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5-moves and Montesinos links

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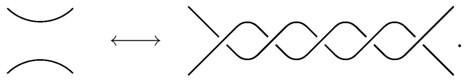
Abstract. We describe several classes of Montesinos links up to mutation and 5-move equivalence, and obtain from this a Jones and Kauffman polynomial test for a Montesinos link.

1. Introduction.

k-moves are a family of natural operations on knot and link diagrams. They were introduced by Nakanishi, and were studied first more systematically by Przytycki [**P**]. The following picture shows as an example the move for $k = 5$:



(1)

We call two links *k-move equivalent*, or simpler *k-equivalent*, if they are related by a sequence of k-moves (or their inverses). The case $k = 2$ is the usual crossing change. The cases $k = 3, 4$ were connected to two long-standing problems. The 3-move conjecture stated that all links are 3-move equivalent to a trivial link. It was refuted only recently in [**DP**]. The conjecture is known to be true for many links, for example 3-algebraic links [**PT**], links of braid index 4 and 5 [**Ch**] (latter with the exception of one equivalence class, which later provided a counterexample), and knots of weak genus two [**St3**]. The 4-move conjecture states that all knots are 4-move equivalent to the unknot. This conjecture remains open, though partial confirmations exist, and (in the general case) counterexamples are suspected (see [**As**], [**P2**], and also [**St3**]).

The difficulty with the cases $k = 3, 4$ is that, while equivalence classes are rather few and large, (possibly exactly because of this) it is impossible to obtain essential information on them using polynomial (or other easy to compute) invariants.

However, the case $k = 5$ is different. Now there are some invariants that come from the Jones V and Kauffman F polynomial. Unfortunately, the equivalence classes become many, and their intersection with a meaningful family of links is difficult to describe. Recently, Przytycki-Dąbkowski-Ishiwata (see [**I**]) succeeded in determining the classes for rational (or 2-bridge) links. The aim of this paper is to extend their result to

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Montesinos links (up to mutation). Theorem 3.2 gives representatives of all equivalence classes of Montesinos links L, and in Theorem 4.2 we distinguish most of these classes. In particular, we can completely determine the 5-move equivalence classes under the condition that $V(4_1)\nmid V(L)$, or when L is a pretzel link (see Remark 4.2).

The motivation for our work is to gain a condition on the polynomial invariants of Montesinos links. A few such conditions were obtained for partial classes, but all conditions for a general Montesinos link we have so far are inefficient, or not easy to test.

In [**LT**] semiadequate links were introduced. It was observed that Montesinos links are semiadequate, and that for such links one of the leading or trailing coefficients of the Jones polynomial V must be ± 1. (In [**St4**] we understood also coefficients 2 and 3, and in particular proved that the Jones polynomial is non-trivial.) However, this property is not helpful as a Montesinos link test, because it is satisfied for many (other) links. Similarly impractical is the semiadequacy condition on the Kauffman F polynomial of [**Th**]. (The simplest knots whose polynomial shows negative "critical line" coefficients on either side have 15 crossings; see [**St4**].) For the Alexander polynomial no conditions whatsoever are known, that is, it is possible that every admissible Alexander polynomial is realized by a Montesinos link. In [**LT**] it was shown how to determine the crossing number of a Montesinos link from the V and F polynomial, but a(n extensive, though systematic) diagram verification is not promising either as a Montesinos link test.

Now, in contrast, since we can evaluate the 5-move invariants of the Jones and Kauffman polynomial on the classes of Montesinos links we obtain (Theorems 4.1 and 6.1), we gain a condition on these polynomials, which turns out easy to verify. (It makes no assumption on the diagram we perform it on, except, of course, that one can evaluate the polynomials.) We will give some examples that show how to apply our test. We also show how it can be sharpened for unknotting number one knots, using the result of Motegi [**Mo**].

2. Preliminaries.

2.1. Polynomial invariants.

Let V be the Jones polynomial [**J**], Q the BLMH polynomial and F the Kauffman polynomial [**Kf**]. We give a basic description of these invariants.

Recall first the construction of the Kauffman bracket in [**Kf 2**]. The Kauffman bracket $[D]$ of a diagram D is a Laurent polynomial in a variable A, obtained by summing over all states the terms

$$A^{\#A-\#B}(-A^2-A^{-2})^{|S|-1}. \tag{2}$$

A state is a choice of splittings of type A or B for any single crossing (see Figure 1), $\#A$ and $\#B$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in a state.

The Jones polynomial of a link L is related to the Kauffman bracket of some diagram D of L by

$$V_L(t) = (-t^{-3/4})^{-w(D)}\,[D]\Big|_{A=t^{-1/4}}. \tag{3}$$

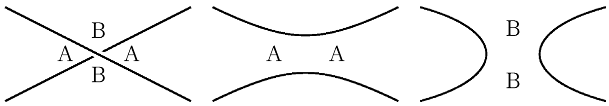
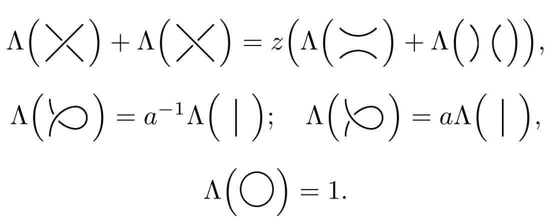


Figure 1. The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counter-clockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.

The *Kauffman polynomial* [**Kf**] F is usually defined via a regular isotopy invariant $\Lambda(a, z)$ of unoriented links.

We use here a slightly different convention for the variables in F, differing from [**Kf**], [**Th**] by the interchange of a and a^{-1}. Thus in particular we have the relation $F(D)(a, z) = a^{w(D)}\Lambda(D)(a, z)$, where $w(D)$ is the writhe of a link diagram D, and $\Lambda(D)$ is the writhe-unnormalized version of F. Λ is given in our convention by the properties



(4)

The BLMH polynomial Q is most easily specified by $Q(z) = F(1, z)$.

2.2. Families of links.

In the following we define rational, pretzel and Montesinos links according to Conway [**Co**].

DEFINITION 2.1. A *tangle diagram* is a diagram consisting of strands crossing each other, and having 4 ends. A *rational* tangle diagram is the one that can be obtained from the primitive Conway tangle diagrams by iterated left-associative product in the

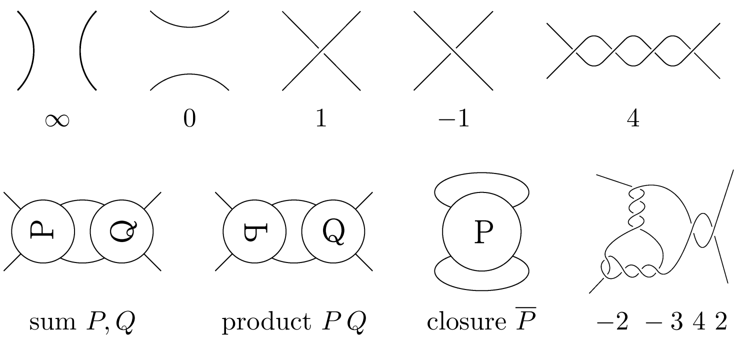


Figure 2. Conway's tangles and operations with them. (The designation 'product' is very unlucky, as this operation is neither commutative, nor associative, nor is it distributive with 'sum'. Also, 'sum' is associative, but not commutative.)

way displayed in Figure 2. (A simple but typical example is shown in the figure.)

Let the *continued* (or *iterated*) *fraction* $[[s_1,\dots,s_r]]$ for integers s_i be defined inductively by $[[s]]=s$ and

$$[[s_1,\dots,s_{r-1},s_r]] = s_r + \frac{1}{[[s_1,\dots,s_{r-1}]]}.$$

The rational tangle $T(p/q)$ is the one with Conway notation $c_1\ c_2\dots c_n$, when the c_i are chosen so that

$$[[c_1,c_2,c_3,\dots,c_n]] = \frac{p}{q}. \tag{5}$$

One can assume without loss of generality that $(p,q)=1$, and $0<|q|<p$. A *rational* (*or* 2-*bridge*) *link* $S(p,q)$ is the closure of $T(p/q)$.

Montesinos links (see e.g. [**LT**]) are generalizations of pretzel and rational links and special types of arborescent links. They are denoted in the form $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$, where e,p_i,q_i are integers, $(p_i,q_i)=1$ and $0<|q_i|<p_i$. Sometimes e is called the *integer part*, and the $\frac{q_i}{p_i}$ are called *fractional parts*. They both together form the *entries*. If $e=0$, it is omitted in the notation. A *pretzel link* is a Montesinos link with all $|q_i|=1$.

To visualize the Montesinos link from a notation, let p_i/q_i be continued fractions of rational tangles $c_{1,i}\dots c_{n_i,i}$ with $[[c_{1,i},c_{2,i},c_{3,i},\dots,c_{l_i,i}]]=\frac{p_i}{q_i}$. Then $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$ is the link that corresponds to the Conway notation

$$(c_{1,1}\dots c_{l_1,1}),(c_{1,2}\dots c_{l_2,2}),\dots,(c_{1,n}\dots c_{l_n,n}),e0.$$

The defining convention is that all $q_i>0$ and if $p_i<0$, then the tangle is composed so as to give a non-alternating sum with a tangle with $p_{i\pm1}>0$. This defines the diagram up to mirroring.

An easy exercise shows that if $q_i>0$ resp. $q_i<0$, then

$$M(\dots,q_i/p_i,\dots,e) = M(\dots,(q_i\mp p_i)/p_i,\dots,e\pm 1), \tag{6}$$

i.e. both forms represent the same link (up to mirroring).

Note that our notation *may differ* from other authors' by the sign of e and/or multiplicative inversion of the fractional parts. For example $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$ is denoted as $\mathfrak{m}(e;\frac{p_1}{q_1},\dots,\frac{p_n}{q_n})$ in [**BZ**, Definition 12.28] and as $M(-e;(p_1,q_1),\dots,(p_n,q_n))$ in [**Mo**] and the tables of [**Kw**].

Our convention chosen here appears more natural – the identity (6) preserves the sum of all entries, and an integer entry can be formally regarded as a fractional part. Theorem 12.29 in [**BZ**] asserts that the entry sum, together with the vector of the fractional parts, modulo $\boldsymbol{Z}$ and up to cyclic permutations and reversal, determine the isotopy class of a Montesinos link L. So the number n of fractional parts is an invariant of L; we call it the *length* of L.

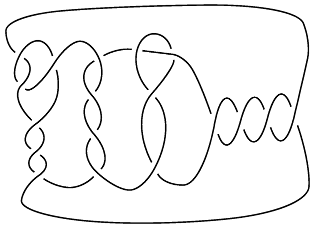


Figure 3. The Montesinos knot $M(3/11,-1/4,2/5,4)$ with Conway notation $(213,214,22,40)$.

If the length $n<3$, an easy observation shows that the Montesinos link is in fact a rational link. Let us then write rational links as Montesinos links of length 1. For example, $M(1)=M(\infty)$ is the unknot, and $M(0)$ is the 2-component unlink, while $M(2/5)=M(5/2)$ is the Figure-8 knot. We can also formally incorporate the connected sum of some number of rational links into our Montesinos link notation by replacing q_n/p_n by ∞. So for example $M(1/2,\infty,\infty)=M(1/2,\infty,\infty,1)=M(1/2)\# M(\infty)$, which is a Hopf link with an extra trivial split component. While such links fit into our notation, they are nonetheless not to be regarded as genuine Montesinos links.

In the following the mirroring convention in the notation $M(q_1/p_1,\ldots,q_n/p_n,e)$ will be so that a $+1$ integer twist in e is a crossing whose A-splicing gives an ∞-tangle. So for example, the positive (right-hand) trefoil is $M(-3)$. (If $e=0$, use (6) to make it ±1, and specify the mirroring accordingly.)

Geometric properties of Montesinos links are discussed in detail in [**BZ**]. A typical example is shown on Figure 3.

3. 5-equivalence of Montesinos links.

Definition 3.1. We denote the equivalence of polynomials in $\boldsymbol{Z}[t^{\pm1/2}]$ modulo $X=(t^5+1)/(t+1)$ and multiplication with $\pm t^{k/2}$ by $\doteq$. By $\bar V$ we denote the equivalence class of a polynomial V under $\doteq$. For $c\in\boldsymbol{C}$, we denote by $\tilde c$ the set of complex numbers obtained from c by multiplication with a 20-th root of unity. If $\tilde c_1=\tilde c_2$, we write also $c_1\simeq c_2$.

Proposition 3.1 (Przytycki-Ishiwata [**I**]). *$\bar V$ is an invariant of 5-moves.*

The roots of $X=(t^5+1)/(t+1)$ are the primitive 10-th roots of unity. These are two pairs of conjugate complex numbers. Since, when working with real coefficients, conjugate complex numbers are equivalent, we have two roots to work with, $e^{\pi i/5}$ and $e^{3\pi i/5}$. In particular we obtain from the previous proposition:

Corollary 3.1 (Przytycki-Ishiwata). *For a link L and $n=1,3$, let $x_n(L)=V_L(e^{n\pi i/5})$. Then the quantities $\gamma_n=\widetilde{x_n}$, and in particular $v_n=|x_n|$ are invariants of 5-moves.*

Remark 3.1. The powers of t in V are either all integral (for odd number of components) or half-integral (for even number). This implies that if $x_n(L)=x_n(L')\cdot$

$e^{k\pi i/10} \neq 0$ for $n = 1$ or 3 (and $k \in \boldsymbol{Z}$), then k is even resp. odd if L, L' have the same resp. opposite component number parity. So the indeterminacy of $\widetilde{x_n}$ is really just up to 10-th roots of unity.

Concerning Q, we have the following easy to show fact (see for example Lemma 6.1 below).

Proposition 3.2. *$Q(2\cos 2\pi/5)$ and $Q(2\cos 4\pi/5)$ are invariants of 5-moves.*

These values of Q are studied by Jones [**J2**] and Rong [**R**], who show that they are equal to $\pm\sqrt{5}^{d(L)}$, where $d(L) := \dim_{\boldsymbol{Z}_5} H_1(D_L, \boldsymbol{Z}_5)$ is the number of torsion numbers divisible by 5 of the homology of the 2-branched cover D_L. This special form makes the Q values of limited use as detectors of 5-move (in)equivalence, but nonetheless one should not write them off. In particular the sign can be useful in some cases.

Ishiwata [**I**] succeeded in determining the 5-move equivalence classes of rational tangles.

Theorem 3.1 (Ishiwata). *Rational tangles are 5-move equivalent to one of the 12 basic tangles*

$$0,\ \infty,\ \pm1,\ \pm2,\ \pm1/2,\ \pm3/2,\ 2/5,\ 5/2.$$

One obtains immediately from that the result for rational (2-bridge) links.

Corollary 3.2 (Ishiwata). *Rational links are 5-move equivalent to one of the unknot, 2-component unlink, Figure-8 knot or Hopf link.*

This means in particular that v_1 and v_3 take only 4 values on rational links. (They are indeed different for the 4 classes, so the classes are distinct.) Nonetheless, some very strong conditions on polynomial invariants of 2-bridge links are known. For the Alexander polynomial such a condition is formulated in [**Mu**]. The test of Kanenobu [**K**] using V and Q is particularly efficient; see also [**St**]. Here we use some of Ishiwata's work to extend her result to Montesinos links. Our goal is to obtain conditions on the polynomial invariants of Montesinos links (which are much fewer in previous literature, and less efficient).

Theorem 3.2. *A Montesinos link is equivalent up to 5-moves and mutations to one of*

1. *$M(-1/2,\dots,-1/2,1/2,\dots,1/2)$ with $k \geq 0$ entries $1/2$ and $0 \leq l \leq 4$ entries $-1/2$, s.t. $l+k \geq 3$,*
2. *$M(1/2,\dots,1/2,2/5,\dots,2/5)$ with $k \geq 0$ entries $1/2$, and $1 \leq l$ entries $2/5$, s.t. $l+k \geq 3$,*
3. *the connected sum of some number (possibly no) of Hopf links, with possible trivial split components,*
4. *the connected sum of some number (at least one) of Figure-8 knots with a link of Form 3, or*
5. *$M(1/2,1/2,1/2,1)$.*

Proof. We apply first Theorem 3.1 to reduce all tangles to the 12 basic ones. If one of the tangles is ∞, we are down to the connected sum of rational links, and with Corollary 3.2 have the third or fourth form.

So we assume henceforth that we have no ∞ tangle. Then the 11 remaining basic tangles have non-integer parts $\pm 1/2$ and $2/5$. We assume that at least three non-integer parts occur, otherwise we have again a rational link.

Now assume that at least one tangle $2/5$ occurs. Then we use the observation of Ishiwata (made in the proof of Theorem 3.1) that the $2/5$ tangle acts (up to 5-moves) as an "annihilator" of integer twists. Since one can make all $-1/2$ to $1/2$ up to integer twists, we have the second form.

So assume below that we have only $\pm 1/2$ and possible integer twists. It is an easy observation that 5 copies of $1/2$ are 5-move equivalent to 5 copies of $-1/2$, so we can always make the $-1/2$ to be at most 4. Moreover, the normal form of Montesinos links teaches that if have $n \neq 0$ integer twists, then we can assume that all $\pm 1/2$ have the sign of n. (If $n = 0$ we have the first form.)

Assume first $n > 0$. Clearly we can achieve that $n < 5$. Now if the number l of '$1/2$' satisfies $l \geq 5 - n$, we can make the n integer twists to $5 - n$ of opposite sign, and annihilate them making $5 - n$ of the l copies of $1/2$ to $-1/2$. Since $0 < n < 5$ and $l \geq 3$, there is only one case where this procedure does not work, namely when $l = 3$ and $n = 1$. In this case we have the link of the fifth form. Otherwise we achieve the first form.

If $n < 0$ we can apply the same argument, since the family of links of the first form is invariant (by making the $-1/2$ to be < 5 under 5-equivalence) to the family of their mirror images. Also, the link in the fifth form (as well as all links of Forms 2, 3 and 4) is equivalent to its mirror image. $\square$

Remark 3.2. Note that for pretzel links, Form 2 does not occur, and the only link of Form 4 is 4_1.

Remark 3.3. The connected sum (of links) in Forms 3 and 4 is generally not well-determined, but its ambiguity is removed under 5-equivalence. If L is a 2-component rational link factor, make it into a knot by a 5-move, slide the other connected sum factors into the (part of the former) other component, and undo the initial 5-move. The (not very natural) distinction of Forms 3 and 4 will become clear with Theorem 4.1 below.

Remark 3.4. Taking into account in particular the previous remark, the problem to remove mutation in the description of Theorem 3.2 is easily located to lie within the links of Form 2. It is the permutability of the entries $1/2, 2/5$, which is (basically) the question whether the 1/2,2/5 Montesinos tangle (the one with Conway Notation 2,22) is 5-equivalent to its (mutant) 2/5,1/2 Montesinos tangle.

4. Jones polynomial of Montesinos links.

4.1. Formulas for the Jones polynomial.

We succeeded in evaluating $\bar{V}$ on the representatives in Theorem 3.2, and so we have

Theorem 4.1. *For a Montesinos link L, the reduced Jones polynomial $\bar{V}_L$ equals* *one of*

1. *Form* 1, *with* $k+l \geq 3$, $k,l \geq 0$, $l \leq 4$:

$$V_1(k,l) = \frac{(-t-1/t)^{k+l} + (-1/t)^l(1-t)^{k+l}(t+1+1/t)}{-\sqrt{t}-\frac{1}{\sqrt{t}}}$$

2. *Form* 2, *with* $k+l \geq 3$, $k, l-1 \geq 0$:

$$V_2(k,l) = \frac{(1-t^2)^l(1-t)^k(t+1+1/t)}{-\sqrt{t}-\frac{1}{\sqrt{t}}}$$

3. *Form* 3, *with* $k,l \geq 0$:

$$V_3(k,l) = \left(-\sqrt{t}-\frac{1}{\sqrt{t}}\right)^l\left(-t-\frac{1}{t}\right)^k$$

4. *Form* 4: 0
5. *Form* 5:

$$V_5 = \frac{(-t-1/t)^3 - (1-t)^3t(t+1+1/t)}{-\sqrt{t}-\frac{1}{\sqrt{t}}}$$

The proof of this theorem bases on routine Kauffman bracket skein module calculation, and we omit it. (The reader may consult [**St2**] for some explanation on this kind of calculation.) The given polynomials correspond (up to units) to the Jones polynomials of the knots, except for Form 2, where the numerator was simplified by reducing modulo $X = (t^5+1)/(t+1)$. (For the calculation, we also used $M(-1/2,\dots,-1/2,2/5,\dots,2/5)$, which is equivalent.) The fraction may therefore not be a genuine polynomial in $\boldsymbol{Z}[t^{\pm1}]$. Instead it should be considered lying in the field $\boldsymbol{Z}[t^{\pm1}]/\langle X\rangle$ (where the division is possible).

REMARK 4.1. We should remark the following relation of these forms to $d(L)$.

1. Form 1: $d(L) \leq 1$ (see the argument for Form 2 below). Whether $d(L) = 0$ or 1 can then be decided by looking at the (5-divisibility of the) determinant. By a simple calculation we find

$$d(L) = \begin{cases} 0 & \text{if } 5 \nmid k-l \\ 1 & \text{if } 5 \mid k-l \end{cases}$$

2. Form 2: $|d(L)-l| \leq 1$. This follows because by a change from a 0 tangle to an infinity tangle, $d(L)$ is altered by at most ±1, and we can obtain the connected sum L' of k Hopf links and l Figure-8-knots, and $d(L') = l$.
3. Form 3: $l = d(L)$
4. Form 4: $d(L) > 0$

5. Form 5: By direct calculation the determinant is 20, so $d(L) = 1$.

For the following calculations, it is useful to compile a few norms of complex numbers that will repeatedly occur.

norm of	$t = e^{\pi i/5}$	$t = e^{3\pi i/5}$
$1-t$	$\frac{\sqrt{5}-1}{2} \approx 0.618034$	$\frac{1+\sqrt{5}}{2} \approx 1.618034$
$1+t$	$\frac{\sqrt{2\sqrt{5}+10}}{2} \approx 1.902113$	$\sqrt{\frac{5-\sqrt{5}}{2}} \approx 1.175571$
$1+t^2$	$\frac{\sqrt{5}+1}{2} \approx 1.618034$	$\frac{\sqrt{5}-1}{2} \approx 0.618034$
$1-t^2$	$\sqrt{\frac{5-\sqrt{5}}{2}} \approx 1.175571$	$\frac{\sqrt{2\sqrt{5}+10}}{2} \approx 1.902113$
$1+t+t^2$	$\frac{\sqrt{5}+3}{2} \approx 2.618034$	$\frac{3-\sqrt{5}}{2} \approx 0.381966$

(7)

Using this table, one easily sees

Corollary 4.1. *The set*

$$\big\{\big(|V(e^{\pi i/5})|, |V(e^{3\pi i/5})|\big) : L \text{ is a Montesinos link}\big\}$$

as a subset of $\boldsymbol{R}^2$ *is discrete.*

Discrete is to be understood here so that the intersection with any ball is finite. (It is in fact in size at most logarithmic in the radius of a ball centered at the origin $(0,0)$.)

Proof. We can treat the families $V_{1,2,3}$ separately. In case of $V_{1,3}$, the invariant v_1 diverges (exponentially) when $k + l \to \infty$. For V_2 take v_3. □

The pretzel links are those for which Form 2 does not occur, and so we have

Corollary 4.2. *The set*

$$\big\{|V(e^{\pi i/5})| : L \text{ is a pretzel link}\big\} \subset \boldsymbol{R}$$

is discrete.

Note that this gives a very strong condition on the polynomials of pretzel or Montesinos links. For a general link, there is no particular feature of v_1 or v_3 to be expected. For arbitrary links they will be dense in $\boldsymbol{R}_+$. Jones claimed in [**J**] (and I wrote down an argument in [**St3**]) that v_1 is dense in an interval on closed 4-braids. On the opposite hand, while several other conditions on invariants of Montesinos links are known, these are all difficult to test and/or assume additional properties of the diagram.

For example, for a Montesinos link L some properties of $\pi_1(D_L)$ are known, where D_L is the double branched cover [**BZ**]. While it is easy to gain a presentation of $\pi_1(D_L)$ from any diagram of L, the decision problem of such properties from this presentation seems highly difficult.

Also, the work in [**LT**], [**Th**] implies that any Montesinos link has a minimal (in crossing number) Montesinos diagram. However, it is not clear how to find this diagram starting from a given (not mandatorily minimal) one. If the link is alternating, all minimal diagrams will be Montesinos diagrams. However, for non-alternating knots there may be even non-Montesinos minimal crossing diagrams; the knots 10_{145}, 10_{146} and 10_{147} are examples.

In contrast, our result allows for a (not completely exact but) efficient test of the Montesinos property. It is in fact this condition, arising from Ishiwata's result and the detection of 5-moves by the polynomials, that motivated the present work. We give some examples of the application of our condition in Section 5. First we show, however, that the invariants we have in hand suffice to distinguish most equivalence classes in Theorem 3.2.

4.2. Distinction of Forms.

Theorem 4.2. *The links in Theorem* 3.2 *determine exactly the equivalence classes of Montesinos links under mutation and* 5*-moves, except possibly* 5*-move equivalence of different links of Form* 4*.*

Links of Form 4 (originally overlooked by myself) seem difficult to distinguish, when the Jones-Rong values of Q in Proposition 3.2 (and in particular d) fail. The simplest examples are the pairs $(4_1, 4_1\#H\#H)$ and $(4_1\#4_1,\ 4_1\#U_2\#H)$, where U_2 is the 2-component unlink and H the Hopf link. (The invariants of Section 6 seem not helpful either.)

Mutations cause similar difficulties. The massive failure of polynomial invariants to detect mutation results in a serious lack of tools to examine 5-equivalence of mutants. While it is almost certain that such examples exist, probably no simpler invariants than the (almost incomputable) Burnside groups [**DP**] could be applied to show it. However, note that at least it is known that any mutant of a Montesinos link is again a Montesinos link, so that the study of 5-equivalence classes of mutants among Montesinos links would reduce to the mutants of the representative links in Theorem 4.1, as explained in Remark 3.4.

Remark 4.2. Note that by the Theorem, the links L of Form 4 are exactly those whose Jones polynomial is divisible by the polynomial of the Figure-8 knot. So ambiguities will disappear if we assume $V(4_1) \nmid V(L)$. The problems with both Form 4 and mutations become also trivial by Remark 3.2, when restricting oneself to pretzel links. See Remark 5.1 for one more case.

Proof of Theorem 4.2. We check case by case possible duplications of $\bar{V}_L$ and $d(L)$. Often v_1 and v_3, or sometimes at least γ_1 and γ_3, already suffice to distinguish the forms. We will show that all these invariants coincide only if the links are identical (except Form 4, and except the final case, where we are lead to apply the Q polynomial to arrive at this conclusion). We remark also that $V_5 = V_1(4, -1)$. Although the value $l = -1$ in Form 1 does not make sense knot-theoretically, from the point of view of formal algebra this identity will allow us to handle Form 5 often in the same way as Form 1.

We also meet the convention that in a comparison 'Form x vs Form y', the integers k, l are parameters that correspond to 'Form x', while k', l' correspond to 'Form y'.

- Form 1 vs Form 1: We need to deal with

$$V_1(k,l) \doteq V_1(k',l'). \tag{8}$$

Looking at $t = e^{\pi i/5}$, we find using the above values and $k+l \geq 3$,

$$\left|(-1/t)^l(1-t)^{k+l}(t+1+1/t)\right| \leq \left(\frac{\sqrt{5}-1}{2}\right)^3 \frac{\sqrt{5}+3}{2} = \frac{\sqrt{5}-1}{2}.$$

Using this inequality and the analogous one with k,l replaced by k',l', and taking norms in (8) we find

$$\left|\left|-t-\frac{1}{t}\right|^{k+l} - \left|-t-\frac{1}{t}\right|^{k'+l'}\right| \leq \sqrt{5}-1. \tag{9}$$

But $\left|-t-\frac{1}{t}\right| = \frac{\sqrt{5}+1}{2}$, and $k+l, k'+l' \geq 3$, so we see that $k+l = k'+l'$. We will argue that then (8) gives $5 \mid l-l'$, so $(k,l) = (k',l')$, as desired.

Indeed, set $\tilde{k} = k+l = k'+l'$. Now it is more convenient to use γ_n instead of v_n. So assume there are two different values $l_{1,2}$ of $l = 0,\dots,4$, such that $V_1(\tilde{k}-l,l)$ (are different but) differ by a 20-th root of unity. Now

$$|t+1| \cdot \left|V_1(\tilde{k}-l,l)\right| \geq \left|-t-\frac{1}{t}\right|^{\tilde{k}} - |1-t|^{\tilde{k}}\left|1+t+\frac{1}{t}\right|.$$

So $V_1(\tilde{k}-l_1,l_1) \simeq V_1(\tilde{k}-l_2,l_2)$ implies that

$$|t+1|\cdot\left|V_1(\tilde{k}-l_1,l_1)-V_1(\tilde{k}-l_2,l_2)\right| \geq \left|1-e^{\pi i/10}\right|\cdot\left[\left|-t-\frac{1}{t}\right|^{\tilde{k}}-|1-t|^{\tilde{k}}\left|1+t+\frac{1}{t}\right|\right].$$

On the other hand, by the definition of V_1,

$$|t+1|\cdot\left|V_1(\tilde{k}-l_1,l_1)-V_1(\tilde{k}-l_2,l_2)\right| \leq \frac{\sqrt{2\sqrt{5}+10}}{2}|1-t|^{\tilde{k}}\left|1+t+\frac{1}{t}\right|,$$

where the first factor on the right stands for the largest distance between two 5-th roots of unity (which are $-1/t$). Setting $t = e^{\pi i/5}$, and combining the last two inequalities, we obtain

$$\begin{aligned}&\frac{\sqrt{2\sqrt{5}+10}}{2}\cdot\left(\frac{\sqrt{5}-1}{2}\right)^{\tilde{k}}\frac{\sqrt{5}+3}{2}\\ &\quad\geq \left|1-e^{\pi i/10}\right|\cdot\left[\left(\frac{\sqrt{5}+1}{2}\right)^{\tilde{k}}-\left(\frac{\sqrt{5}-1}{2}\right)^{\tilde{k}}\cdot\frac{\sqrt{5}+3}{2}\right].\end{aligned}$$

So with $|1-e^{\pi i/10}| = \sqrt{2-\sqrt{\frac{\sqrt{5}+5}{2}}}$, we have

$$\left(\frac{\sqrt{5}+1}{2}\right)^{\tilde{k}} \leq \left(\frac{\sqrt{5}-1}{2}\right)^{\tilde{k}-2} \left[1+\frac{\sqrt{2\sqrt{5}+10}}{2\sqrt{2-\sqrt{\frac{\sqrt{5}+5}{2}}}}\right].$$

Evaluating the second factor on the right, we have

$$\left(\frac{\sqrt{5}+1}{2}\right)^{2\tilde{k}-2} \leq 7.07958\ldots,$$

so $\tilde{k} \leq 3$, that is, $\tilde{k} = 3$. These cases are easily checked directly. By direct calculation, we find that v_1 and v_3 distinguish the cases $l = 0, 3$ from those $l = 1, 2$. Then one verifies that

$$\left(\frac{V_1(l, 3-l)(e^{\pi i/5})}{V_1(3-l, l)(e^{\pi i/5})}\right)^{20} \neq 1$$

for $l = 0, 1$, and this case is finished.

- Form 1 vs Form 2: We assume

$$V_1(k, l) \doteq V_2(k', l'), \tag{10}$$

for $k + l \geq 3$, $\tilde{k} := k' + l' \geq 3$, which means

$$\begin{aligned} \left(-t-\frac{1}{t}\right)^{k+l} &+ \left(-\frac{1}{t}\right)^{l} (1-t)^{k+l}\left(t+1+\frac{1}{t}\right) \\ &\doteq (1-t)^{\tilde{k}}\left(t+1+\frac{1}{t}\right)\left(\frac{1-t^2}{1-t}\right)^{l'}. \end{aligned} \tag{11}$$

Using Remark 4.1, we see that for a link L fitting into both forms we have $d(L) \leq 1$, and so $l' \leq 2$. For $t = e^{\pi i/5}$, the base of the rightmost exponent in (11) is of norm > 1. We take norms in (11) and bring the second summand on the left hand side to the right. We obtain analogously to (9)

$$\left|-t-\frac{1}{t}\right|^{k+l} \leq \frac{\sqrt{5}-1}{2}\left(1+\frac{2\sqrt{5}+10}{4}\right) \approx 2.85\cdots < \left(\frac{\sqrt{5}+1}{2}\right)^{3},$$

which is impossible.

- Form 1 vs Form 3: Let $\tilde{k} = k' + l' \geq 0$. We must consider

$$\Big(-t-\frac{1}{t}\Big)^{k+l}+\Big(-\frac{1}{t}\Big)^{l}(1-t)^{k+l}\Big(t+1+\frac{1}{t}\Big)$$
$$\doteq\Big(-t-\frac{1}{t}\Big)^{\tilde{k}}\Big(-\sqrt{t}-\frac{1}{\sqrt{t}}\Big)\left(\frac{-\sqrt{t}-\frac{1}{\sqrt{t}}}{-t-\frac{1}{t}}\right)^{l'}. \tag{12}$$

By Remark 4.1 we have $l'=0$ if $5\nmid k-l$ and $l'=1$ otherwise. So

$$|1-t|^{k+l}\left|1+t+\frac{1}{t}\right|\le|t^2+1|^{k+l}+|t+1|\cdot|t^2+1|^{\tilde{k}}\max\left(1,\frac{|t+1|}{|t^2+1|}\right).$$

Look first at $t=e^{3\pi i/5}$. Since $|t+1|>1>|t^2+1|$, we have the second alternative in the maximum. Then

$$\left(\frac{\sqrt{5}+1}{2}\right)^{k+l}\cdot\frac{3-\sqrt{5}}{2}\le\left(\frac{\sqrt{5}-1}{2}\right)^{k+l}+\left(\frac{\sqrt{5}-1}{2}\right)^{\tilde{k}-1}\cdot\frac{5-\sqrt{5}}{2}.$$

So

$$\begin{aligned}\left(\frac{\sqrt{5}+1}{2}\right)^{k+l}&\le\left(\frac{\sqrt{5}-1}{2}\right)^{k+l-2}+\left(\frac{\sqrt{5}-1}{2}\right)^{\tilde{k}-3}\cdot\frac{5-\sqrt{5}}{2}\\&\le\frac{\sqrt{5}-1}{2}+\frac{5+3\sqrt{5}}{2}=2+2\sqrt{5}.\end{aligned}$$

The last inequality uses $\tilde{k}\ge 0$, $k+l\ge 3$. Then by calculation $k+l\le 3$, and so $k+l=3$. Now look at $t=e^{\pi i/5}$. Taking norms in (12), and this time minimizing with respect to $l'=0,1$, we find

$$\begin{aligned}&\left(\frac{\sqrt{5}+1}{2}\right)^{3}+\left(\frac{\sqrt{5}-1}{2}\right)^{3}\frac{\sqrt{5}+3}{2}\\&\quad=\left(\frac{\sqrt{5}+1}{2}\right)^{3}+\left(\frac{\sqrt{5}+1}{2}\right)^{-1}\ge\left(\frac{\sqrt{5}+1}{2}\right)^{\tilde{k}}\frac{\sqrt{2\sqrt{5}+10}}{2}.\end{aligned}$$

The left hand side evaluates to $(3+3\sqrt{5})/2$, and then

$$\left(\frac{\sqrt{5}+1}{2}\right)^{\tilde{k}}\le\frac{3+3\sqrt{5}}{\sqrt{2\sqrt{5}+10}}\approx 2.55,$$

so $\tilde{k}\le 1$. Thus it remains to check the cases $k'+l'\le 1$, $k+l=3$ (with $k,l,k'\ge 0$, $l\le 4$; and $l'=1$ if $5\mid k-l$ and $l'=0$ otherwise). It is easy to perform (still better by computer) these handful of comparisons; v_3 distinguishes all such $V_1(k,l)$ and $V_3(k',l')$.

- Form 1 vs Form 4: The same estimate as with 'Form 1 vs Form 1' works.
- Form 1 vs Form 5: We can apply the same argument as with 'Form 1 vs Form 1', since we did not use that $l > 0$ there. Then again we need to check only $\tilde{k} = 3$. Direct calculation shows that v_1 and v_3 distinguish the case $l = -1$ (in Form 5) from $l = 0, 3$ and $l = 1, 2$.
- Form 2 vs Form 2:

$$(1-t^2)^l(1-t)^k \doteq (1-t^2)^{l'}(1-t)^{k'}.$$

Since $|l - d(L)|, |l' - d(L)| \leq 1$, we have $|l - l'| \leq 2$. Since for $l = l'$ we are easily done, assume with loss of generality $l - l' \in \{1, 2\}$. Then

$$|1-t|^{k-k'} \in \left\{ \frac{1}{|1-t^2|}, \frac{1}{|1-t^2|^2} \right\}.$$

For $t = e^{\pi i/5}$ the numbers on the right are $0.850\dots$, $0.723\dots$, while the sequence on the left is 1, $0.618\dots$, $0.381\dots$, $0.236\dots$ etc.
- Form 2 vs Form 3: We have to check

$$(1-t^2)^l(1-t)^k\left(t+1+\frac{1}{t}\right) \doteq \left(-\sqrt{t}-\frac{1}{\sqrt{t}}\right)^{l'+1}\left(-t-\frac{1}{t}\right)^{k'}. \tag{13}$$

Now by Remark 4.1, $|l - l'| \leq 1$, so $\tilde{l} := l' + 1 - l = 0, 1, 2$. Taking norms in (13) and using $\tilde{l} = 0, 1, 2$, we have

$$|1-t|^{l+k} = \frac{|t+1|^{\tilde{l}}|t^2+1|^{k'}}{\left|1+t+\frac{1}{t}\right|}.$$

Now for $t = e^{\pi i/5}$, we get

$$\left(\frac{\sqrt{5}-1}{2}\right)^{k+l} = \frac{\left(\frac{\sqrt{5}+1}{2}\right)^{k'} \cdot \left(\frac{\sqrt{2\sqrt{5}+10}}{2}\right)^{\tilde{l}}}{\frac{\sqrt{5}+3}{2}} = \left(\frac{\sqrt{5}+1}{2}\right)^{k'-2} \cdot \left(\frac{\sqrt{2\sqrt{5}+10}}{2}\right)^{\tilde{l}}.$$

Now the right hand side is at least $\left(\frac{\sqrt{5}-1}{2}\right)^2$, while the left hand side for $k + l \geq 3$ is at most $\left(\frac{\sqrt{5}-1}{2}\right)^3$, a contradiction.
- Form 2 vs Form 4: trivial
- Form 5 vs Form 2: We apply the same argument as with 'Form 1 vs Form 2'.
- Form 3 vs Form 3: If $V_3(k,l) \doteq V_3(k',l')$, we have $l = l'$ by Remark 4.1, and then $k = k'$ is easy to see using v_1, v_3.
- Form 3 vs Form 4: trivial
- Form 5 vs Form 3: By the same norm estimate as in 'Form 1 vs Form 3', we are left with $k' + l' \leq 1$, and since $d(L) = 1$ in Form 5, also $l' = 1$, so $k' = 0$. This case, however, is not ruled out by v_1 or v_3, and in fact not even by $\bar{V}$. It is the

question of 5-equivalence of the link 7_1^3 in [**Ro**, appendix] (which is the last link in Theorem 4.1) and the 2-component unlink. This problem was encountered also in Ishiwata's tabulation [**I**] of 5-equivalence of links up to 9 crossings. The problem is now resolved using the Q polynomial. Clearly for both links $d(L) = 1$. However, exactly the sign, which $Q(2\cos 2\pi/5)$ contains additionally, manages to distinguish the two links.

- In Form 4 vs Form 4, Form 4 vs Form 5 and Form 5 vs Form 5 there is nothing to do, and so our proof is complete. □

5. Applications.

The preceding discussion explains how to proceed to test the Montesinos property of some link L using the Jones polynomial.

For Form 1, one should take $t = e^{\pi i/5}$. Then $|t^2+1| > 1 > |1-t|$. Increase $m = k+l$ from 3 on, as long as

$$|t^2+1|^m \leq |t+1| \cdot v_1(L) + |1-t|^3 \cdot |t+1+1/t|.$$

If

$$\big||t^2+1|^m - |t+1| \cdot v_1(L)\big| \leq |1-t|^m \cdot |t+1+1/t|,$$

then compare $\bar{V}$ with $\bar{V}_1(k,l) = \bar{V}_1(m-l,l)$ for $0 \leq l \leq 4$. Note that we have a restriction on l from $d(L)$. While we cannot determine $d(L)$ from the Jones polynomial, it still leaves a "trace" in form of the condition that $5^{d(L)}$ divides $\det(L) = |V_L(-1)|$. So among the 5 possible l we can exclude the cases where 5 divides exactly one of $k-l = m-2l$ and $V_L(-1)$, but not the other.

For Form 2, one should take $t = e^{3\pi i/5}$. Then $|t^2-1| > |1-t| > 1$. Now we increase first l from 1 onward as long as $5^{l-1} \mid V_L(-1)$ (because of the relation to $d(L)$), and

$$|t^2-1|^l|t+1+1/t| \leq |t+1| \cdot v_3(L).$$

For such l, iterate k from $\max(0, 3-l)$ onward, as long as

$$|t^2-1|^l|1-t|^k|t+1+1/t| \leq |t+1| \cdot v_3(L).$$

If equality holds, compare $\bar{V}$ with $\bar{V}_2(k,l)$.

With a similar procedure one tests Form 3, now using $t = e^{\pi i/5}$. (Then $|t^2+1|$, $|1+t| > 1$, and we must have $5^l \mid \det(L)$.) Forms 4 and 5 (latter actually being redundant, since equivalent, as far as $\bar{V}$ can tell, to Form 3 for $k = 0, l = 1$) are tested by direct comparison.

If one fails to find $\bar{V}$ in these forms, one can conclude that L is not (even 5-equivalent to) a Montesinos link. I wrote a computer program that performs this test, and show its output on (say) non-alternating 10 crossing knots[1]:

[1] In this table, we use the numbering of [**Ro**, appendix], but the mirroring convention in [**HT**]; a conversion to Rolfsen's mirroring can be found on my website.

```
10 124 match form 3 for k=2, l=0
10 125 match form 3 for k=2, l=0
10 126 match form 3 for k=2, l=0
10 127 match form 3 for k=2, l=0
10 128 match form 1 for k=2, l=1
10 129 form 5
10 130 match form 1 for k=0, l=3
10 131 match form 1 for k=1, l=2
10 132 form 5
10 133 match form 1 for k=2, l=1
10 134 match form 1 for k=3, l=0
10 135 match form 1 for k=3, l=0
10 136 match form 2 for k=1, l=2
10 137 match form 2 for k=1, l=2
10 138 match form 2 for k=1, l=2
10 139 match form 3 for k=1, l=0
10 140 match form 3 for k=0, l=0
10 141 match form 3 for k=0, l=0
10 142 form 5
10 143 match form 3 for k=1, l=0
10 144 match form 3 for k=0, l=0
10 145 match form 2 for k=2, l=1
10 146 match form 2 for k=2, l=1
10 147 match form 2 for k=2, l=1
10 148 match form 1 for k=1, l=2
10 149 match form 1 for k=1, l=2
10 150
10 151
10 152
10 153
10 154
10 155 match form 2 for k=1, l=2
10 156 match form 2 for k=1, l=2
10 157
10 158 match form 2 for k=1, l=2
10 159
10 160
10 161 match form 2 for k=1, l=2
10 162 form 5
10 163 match form 2 for k=3, l=1
10 164 match form 2 for k=1, l=2
10 165
```

Whenever no form is found, the Montesinos property is ruled out. This happens for 9 of the last 18 knots; the first 24 knots are Montesinos. Our test thus seems relatively efficient. It can surely not be perfect, since it is invariant under 5-moves, and also sporadic duplications of Jones polynomials occur. However, it seems easier to perform than all previously known Montesinos link tests (at least for general diagrams, and as long as the Jones polynomial can be calculated).

Let us make a brief comment on a result of Motegi [**Mo**]. He shows

THEOREM 5.1 (Motegi). *Montesinos knots/links of length* > 3 *have unknotting/unlinking number* > 1.

This means that for an unknotting number one knot or link we can sharpen the Montesinos property test by demanding $k + l \leq 3$ in Forms 1, 2 and 3. For example, the knot 10_{88} has a Jones polynomial that matches Form 2 for $k = 3$ and $l = 1$. I.e. 10_{88} could be 5-equivalent to $M(1/2, 1/2, 1/2, 2/5)$, and then only to its class, since either's d-invariants coincide (see the proof of Theorem 4.2). But having unknotting number 1, we see that 10_{88} cannot be a Montesinos knot.

More generally, we have the following. Let us again write a rational link as a Montesinos link of length 1, and use '∞' to formalize connected sums of rational links (see end of Section 2.2).

COROLLARY 5.1. *The* 5*-move equivalence classes of unknotting number* 1 *Montesinos* knots *are contained in the following* 12 *classes*:

$$\begin{array}{ccc} M(1/2,1/2,1/2) & M(1/2,-1/2,-1/2) & M(1/2,2/5,2/5) \\ M(1/2,1/2,-1/2) & M(-1/2,-1/2,-1/2) & M(1/2,1/2,2/5) \\ M(1) & M(2/5) & M(1/2,\infty,\infty) \\ M(1/2) & M(1/2,1/2,\infty) & M(1/2,1/2,1/2,1). \end{array}$$

A look at the first 24 non-alternating 10 crossing knots in the above calculation and in the tables of [**Kw**] (with the supplement that $u(10_{131}) = 1$; see [**St5**]) helps to see that 10 of these 12 classes are realized; the other 2 are $M(1/2,1/2,\infty)$ and $M(1/2,\infty,\infty)$, and we do not know about them.

PROOF. Since reducing a Montesinos knot (or link) K by 5-moves gives no representative of larger length, we need to look in Forms 1 and 2 only at the links with $k+l \le 3$ and in Forms 3 and 4 at those with $k+l \le 2$. Latter two forms occur when some rational tangle reduces to an ∞ tangle, and we write them as a length-3 notation with a fractional part ∞. If at most 2 fractional parts occur, we have a rational link. If all three fractional parts are ∞ or $2/5$, then $u(K) \ge d(K) \ge 2$. The classes $M(0)$ and $M(1/2,2/5,\infty)$ are ruled out using the Jones-Rong value of Q, see [**St5**]. □

REMARK 5.1. In addition to Remark 4.2, by Motegi's result, the problems to distinguish 5-move equivalence classes in Theorem 4.2 are remedied also when considering unknotting/linking number one knots/links. Mutations become trivial for Montesinos links of length 3, and one easily observes that in each pair of links of Form 4 with equal Jones-Rong value, at least one link has 3 connected sum factors. Such a link does not occur when reducing a Montesinos link of length 3 by 5-moves.

6. Invariants of the Kauffman polynomial.

Now we study the 5-move invariants of the Kauffman polynomial to enhance the test. The following is easy to see:

LEMMA 6.1. *Let $n, m \in \boldsymbol{Z}_k$, so that $n \ne \pm m$ and $w = e^{2\pi i n/k} \ne \pm 1, \pm i$. Then $F(a,z)$ for $a = e^{2\pi i m/k}$ and $z = w + w^{-1} = 2\cos 2\pi n/k$, up to multiplication with (powers of) a, is a k-move invariant.*

PROOF (sketch). Consider the generating function

$$f(a,z,x) = \sum_{j=0}^{\infty} \Lambda(A_j)(a,z)x^j,$$

where A_j are link diagrams with a twist tangle of j crossings (that is, a tangle with Conway notation j, as on the right of (1) for $j=5$). Use the relations (4) to rewrite f as a rational function in x, determined by $A_{0,1,\infty}$. Finally analyze for what values of a and z (for which $F(a,z)$ makes sense), the zeros x of the denominator polynomial are distinct k-th roots of unity. □

In the case of $k = 5$, Lemma 6.1 gives up to complex conjugacy four invariant evaluations of F. We finish the paper with a uniform calculation of them, and some examples in application of the corresponding formulas. Despite that these values mainly coincide (as we will note below) with the invariants we already treated, their formulas make still some sense, at least because the (sign in) the Jones-Rong value can be useful.

To evaluate F on the links in Theorem 3.2, we make again a skein module calculation, this time using the Kauffman polynomial (*not* to be confused with the Kauffman bracket) skein relation (4). In the following we assume that a and z are as specified in Lemma 6.1 for $k = 5$, and consider $F(a, z)$ up to powers of a.

We start with determining the coefficients

$$\langle 1/2\rangle = A\langle 0\rangle + B\langle 1\rangle + C\langle\infty\rangle$$
$$\langle 2/5\rangle = \mathrm{D}\langle 0\rangle + \mathrm{E}\langle 1\rangle + \mathrm{F}\langle\infty\rangle$$

of the tangles $T = \langle 1/2\rangle$ and $T' = \langle 2/5\rangle$ in the Kauffman skein module (not to be confused with the Kauffman bracket skein module).

By taking the closure of the sum of the tangle $T = \langle 1/2\rangle$ resp. $T' = \langle 2/5\rangle$ with the 0, ∞ and -1 tangles, we obtain a linear equation system that determines the Kauffman skein module coefficients of T and T'.

Let first $a_1 = \frac{1}{a} + a$, and

$$\begin{aligned} \mathrm{T}_2 &= -1 + \frac{a_1}{z} \\ \mathrm{H} &= za_1 - \mathrm{T}_2 \\ \mathrm{G}_4 &= (1 - a_1^2) - za_1 + z^2a_1^2 + z^3a_1 \\ \mathrm{G}_3 &= (-1/a - 2a) + za_1/a + z^2a_1 \end{aligned}$$

be the writhe-unnormalized polynomials of the 2-component unlink, Hopf link, Figure-8 knot, and negative trefoil, respectively.

Let

$$M = \begin{pmatrix} \mathrm{T}_2 & a & 1 \\ 1 & \frac{1}{a} & \mathrm{T}_2 \\ \frac{1}{a} & \mathrm{T}_2 & a \end{pmatrix}$$

be the matrix that represents the closures of the three skein module generating tangles $\langle 0\rangle$, $\langle 1\rangle$ and $\langle\infty\rangle$, and

$$\mathrm{v} = \begin{pmatrix} a^2 & a^2\,\mathrm{H} \\ \mathrm{H} & \mathrm{G}_4 \\ \frac{1}{a} & \mathrm{G}_3 \end{pmatrix}$$

be the to-result polynomials for the closures of the the tangles T and T'. Then

$$\begin{pmatrix} A & B & C \\ \mathrm{D} & \mathrm{E} & \mathrm{F} \end{pmatrix} = (M^{-1} \cdot \mathrm{v})^T,$$

and by calculation we find

$$(A, B, C) = (za,\ z,\ -1)$$
$$(\mathrm{D}, \mathrm{E}, \mathrm{F}) = \Big(-a^2 + z^2 + a^2 z^2 + a z^3,\ -z + \frac{z^2}{a} + z^3,\ -\frac{z}{a} - z^2 \Big).$$

A routine (and so omitted) further calculation leads to the formulas we wish (the link in Form 5 can be evaluated directly).

THEOREM 6.1. *For a and z as in Lemma 6.1 for $k = 5$, the values $F(a, z)$ of the links in Theorem 3.2 are given, up to powers of a, as follows.*

For Form 1 we have with $k = n$, $l = n'$:

$$F_1(n, n') = \frac{1}{\mathrm{T}_2} \Big[\mathrm{H}^{n+n'} + a^{n-n'} z^{n+n'} \Big(\sum_{j=0}^{n} \sum_{j'=0}^{n'} \binom{n}{j} \binom{n'}{j'} W_1\big((j - j') \bmod 5\big) \Big) \Big],$$

where

$$W_1(0) = -1 + {\mathrm{T}_2}^2,$$
$$W_1(\pm 1) = -a^{\mp 2} + \mathrm{T}_2,$$
$$W_1(\pm 2) = -a^{\pm 1} + a^{\mp 2}\, \mathrm{H}\, \mathrm{T}_2\,.$$

For Form 2 we have

$$F_2(n, n') = \frac{1}{\mathrm{T}_2} \Big[{\mathrm{G}_4}^{n'}\, \mathrm{H}^n + \sum_{j=0}^{n} \sum_{j'=0}^{n'} \binom{n}{j} \binom{n'}{j'} A^{n-j} B^j\, \mathrm{D}^{n'-j'}\, \mathrm{E}^{j'}\, W_2\big((j + j') \bmod 5\big) \Big],$$

where

$$W_2(0) = -1 + {\mathrm{T}_2}^2,$$
$$W_2(\pm 1) = -a^{\mp 1} + a^{\pm 1}\, \mathrm{T}_2,$$
$$W_2(\pm 2) = -a^{\mp 2} + \mathrm{H}\, \mathrm{T}_2\,.$$

For Form 3 and Form 4,

$$F_3(n, n', n'') := {\mathrm{G}_4}^n\, \mathrm{H}^{n'}\, {\mathrm{T}_2}^{n''}\,.$$

If $n > 0$, we have Form 4; if $n = 0$, Form 3 with $k = n'$, $l = n''$.

These expressions look slightly more complicated than those for V, but are still straightforward to evaluate. Using the formulas, we can for example rule out 10_{155} and 10_{161} from being Montesinos. Since 10_{155} with $d = 2$ satisfies the condition of Remark 4.1 for Form 2 and $l = 2$, the proof of Theorem 4.2 and the calculation in Section 5 show that if 10_{155} were Montesinos, it must be in the 5-equivalence class of a Montesinos link corresponding to the polynomial $F_2(1, 2)$ (for example 10_{136}). But the F polynomial invariants of $F(10_{155})$ are different from those of $F_2(1, 2)$. The same argument rules out 10_{161}. The other 7 undecided knots remain, and it is suspectable that they are 5-equivalent to Montesinos links.

Finally, let us clarify the status of the four values in Lemma 6.1. For $m = 1$ we have the two (equivalent under the interchange of $\pm\sqrt{5}$) Jones-Rong values of Q in Proposition 3.2. The other two values (also equivalent up to $\pm\sqrt{5}$) are

$$w_2^{\pm} = F\big(e^{\pm 2\pi i/5}, 2\cos 4\pi/5\big) \quad \text{and} \quad w_4^{\pm} = F\big(e^{\pm 4\pi i/5}, 2\cos 2\pi/5\big).$$

Unfortunately, it turns out that we already know them, too. Proposition 16.6 of [**Li**] shows that $w_{4,2}^{+} = \pm x_{1,3}^2$ mainly identify with the invariants $x_{1,3}$ of Corollary 3.1. Here the sign '$\pm$' depends on the parity of link components, and $w_{2,4}^{+}$ are determined up to multiplication by fifth roots of unity, so from Definition 3.1 and Remark 3.1 we see $w_{4,2}$ to be equivalent to $\widetilde{x_{1,3}}$.

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