the aperture is prescribed.

The L^p-L^q estimates of the Stokes semigroup have been already studied by many authors in some cases of other domains. In fact, when Ω is the whole space, applying the Young inequality to the concrete solution formula, we have (1.1) and (1.2) for $1 \le p \le q \le \infty$ $(p \neq \infty,\ q \neq 1)$. When Ω is the half-space, it is proved by Ukai [**39**] and Borchers and Miyakawa [**5**] that (1.1) and (1.2) hold for $1 \le p \le q \le \infty$ $(p \neq \infty,\ q \neq 1)$ (cf. Desch, Hieber and Prüss [**12**]). When Ω is an infinite layer case, Abe and Shibata [**1**] proved that (1.1) and (1.2) hold for $1 < p \le q < \infty$. When Ω is a bounded domain, (1.1) and (1.2) for $1 < p \le q < \infty$ follow from the result of Giga [**20**] on a characterization of the domains of fractional powers of the Stokes operator. In an infinite layer case and a bounded domain case, an exponential decay property of the semigroup is available.

When Ω is an exterior domain, (1.1) holds for $1 \le p \le q \le \infty$ $(p \neq \infty,\ q \neq 1)$ but (1.2) holds only for $1 \le p \le q \le n$ $(q \neq 1)$. At first Iwashita [**24**] proved that (1.1) holds for $1 < p \le q < \infty$ and (1.2) for $1 < p \le q \le n$ when $n \ge 3$. The refinement of his result was done by the following authors: Chen [**8**] $(n = 3,\ q = \infty)$, Shibata [**36**] $(n = 3,\ q = \infty)$, Borchers and Varnhorn [**7**] $(n = 2$, (1.1) for $p = q)$, Dan and Shibata [**9**], [**10**] $(n = 2)$, Dan, Kobayashi and Shibata [**11**] $(n = 2, 3)$, and Maremonti and Solonnikov [**31**] $(n \ge 2)$. Especially, it was shown by Maremonti and Solonnikov [**31**] that Iwashita's restriction $q \le n$ in (1.2) is unavoidable.

When Ω is a perturbed half-space, as was mentioned, Kubo and Shibata [**30**] proved (1.1) for $1 \le p \le q \le \infty$ $(p \neq \infty,\ q \neq 1)$ and (1.2) for $1 \le p \le q < \infty (q \neq 1)$ when $n \ge 2$.

It is well-known that we can prove the global existence of the solution to the Navier-Stokes problem with small L^n data as an application of the L^p-L^q estimate of the Stokes semigroup. In fact, the time-global existence was proved by many authors in the following domain cases: Giga and Miyakawa [**21**] for bounded domains, Kato [**25**] for the whole space, Ukai [**39**] and Kozono [**26**] for the half-space, Iwashita [**24**] and Wiegner [**40**] for the exterior domain, Abe and Shibata [**1**] for the infinite layer, Kubo and Shibata [**30**] for the perturbed half-space and Hishida [**22**] for the aperture domain.

On the other hand, concerning the local existence of strong solutions with a non-trivial flux in an aperture domain, we refer to Heywood [**23**] and Franzke [**18**], both of which are L^2 theory.

This paper consists of five sections. In the next section, after notation is fixed, we present the statement of our main results: Theorem 2.1 on the local energy decay estimates of the Stokes semigroup, Theorem 2.2 on the L^p-L^q estimates of the Stokes semigroup, Theorem 2.3 on the global existence and decay properties of the Navier-Stokes flow with $\phi(u) \equiv 0$, Theorem 2.4 on some further asymptotic behaviors of the obtained flow under additional summability assumption on the initial data and Theorem 2.6 on L^p theory of time-local solution to the Navier-Stokes problem with a non-trivial flux.

In Section 3, we consider the Stokes resolvent for the half-space $H = H_+$ or H_-. Let $(R(\lambda), \Pi(\lambda))$ be the solution operator to the Stokes resolvent problem in H. We provide the expansion formula of $(R(\lambda), \Pi(\lambda))$ near the origin, which was proved by Kubo and Shibata [**29**] and plays a crucial role in the proof of L^p-L^q estimate.

In Section 4, we prove the local energy decay estimate (Theorem 2.1) and the L^p-L^q estimates in an aperture domain (Theorem 2.2). In order to prove the local energy decay estimate in the same way as Iwashita [**24**], we need the expansion formula of the solution operator near the origin. Constructing a parametrix to the resolvent problem in an aperture domain, we can derive from the results for the half-space that the resolvent operator $(\lambda + A)^{-1}$ has the expansion formula of the same type near the origin as in the half-space. Here what is important is that the order of asymptotic expansion of the Stokes resolvent near the origin is one half better than that for the exterior domain case, because we have the reflection principle on the boundary in the half-space case unlike the whole space case. And then, such better asymptotics near the boundary is also obtained in the aperture domain by perturbation argument. Next we show the L^p-L^q estimates of the Stokes semigroup (Theorem 2.2). Our proof is based on the local energy decay estimate. Here the remarkable points are that we can remove the restriction on (p, q) in (1.2), and that (1.1) and (1.2) hold for the two dimensional case. The reason why we can prove them is that the rate of local energy decay for the aperture domain case is one half better than that for the exterior domain case.

In the final section, we derive various decay properties of the global strong solution as $t \to \infty$ to prove Theorem 2.3 and Theorem 2.4. These theorems is proved in the same way as [**22**]. But differently from [**22**], we can prove $\|\nabla u(t)\|_{L^r} = o(t^{-1+\frac{n}{2r}})$ for $n < r < \infty$, because we have the L^p-L^q estimate of $\nabla T(t)$ without the restriction on (p, q). Moreover we also consider the case where the flux is non-trivial. In a similar way to Kato [**25**] with the aid of an auxiliary function (flux carrier) of Heywood [**23**], we prove that the time-local existence of the unique strong solution to (NS) when the flux is non-trivial.

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2. Main theorems and notations.

First of all, in order to discuss our results more precisely we outline our notation used throughout this paper. We define upper and lower half-spaces by $H_\pm = \{x \in \boldsymbol{R}^n \mid \pm x_n > 1\}$, and sometimes write $H = H_+$ or H_- to describe some assertions for the half-space. To denote the special sets we use the following symbols:

$$\begin{aligned} B_R &= \{x \in \boldsymbol{R}^n \mid |x| < R\}, \quad \Omega_R = \Omega \cap B_R, \\ D_R^\pm &= \{x \in H_\pm \mid R < |x| < R+1\}, \\ C_R^\pm &= \{x \in H_\pm \mid |x'| \le R,\ \pm x_n \le R\}, \end{aligned} \tag{2.1}$$

where $x' = (x_1, \ldots, x_{n-1})$. Let $\Omega \subset \boldsymbol{R}^n$ be an aperture domain with smooth enough boundary $\partial\Omega$, namely, there is a positive number R_0 such that

$$\Omega \backslash B_{R_0} = (H_+ \cup H_-) \backslash B_{R_0}. \tag{2.2}$$

In what follows we fix such R_0. Ω is divided into some upper domain Ω_+, some lower domain Ω_- and some smooth $(n-1)$-dimensional manifold M in the hole such that $\Omega = \Omega_+ \cup M \cup \Omega_-$, $\Omega_\pm \backslash B_{R_0} = H_\pm \backslash B_{R_0}$ and $M \cup \partial M = \partial\Omega_+ \cap \partial\Omega_- \subset \overline{B_{R_0}}$.

For a domain $G \subset \boldsymbol{R}^n$ we will use the standard symbols: for example, $L^p(G)$ denotes the Lebesgue space with norm $\|\cdot\|_{L^p(G)}$ and $W^{m,p}(G)$ denotes the Sobolev space with norm $\|\cdot\|_{W^{m,p}(G)}$. We set

$$\begin{aligned} L_R^p(G) &= \{f \in L^p(G) \mid f(x) = 0 \text{ for } |x| > R\}, \\ W_0^{N,p}(G) &= \{f \in W^{N,p}(G) \mid \partial_x^\alpha f|_{\partial G} = 0 \text{ for } |\alpha| \le N-1\}, \quad N \ge 1, \\ \dot{W}^{N,p}(G) &= \left\{ f \in W_0^{N,p}(G) \,\middle|\, \int_G f\,dx = 0 \right\}, \quad N \ge 1, \\ \dot{W}^{0,p}(G) &= \left\{ f \in L^p(G) \,\middle|\, \int_G f\,dx = 0 \right\}. \end{aligned}$$

We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion.

For Banach spaces X and Y, $\mathscr{L}(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write $\mathscr{L}(X) = \mathscr{L}(X,X)$. $\mathscr{B}(U;X)$ denotes the set of all X-valued bounded holomorphic functions on U. And $BC([0,T);X)$ denotes the class of X-valued bounded continuous function on $[0,T)$.

Given $R \ge R_0$, we take (and fix) the cut-off function $\psi_{\pm,R} \in C^\infty(\boldsymbol{R}^n;[0,1])$ satisfying

$$\psi_{\pm,R} = \begin{cases} 1 & \text{for } H_\pm \backslash B_{R+1}, \\ 0 & \text{for } H_\mp \cup B_R. \end{cases} \tag{2.3}$$

When we construct a parametrix in Ω with use of the cut-off functions $\psi_{\pm,R}$, the bounded domain $D_R^{\pm}$ appears. In order to recover the solenoidal condition in the cut-off procedure, we need the Bogovskiĭ operators $\boldsymbol{B}_{\pm,R}$ which has the following properties respectively: there are linear operators $\boldsymbol{B}_{\pm,R}$ from $\dot{W}^{N,p}(D_R^{\pm})$ into $W^{N+1,p}(\boldsymbol{R}^n)$ such that

$$\nabla \cdot \boldsymbol{B}_{\pm,R} f = f, \quad \|\boldsymbol{B}_{\pm,R} f\|_{W^{N+1,p}(\boldsymbol{R}^n)} \leq C_{N,p,R} \|f\|_{W^{N,p}(D_R^{\pm})}, \quad \mathrm{supp}\, \boldsymbol{B}_{\pm,R} f \subset \overline{D_R^{\pm}} \tag{2.4}$$

for $1 < p < \infty$, $f \in \dot{W}^{N,p}(D_R^{\pm})$ and a nonnegative integer N (see Bogovskiĭ [**4**], Borchers and Sohr [**6**] and Galdi [**19**] for detail).

When Ω is the half-space or the aperture domain, the space $L^p(\Omega)$ admits the Helmholtz decomposition

$$L^p(\Omega) = J^p(\Omega) \oplus G^p(\Omega)$$

for $1 < p < \infty$ and $n \geq 2$, where $J^p(\Omega)$ and $G^p(\Omega)$ are defined by the following relation respectively:

$$\begin{aligned} J^p(\Omega) &= \overline{\{u \in C_0^\infty(\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)}}, \\ G^p(\Omega) &= \{\nabla \pi \in L^p(\Omega) \mid \pi \in L^p_{loc}(\overline{\Omega})\}. \end{aligned}$$

Let $P_{p,\Omega}$ be a continuous projection from $L^p(\Omega)$ to $J^p(\Omega)$ associated with the Helmholtz decomposition. The Stokes operator $A_{p,\Omega}$ is defined by $A_{p,\Omega} = -P_{p,\Omega}\Delta$ with a domain

$$D(A_{p,\Omega}) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap J^p(\Omega),$$

where $1 < p < \infty$. For simplicity we use the abbreviations P_p for $P_{p,\Omega}$ and A_p for $A_{p,\Omega}$ when Ω is an aperture domain and the subscript p is also often omitted if there is no confusion. The Stokes operator satisfies the parabolic resolvent estimate

$$\|(\lambda + A_\Omega)^{-1}\|_{\mathscr{L}(J^p(\Omega))} \leq \frac{C_\varepsilon}{|\lambda|} \tag{2.5}$$

for $|\arg \lambda| \leq \pi - \varepsilon$ $(\lambda \neq 0)$, where $\varepsilon > 0$ is arbitrary (see Farwig [**15**] and Farwig and Sohr [**17**] for the aperture domain, McCracken [**32**] and Farwig and Sohr [**16**] for the half-space). Estimate (2.5) implies that $-A_\Omega$ generates a bounded analytic semigroup $T_{A_\Omega}(t)$ of class C_0 in each $J^p(\Omega)$. We write $E_\pm(t) = T_{A_{H_\pm}}(t)$ and $T(t) = T_A(t)$ as the Stokes semigroup for the half-space and the one for the aperture domain respectively.

To denote various constants we use the same letter C, and $C_{A,B,\cdots}$ and $C(A, B, \cdots)$ denote the constant depending on the quantities $A, B, \cdots$, respectively. The constants C and $C_{A,B,\cdots}$ may change from line to line.

The following theorems provide the decay estimates of Stokes semigroup $T(t)$ for the aperture domain.

THEOREM 2.1 (local energy decay). *Let $n \geq 2$, $1 < p < \infty$, m be a nonnegative integer and $R > R_0$, where R_0 is satisfied with* (2.2). *Then there exists a positive constant $C_{p,m}$ such that*

$$\left\|\partial_t^m T(t)Pf\right\|_{W^{2,p}(\Omega_R)} \leq C_{p,m} t^{-\frac{n+1}{2}-m}\|f\|_{L^p(\Omega)} \tag{2.6}$$

for any $t \geq 1$ and $f \in L^p_R(\Omega)$.

THEOREM 2.2 (L^p-L^q estimates). *Let $n \geq 2$.*

(i) *Let $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). There exists a positive constant $C_{p,q}$ such that*

$$\|T(t)f\|_{L^q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)} \tag{2.7}$$

for all $t > 0$ and $f \in J^p(\Omega)$. When $p = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap J^s(\Omega)$ for some $s \in (1, \infty)$.

(ii) *Let $1 \leq p \leq q < \infty$ ($q \neq 1$), there holds the estimate*:

$$\|\nabla T(t)f\|_{L^q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{2.8}$$

for all $t > 0$ and $f \in J^p(\Omega)$. When $p = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap J^s(\Omega)$ for some $s \in (1, \infty)$.

Next we consider the application of the L^p-L^q estimates to the Navier-Stokes initial value problem (NS). Applying the solenoidal projection P to (NS), we can rewrite (NS) with $\phi(u) = 0$ as follows:

$$\partial_t u + Au + P((u \cdot \nabla)u) = 0, \quad u(0) = a, \tag{PNS}$$

where $A = -P\Delta$ is the Stokes operator.

For given $a \in J^n(\Omega)$ and $0 < T \leq \infty$, a measurable function u defined on $\Omega \times (0, T)$ is called a strong solution of (NS) on $(0, T)$ satisfying $\phi(u) = 0$ if u belongs to

$$u \in C([0, T); J^n(\Omega)) \cap C((0, T); D(A)) \cap C^1((0, T); J^n(\Omega))$$

together with $\lim_{t\to 0} \|u(t) - a\|_{L^n} = 0$ and satisfies (PNS) for $0 < t < T$ in $J^n(\Omega)$.

In the same way as Hishida's argument [**22**], we can show the following theorem which tells us the global existence of a strong solution to (NS) with $\phi(u) = 0$ and several decay properties when the initial data a are small:

THEOREM 2.3. *Let $n \geq 2$. There exists a constant $\delta = \delta(\Omega, n) > 0$ with the following property*: *if $a \in J^n(\Omega)$ satisfies $\|a\|_{L^n} \leq \delta$, the problem* (NS) *with $\phi(u) = 0$ admits a unique strong solution $u(t)$ on $(0, \infty)$. Moreover as $t \to \infty$,*

$$\|u(t)\|_{L^r} = o\big(t^{-\frac{1}{2}+\frac{n}{2r}}\big) \quad \textit{for} \ \ n \le r \le \infty, \tag{2.9}$$

$$\|\nabla u(t)\|_{L^r} = o\big(t^{-1+\frac{n}{2r}}\big) \quad \textit{for} \ \ n \le r < \infty, \tag{2.10}$$

$$\|\partial_t u(t)\|_{L^r} + \|Au(t)\|_{L^r} = o\big(t^{-\frac{3}{2}+\frac{n}{2r}}\big) \quad \textit{for} \ \ n \le r < \infty. \tag{2.11}$$

For $n = 2$*, the smallness of* $\|a\|_{L^2(\Omega)}$ *is redundant.*

Moreover if $a \in L^1(\Omega) \cap J^n(\Omega)$ has small $\|a\|_{L^n}$, then we can show the following theorem. For the case $n \ge 3$, the results are exactly the same as those in [**22**].

THEOREM 2.4. *Let* $n \ge 2$*. There exists a constant* $\eta = \eta(\Omega, n) \in (0, \delta]$ *with the following property*: *if* $a \in L^1(\Omega) \cap J^n(\Omega)$ *satisfies* $\|a\|_{L^n} \le \eta$*, then the solution* $u(t)$ *obtained in Theorem* 2.3 *enjoys*

$$\|u(t)\|_{L^r} = O\big(t^{-\frac{n}{2}(1-\frac{1}{r})}\big) \qquad \textit{for} \ \ 1 < r \le \infty, \tag{2.12}$$

$$\|\nabla u(t)\|_{L^r} = O\big(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}\big) \quad \textit{for} \ \ 1 < r < \infty, \tag{2.13}$$

$$\|\partial_t u(t)\|_{L^r} + \|Au(t)\|_{L^r} = O\big(t^{-\frac{n}{2}(1-\frac{1}{r})-1}\big) \quad \textit{for} \ \ 1 < r < \infty, \tag{2.14}$$

$$\|\nabla^2 u(t)\|_{L^r} + \|\nabla \pi(t)\|_{L^r} = O\big(t^{-\frac{n}{2}(1-\frac{1}{r})-1}\big) \quad \textit{for} \ \ 1 < r < n \tag{2.15}$$

as $t \to \infty$*. Moreover, for each* $t > 0$ *there exist two constants* $\pi_\pm(t) \in \boldsymbol{R}$ *such that* $\pi(t) - \pi_\pm(t) \in L^r(\Omega_\pm)$ *with*

$$\|\pi(t) - \pi_\pm\|_{L^r(\Omega_\pm)} + |[\pi(t)]| = O\big(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}\big) \quad \textit{for} \ n' < r < \infty \tag{2.16}$$

as $t \to \infty$ *where* $[\pi(t)] = \pi_+(t) - \pi_-(t)$*. For* $n = 2$*, the smallness of* $\|a\|_{L^2(\Omega)}$ *is redundant.*

REMARK 2.5. In the two dimensional case, Kozono and Ogawa [**27**] established the global existence result without the smallness of $\|a\|_{L^2}$ for an arbitrary unbounded domain, which covers the aperture domain with the hidden flux condition $\phi(u) = 0$. But (2.9) with $r = \infty$ was not obtained in [**27**]. In [**28**] they derived various decay properties of the global solution when $a \in L^p(\Omega) \cap J^2(\Omega)$ with $1 < p < 2$.

Next, we shall consider the case where the flux is non-trivial. We fix an auxiliary function $\chi \in C^\infty(\Omega) \cap W^{2,p}(\Omega)(n' < p < \infty)$ satisfying

$$\chi|_{\partial\Omega} = 0, \ \nabla \cdot \chi = 0, \ \phi(\chi) = 1$$

(see Heywood [**23**, Lemma 11] and Galdi [**19**, III.4.3]). Given a flux $\phi(v(t)) = \alpha(t)$, we study the problem (NSf) (see Section 5). We set $u(t,x) = v(t,x) - \alpha(t)\chi(x)$ and reduce (NSf) to (NS′) with vanishing flux condition (see Section 5). For $n \ge 3$, the notion of strong solution u to (NS′) with $\phi(u) = 0$ is defined similarly to that given above for (NS) with $\phi(u) = 0$. For $n = 2$, the auxiliary function χ does not belong to $L^2(\Omega)$ and the force term includes $\alpha'\chi$ in (NS′); thus, the solution u to (NS′) can't belong to $J^2(\Omega)$. Therefore we must change the definition of the strong solution $u(t)$ to (NS′) with

$\phi(u) = 0$ as follows: for given $a \in J^p(\Omega)$ $(n = 2 < p < \infty)$ and $0 < T \leq \infty$, a measurable function u defined on $\Omega \times (0, T)$ is called a strong solution of (NS$'$) on $(0, T)$ satisfying $\phi(u) = 0$ if u belongs to

$$u \in C([0, T); J^p(\Omega)) \cap C((0, T); D(A)) \cap C^1((0, T); J^p(\Omega))$$

together with $\lim_{t \to 0} \|u(t) - a\|_{L^p} = 0$ and satisfies (PNS$'$) for $0 < t < T$ in $J^p(\Omega)$.

The following theorem gives us the time-local solution to the Navier-Stokes problem with a non-flux condition:

THEOREM 2.6. *Suppose that the flux $\phi(v(t)) = \alpha(t)$ belongs to $C^{1,\theta}([0, T])$ with some $T > 0$ and $0 < \theta < 1$ in the problem* (NSf).

(i) *Let $n \geq 3$. If $a - \alpha(0)\chi \in J^n(\Omega)$, then there exists $T_* \in (0, T]$ such that the reduced problem* (NS$'$) *admits a unique strong solution $u(t)$ on $(0, T_*)$. Moreover the solution $u(t)$ satisfies*

$$t^{\frac{1}{2} - \frac{n}{2r}} u \in BC([0, T_*); J^r(\Omega)) \quad \text{for} \ \ n \leq r \leq \infty, \tag{2.17}$$

$$t^{1 - \frac{n}{2r}} \nabla u \in BC([0, T_*); L^r(\Omega)) \quad \text{for} \ \ n \leq r < \infty. \tag{2.18}$$

The values of $t^{\frac{1}{2} - \frac{n}{2r}} u(t)$ and $t^{1 - \frac{n}{2r}} \nabla u(t)$ at $t = 0$ vanish except for $r = n$ in (2.17), *in which $u(0) = a - \alpha(0)\chi$.*

(ii) *Let $n = 2 < p < \infty$. If $a - \alpha(0)\chi \in J^p(\Omega)$, then there is $T_* \in (0, T]$ such that the reduced problem* (NS$'$) *admits a unique strong solution $u(t)$ on $(0, T_*)$. Moreover the solution $u(t)$ satisfies*

$$t^{\frac{1}{p} - \frac{1}{r}} u \in BC([0, T_*); J^r(\Omega)) \quad \text{for} \ \ p \leq r \leq \infty, \tag{2.19}$$

$$t^{\frac{1}{p} - \frac{1}{r} + \frac{1}{2}} \nabla u \in BC([0, T_*); L^r(\Omega)) \quad \text{for} \ \ p \leq r < \infty. \tag{2.20}$$

The values of $t^{\frac{1}{p} - \frac{1}{r}} u(t)$ and $t^{\frac{1}{p} - \frac{1}{r} + \frac{1}{2}} \nabla u(t)$ at $t = 0$ vanish except for $r = p$ in (2.19), *in which $u(0) = a - \alpha(0)\chi$.*

3. Preliminaries.

We shall consider the Stokes resolvent problem in the half-space $H = H_+$ or H_-:

$$\begin{cases} (\lambda - \Delta)u + \nabla \pi = f, \quad \nabla \cdot u = 0 & \text{in } H, \\ u = 0 & \text{on } \partial H. \end{cases} \tag{3.1}$$

Let $(R(\lambda), \Pi(\lambda))$ be the solution operator to (3.1). In [**29**], we proved two theorems concerning the property of $(R(\lambda)f, \Pi(\lambda)f)$ near the origin when f has compact support. One of them gives us the expansion formula of $(R(\lambda), \Pi(\lambda))$ near the origin and the other gives us the continuous property of $(R(\lambda), \Pi(\lambda))$ at the origin.

Theorem 3.1. *Let $n \geq 2$. We set $U_r = \{\lambda \in \boldsymbol{C} \mid |\lambda| < r\}$, $\dot{U}_r = U_r \backslash (-\infty, 0]$ and*

$$B_H = \mathscr{L}\big(L^p_R(H), W^{2,p}(H \cap B_R) \times W^{1,p}(H \cap B_R)\big).$$

Then $(R(\lambda), \Pi(\lambda))$ has the following expansion formula with respect to $\lambda \in \dot{U}_{1/2}$:

$$(R(\lambda), \Pi(\lambda)) = \begin{cases} G_1(\lambda)\lambda^{\frac{n-1}{2}} + G_2(\lambda)\lambda^{\frac{n}{2}} \log \lambda + G_3(\lambda) & \text{where } n \text{ is even,} \\ G_1(\lambda)\lambda^{\frac{n}{2}} + G_2(\lambda)\lambda^{\frac{n-1}{2}} \log \lambda + G_3(\lambda) & \text{where } n \text{ is odd,} \end{cases} \tag{3.2}$$

where $G_1(\lambda), G_2(\lambda)$ and $G_3(\lambda)$ are B_H-valued holomorphic functions in $U_{1/2}$.

Theorem 3.2. *Let $n \geq 2$, $1 < p < \infty$ and $f \in L^p_R(H)$. Let $R(\lambda)$ and $\Pi(\lambda)$ be the solution operators to (3.1) for $\lambda \in \boldsymbol{C} \backslash (-\infty, 0]$. Then there exist operators $R(0) : L^p_R(H) \to W^{2,p}_{loc}(H)$ and $\Pi(0) : L^p_R(H) \to W^{1,p}_{loc}(H)$ which have the following properties:*

(i) *If we set $u = R(0)f$ and $\pi = \Pi(0)f$, then (u, π) satisfies the equation:*

$$-\Delta u + \nabla \pi = f, \ \nabla \cdot u = 0 \quad \text{in } H, \qquad u|_{\partial H} = 0.$$

(ii) *(u, π) satisfies the estimates:*

$$\|u\|_{W^{2,p}(H \cap B_L)} + \|\pi\|_{W^{1,p}(H \cap B_L)} \leq C_{R,L}\|f\|_{L^p(H)} \quad \text{for} \quad L > 0,$$
$$\sup_{|x| \geq R,\ x \in H} \big[|x|^{n-1}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|\big] \leq C_R\|f\|_{L^p(H)}$$

and the formula:

$$\|R(\lambda)f - R(0)f\|_{W^{2,p}(H \cap B_L)} \leq C\, p(|\lambda|)\|f\|_{L^p(H)},$$
$$\|\Pi(\lambda)f - \Pi(0)f\|_{W^{1,p}(H \cap B_L)} \leq C\, p(|\lambda|)\|f\|_{L^p(H)}$$

for any $L > 0$ and any $\lambda \in \dot{U}_{1/2}$ where $p(t) = \max(t,\ t^{\frac{n-1}{2}}|\log t|)$ for $t \in [0, 1)$.

Moreover we can show the following lemma concerning the uniqueness in the same way as in [**30**, Lemma 2.3]:

Lemma 3.3. *Let $n \geq 2$ and $1 < p < \infty$. Let Ω be the half-space H or an aperture domain. Let $u \in W^{2,p}_{loc}(\Omega)$ and $\pi \in W^{1,p}_{loc}(\Omega)$ enjoy*

$$-\Delta u + \nabla \pi = 0, \ \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0. \tag{3.3}$$

Moreover

$$\sup_{x \in \Omega,\ |x| \geq R+3} \big[|x|^{n-1}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|\big] < \infty. \tag{3.4}$$

Then $u \equiv 0$ and $\pi \equiv 0$.

4. The proof of Theorem 2.1 and 2.2.

In this section we shall present the outline of the proof of Theorem 2.1 and Theorem 2.2. We refer to Kubo and Shibata [**30**] for details. First we shall show the following theorem, which is the case $m = 0$ in Theorem 2.1. Higher order derivatives of $T(t)Pf$ in t and x are discussed similarly.

THEOREM 4.1. *Let $n \geq 2$, $1 < p < \infty$ and $R > R_0$. Then there exists a positive constant C_p such that the inequality*

$$\|T(t)Pf\|_{L^p(\Omega_R)} \leq C_p t^{-\frac{n}{2}-\frac{1}{2}} \|f\|_{L^p(\Omega)} \tag{4.1}$$

is valid for any $f \in L^p_R(\Omega)$ and $t \geq 1$.

In order to show Theorem 4.1 in the same way as in Iwashita [**24**], we need the following theorem which gives us the expansion formula of the solution operator to the Stokes resolvent problem in the aperture domain Ω.

THEOREM 4.2. *Let $1 < p < \infty$ and $R > R_0$. Set $B_\Omega = \mathscr{L}(L^p_R(\Omega), W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R))$. Then there exists a constant $\lambda_0 > 0$ and $(U(\lambda), \Theta(\lambda))$ such that*

$$U(\lambda)f = (\lambda + A)^{-1} Pf$$

for $f \in L^p_R(\Omega)$ and $\lambda \in U_{\lambda_0}$, and

$$(U(\lambda), \Theta(\lambda)) = \begin{cases} H_1(\lambda)\lambda^{\frac{n-1}{2}} + H_2(\lambda)\lambda^{\frac{n}{2}} \log\lambda + H_3(\lambda) & \text{where } n \text{ is even,} \\ H_1(\lambda)\lambda^{\frac{n}{2}} + H_2(\lambda)\lambda^{\frac{n-1}{2}} \log\lambda + H_3(\lambda) & \text{where } n \text{ is odd} \end{cases}$$

for any $\lambda \in \dot{U}_{\lambda_0}$, where $H_j \in \mathscr{B}(\dot{U}_{\lambda_0}; B_\Omega)(j = 1, 2)$ and $H_3 \in \mathscr{B}(U_{\lambda_0}; B_\Omega)$.

In order to show Theorem 4.2 we shall introduce the notation which is used to construct a parametrix. Let E_R be a bounded domain with smooth boundary ∂E_R such that $E_R \subset \Omega \cap B_{R+5}$ and $E_R \cap B_{R+4} = \Omega \cap B_{R+4}$. In particular we have $D^\pm_{R+1} \subset \Omega \cap B_{R+3} \subset E_R$, where $D^\pm_R$ is defined by (2.1).

Given $f \in L^p_{R+3}(\Omega)$, we set $Af = w$ and $\Phi f = \theta$, where w and θ are the solution to the following equations:

$$-\Delta w + \nabla\theta = f, \qquad \nabla \cdot w = 0 \quad \text{in } E_R, \qquad w|_{\partial E_R} = 0.$$

Let $R_\pm(\lambda)$ and $\Pi_\pm(\lambda)$ be solution operators to (3.1). Set $f_0(x) = f(x)$ for $|x| > R$ and $f_0(x) = 0$ for $|x| \leq R$. It follows from Theorem 3.2 that $w_\pm = R_\pm(\lambda)f_0$ and $\theta_\pm = \Pi_\pm(\lambda)f_0$ satisfy the following equations:

$$(\lambda - \Delta)w_\pm + \nabla\theta_\pm = f_0, \ \nabla \cdot w_\pm = 0 \quad \text{in} \quad H_\pm, \qquad w_\pm|_{\partial H_\pm} = 0,$$

where $\lambda \in \Sigma_\varepsilon = \{\lambda \in \boldsymbol{C} \mid |\arg \lambda| \leq \pi - \varepsilon\}$ for $0 < \varepsilon < \pi/2$. By addition of some constant to $\Pi_\pm(\lambda) f_0$, we may assume that

$$\int_{D_R^\pm} (\Phi f - \Pi_\pm(\lambda) f_0) dx = 0.$$

In the course of this proof, for simplicity, we use the abbreviations $\psi_\pm$ for the cut-off functions $\psi_{\pm,R}$ (given by (2.3)) and $\boldsymbol{B}_\pm$ for the Bogovskiĭ operators $\boldsymbol{B}_{\pm,R}$ (given by (2.4)). We set

$$\begin{aligned} U(\lambda)f &= \psi_+ R_+(\lambda)f + \psi_- R_-(\lambda)f + (1 - \psi_+ - \psi_-)Af \\ &\qquad - \boldsymbol{B}_+\big[(\nabla\psi_+)\cdot(R_+(\lambda)f - Af)\big] - \boldsymbol{B}_-\big[(\nabla\psi_-)\cdot(R_-(\lambda)f - Af)\big], \\ \Theta(\lambda)f &= \psi_+\Pi_+(\lambda)f + \psi_-\Pi_-(\lambda)f + (1 - \psi_+ - \psi_-)\Phi f. \end{aligned}$$

And then we see $U(\lambda)f \in W^{2,p}_{loc}(\Omega)$ and $\Theta(\lambda)f \in W^{1,p}_{loc}(\Omega)$ satisfying

$$(\lambda - \Delta)U(\lambda)f + \nabla\Theta(\lambda)f = (1 + S_\lambda)f, \qquad \nabla \cdot U(\lambda)f = 0$$

in Ω subject to $U(\lambda)f|_{\partial\Omega} = 0$ and

$$\phi(U(\lambda)f) = \int_M N \cdot Af d\sigma = \int_{\Omega_+ \cap E_R} \nabla \cdot Af dx = 0,$$

where

$$\begin{aligned} S_\lambda f = &- 2(\nabla\psi_+)\cdot(\nabla(R_+(\lambda)f - Af)) - 2(\nabla\psi_-)\cdot(\nabla(R_-(\lambda)f - Af)) \\ &- (\Delta\psi_+)(R_+(\lambda)f - Af) - (\Delta\psi_-)(R_-(\lambda)f - Af) \\ &- (\lambda - \Delta)\boldsymbol{B}_+\big[(\nabla\psi_+)\cdot(R_+(\lambda)f - Af)\big] - (\lambda - \Delta)\boldsymbol{B}_-\big[(\nabla\psi_-)\cdot(R_-(\lambda)f - Af)\big] \\ &+ \lambda(1 - \psi_+ - \psi_-)Af - (\nabla\psi_+)(\Pi_+(\lambda)f - \Phi f) + (\nabla\psi_-)(\Pi_-(\lambda)f - \Phi f). \quad (4.2) \end{aligned}$$

To obtain Theorem 4.2, we need the fact that $(1+S_\lambda)^{-1}$ is a bounded linear operator on $L^p_{R+3}(\Omega)$. We know that $S_\lambda : L^p_{R+3}(\Omega) \to L^p_{R+3}(\Omega)$ is a compact operator (see Abels [**2**] and Hishida [**22**]). Moreover from Theorem 3.2, we see that

$$\|S_\lambda - S_0\|_{\mathscr{L}(L^p_{R+3}(\Omega))} \leq Cp(|\lambda|) \tag{4.3}$$

for any $\lambda \in \dot{U}_{1/2}$ and $1 < p < \infty$, where $p(t)$ is the same as in Theorem 3.2. Therefore it is sufficient to prove the fact:

$$(1 + S_0)^{-1} \in \mathscr{L}\big(L^p_{R+3}(\Omega)\big). \tag{4.4}$$

Now we shall show (4.4). We see that S_0 is a compact operator in $L^p_{R+3}(\Omega)$. Therefore owing to the Fredholm theorem, it is sufficient to show that $1+S_0$ is injective in $L^p_{R+3}(\Omega)$. To this end, let $f \in L^p_{R+3}(\Omega)$ satisfy $(1+S_0)f = 0$. By using the fact $(U(0)f, \Theta(0)f) = (R_\pm(0)f, \Pi_\pm(0)f)$ for $x \in \Omega_\pm \backslash B_{R+1}$ and the estimate obtained by Theorem 3.2:

$$\sup_{|x|\geq R,\ x\in H_\pm} \big(|x|^{n-1}|R_\pm(0)f| + |x|^{n-1}|\nabla R_\pm(0)f| + |x|^{n-1}|\Pi_\pm(0)f|\big) \leq C_R\|f\|_{L^p(\Omega)}, \tag{4.5}$$

we see that $(U(0)f, \Theta(0)f)$ satisfies all the assumption of Lemma 3.3. Therefore we obtain $U(0)f = 0$ and $\Theta(0)f = 0$. We have

$$\begin{cases} \psi_+ R_+(0)f + \psi_- R_-(0)f + (1-\psi_+-\psi_-)Af \\ \quad -\boldsymbol{B}_+\big[(\nabla\psi_+)\cdot(R_+(0)f - Af)\big] - \boldsymbol{B}_-\big[(\nabla\psi_-)\cdot(R_-(0)f - Af)\big] = 0 & \text{in } \Omega, \\ \psi_+\Pi_+(0)f + \psi_-\Pi_-(0)f + (1-\psi_+-\psi_-)\Phi f = 0 & \text{in } \Omega. \end{cases} \tag{4.6}$$

Following the argument due to [**30**], we can show that $f = 0$ in Ω. Therefore we obtain (4.4). We get the following lemma from (4.3) and (4.4).

LEMMA 4.3. *There exists a positive constant λ_0 such that for $\lambda \in \Sigma_\varepsilon \cup \{0\}$ with $|\lambda| \leq \lambda_0$, the following relations hold*:

$$(1+S_\lambda)^{-1} \in \mathscr{L}\big(L^p_{R+3}(\Omega)\big), \qquad \big\|(1+S_\lambda)^{-1}\big\|_{\mathscr{L}(L^p_{R+3}(\Omega))} \leq C.$$

By Lemma 4.3, we can describe the solution (u,π) to (3.1) as follows:

$$u(x) = U(\lambda)(1+S_\lambda)^{-1}f, \quad \pi(x) = \Theta(\lambda)(1+S_\lambda)^{-1}f,$$

where

$$(1+S_\lambda)^{-1}f = (1+S_0)^{-1}\sum_{j=0}^{\infty}\big[(-1)(S_\lambda - S_0)(1+S_0)^{-1}\big]^j f.$$

When n is even, by (4.2) and Theorem 3.1, we have

$$S_\lambda - S_0 = \widetilde{G_1}(\lambda)\lambda^{\frac{n-1}{2}} + \widetilde{G_2}(\lambda)\lambda^{\frac{n}{2}}\log\lambda + \widetilde{G_4}(\lambda)\lambda,$$

where $\widetilde{G_j}(\lambda)$ $(j = 1,2,4) \in \mathscr{B}(U_{\lambda_0}, B_\Omega)$. We have

$$(1+S_\lambda)^{-1} = \widehat{H_1}(\lambda)\lambda^{\frac{n-1}{2}} + \widehat{H_2}(\lambda)\lambda^{\frac{n}{2}}\log\lambda + \widehat{H_3}(\lambda),$$

where $\widehat{H_j} \in \mathscr{B}(\dot{U}_{\lambda_0}, B_\Omega)(j = 1,2)$ and $\widehat{H_3} \in \mathscr{B}(U_{\lambda_0}, B_\Omega)$. Since we can show the expansion formula for the odd dimensional case in the same way as the even dimensional case, we obtain Theorem 4.2.

In the same way as [**24**], by using the Dunford integral representation of the Stokes semigroup in terms of the resolvent together with a formula of the gamma function, we can obtain Theorem 4.1.

Remark 4.4. For the exterior domain case, Iwashita [**24**] proved that there holds the estimate:

$$\|T(t)Pf\|_{L^p(\Omega_R)} \le Ct^{-\frac{n}{2}}\|f\|_{L^p(\Omega)}.$$

The reason why the rate of decay for the aperture domain case is one-half better than the one for the exterior domain case is that the worst term in expansion is canceled out by the reflection at the boundary.

Next we shall go on showing the L^p-L^q estimate in an aperture domain Ω by using the cut-off technique. First we show the decay estimate of the Stokes semigroup in Ω_R for general data. By using Theorem 2.1 and the L^p-L^q estimate of Stokes semigroup $E_\pm(t)$ in the half-space proved by Ukai [**39**] and Borchers and Miyakawa [**5**], together with a Poincare type inequality:

$$\|E_\pm(t)f\|_{L^p(C_R^\pm)} \le R\|\nabla E_\pm(t)\|_{L^p(C_R^\pm)} \tag{4.7}$$

for the cylinder $C_R^\pm$ defined by (2.1), we obtain the following lemma:

Lemma 4.5. *Let $n \ge 2$, $1 < p < \infty$ and $R \ge R_0$. Then there exists a positive number $C = C(\Omega, n, p, R)$ such that*

$$\|\partial_t T(t)f\|_{W^{1,p}(\Omega_R)} + \|T(t)f\|_{W^{1,p}(\Omega_R)} \le Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ and $t \ge 2$.

We know that in the exterior domain, there holds the following estimate:

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \le Ct^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}.$$

The reason why the rate of decay for the aperture domain case is one half better than the one for the exterior domain case is that the better decay obtained in Theorem 2.1 and the Poincare type inequality (4.7) hold.

Secondly we show the L^p-L^q estimates of Stokes semigroup in $\Omega_\pm \backslash \Omega_R$. By using the cut-off technique and the L^p-L^q estimates of Stokes semigroup $E(t)$ in the half-space, we obtain the following lemma:

Lemma 4.6.

(i) *Let $1 < p \le q \le \infty$ $(p \ne \infty)$ with $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$. Then there exists a positive number $C = C(p, q, R)$ such that*

$$\|T(t)f\|_{L^q(\Omega_\pm \backslash \Omega_R)} \le Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ *and* $t \geq 2$.

(ii) *Let* $1 < p < \infty$. *Then there exists a positive number* $C = C(p, R)$ *such that*

$$\|\nabla T(t)f\|_{L^p(\Omega_\pm \setminus \Omega_R)} \leq Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ *and* $t \geq 2$.

Thirdly we prove the L^p-L^q estimates of Stokes semigroup $T(t)$ in the aperture domain near $t = 0$. By using the interpolation theory and the resolvent estimate of Stokes semigroup, we obtain the following lemma:

LEMMA 4.7. *Let* $1 < p \leq q \leq \infty$ $(p \neq \infty)$ *with* $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$. *Then there exists a positive number* $C = C(p, q, R)$ *such that*

$$\|T(t)f\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)},$$

$$\|\nabla T(t)f\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ *and* $0 < t < 2$.

We can immediately show Theorem 2.2 from the three lemmas above.

5. The Navier-Stokes flow in an aperture domain.

In this section, we shall apply the L^p-L^q estimate to the Navier-Stokes equation. We begin with the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. By means of a standard contraction mapping principle in the same way as Kato [**25**], we can construct a unique global solution $u(t)$ of the integral equation

$$u(t) = T(t)a - \int_0^t T(t-\tau)P((u \cdot \nabla)u)(\tau)d\tau,$$

provided that $\|a\|_{L^n} \leq \delta_0$, where $\delta_0 = \delta_0(\Omega, n)$ is a positive constant. The solution $u(t)$ enjoys

$$\|u(t)\|_{L^r} \leq Ct^{-\frac{1}{2}+\frac{n}{2r}}\|a\|_{L^n} \quad \text{for} \quad n \leq r \leq \infty, \tag{5.1}$$

$$\|\nabla u(t)\|_{L^r} \leq Ct^{-1+\frac{n}{2r}}\|a\|_{L^n} \quad \text{for} \quad n \leq r < \infty \tag{5.2}$$

for $t > 0$, which imply the Hölder estimate:

$$\|u(t) - u(\tau)\|_{L^\infty} + \|\nabla u(t) - \nabla u(\tau)\|_{L^n} \leq C(t-\tau)^\theta \tau^{-\frac{1}{2}-\theta}\|a\|_{L^n} \tag{5.3}$$

for $0 < \tau < t$ and $0 < \theta < \frac{1}{2}$. Due to the Hölder estimate, the solution $u(t)$ becomes actually a strong one of (NS) (see Tanabe [**38**]).

Furthermore, in the same way as Hishida [**22**], we can obtain the decay properties (2.9) and (2.10) for $r = n$. We also find (2.10) for $n < r < \infty$, which follows from

$$\begin{aligned}
\|\nabla u(t)\|_{L^r} &\le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})-\frac{1}{2}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^n} + C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})-\frac{1}{2}}\|P(u\cdot\nabla u)(\tau)\|_{L^n}d\tau \\
&\le Ct^{-1+\frac{n}{2r}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^n} + C\int_{\frac{t}{2}}^{t}(t-\tau)^{-1+\frac{n}{2r}}\tau^{-1}d\tau\|a\|_{L^n}\left(\sup_{\frac{t}{2}\le\tau\le t}\tau^{\frac{1}{2}}\|u(\tau)\|_{L^\infty}\right) \\
&\le Ct^{-1+\frac{n}{2r}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^n} + Ct^{-1+\frac{n}{2r}}\|a\|_{L^n}\left(\sup_{\frac{t}{2}\le\tau\le t}\tau^{\frac{1}{2}}\|u(\tau)\|_{L^\infty}\right)
\end{aligned}$$

together with (2.9). Finally, as in [**22**], by (2.9) and (2.10) combining with (5.3), we obtain (2.11). □

Since Hishida [**22**] proved Theorem 2.4 for $n \ge 3$, we have only to give a comment on the case $n = 2$. The key of his proof is to show the following lemma.

Lemma 5.1. *Let $n \ge 2$ and $a \in L^1(\Omega) \cap J^n(\Omega)$. When $n \ge 3$, for any small $\varepsilon > 0$ there are constants $\eta_* = \eta_*(\Omega, n, \varepsilon) \in (0, \delta]$ and $C = C(\Omega, n, \|a\|_{L^1}, \|a\|_{L^n}, \varepsilon)$ such that if $\|a\|_{L^n} \le \eta_*$, then the solution $u(t)$ obtained in Theorem 2.3 satisfies*

$$\|u(t)\|_{L^{\frac{n}{n-1}}} \le C(1+t)^{-\frac{1}{2}+\varepsilon}, \tag{5.4}$$

$$\|u(t)\|_{L^{2n}} \le Ct^{-\frac{1}{4}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon}, \tag{5.5}$$

$$\|\nabla u(t)\|_{L^n} \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon} \tag{5.6}$$

for $t > 0$. When $n = 2$, without the assumption that a is small, the solution $u(t)$ obtained in Theorem 2.3 satisfies (5.4)–(5.6).

When $n = 2$, Kozono and Ogawa [**28**] proved that if $a \in J^2(\Omega) \cap L^p(\Omega)$ with $p = 1/(1-\varepsilon)$, then the solution $u(t)$ obtained in Theorem 2.3 enjoys (5.4)–(5.6) for $t \ge 1$ without any smallness condition on the initial data. We thus obtain Lemma 5.1 for $n = 2$.

Next we shall show the time-local existence of the strong solution $v(t)$ to the following Navier-Stokes problem with the non-trivial flux $\alpha(t) \not\equiv 0$ in $[0, \infty)$:

$$\begin{cases}
\partial_t v - \Delta v + (v\cdot\nabla)v + \nabla\pi = 0 & \text{in } \Omega\times(0,\infty), \\
\nabla\cdot v = 0 & \text{in } \Omega\times(0,\infty), \\
v(x,t) = 0 & \text{on } \partial\Omega\times(0,\infty), \\
v(x,0) = a(x) & \text{in } \Omega, \\
\phi(v) = \alpha(t).
\end{cases} \tag{NSf}$$

To this end, we prepare the auxiliary function. Heywood [**23**] showed that there exists $\chi = \chi(x) \in C^\infty(\Omega) \cap W^{2,q}(\Omega)$ $(n' < q < \infty)$ enjoying the following equations:

$$\chi|_{\partial\Omega} = 0, \quad \nabla \cdot \chi = 0, \quad \phi(\chi) = 1. \tag{5.7}$$

Now by using the auxiliary function above, we set $u(x,t) = v(x,t) - \alpha(t)\chi(x)$. We see that u enjoys the following equations:

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = -F(u) + G(\alpha, \chi), \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, \infty) \tag{NS$'$}$$

subject to $u|_{\partial\Omega} = 0$, $\phi(u) = 0$ and $u(0) = v(0) - \alpha(0)\chi$, where

$$F(u) = \alpha(\chi \cdot \nabla)u + \alpha(u \cdot \nabla)\chi, \quad G(\alpha, \chi) = -\alpha'\chi + \alpha\Delta\chi - \alpha^2(\chi \cdot \nabla)\chi.$$

Applying the solenoidal projection P to (NS$'$), we can rewrite (NS$'$) as follows:

$$\partial_t u + Au = -P((u \cdot \nabla)u) - PF(u) + PG(\alpha, \chi), \quad u(0) = v(0) - \alpha(0)\chi, \tag{PNS$'$}$$

where $A = -P\Delta$ is the Stokes operator. This is further transformed into the nonlinear integral equation:

$$\begin{aligned} u(t) = T(t)u(0) &- \int_0^t T(t-s)P((u \cdot \nabla)u)(s)ds \\ &- \int_0^t T(t-s)PF(u)(s)ds + \int_0^t T(t-s)PG(\alpha, \chi)(s)ds. \end{aligned} \tag{IE}$$

We shall construct a unique time-local solution $u(t)$ of the integral solution (IE) by successive approximation, according to the following scheme:

$$\begin{aligned} u_0(t) &= T(t)u(0) + \int_0^t T(t-s)PG(\alpha, \chi)(s)ds, \\ u_{m+1}(t) &= u_0(t) - \int_0^t T(t-s)P((u_m \cdot \nabla)u_m)(s)ds - \int_0^t T(t-s)PF(u_m)(s)ds. \end{aligned} \tag{INT}$$

Before we estimate $u_0(t)$ and $u_{m+1}(t)$, we ready for the following proposition which is proved by elementary calculation.

PROPOSITION 5.2. *Let $1 < q \le r < \infty$ such that $1/q - 1/r < 1/n$. There holds the following estimate*:

$$\int_0^t \|\nabla^j T(t-s)P(g(s)f(\cdot))\|_{L^r} ds \le C_{q,r}\mathscr{A}\|f\|_{L^q} B\Big(-\frac{n}{2q} + \frac{n}{2r} + 1 - \frac{j}{2}, 1\Big) t^{-\frac{n}{2q} + \frac{n}{2r} + 1 - \frac{j}{2}}$$

for $f \in L^q(\Omega)$ and g with $\sup_{0<s<t} |g(s)| \le \mathscr{A}$, where $B(\cdot,\cdot)$ denotes the beta function.

Proof. We have

$$\int_0^t \|\nabla^j T(t-s)P(g(s)f(\cdot))\|_{L^r} ds \le C_{q,r} \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{j}{2}} \sup_{0<s<t}|g(s)|\|f\|_{L^q} ds$$
$$\le C_{q,r}\mathscr{A}\|f\|_{L^q}\int_0^1 (1-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{j}{2}} d\tau\ t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})+1-\frac{j}{2}}$$
$$\le C_{q,r}\mathscr{A}\|f\|_{L^q} B\Big(-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2},1\Big)t^{-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2}}.$$

□

Proof of Theorem 2.6.

(i) We shall solve (INT) for $n \ge 3$ by successive approximation. To this end we show by induction that the u_m exist and satisfy the following relations:

$$t^{\frac{1}{4}} u_m \in BC([0,T]; J^{2n}(\Omega)), \tag{5.8}$$

$$t^{\frac{1}{2}} \nabla u_m \in BC([0,T]; L^n(\Omega)) \tag{5.9}$$

with value zero at $t=0$ and

$$\sup_{0<t\le T}\big(t^{\frac{1}{4}}\|u_m(t)\|_{L^{2n}} + t^{\frac{1}{2}}\|\nabla u_m(t)\|_{L^n}\big) \le K_m. \tag{5.10}$$

In order to estimate $u_0(t)$, we set

$$\begin{aligned} u_0(t) &= T(t)u(0) + \int_0^t T(t-s)P(\alpha\Delta\chi)(s)ds \\ &\quad - \int_0^t T(t-s)P(\alpha^2(\chi\cdot\nabla)\chi)ds - \int_0^t T(t-s)P(\alpha'\chi)(s)ds \\ &=: T(t)u(0) + u_0^1(t) + u_0^2(t) + u_0^3(t). \end{aligned} \tag{5.11}$$

We shall show the estimate of u_0^j $(j=1,2,3)$. Setting

$$\mathscr{A} = \max\Big(\max_{0\le t\le T}|\alpha(t)|,\ \max_{0\le t\le T}|\alpha'(t)|\Big), \quad \mathscr{B}^j_{q,r} = B\Big(-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2},1\Big)$$

and using Proposition 5.2, we have

$$\big\|\nabla^j u_0^1(t)\big\|_{L^r} \le C_{q,r}\mathscr{A}\mathscr{B}^j_{q,r}\|\Delta\chi\|_{L^q} t^{-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2}}, \tag{5.12}$$

$$\big\|\nabla^j u_0^2(t)\big\|_{L^r} \le C_{q,r}\mathscr{A}^2\mathscr{B}^j_{q,r}\|\chi\|_{L^{2q}}\|\nabla\chi\|_{L^{2q}} t^{-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2}}, \tag{5.13}$$

$$\big\|\nabla^j u_0^3(t)\big\|_{L^r} \le C_{q,r}\mathscr{A}\mathscr{B}^j_{q,r}\|\chi\|_{L^q} t^{-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2}}, \tag{5.14}$$

where $n' < q \le r < \infty$ with $\frac{1}{q} - \frac{1}{r} < \frac{1}{n}$ for $j = 0, 1$.

By (5.11)–(5.14), for $n' \le \frac{n}{2} < q \le n$, there exists the positive number K_0 enjoying the following inequality:

$$\sup_{0<t\le T} \left(t^{\frac{1}{4}} \|u_0(t)\|_{L^{2n}} + t^{\frac{1}{2}} \|\nabla u_0(t)\|_{L^n} \right) \le K_0 \tag{5.15}$$

with

$$K_0 = K_0(T) = \sup_{0<t\le T} \left(t^{\frac{1}{4}} \|T(t)u(0)\|_{L^{2n}} + t^{\frac{1}{2}} \|\nabla T(t)u(0)\|_{L^n} \right)$$
$$+ C_{q,n} \mathscr{A} \left(\mathscr{B}^0_{q,2n} + \mathscr{B}^1_{q,n} \right) \left(\|\Delta\chi\|_{L^q} + \|\chi\|_{L^q} + \mathscr{A} \|\chi\|_{L^{2q}} \|\nabla\chi\|_{L^{2q}} \right) T^{\frac{3}{2} - \frac{n}{2q}}.$$

Note that we can take small $K_0 = K_0(T_*)$ when we restrict the time to some short interval $[0, T_*]$ since $u(0) \in J^n(\Omega)$.

The continuity at $t = 0$, with value zero, of the function (5.8) with $n = 0$ follows from the facts that the operator $t^{\frac{1}{4}} T(t)$ is uniformly bounded from J^n to J^{2n} and tends to zero strongly as $t \to 0$. A similar continuous property of (5.9) is shown similarly.

We shall proceed to the next step. Assuming now that (5.8) and (5.9) with (5.10) are true for m, we shall show those for $m + 1$. For simplicity, we set

$$\begin{aligned} u_{m+1}(t) &= u_0(t) - \int_0^t T(t-s) P((u_m \cdot \nabla) u_m)(s) ds \\ &\quad - \int_0^t T(t-s) P(\alpha(\chi \cdot \nabla) u_m)(s) ds - \int_0^t T(t-s) P(\alpha(u_m \cdot \nabla)\chi)(s) ds \\ &=: u_0(t) + u^1_{m+1}(t) + u^2_{m+1}(t) + u^3_{m+1}(t). \end{aligned} \tag{5.16}$$

Since we have already estimated the first term $u_0(t)$ in (5.16), we shall begin to estimate the second term $u^1_{m+1}(t)$. We have

$$\begin{aligned} \left\| \nabla^j u^1_{m+1}(t) \right\|_{L^r} &\le \int_0^t \left\| \nabla^j T(t-s) P((u_m \cdot \nabla) u_m)(s) \right\|_{L^r} ds \\ &\le C_{n,r} \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{2n} + \frac{1}{n} - \frac{1}{r}) - \frac{j}{2}} \|u_m(s)\|_{L^{2n}} \|\nabla u_m(s)\|_{L^n} ds \\ &\le C_{n,r} K_m^2 \int_0^t (t-s)^{-\frac{n}{2}(\frac{3}{2n} - \frac{1}{r}) - \frac{j}{2}} s^{-\frac{3}{4}} ds \\ &= C_{n,r} K_m^2 B\left(\frac{1}{4} + \frac{n}{2r} - \frac{j}{2}, \frac{1}{4} \right) t^{\frac{n}{2r} - \frac{1}{2} - \frac{j}{2}}. \end{aligned}$$

Therefore we obtain

$$\sup_{0<t\le T} \left(t^{\frac{1}{4}} \|u^1_{m+1}(t)\|_{L^{2n}} + t^{\frac{1}{2}} \|\nabla u^1_{m+1}(t)\|_{L^n} \right) \le C_n K_m^2 \left(B\left(\frac{1}{2}, \frac{1}{4} \right) + B\left(\frac{1}{4}, \frac{1}{4} \right) \right). \tag{5.17}$$

Next we fix $n < \widetilde{q} \le 2n$ and go on estimating the $u^2_{m+1}(t)$ and $u^3_{m+1}(t)$ in a similar way to $u^1_{m+1}(t)$. We see

$$\sup_{0<t\le T} \left(t^{\frac14}\|u^2_{m+1}(t)\|_{L^{2n}} + t^{\frac12}\|\nabla u^2_{m+1}(t)\|_{L^n}\right) \le C_{\widetilde{q}}\mathscr{A}K_m\|\chi\|_{L^{\widetilde{q}}}\left(B\left(\frac34 - \frac{n}{2\widetilde{q}}, \frac12\right) + B\left(\frac12 - \frac{n}{2\widetilde{q}}, \frac12\right)\right)T^{\frac12 - \frac{n}{2\widetilde{q}}} \tag{5.18}$$

and

$$\sup_{0<t\le T} \left(t^{\frac14}\|u^3_{m+1}(t)\|_{L^{2n}} + t^{\frac12}\|\nabla u^3_{m+1}(t)\|_{L^n}\right) \le C_{\widetilde{q}}\mathscr{A}K_m\|\nabla\chi\|_{L^{\widetilde{q}}}\left(B\left(1 - \frac{n}{2\widetilde{q}}, \frac34\right) + B\left(\frac34 - \frac{n}{2\widetilde{q}}, \frac34\right)\right)T^{1 - \frac{n}{2\widetilde{q}}}. \tag{5.19}$$

By (5.17)–(5.19), we have

$$\sup_{0<t\le T} \left(t^{\frac14}\|u_{m+1}(t)\|_{L^{2n}} + t^{\frac12}\|\nabla u_m(t)\|_{L^n}\right) \le K_{m+1}$$

with

$$K_{m+1} = K_0 + LK_m + NK_m^2,$$

where

$$\begin{aligned} L &= C_{\widetilde{q}}\mathscr{A}\|\chi\|_{L^{\widetilde{q}}}\left(B\left(\frac34 - \frac{n}{2\widetilde{q}},\ \frac12\right) + B\left(\frac12 - \frac{n}{2\widetilde{q}},\ \frac12\right)\right)T^{\frac12 - \frac{n}{2\widetilde{q}}} \\ &\quad + C_{\widetilde{q}}\mathscr{A}\|\nabla\chi\|_{L^{\widetilde{q}}}\left(B\left(1 - \frac{n}{2\widetilde{q}},\ \frac34\right) + B\left(\frac34 - \frac{n}{2\widetilde{q}},\ \frac34\right)\right)T^{1 - \frac{n}{2\widetilde{q}}}, \\ N &= C_{n,r}\left(B\left(\frac12,\ \frac14\right) + B\left(\frac14,\ \frac14\right)\right). \end{aligned}$$

One can replace T by some small $T_* \in (0, T]$ so that $L < 1$ and $K_0 < \frac{(1-L)^2}{4N}$. Set

$$K := \frac{(1-L) - \sqrt{(1-L)^2 - 4NK_0}}{2N}.$$

We easily find that $K_0 < K$ and that $K_m \le K$ implies

$$K_{m+1} \le K_0 + LK + NK^2 = K.$$

We thus obtain

$$\sup_{0<t\le T_*}\left(t^{\frac14}\|u_m(t)\|_{L^{2n}}+t^{\frac12}\|\nabla u_m(t)\|_{L^n}\right)\le K$$

for all m. This together with the same calculations for

$$\gamma_m(T_*):=\sup_{0<t\le T_*}\left(t^{\frac14}\|u_m(t)-u_{m-1}(t)\|_{L^{2n}}+t^{\frac12}\|\nabla u_m(t)-\nabla u_{m-1}(t)\|_{L^n}\right)$$

as above yields

$$\gamma_{m+1}(T_*)\le\left\{CK_0(T_*)+C_{\tilde q}\mathscr{A}\left(\|\chi\|_{L^{\tilde q}}T_*^{\frac12-\frac{n}{2\tilde q}}+\|\nabla\chi\|_{L^{\tilde q}}T_*^{1-\frac{n}{2\tilde q}}\right)\right\}\gamma_m(T_*)$$

for all m. When we take still smaller T_* (if necessary), we see that the sequence $\{u_m\}$ converges uniformly in t as $m\to\infty$ to a function u, which satisfies (IE) for $0<t\le T_*$ and is of class

$$t^{\frac14}u\in BC([0,T_*];J^{2n}(\Omega)),\quad t^{\frac12}\nabla u\in BC([0,T_*];L^n(\Omega))$$

with

$$\sup_{0<t\le T_*}\left(t^{\frac14}\|u(t)\|_{L^{2n}}+t^{\frac12}\|\nabla u(t)\|_{L^n}\right)\le K.$$

By use of this we estimate (IE) to obtain (2.17) for $n\le r\le\infty$ with initial condition and (2.18) for $n\le r<2n$; and then, a bootstrap argument yields (2.18) for any $r<\infty$. This leads to a local solution $u(t)$ to (IE) with desired estimates. Since $\alpha\in C^{1,\theta}$, the solution $u(t)$ actually becomes a strong one (see Tanabe [**38**]). We thus complete the proof of Theorem 2.6 for $n\ge 3$.

(ii) We shall show the outline of the proof. Let $n=2<p<\infty$ and $u(0)\in J^p(\Omega)$. Then, by using successive approximation scheme (INT) again, we can show the existence of a unique solution u to (IE), which satisfies

$$t^{\frac{1}{2p}}u\in BC([0,T_*];J^{2p}(\Omega)),\quad t^{\frac12}\nabla u\in BC([0,T_*];L^p(\Omega))$$

with

$$\sup_{0<t\le T_*}\left(t^{\frac{1}{2p}}\|u(t)\|_{L^{2p}}+t^{\frac12}\|\nabla u(t)\|_{L^p}\right)\le K.$$

Theorem 2.6 (ii) is thus proved in the same way as the case where $n\ge 3$. □

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