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Invariants of two-dimensional projectively Anosov diffeomorphisms and their applications

By Masayuki Asaoka

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Abstract. We define invariants of two dimensional C^2 projectively Anosov diffeomorphisms. The invariants are defined by the topology of the space of circles tangent to an invariant subbundle and are preserved under homotopy of projectively Anosov diffeomorphisms. As an application, we show that the invariant subbundle is not uniquely integrable and two distinct periodic orbits exist if certain invariants do not vanish.

1. Introduction.

Let M be a smooth closed manifold and fix a norm $\|\cdot\|$ on the tangent bundle TM. For a diffeomorphism f on M and an invariant set Λ of f, we say a continuous splitting $TM|_\Lambda = E^u \oplus E^s$ is *a dominated splitting* associated to f on Λ when $Df(E^u) = E^u$, $Df(E^s) = E^s$, and there exist two constants $K > 0$ and $0 < \lambda < 1$ such that

$$\left\|Df^n\,|_{E^s(z)}\,\right\| \cdot \left\|(Df^n\,|_{E^u(z)})^{-1}\right\| < K\lambda^n$$

for all $z \in \Lambda$ and $n \geq 1$. A non-trivial dominated splitting $TM = E^u_f \oplus E^s_f$ on the whole manifold is called *a projectively Anosov splitting* (or simply a $\boldsymbol{P}$A splitting) associated to f. We say a diffeomorphism f is *projectively Anosov* (or simply $\boldsymbol{P}$A) when f admits a $\boldsymbol{P}$A splitting. Remark that if a dominated splitting satisfies stronger inequalities $\|Df^n|_{E^s(z)}\| < K\lambda^n$ and $\|(Df^n|_{E^u(z)})^{-1}\| < K\lambda^n$ for all $z \in \Lambda$ and $n \geq 1$, then it is called *a hyperbolic splitting*. We say a diffeomorphism is *Anosov* when it admits a hyperbolic splitting on the whole manifold.

A continuous family $\{f_\lambda\}_{\lambda\in[0,1]}$ of C^1 diffeomorphisms is called *a* $\boldsymbol{P}$A *homotopy* if all f_λ are $\boldsymbol{P}$A diffeomorphisms. The main aim of this paper is to define $\boldsymbol{P}$A homotopy invariants of two-dimensional $\boldsymbol{P}$A diffeomorphisms. Since the two-dimensional torus $\boldsymbol{T}^2$ is the only orientable surface which admits a $\boldsymbol{P}$A diffeomorphism, we focus our attention on $\boldsymbol{P}$A diffeomorphisms on $\boldsymbol{T}^2$.

Originally, the concept of $\boldsymbol{P}$A systems was introduced by Mitsumatsu [**4**] and by Eliashberg and Thurston [**2**] for three-dimensional flows in order to study contact structures on three-dimensional manifolds (in [**2**], Eliashberg and Thurston called them *conformally Anosov systems*). They gave a natural correspondence between three-dimensional

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$\boldsymbol{P}$A flows and bi-contact structures, which are pairs of mutually transverse positive and negative contact structures. It induces a one-to-one correspondence between homotopy classes of $\boldsymbol{P}$A flows and bi-contact structures. In this view point, the study of $\boldsymbol{P}$A homotopy invariants of $\boldsymbol{P}$A diffeomorphisms on $\boldsymbol{T}^2$ is a first step to study the homotopy classes of three-dimensional $\boldsymbol{P}$A flows and bi-contact structures.

In this paper, we also give two applications of the invariants. It is known that any C^2 Anosov diffeomorphism on $\boldsymbol{T}^2$ admits a C^1 $\boldsymbol{P}$A splitting. However, some C^2 $\boldsymbol{P}$A diffeomorphisms do not in general. For example, Eliashberg and Thurston [**2**, Example 2.2.9] constructed an example without C^1 $\boldsymbol{P}$A splitting. It is natural to ask which $\boldsymbol{P}$A homotopy class contains a $\boldsymbol{P}$A system with a C^1 $\boldsymbol{P}$A splitting or not. The first application shows that our invariants are obstructions to admit a C^1 $\boldsymbol{P}$A splitting. In fact, we show that the non-vanishing of certain cohomology invariants implies that the $\boldsymbol{P}$A splitting is not uniquely integrable, and hence, is not of class C^1. The second application is a kind of fixed point theorem. We show that the non-vanishing of certain cohomology invariants implies the existence of at least two distinct periodic orbits.

To state our results more precisely, we introduce some definitions. Let $\mathrm{Diff}^r(\boldsymbol{T}^2)$ be the space of C^r diffeomorphisms of $\boldsymbol{T}^2$ with C^r-topology and $\boldsymbol{P}\mathrm{A}^r(\boldsymbol{T}^2)$ the subset of $\mathrm{Diff}^r(\boldsymbol{T}^2)$ consisting of C^r $\boldsymbol{P}$A diffeomorphisms. As we see in Subsection 3.1, any $f \in \boldsymbol{P}\mathrm{A}^1(\boldsymbol{T}^2)$ admits a unique $\boldsymbol{P}$A splitting and the existence of such a splitting is persistent under C^1-perturbation of f. In particular, $\boldsymbol{P}\mathrm{A}^r(\boldsymbol{T}^2)$ is an open subset of $\mathrm{Diff}^r(\boldsymbol{T}^2)$ for any $r \geq 1$. Let $E^u_f \oplus E^s_f$ denote the $\boldsymbol{P}$A splitting associated to f. We say $f \in \boldsymbol{P}\mathrm{A}^1(\boldsymbol{T}^2)$ is *orientable* when both E^u_f and E^s_f are orientable and Df preserves their orientations.

A periodic point p of a diffeomorphism f of $\boldsymbol{T}^2$ is called *hyperbolic* if Df^n_p has no eigenvalues of absolute value one, where n is the period of p. We say a hyperbolic periodic point p is *attracting*, *repelling*, or *of saddle-type* when all eigenvalues of Df^n_p are of absolute value less than one, greater than one, or otherwise, respectively. A diffeomorphism is called *non-degenerate* if all periodic points are hyperbolic. Such diffeomorphisms are generic in $\boldsymbol{P}\mathrm{A}^r(\boldsymbol{T}^2)$ by the Kupka-Smale theorem.

Let S^1 denote the circle. Let $C^1(S^1, \boldsymbol{T}^2)$ be the space of C^1 maps from S^1 to $\boldsymbol{T}^2$ with the C^1 topology and $\mathrm{Diff}^1_+(S^1)$ the group of orientation preserving diffeomorphisms of S^1. The group $\mathrm{Diff}^1_+(S^1)$ acts on $C^1(S^1, \boldsymbol{T}^2)$ from the right by composition. For a $\boldsymbol{P}$A diffeomorphism f of $\boldsymbol{T}^2$, we define a $\mathrm{Diff}^1_+(S^1)$-invariant subspace $\widetilde{\mathscr{C}}(E^u_f)$ of $C^1(S^1, \boldsymbol{T}^2)$ by

$$\widetilde{\mathscr{C}}(E^u_f) = \left\{ \gamma \in C^1(S^1, \boldsymbol{T}^2) \;\middle|\; \left(\frac{d\gamma}{dt}\right)(t) \in E^u_f(\gamma(t)) \setminus \{0\} \text{ for any } t \in S^1 \right\}.$$

Let $\mathscr{C}(E^u_f)$ denote a quotient space $\widetilde{\mathscr{C}}(E^u_f)/\mathrm{Diff}^1_+(S^1)$. As we see in Subsection 2.2, the space $\mathscr{C}(E^u_f)$ is metrizable. Let $[\gamma]_c$ denote the equivalence class of $\gamma \in \widetilde{\mathscr{C}}(E^u_f)$ in $\mathscr{C}(E^u_f)$. For an integral homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, let $\mathscr{C}_a(E^u_f)$ denote the set consisting of elements $[\gamma]_c$ of $\mathscr{C}(E^u_f)$ such that γ represents the class a.

We call $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$ *a prime homology class* if $a \neq na'$ for any $n \geq 2$ and $a' \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$. For a topological space X, let $X \cup \{\infty\}$ denote the one point compactification of X and $H_c^*(X)$ denote the compactly supported cohomology groups of X.

THEOREM A (Invariance of homotopy type). *Let f_0 and f_1 be C^2 $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$ which are orientable, non-degenerate, and mutually $\boldsymbol{P}$A homotopic. Then, pointed spaces $(\mathscr{C}_a(E^u_{f_0}) \cup \{\infty\}, \infty)$ and $(\mathscr{C}_a(E^u_{f_1}) \cup \{\infty\}, \infty)$ have the same homotopy type for any prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$. In particular, $H_c^*(\mathscr{C}_a(E^u_{f_0}))$ is isomorphic to $H_c^*(\mathscr{C}_a(E^u_{f_1}))$.*

Any $\boldsymbol{P}$A diffeomorphism f on $\boldsymbol{T}^2$ induces a natural homeomorphism $\mathscr{C}(f)$ on $\mathscr{C}(E^u_f)$ by $\mathscr{C}(f)([\gamma]_c) = [f \circ \gamma]_c$. We define *the index* of a periodic point $[\gamma]_c$ of $\mathscr{C}(f)$ by the number of $t \in S^1$ such that $\gamma(t)$ is a repelling periodic point of f. It does not depend on the choice of the representative γ and we denote it by ind $[\gamma]_c$. We define the unstable set $W^u([\gamma]_c; \mathscr{C}(f))$ of a fixed point $[\gamma]_c$ of $\mathscr{C}(f)^n$ by

$$W^u([\gamma]_c; \mathscr{C}(f)) = \Big\{ [\gamma']_c \in \mathscr{C}(E^u_f) \;\Big|\; \lim_{k\to\infty} \mathscr{C}(f)^{-kn}([\gamma']_c) = [\gamma]_c \Big\}.$$

THEOREM B (Morse decomposition). *Let f be a C^2 non-degenerate orientable $\boldsymbol{P}$A diffeomorphism of $\boldsymbol{T}^2$. For any prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, the decomposition*

$$\mathscr{C}_a(E^u_f) = \bigsqcup W^u([\gamma]_c; \mathscr{C}(f))$$

gives a structure of CW complex to $\mathscr{C}_a(E^u_f)$ with $\dim W^u([\gamma]_c; \mathscr{C}(f)) = \operatorname{ind}\,[\gamma]_c$, where the union runs over all periodic points $[\gamma]_c$ of $\mathscr{C}(f)$ in $\mathscr{C}_a(E^u_f)$.

The followings are applications of Theorems A and B.

THEOREM C (Non-smoothness). *Let f be a C^2 non-degenerate orientable $\boldsymbol{P}$A diffeomorphism of $\boldsymbol{T}^2$. Suppose that there exist prime homology classes $a_1, a_2 \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$ with $a_1 \neq \pm a_2$ and integers $k_1, k_2 \geq 0$ satisfying $H_c^{k_i}(\mathscr{C}_{a_i}(E^u_f)) \neq \{0\}$ for $i = 1, 2$. Then, E^u_g is not uniquely integrable for any C^1 $\boldsymbol{P}$A diffeomorphism g which is $\boldsymbol{P}$A homotopic to f. In particular, it is not a C^1 subbundle of $T\boldsymbol{T}^2$.*

THEOREM D (Periodic orbits). *Let f be a C^2 non-degenerate orientable $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$. Suppose that there exist a prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$ and an integer $k \geq 1$ satisfying $H_c^k(\mathscr{C}_a(E^u_f)) \neq \{0\}$. Then, any C^1 $\boldsymbol{P}$A diffeomorphism which is $\boldsymbol{P}$A homotopic to f has at least two distinct periodic orbits.*

The author recommends that the readers should refer to Subsection 2.3 of [**1**], which provides examples of $\boldsymbol{P}$A diffeomorphisms on $\boldsymbol{T}^2$ and some applications of the above theorems to them.

The outline of the proofs of the main results is as follows. More detailed outlines are given at the beginning of each section. In Section 2, we prepare terminology on the space of curves tangent to a line field. We study general $\boldsymbol{P}$A diffeomorphisms in Section 3. In particular, we give a combinatorial description of invariant segments in Subsection 3.2. In Section 4, we investigate invariant foliations of non-degenerate $\boldsymbol{P}$A diffeomorphisms and prove Theorem B. In Section 5, we begin the study of $\boldsymbol{P}$A homotopy and reduce the proof of Theorem A to the case of a $\boldsymbol{P}$A homotopy with a simple bifurcation. A keystone is the compactness of the spaces of invariant embedded circles or intervals, which is given in Subsection 5.3. We study the bifurcation of combinatorics of invariant segments in Section 6 and prove Theorem A in Section 7. In Section 8, we prove Theorems C and D.

To end the introduction, we pose a question. It is easy to see that if f is an Anosov diffeomorphism then $H_c^*(\mathscr{C}_a(E_f^u))$ vanishes for any $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$. By Theorem A, the same holds for any C^2 non-degenerate $\boldsymbol{P}$A diffeomorphism g which is $\boldsymbol{P}$A homotopic to f. Our question is whether the converse is true or not.

Question. Suppose that $H_c^*(\mathscr{C}_a(E_f^u))$ vanishes for all $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$. Is the map f $\boldsymbol{P}$A homotopic to an Anosov diffeomorphism?

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1.1. Notations.

Let $\boldsymbol{Z}, \boldsymbol{R}$ denote the set of all integers and real numbers respectively. Let $\#X$ denote the cardinality of a set X. For real numbers a and b with $a < b$, let $[a, b],]a, b], [a, b[$, and $]a, b[$ denote the intervals $\{a \leq x \leq b\}$, $\{a < x \leq b\}$, $\{a \leq x < b\}$, and $\{a < x < b\}$ respectively. For a subspace Y of a topological space X, let $\overline{Y}$ denote the closure of Y.

For manifolds M_1 and M_2, let $C^r(M_1, M_2)$ denote the space of all C^r maps from M_1 to M_2 and $\mathrm{Diff}^r(M_1)$ the group of all C^r diffeomorphisms of M_1 with the C^r-topology. If M_1 is oriented, $\mathrm{Diff}^r_+(M_1)$ denotes the subgroup of $\mathrm{Diff}^r(M_1)$ consisting of all orientation preserving diffeomorphisms.

For a homeomorphism g on a Hausdorff space X, we write $\mathscr{O}(z; g)$ for *the orbit* $\{g^n(z) \mid n \in \boldsymbol{Z}\}$ of $z \in X$. Let $\mathrm{Fix}(g)$ denote the set of fixed points and $\mathrm{Per}(g)$ the set of periodic points of g. For $p \in \mathrm{Per}(g)$ with period k, we define *the stable set* $W^s(p; g)$ by the set of $z \in X$ such that $g^{kn}(z)$ converges to p as $n \to \infty$. We also define *the unstable set* $W^u(p; g)$ by $W^u(p; g) = W^s(p; g^{-1})$.

2. The space of curves.

Our proof of main theorems is based on the study of the spaces of curves tangent to E^u at constant speed. In this section, we show basic properties of such spaces in a general setting. We study the space of segments in Subsection 2.1 and that of circles in Subsection 2.2.

2.1. Segments tangent to a unit vector field.

Let X be one of $[0, 1]$, $]0, 1[$, or $S^1 = \boldsymbol{R}/\boldsymbol{Z}$. For $\xi \in C^1(X, \boldsymbol{T}^2)$, we define *the length* $|\xi|$ by $|\xi| = \int_X \|(d\xi/dt)(t)\| dt$. Remark that $|\xi \circ \eta| = |\xi|$ for any $\eta \in \mathrm{Diff}^1_+(X)$. Let $\mathrm{Im}\, \xi$ and $\mathrm{Int}\, \xi$ denote the subsets $\xi(X)$ and $\xi(]0, 1[)$ of $\boldsymbol{T}^2$ respectively.

Let $\mathscr{S}(\boldsymbol{T}^2)$ be the subset of $C^1([0, 1], \boldsymbol{T}^2)$ consisting of all ξ with the constant speed $|d\xi/dt|$. For a C^1 immersion $\xi : [0, 1] \to \boldsymbol{T}^2$, we define *the normalizer* $\eta_\xi \in \mathrm{Diff}^1_+([0, 1])$

of ξ by $\eta_\xi(t) = |\xi|^{-1} \int_0^t \|(d\xi/dt)(t)\| dt$. The map η_ξ is characterized as the unique orientation preserving map with $\xi \circ \eta_\xi^{-1} \in \mathscr{S}(\boldsymbol{T}^2)$. Remark that η_ξ depends continuously on ξ with respect to the C^1-topology.

Any $f \in \mathrm{Diff}^1(\boldsymbol{T}^2)$ induces a self-map $\mathscr{S}(f)$ on $\mathscr{S}(\boldsymbol{T}^2)$ by $\mathscr{S}(f)(\xi) = f \circ \xi$ if $|\xi| = 0$ and $\mathscr{S}(f)(\xi) = (f \circ \xi) \circ \eta_{f\circ\xi}^{-1}$ otherwise, where $\eta_{f\circ\eta}$ is the normalizer of $f\circ\xi$. It is easy to see that $\mathscr{S}(f)(\xi)$ depends continuously on f and ξ, and the equation $\mathscr{S}(f\circ g) = \mathscr{S}(f) \circ \mathscr{S}(g)$ holds. In particular, $\mathscr{S}(f)$ is a homeomorphism on $\mathscr{S}(\boldsymbol{T}^2)$.

Let $\mathscr{X}_1^0(\boldsymbol{T}^2)$ denote the set of continuous unit vector fields on $\boldsymbol{T}^2$. For $e \in \mathscr{X}_1^0(\boldsymbol{T}^2)$, we define a subset $\mathscr{S}(e)$ of $\mathscr{S}(\boldsymbol{T}^2)$ by

$$\mathscr{S}(e) = \left\{ \xi \in \mathscr{S}(\boldsymbol{T}^2) \,\middle|\, \frac{d\xi}{dt}(t) = |\xi| \cdot e(\xi(t)) \text{ for any } t \in [0,1] \right\}.$$

LEMMA 2.1. *Let $\{e_i \in \mathscr{X}_1^0(\boldsymbol{T}^2)\}_{i=1}^\infty$ be a sequence which converges to $e_* \in \mathscr{X}_1^0(\boldsymbol{T}^2)$. Any sequence $\{\xi_i \in \mathscr{S}(e_i)\}_{i=1}^\infty$ with $\sup_{i\geq 1} |\xi_i| < \infty$ has a subsequence which converges to an element of $\mathscr{S}(e_*)$.*

PROOF. Without loss of generality, we assume that $|\xi_i|$ converges to a real number. Since $(d\xi_i/dt)(t) = |\xi_i| e_i(\xi_i(t))$, the sequence $\{\xi_i\}_{i\geq 1}$ is equi-continuous. By the Ascoli-Arzelá theorem, it has a subsequence $\{\xi_{i_k}\}_{k\geq 1}$ which converges with respect to the C^0-topology. The convergence of e_i and $|\xi_i|$ implies that the sequence $\{\xi_{i_k}\}$ converges to an element of $\mathscr{S}(e_*)$ with respect to the C^1-topology. $\square$

Fix $e \in \mathscr{X}_1^0(\boldsymbol{T}^2)$. For $\xi, \xi' \in \mathscr{S}(e)$ with $\xi(1) = \xi'(0)$, we define the composition $\xi * \xi' \in \mathscr{S}(e)$ by $\xi * \xi' = \xi$ if $|\xi'| = 0$, $\xi * \xi' = \xi'$ if $|\xi| = 0$, and

$$(\xi * \xi')(t) = \begin{cases} \xi\left(\dfrac{|\xi| + |\xi'|}{|\xi|} t\right) & \text{for } t \in \left[0, \dfrac{|\xi|}{|\xi| + |\xi'|}\right] \\[2ex] \xi'\left(\dfrac{|\xi| + |\xi'|}{|\xi'|} t - \dfrac{|\xi|}{|\xi'|}\right) & \text{for } t \in \left]\dfrac{|\xi|}{|\xi| + |\xi'|}, 1\right] \end{cases}$$

otherwise. It is easy to check $|\xi * \xi'| = |\xi| + |\xi'|$ and $\mathscr{S}(f)(\xi * \xi') = \mathscr{S}(f)(\xi) * \mathscr{S}(f)(\xi')$. We also see that $\xi * \xi'$ depends continuously on ξ and ξ'.

For a finite subset Λ of $\boldsymbol{T}^2$, we put

$$\mathscr{S}(e, \Lambda) = \{\xi \in \mathscr{S}(e) \mid \xi(0), \xi(1) \in \Lambda\}.$$

We say a decomposition $\xi = \xi_1 * \cdots * \xi_k$ of $\xi \in \mathscr{S}(e,\Lambda)$ is *irreducible* with respect to Λ if $\xi_i \in \mathscr{S}(e,\Lambda)$ and $\mathrm{Int}\, \xi_i \cap \Lambda = \varnothing$ for all $i = 1, 2, \ldots, k$. It is easy to see that any $\xi \in \mathscr{S}(e,\Lambda)$ with $|\xi| \neq 0$ has a unique irreducible decomposition since $\xi^{-1}(\Lambda)$ is a finite set.

We say an injectively immersed one-dimensional submanifold L of $\boldsymbol{T}^2$ *is tangent to* $e \in \mathscr{X}_1^0(\boldsymbol{T}^2)$ when $e(z) \in T_z L$ for all $z \in L$. For a one-dimensional manifold X, let $\mathscr{E}(X, e)$ be the collection of C^1 injectively immersed submanifolds of $\boldsymbol{T}^2$ that are diffeomorphic to X and are tangent to e. We write $|L|$ for the length of $L \in \mathscr{E}(X, e)$. Let

$\mathscr{E}(\sqcup_{<\infty}X, e)$ be the collection of subset of M that are disjoint unions of finitely many elements of $\mathscr{E}(X, e)$.

Fix $L \in \mathscr{E}(]0,1[, e)$ and a C^1 immersion $\xi :]-1,1[\to \boldsymbol{T}^2$ so that $\operatorname{Im}\xi = L$, $\xi(0) = p$ and $(d\xi/dt)(0) \in \{a \cdot e(p) \mid a > 0\}$. We call $\xi(]0,1[)$ and $\xi(]-1,0[)$ *the positive* and *the negative components* of $L \setminus \{p\}$ respectively. If $\xi(t)$ converges as $t \to 1$ or $t \to -1$, we call the limit point *the positive* or *the negative boundary point* of L, and write $\partial_+ L$ or $\partial_- L$ for it. If $\xi(t)$ does not converge, we do not define $\partial_\pm L$. Remark that the positive and negative components of $L \setminus \{p\}$ do not depend on the choice of ξ and $\partial_\pm L$ does not depend on the choice of p and ξ. We also remark that $|L|$ is finite if and only if both $\partial_+ L$ and $\partial_- L$ exist. In such a case, there exists a unique $\xi \in \mathscr{S}(e)$ satisfying $L = \operatorname{Int}\xi$.

2.2. Circles tangent to a unit vector field.

Recall that the group $\mathrm{Diff}^1_+(S^1)$ acts on $C^1(S^1, \boldsymbol{T}^2)$ from the right. Let π_c denote the quotient map from $C^1(S^1, \boldsymbol{T}^2)$ to $C^1(S^1, \boldsymbol{T}^2)/\mathrm{Diff}^1_+(S^1)$. A C^1 diffeomorphism f of $\boldsymbol{T}^2$ induces a homeomorphism $\mathscr{C}(f)$ of $C^1(S^1, \boldsymbol{T}^2)/\mathrm{Diff}^1_+(S^1)$ by $\pi_c(f \circ \gamma) = \mathscr{C}(f)(\pi_c(\gamma))$. We define the image $\operatorname{Im} c$ and the length $|c|$ of $c = \pi_c(\gamma) \in C^1(S^1, \boldsymbol{T}^2)/\mathrm{Diff}^1_+(S^1)$ by $\operatorname{Im} c = \operatorname{Im}\gamma$ and $|c| = |\gamma|$.

Fix $e \in \mathscr{X}^0_1(M)$ and define a subspace $\widetilde{\mathscr{C}}(e)$ of $C^1(S^1, \boldsymbol{T}^2)$ by

$$\widetilde{\mathscr{C}}(e) = \left\{ \gamma \in C^1(S^1, \boldsymbol{T}^2) \,\middle|\, |\gamma| \neq 0,\ \frac{d\gamma}{dt}(t) = |\gamma| \cdot e(\gamma(t)) \text{ for any } t \in S^1 \right\}.$$

In the below, we regard $\widetilde{\mathscr{C}}(e)$ as a subspace of both $C^1(S^1, \boldsymbol{T}^2)$ and $\mathscr{S}(e)$. Let $\mathscr{C}(e)$ denote the set $\pi_c(\widetilde{\mathscr{C}}(e))$.

Lemma 2.2. *The restriction of π_c to $\widetilde{\mathscr{C}}(e)$ is a proper map. In particular, the space $\mathscr{C}(e)$ is metrizable.*

Proof. Notice that $|\pi_c(\gamma)| = |\gamma|$ for any $\gamma \in \widetilde{\mathscr{C}}(e)$ and it depends continuously on γ. It implies the set $\{|\gamma| \mid \gamma \in \widetilde{\mathscr{C}}(e), \pi_c(\gamma) \in S\}$ is bounded for any compact subset S of $\mathscr{C}(e)$. By Lemma 2.1, $\pi_c^{-1}(S) \cap \widetilde{\mathscr{C}}(e)$ is a compact subset of $\widetilde{\mathscr{C}}(e)$.

Since $\widetilde{\mathscr{C}}(e)$ admits a natural metric as a subspace of $C^1([0,1], \boldsymbol{T}^2)$, the space $\mathscr{C}(e)$ is metrizable. □

It is important to remark that there exists $K_c > 0$ such that $|c| \geq K_c$ for any $e \in \mathscr{X}^0_1(\boldsymbol{T}^2)$ and any $c \in \mathscr{C}(e)$. In fact, the Poincare-Bendixon theorem implies no $c \in \mathscr{C}(e)$ is null-homotopic. Hence, $|c|$ is not less than the injectivity radius of $\boldsymbol{T}^2$.

For $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$ and a finite subset Λ of M, we put

$$\mathscr{C}_a(e) = \{\pi_c(\gamma) \in \mathscr{C}(e) \mid [\gamma] = a \text{ in } H_1(\boldsymbol{T}^2, \boldsymbol{Z})\},$$

$$\widetilde{\mathscr{C}}_a(e, \Lambda) = \{\gamma \in \widetilde{\mathscr{C}}(e) \cap \mathscr{S}(e, \Lambda) \mid [\gamma] = a \text{ in } H_1(\boldsymbol{T}^2, \boldsymbol{Z})\},$$

$$\mathscr{C}_a(e, \Lambda) = \pi_c(\widetilde{\mathscr{C}}_a(e, \Lambda)) = \{c \in \mathscr{C}_a(e) \mid \operatorname{Im} c \cap \Lambda \neq \varnothing\}.$$

It is easy to see that $\xi_1, \xi_2 \in \widetilde{\mathscr{C}}_a(e, \Lambda)$ satisfy $\pi_c(\xi_1) = \pi_c(\xi_2)$ if and only if $\xi_1 = \xi * \xi'$ and

$\xi_2 = \xi' * \xi$ for some $\xi, \xi' \in \mathscr{S}(e, \Lambda)$.

3. Framed $\boldsymbol{P}$A diffeomorphisms.

We begin to study the space of curves tangent to the invariant subbundles of a $\boldsymbol{P}$A diffeomorphism. In Subsection 3.1, we introduce framed $\boldsymbol{P}$A diffeomorphisms and canonical coordinates associated to them. In Subsections 3.2 and 3.3, we give a combinatorial description of invariant segments. A keystone is the uniqueness of a segment connecting a pair of periodic points, which is proved in Proposition 3.4. In Subsection 3.4, we study invariant embedded circles without hyperbolic periodic points. Such a circle may exhibit complicated bifurcation under perturbation. However, their normal hyperbolicity and finiteness, proved in Proposition 3.13, imply that they are tame in the context of the topology of $\mathscr{C}(e^u)$.

3.1. Framed $\boldsymbol{P}$A diffeomorphisms and canonical coordinates.

First, we show the uniqueness and the persistence of $\boldsymbol{P}$A splittings.

LEMMA 3.1. *Any $f \in \boldsymbol{P}\mathrm{A}^1(\boldsymbol{T}^2)$ admits a unique $\boldsymbol{P}$A splitting.*

PROOF. Suppose that f admits two $\boldsymbol{P}$A splittings $E_1^u \oplus E_1^s$ and $E_2^u \oplus E_2^s$. It is sufficient to show that each $E_2^u(z)$ and $E_2^s(z)$ coincides with either $E_1^u(z)$ or $E_1^s(z)$ for any $z \in \boldsymbol{T}^2$. In fact, it implies $\{E_2^u(z), E_2^s(z)\} = \{E_1^u(z), E_1^s(z)\}$ since $E_2^u(z) \oplus E_2^s(z)$ is a splitting of $T_z\boldsymbol{T}^2$. By the domination property of the splittings, we obtain $E_2^u(z) = E_1^u(z)$ and $E_2^s(z) = E_1^s(z)$.

Suppose that $E_2^u(z)$ coincides with neither $E_1^u(z)$ nor $E_1^s(z)$. By the domination property of $E_2^u \oplus E_2^s$ the angle between $E_2^u(f^n(z)) = Df^n(E_2^u(z))$ and $E_1^\sigma(f^n(z)) = Df^n(E_1^\sigma(z))$ converges to zero as $n \to \infty$ for $\sigma = u, s$. The continuity of the splittings implies that $E_1^u(z_*) = E_1^s(z_*) = E_2^u(z_*)$ for any accumulation point of $\{f^n(z) \mid n \geq 0\}$. It contradicts that $E_1^u(z_*) \oplus E_1^s(z_*)$ is a splitting of $T_{z_*}\boldsymbol{T}^2$. $\square$

We write $E_f^u \oplus E_f^s$ for the unique $\boldsymbol{P}$A splitting associated to a $\boldsymbol{P}$A diffeomorphism f of $\boldsymbol{T}^2$. By the same argument as in the case of hyperbolic splittings (see, *e.g.*, Subsection 6.4 of [**3**]), we can show the persistence of $\boldsymbol{P}$A splittings. Namely, $\boldsymbol{P}\mathrm{A}^r(\boldsymbol{T}^2)$ is an open subset of $\mathrm{Diff}^r(\boldsymbol{T}^2)$ for any $r \geq 1$ and $\boldsymbol{P}$A splittings depend continuously on $\boldsymbol{P}$A diffeomorphisms.

We call an element (f, e^u, e^s) of $\boldsymbol{P}\mathrm{A}^r(\boldsymbol{T}^2) \times (\mathscr{X}_1^0(\boldsymbol{T}^2))^2$ *a C^r framed $\boldsymbol{P}$A diffeomorphism* if $e^u(z) \in E_f^u(z)$ and $e^s(z) \in E_f^s(z)$ for any $z \in \boldsymbol{T}^2$. Remark that f is orientable if and only if (f, e^u, e^s) is a framed $\boldsymbol{P}$A diffeomorphism for some $e^u, e^s \in \mathscr{X}_1^0(\boldsymbol{T}^2)$.

We call a continuous family $\{(f_\lambda, e_\lambda^u, e_\lambda^s)\}_{\lambda \in [0,1]}$ of C^1 framed $\boldsymbol{P}$A diffeomorphisms on $\boldsymbol{T}^2$ *a framed $\boldsymbol{P}$A homotopy*. For a C^1 framed $\boldsymbol{P}$A diffeomorphism (f, e^u, e^s) and a $\boldsymbol{P}$A homotopy $\{f_\lambda\}_{\lambda \in I}$ with $f_0 = f$, the persistence of $\boldsymbol{P}$A splittings implies that there exists a unique framed $\boldsymbol{P}$A homotopy $\{(f_\lambda, e_\lambda^u, e_\lambda^s)\}_{\lambda \in I}$ with $(f_0, e_0^u, e_0^s) = (f, e^u, e^s)$. It is easy to see that $\mathscr{C}_a(E_f^u)$ is homeomorphic to $\mathscr{C}_a(e^u) \sqcup \mathscr{C}_{-a}(e^u)$ for any $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z}) \setminus \{0\}$. Hence, in order to prove Theorems A and B, it is sufficient to show the corresponding results for $\mathscr{C}_a(e^u)$.

Next, we introduce coordinates compatible with a framed $\boldsymbol{P}$A diffeomorphism (f, e^u, e^s). Let $\{e_x(w), e_y(w)\}$ denote the standard basis of $T_w\boldsymbol{R}^2$ for $w \in \boldsymbol{R}^2$. For

$\epsilon > 0$, a C^1 embedding φ of $[-1,1]^2$ into $\boldsymbol{T}^2$ is called *an ϵ-canonical coordinate* for (e^u, e^s) if

$$e^u(\varphi(w)) \in D\varphi(\{ae_x(w) + be_y(w) \mid |b| < \epsilon a\})$$
$$e^s(\varphi(w)) \in D\varphi(\{ae_x(w) + be_y(w) \mid |a| < \epsilon b\})$$

for any $w \in [-1,1]^2$. For any $z \in \boldsymbol{T}^2$ and any neighborhood U of z, we can take an ϵ-canonical coordinate φ so that $\varphi(0,0) = z$ and $\mathrm{Im}\ \varphi \subset U$. In fact, we can construct such an ϵ-canonical coordinate by using foliations along C^1 approximations of e^u and e^s. If e^u is of class C^1 on a neighborhood of z then we can take an ϵ-canonical coordinate φ so that $\varphi([-1,1] \times \{y\})$ is tangent to e^u for any $y \in [-1,1]$.

Let φ be an ϵ-canonical coordinate with $\epsilon \in]0, 1/4[$. Take $\xi \in \mathscr{S}(e^u)$ satisfying $\mathrm{Im}\ \xi \subset \mathrm{Im}\ \varphi$. Then, there exist C^1 function h on $[-1,1]$ and an interval $[a,b]$ such that $\mathrm{Im}\ \xi = \varphi(\{(x, h(x)) \mid x \in [a,b]\})$ and $|dh/dx| < \epsilon$. If $\xi(0) = \varphi(x_0, y_0)$ for some $(x_0, y_0) \in [-1,1] \times [-1+2\epsilon, 1-2\epsilon]$ and $\xi(1) \in \partial(\mathrm{Im}\ \varphi)$, then $|h(x)| < |y_0| + \epsilon(1 + |x_0|)$ for any $x \in [a,1]$ and $\mathrm{Im}\ \xi = \varphi(\{(x, h(x)) \mid x \in [a,1]\})$.

Recall that $W^u(p) = W^u(p; f)$ and $W^s(p) = W^s(p; f)$ denote the unstable set and the stable set of a periodic point p of f respectively. We will make repeated use of the following lemma.

Lemma 3.2. *Let L^s and L^u be subsets of $\boldsymbol{T}^2$ with $L^s \in \mathscr{E}(\sqcup_{<\infty}[0,1], e^s) \cup \mathscr{E}(\sqcup_{<\infty} S^1, e^s)$ and $L^u \in \mathscr{E}(]0,1[, e^u)$. If $L^s \cap L^u \neq \emptyset$, $f(L^s) = L^s$, and $L^u \subset W^u(r)$ for $r \in \mathrm{Per}(f)$, then $L^s \cap L^u$ consists of exactly one point. If $f(L^u) = L^u$ in addition, then we have $L^s \cap L^u = \{r\}$.*

Proof. Suppose $L^u \cap L^s$ contains at least two points. Then, there exists a compact subinterval I^u of L^u which contains at least two points of L^s.

Since L^s is a compact f-invariant set and intersects with $L^u \subset W^u(r)$, it contains r. Take a $1/4$-canonical coordinate φ so that $\varphi(0,0) = r$ and $L^s \cap \mathrm{Im}\ \varphi = \varphi(0 \times [-1,1])$. Since $I^u \subset W^u(r)$, there exist $n \geq 1$ such that $f^{-n}(I^u) \subset \mathrm{Im}\ \varphi$. Then, we can take a function h^u on $[-1,1]$ so that $f^{-n}(I^u) \subset \varphi(\{(x, h^u(x)) \mid x \in [-1,1]\})$. In particular, $L^s \cap f^{-n}(I^u)$ contains at most one point. However, it contradicts that I^u contains two points of L^s.

If $f(L^u) = L^u$, then the unique intersection point of L^u and L^s must be a fixed point of f. Since $W^u(r) \cap \mathrm{Fix}(f) = \{r\}$, it coincides with r. □

By replacing f with f^{-1}, we also obtain the following.

Lemma 3.3. *Let L^u and L^s be subsets of $\boldsymbol{T}^2$ with $L^u \in \mathscr{E}(\sqcup_{<\infty}[0,1], e^u) \cup \mathscr{E}(\sqcup_{<\infty} S^1, e^u)$ and $L^s \in \mathscr{E}(]0,1[, e^s)$. If $L^u \cap L^s \neq \emptyset$, $f(L^u) = L^u$, and $L^s \subset W^s(r)$ for $r \in \mathrm{Per}(f)$, then $L^u \cap L^s$ consists of exactly one point. If $f(L^s) = L^s$ in addition, then we have $L^u \cap L^s = \{r\}$.* □

3.2. Connecting segments.

In this subsection, we fix a framed C^1 $\boldsymbol{P}$A diffeomorphism (f, e^u, e^s) and suppose that $\mathrm{Fix}(f^n)$ is a finite set for all $n \geq 1$.

For $\sigma \in \{u, s\}$ and $p, q \in \mathrm{Per}(f)$, we call $\xi \in \mathscr{S}(e^\sigma)$ *a σ-connecting segment* of (p, q) if $\xi(0) = p$, $\xi(1) = q$, $\xi \in \mathrm{Per}(\mathscr{S}(f))$, and $\mathrm{Int}\ \xi \cap \mathrm{Per}(f) = \varnothing$. For $p \in \mathrm{Per}(f)$, let $D^\sigma_+ p$ denote the set of $q \in \mathrm{Per}(f)$ such that a σ-connecting segment of (p, q) exists. Let $D^\sigma_- p$ also denote the set of $q \in \mathrm{Per}(f)$ such that a σ-connecting segment of (q, p) exists.

For a σ-connecting segment ξ of (p, q) with $\mathscr{S}(f)^k(\xi) = \xi$, the normalizer η of $f^k \circ \xi$ satisfies $f^k \circ \xi = \xi \circ \eta$. Since η has no periodic point, we have either $\eta(t) > t$ for all $t \in]0, 1[$ or $\eta(t) < t$ for all $t \in]0, 1[$. We say ξ is *ascending* in the former case and *descending* in the latter case. Remark that $\mathrm{Int}\ \xi \in \mathscr{E}(]0, 1[, e^u)$. In fact, if it is not, then $\xi(t_1) = \xi(t_2)$ for some $t_1, t_2 \in]0, 1[$ with $t_1 \neq t_2$. The set $f^n \circ \xi([t_1, t_2]) = \xi([\eta^n(t_1), \eta^n(t_2)])$ contains an embedded circle tangent to e^u and its length tends to zero as $n \to +\infty$ or $-\infty$. However, it contradicts that no element of $\mathscr{E}(S^1, e^u)$ is null-homotopic.

Proposition 3.4. *If there exists a σ-connecting segment ξ_1 of (p, q) for $\sigma \in \{u, s\}$ and $p, q \in \mathrm{Per}(f)$, then it is unique. Moreover, if such ξ_1 exists then the followings hold*:

1. *$D^\sigma_+ p = \{q\}$ if $\sigma = u$ and ξ_1 is ascending, or $\sigma = s$ and ξ_1 is descending.*
2. *$D^\sigma_- q = \{p\}$ if $\sigma = u$ and ξ_1 is descending, or $\sigma = s$ and ξ_1 is ascending.*

Proof. To simplify the arguments, we assume that $\sigma = u$ and ξ_1 is ascending. The proof for other cases is similar.

Choose $q' \in D^u_+ p$ and a u-connecting segment ξ_2 of (p, q'). By taking an iteration of f if it is necessary, we assume $\mathscr{S}(f)(\xi_i) = \xi_i$ for $i = 1, 2$. We show that $\xi_1(]0, \epsilon_1[) = \xi_2(]0, \epsilon_2[)$ for some $\epsilon_1, \epsilon_2 \in]0, 1[$. Once it is done, we have $\mathrm{Int}\ \xi_1 = \mathrm{Int}\ \xi_2$. It implies $\xi_1 = \xi_2$ since $\mathrm{Int}\ \xi_1$ and $\mathrm{Int}\ \xi_2$ are elements of $\mathscr{E}(]0, 1[, e^u)$.

Fix a 1/8-canonical coordinate φ so that $\varphi(0, 0) = p$, $D\varphi^{-1}(e^s(p))$ is parallel to $e_y(0, 0)$, $\mathrm{Int}\ \xi_1 \not\subset \mathrm{Im}\ \varphi$, and $\varphi([0, 1] \times \{0\}) \subset \mathrm{Im}\ \xi_2$. There exists a C^1-function h on $[0, 1]$ such that $h(0) = 0$ and $\varphi(\{(x, h(x)) \mid x \in [0, 1]\}) \subset \mathrm{Im}\ \xi_1$. Remark that $|h'(x)| < 1/8$ for any $x \in [0, 1]$. In particular, we have $|h(x)| \leq x/8$.

Fix a map $F = \varphi^{-1} \circ f \circ \varphi$ on a small neighborhood V of $(0, 0)$. We define functions $a(w), b(w), c(w)$, and $d(w)$ on V by $DF_w(e_x(w)) = a(w)e_x(F(w)) + c(w)e_y(F(w))$ and $DF_w(e_y(w)) = b(w)e_x(F(w)) + d(w)e_y(F(w))$. Notice that $0 < d(0, 0) < a(0, 0)$ and $b(0, 0) = c(0, 0) = 0$ since $T_{(0,0)}\boldsymbol{R}^2 = \boldsymbol{R}e_x \oplus \boldsymbol{R}e_y$ is a dominated splitting on $\{(0, 0)\}$ associated to F. By the continuity of DF, there exist $x_0 > 0$ and $\lambda \in]0, 1[$ such that $[-x_0, x_0]^2 \subset V$ and $\sup\{\sqrt{d(w)^2 + b(w)^2} \mid w \in [-x_0, x_0]^2\} < \lambda \inf\{a(w) - |b(w)| \mid w \in [-x_0, x_0]^2\}$. For $x \in]0, x_0[$ and $x' \in]0, 1[$ with $(x', h(x')) = F(x, h(x))$, we have

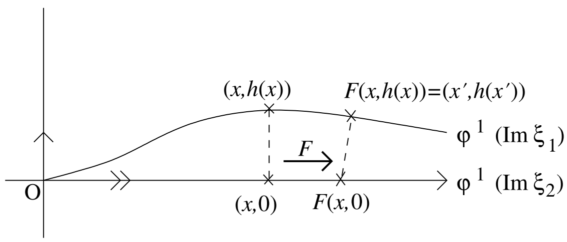

Figure 1. Proof of Proposition 3.4.

$$
\begin{aligned}
|h(x')| &\leq \|F(x,h(x)) - F(x,0)\| \\
&\leq \int_0^{h(x)} \big\|DF_{(x,s)}(e_y(x,s))\big\| ds \\
&\leq |h(x)| \sup\big\{\sqrt{d(w)^2 + b(w)^2} \mid w \in [-x_0, x_0]^2\big\} \\
&\leq \lambda |h(x)| \inf\{a(w) - |b(w)| \mid w \in [-x_0, x_0]^2\} \\
&\leq \lambda |h(x)| \frac{1}{x} \int_0^x \left\{ a(t,h(t)) + \frac{dh}{dt}(t) b(t,h(t)) \right\} dt = \lambda \frac{|h(x)|}{x} x'.
\end{aligned}
$$

It implies

$$
\sup\left\{ \frac{|h(x')|}{x'} \,\middle|\, x' \in (0, x_0') \right\} \leq \lambda \sup\left\{ \frac{|h(x)|}{x} \,\middle|\, x \in (0, x_0) \right\} \leq x_0/8,
$$

where $(x_0', h(x_0')) = F(x_0, h(x_0))$. Notice that $x_0' > x_0$ since ξ_1 is ascending. Hence, we obtain $h(x) = 0$ for any $x \in]0, x_0[$. It implies $\xi_1([0,\epsilon_1]) = \xi_2([0,\epsilon_2])$ for some $\epsilon_1, \epsilon_2 \in]0,1[$. $\square$

Let $[p,q]^\sigma$ denote the unique σ-connecting segment of (p,q) if it exists. By the uniqueness, $[p,q]^\sigma$ is a fixed point of $\mathscr{S}(f)$ if p and q are fixed points of f.

Remark that $D^u_+ p$ may contain several points in general when $[p,q]^u$ is descending. In Subsection 4.2, we see that if p is an attracting fixed point and $D^u_+ p$ contains a repelling fixed point, then $D^u_+ p$ consists of at least three points. In particular, E^u_f is not uniquely integrable at such a point p.

Next, we give a criterion for the existence of connecting segments. Let us recall the definition of the strong unstable manifolds. For $p \in \mathrm{Fix}(f)$ with $\|Df(e^u(p))\| > 1$, fix $\lambda > 1$ so that $\|Df(e^s(p))\| < \lambda < \|Df(e^u(p))\|$. For a given number $\delta > 0$, we define a subset $W^{uu}_\delta(p)$ of $W^u(p)$ by

$$
W^{uu}_\delta(p) = \big\{ z \in \boldsymbol{T}^2 \mid d(f^{-n}(z), p) \leq \lambda^{-n}\delta \text{ for any } n \geq 0 \big\}.
$$

If $\delta > 0$ is sufficiently small, then $W^{uu}_\delta(p)$ is an embedded compact interval with $e^u(p) \in T_p W^{uu}_\delta(p)$. We call $W^{uu}_\delta(p)$ *the local strong unstable manifold* of p. It is known that it depends continuously on p and f as a C^1 embedded manifold. The set $W^{uu}(p) = \bigcup_{n \geq 0} f^n(W^{uu}_\delta(p))$ is called *the strong unstable manifold* of p and it does not depend on the choice of λ and δ (see *e.g.* [**8**]). The manifold $W^{uu}(p)$ is characterized as the unique f-invariant C^1 injectively immersed open interval satisfying $p \in W^{uu}(p) \subset W^u(p)$ and $e^u(p) \in T_p W^{uu}(p)$.

Lemma 3.5. *Let I be a compact embedded interval with $f^{-1}(I) \subset I$ and $e^s(z) \notin T_z I$ for any $z \in I$. Then, I is tangent to e^u. In particular, the strong unstable manifold of a periodic point is tangent to e^u.*

Proof. We define a function α on I by $e^u(z) + \alpha(z) e^s(z) \in T_z I$. By the domination property, there exist $N \geq 1$ and $\lambda \in (0,1)$ such that $|\alpha(z)| \leq \lambda|\alpha(f^{-N}(z))|$ for any $z \in I$. Since $f^{-N}(I) \subset I$ and the function α is bounded, we obtain $\alpha(z) = 0$ for any $z \in I$.

The latter part is an easy consequence of the former. $\square$

Let $W^{uu}_+(p)$ and $W^{uu}_-(p)$ denote the positive and negative components of $W^{uu}(p) \setminus \{p\}$ respectively. It is easy to see that if $\partial_+ W^{uu}(p) = q$ then $D^u_+ p = \{q\}$, $[p,q]^u$ is an ascending u-connecting segment, Int $[p,q]^u = W^{uu}_+(p) \subset W^s(q)$, and $\|Df(e^u(q))\| \leq 1$. We define *the local strong stable manifold* $W^{ss}_\delta(p) = W^{uu}_\delta(p; f^{-1})$, *the strong stable manifold* $W^{ss}(p)$, and the connected components $W^{ss}_\pm(p)$ of $W^{ss}(p) \setminus \{p\}$ by the same way as above.

LEMMA 3.6. *Let p and q be fixed points of f such that $\|Df(e^u(p))\| > 1$ and q is attracting. If $W^{uu}_+(p) \subset W^s(q)$, then $q \in D^u_+ p$. In particular, $[p,q]^u$ is ascending, and hence, $D^u_+ p = \{q\}$.*

PROOF. Suppose $W^{uu}_+(p) \subset W^s(q)$. There exist a compact interval $I \subset W^{uu}_+(p)$ and $\delta > 0$ such that $\bigcup_{n \geq 0} f^n(I) = W^{uu}_+(p) \setminus W^{uu}_\delta(p)$. By the compactness of I, $f^n(I)$ converges to $\{q\}$ as $n \to \infty$ with respect to the Hausdorff metric. It implies $\partial_+ W^{uu}(p) = q$, and hence, $q \in D^u_+ p$. $\square$

Finally, we define a presentation of fixed segments and show that the set $\mathrm{Fix}(\mathscr{S}(f)) \cap \mathscr{S}(e^u)$ is discrete. For $\sigma = u, s$ if a sequence $\{p_i\}_{i=0}^k$ of periodic points of f satisfies $p_i \in D^\sigma_+ p_{i-1}$ for each i, then we define $[p_0, \ldots, p_k]^\sigma \in \mathscr{S}(e^\sigma)$ by the composition $[p_0, p_1]^\sigma * \cdots * [p_{k-1}, p_k]^\sigma$.

PROPOSITION 3.7. *Any $\xi \in \mathscr{S}(e^\sigma) \cap \mathrm{Fix}(\mathscr{S}(f)^n)$ with $|\xi| \neq 0$ has a unique presentation $\xi = [p_0, \ldots, p_k]^\sigma$ and all $p_0, \ldots, p_k$ are fixed points of f^n.*

PROOF. Recall that the normalizer η of $f^n \circ \xi$ is a map in $\mathrm{Diff}^1_+([0,1])$ satisfying $f^n \circ \xi = \xi \circ \eta$. Since ξ is an immersion, $\xi^{-1}(z)$ is a finite set for any $z \in \boldsymbol{T}^2$. Hence, $\xi^{-1}(\mathscr{O}(z; f^n))$ is a periodic orbit of η for any $z \in \mathrm{Per}(f) \cap \mathrm{Im}\, \xi$. It implies $\xi(\mathrm{Per}(\eta)) = \mathrm{Per}(f) \cap \mathrm{Im}\, \xi$.

It is clear that $\mathrm{Per}(\eta) = \mathrm{Fix}(\eta)$, and hence, we have $\mathrm{Per}(f) \cap \mathrm{Im}\, \xi = \mathrm{Fix}(f^n) \cap \mathrm{Im}\, \xi$. By the assumption, $\mathrm{Fix}(f^n)$ is a finite set. Now, the proposition is an easy consequence of the existence and the uniqueness of an irreducible decomposition of ξ associated to $\mathrm{Fix}(f^n)$. $\square$

COROLLARY 3.8. *The set $\{\xi \in \mathscr{S}(e^u) \cap \mathrm{Fix}(\mathscr{S}(f)) \mid |\xi| \leq K\}$ is finite for any given $K > 0$. In particular, $\mathscr{S}(e^u) \cap \mathrm{Fix}(\mathscr{S}(f))$ is a discrete set.*

PROOF. Since $\mathrm{Fix}(f)$ is a finite set, $K_0 = \inf\{|[p,q]^u| \mid p \in \mathrm{Fix}(f), q \in D^u_+ p\}$ is positive. The corollary follows from the proposition and the inequality

$$|[p_0, \ldots, p_k]^u| \geq K_0 \cdot k$$

for any $[p_0, \ldots, p_k]^u \in \mathscr{S}(e^u)$ with $p_0, \ldots, p_k \in \mathrm{Fix}(f)$. $\square$

3.3. A canonical total order on $D^s_{\pm}p$.

Let (f, e^u, e^s) be a framed $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$. For $p \in \mathrm{Per}(f)$, we define a relation $\prec^p_+$ on D^s_+p so that $q_1 \prec^p_+ q_2$ if and only if for any $\epsilon \in]0,1[$ and any neighborhood U of p, there exists $\xi \in S(e^u)$ such that $|\xi| \neq 0$, $\xi(0) \in [p, q_1]^s(]0, \epsilon[)$, $\xi(1) \in [p, q_2]^s(]0, \epsilon[)$, and $\mathrm{Im}\,\xi \subset U$. We write $q_1 \preceq^p_+ q_2$ if $q_1 = q_2$ or $q_1 \prec^p_+ q_2$.

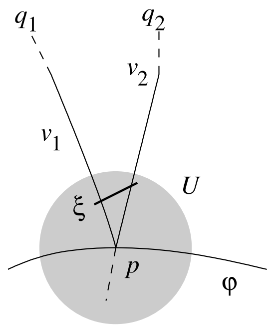

Figure 2. A total order $\prec^p_+$.

Proposition 3.9. *Let p, q_1, q_2 be periodic points of f with $q_1, q_2 \in D^s_+p$. Let φ be a $1/4$-canonical coordinate with $p = \varphi(x_0, y_0) \in \varphi([-1/2, 1/2]^2)$. Suppose that there exist functions v_1 and v_2 on $[y_0, 1]$ such that $v_i(y_0) = x_0$ and $\varphi(\{(v_i(y), y) \mid y \in [y_0, 1]\}) \subset \mathrm{Im}\,[p, q_i]_s$ for each $i = 1, 2$. Then, the following conditions are equivalent*:

1. $q_1 \prec^p_+ q_2$.
2. $v_1(y) < v_2(y)$ *for some* $y \in]y_0, 1]$.
3. $v_1(y) < v_2(y)$ *for any* $y \in]y_0, 1]$.

Proof. Since $\mathrm{Int}\,[p, q_1]^s \cap \mathrm{Int}\,[p, q_2]^s = \varnothing$ if $q_1 \neq q_2$, the latter two conditions are equivalent. Put $I_i = \varphi(\{(v_i(y), y) \mid y \in [y_0, 1]\})$ for $i = 1, 2$.

Take a neighborhood U of p and a number $\epsilon > 0$ so that $U \subset \varphi([-1, 1]^2)$ and $[p, q_i]^s([0, \epsilon]) \subset I_i$. If $q_1 \prec^p_+ q_2$, then there exists $\xi \in \mathscr{S}(e^u)$ such that $\xi(0) \in I_1$, $\xi(1) \in I_2$, and $\mathrm{Im}\,\xi \subset U$. It is easy to see that it implies that $v_1(y) < v_2(y)$ for any $y \in]y_0, 1[$.

Suppose $v_1(y) < v_2(y)$ for any $y \in]y_0, 1[$. It is easy to see that there exists a family $\{\xi_y\}_{y \in]y_0, 1/2[}$ in $\mathscr{S}(e^u)$ such that $\xi_y(0) = \varphi(v_1(y), y)$, $\xi_y(1) \in I_2$, and $\mathrm{Im}\,\xi_y \subset \mathrm{Im}\,\varphi$ for any $y \in]y_0, 1/2[$, and $\mathrm{Im}\,\xi_y$ converges to $\{p\}$ as $y \to y_0$. It implies $q_1 \prec^p_+ q_2$. □

Corollary 3.10. *The relation $\preceq^p_+$ is a total order on D^s_+p.*

Proof. For any $q_1, q_2, q_3 \in D^s_+p$, we can take a $1/4$-canonical coordinate φ with $p = \varphi(x_0, y_0) \in \varphi([-1/2, 1/2]^2)$ and functions v_1, v_2, and v_2 on $[y_0, 1]$ so that $v_i(y_0) = x_0$ and $\varphi(\{(v_i(y), y) \mid y \in [y_0, 1]\}) \subset \mathrm{Im}\,[p, q_i]_s$ for each $i = 1, 2, 3$. Then, it is easy to check that the corollary follows from the proposition. □

We also define a total order $\prec^p_-$ on D^s_-p so that $q_1 \prec^p_- q_2$ if and only if for any $\epsilon \in]0, 1[$ and any neighborhood U of p, there exists $\xi \in S(e^u)$ such that $|\xi| \neq 0$, $\xi(0) \in$

$[q_1, p]^s(]-\epsilon, 0[)$, $\xi(1) \in [q_2, p]^s(]-\epsilon, 0[)$, and $\mathrm{Im}\,\xi \subset U$.

If φ is a 1/4-canonical coordinate, then $f \circ \varphi$ also is. Hence, Proposition 3.9 implies if $q_1 \prec_+^p q_2$ then $f(q_1) \prec_+^{f(p)} f(q_2)$.

LEMMA 3.11. *$D_+^s p \cap \mathscr{O}(q; f) = \{q\}$ for any $q \in D_+^s p$. In particular, p and q have the same period. The similar assertions also hold for $D_-^s p$ and $D_\pm^u p$.*

PROOF. Suppose that $f^k(q) \neq q$ is contained in $D_+^s p$. If $q \prec_+^p f^k(q)$, then $f^{ik}(q) \prec_+^p f^{(i+1)k}(q)$ for any i. Since $\prec_+^p$ is a total order, we obtain that $q \neq f^{ik}(q)$ for any i. It contradicts that q is a periodic point. The same argument shows $f^k(q) \not\prec_+^p q$, and hence, we have $D_+^s p \cap \mathscr{O}(q; f) = \{q\}$.

Now, it is easy to see that p and q have the same period since $D_+^s f^m(p) \cap \mathscr{O}(q; f) = \{f^m(q)\}$ for any $m \geq 0$. $\square$

Remark that $\mathscr{S}(e^u, \mathrm{Fix}(f^k)) \cap \mathrm{Per}(\mathscr{S}(f)) = \mathscr{S}(e^u) \cap \mathrm{Fix}(\mathscr{S}(f)^k)$. In fact, for $\xi = [p_1, \dots, p_m]^u \in \mathscr{S}(e^u, \mathrm{Fix}(f^k)) \cap \mathrm{Per}(\mathscr{S}(f))$, the lemma implies that each p_i is a fixed point of f^k. By the uniqueness of a connecting segment, we have $\mathscr{S}(f)^k(\xi) = [f^k(p_1), \dots, f^k(p_m)]^u = [p_1, \dots, p_m]^u = \xi$.

3.4. Invariant circles without hyperbolic periodic points.

For a C^r diffeomorphism g on a manifold M, let $\mathrm{Fix}_h(g)$ and $\mathrm{Per}_h(g)$ denote the set of hyperbolic fixed points and that of hyperbolic periodic points respectively. By the stable manifold theorem, the unstable set $W^u(p)$ and the stable set $W^s(p)$ are C^r injectively immersed manifolds with $T_pW^u(p) \oplus T_pW^s(p) = T_pM$ for any $p \in \mathrm{Per}_h(g)$. For $k \geq 0$, we define $\mathrm{Fix}_h^k(g)$ and $\mathrm{Per}_h^k(g)$ by

$$\mathrm{Fix}_h^k(g) = \{p \in \mathrm{Fix}_h(g) \mid \dim W^u(p; g) = k\},$$
$$\mathrm{Per}_h^k(g) = \{p \in \mathrm{Per}_h(g) \mid \dim W^u(p; g) = k\}.$$

Recall that a periodic point p is called *attracting* if $p \in \mathrm{Fix}_h^0(g)$, *repelling* if $p \in \mathrm{Fix}_h^{\dim M}(g)$, and *of saddle-type* otherwise.

For $\rho \leq r$, we call an embedded circle C *a ρ-normally attracting circle* for g if $g^m(C) = C$ for some $m \geq 1$ and there exist a dominated splitting $TM|_C = E^u \oplus E^s$, $K > 0$, and $\lambda \in]0, 1[$ such that $T_zC = E^u(z)$ and

$$\|Dg^n|_{E^s(z)}\| \cdot \|(Dg^n|_{E^u(z)})^{-1}\|^\rho < K\lambda^n, \quad \|Dg^n|_{E^s(z)}\| < K\lambda^n$$

for any $n \geq 0$ and $z \in C$. It is known that a ρ-normally attracting circle is a C^ρ submanifold of M, whose existence is persistent under C^ρ-perturbation of g, and which depends continuously on g (see Subsection 12.4 of [**7**]). An embedded circle C' is called *ρ-normally repelling* if it is a ρ-normally attracting circle for g^{-1}. We say a ρ-normally attracting or repelling circle C is *irrational* if the restriction of g^m on C is topologically conjugate to an irrational rotation. Remark that $C = \overline{\mathscr{O}(z; g^m)}$ for any $z \in C$ if C is irrational.

We recall a theorem due to Pujals and Sambarino on the structure of invariant sets of a surface diffeomorphism which admits a dominated splitting. Let $\Omega(g)$ denote the non-wandering set of a diffeomorphism g. That is, a point z of M is contained in $\Omega(g)$ if and only if any neighborhood U of z satisfies $g^n(U) \cap U \neq \varnothing$ for some $n \geq 1$. We say a g-invariant set Λ is *a hyperbolic set of saddle-type* if it admits a hyperbolic splitting $TM|_\Lambda = E^u \oplus E^s$ such that both E^s and E^s are non-trivial.

Theorem 3.12 ([**6**], Theorem B). *Let g be a C^2 diffeomorphism on a compact surface. Suppose that a compact g-invariant set $\Lambda \subset \Omega(f)$ admits a dominated splitting and satisfies $\Lambda \cap \mathrm{Per}(g) \subset \mathrm{Per}_h^1(g)$. Then, Λ is a disjoint union of a hyperbolic g-invariant set of saddle-type and finitely many mutually disjoint 2-normally attracting or repelling irrational circles.*

Fix a C^2 framed $\boldsymbol{P}$A diffeomorphism (f, e^u, e^s). For $\sigma \in \{u, s\}$, we define a subset $\mathscr{C}_*(e^\sigma)$ of $\mathscr{C}(e^\sigma)$ by

$$\mathscr{C}_*(e^\sigma) = \left\{c \in \mathscr{C}(e^\sigma) \cap \mathrm{Per}(\mathscr{C}(f)) \mid \mathrm{Im}\, c \cap \mathrm{Per}_h(f) = \varnothing\right\},$$

and put $\mathscr{C}_{a,*}(e^\sigma) = \mathscr{C}_a(e^\sigma) \cap \mathscr{C}_*(e^\sigma)$. We also define subsets $\Omega^0_*(f)$ and $\Omega^2_*(f)$ of $\boldsymbol{T}^2$ by

$$\Omega^0_*(f) = \bigcup_{c \in \mathscr{C}_*(e^u)} \mathrm{Im}\, c, \quad \Omega^2_*(f) = \bigcup_{c \in \mathscr{C}_*(e^s)} \mathrm{Im}\, c.$$

Proposition 3.13. *Suppose that f has at most one non-hyperbolic periodic orbit. Then, $\Omega^0_*(f)$ is a disjoint union of finitely many 2-normally attracting circles. In particular, $\mathscr{C}_{a,*}(e^u)$ is a finite set for any $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$.*

Proof. The latter is an immediate consequence of the former. We prove the former by assuming the following lemmas, which we show later. Remark that the lemmas hold even if f has more than one non-hyperbolic periodic orbits.

Lemma 3.14. *Put $S_* = \{c \in \mathscr{C}_*(e^u) \mid \mathrm{Im}\, c \cap \mathrm{Per}(f) = \varnothing\}$ and $C_* = \bigcup_{c \in S_*} \mathrm{Im}\, c$. Then, C_* is a disjoint union of finitely many 2-normally attracting irrational circles.*

Lemma 3.15. *Let $p_0, \ldots, p_{k-1}$ are periodic points of f with mutually disjoint orbits and with the same period m. Put $p_k = p_0$ and suppose that there exists a sequence $\{n_i\}_{i=0}^{k-1}$ such that $f^{n_i}(p_{i+1}) \in D^u_+ p_i$ for each $i = 0, \ldots, k-1$. Then, the set $\bigcup_{n=0}^{m-1} \bigcup_{i=0}^{k-1} \mathrm{Im}\, [f^n(p_i), f^{n+n_i}(p_{i+1})]^u$ is an f-invariant element of $\mathscr{E}(\sqcup_{<\infty} S^1, e^u)$.*

Let C_* be the subset of $\Omega^0_*(f)$ given in Lemma 3.14. Put $S = \{c \in \mathscr{C}_*(e^u) \mid \mathrm{Im}\, c \cap \mathrm{Per}(f) \neq \varnothing\}$ and $C = \bigcup_{c \in S} \mathrm{Im}\, c$. It is easy to see that $C \cap C_* = \varnothing$. Hence, it is sufficient to show that C is a disjoint union of finitely many 2-normally attracting circles.

Suppose that C is non-empty. Then, it contains a non-hyperbolic periodic point p_*. Let m be the period of p_*. Put $A = \{(n, n') \in \{0, \ldots, m-1\}^2 \mid f^{n+n'}(p_*) \in D^u_+ f^n(p_*)\}$. For any $z \in C$, there exist non-hyperbolic periodic points q_* and q'_* such that $q'_* \in D^u_+ q_*$ and $z \in \mathrm{Im}\, [q_*, q'_*]^u$. By the assumption, $\mathscr{O}(p_*)$ is the unique non-hyperbolic periodic orbit. Hence, we have $C \subset \bigcup_{(n,n')\in A} \mathrm{Im}\ [f^n(p_*), f^{n+n'}(p_*)]^u$. Fix $(n_0, n_0') \in A$. It is clear that $\{(n, n_0') \mid n = 0, \dots, m-1\}$ is a subset of A. Lemma 3.11 implies $A = \{(n, n_0') \mid n = 0, \dots, m-1\}$. Since C is f-invariant, we have $C = \bigcup_{n=0}^{m-1} \mathrm{Im}\ [f^n(p_*), f^{n+n_0'}(p_*)]^u$. Lemma 3.15 implies that C is a disjoint union of finitely many embedded circles.

Notice that $f^{mn}(z)$ converges to a point of $\mathscr{O}(p_*; f)$ as $n \to \infty$ for any $z \in C$. Since $\|Df^m(e^s(p_*))\| < \|Df^m(e^u(p_*))\| = 1$, both $\|Df^n(e^s(z))\| \cdot \|Df^n(e^u(z))\|^{-2}$ and $\|Df^n(e^s(z))\|$ tends to zero as $n \to \infty$. The compactness of C implies that C is a union of 2-normally attracting circles. $\square$

Proof of Lemma 3.14. Fix $c \in S_*$ and take $\gamma \in \pi_c^{-1}(c) \cap \widetilde{\mathscr{C}}(e^u)$. Then, there exist $n \geq 1$ and $\eta \in \mathrm{Diff}^1_+(S^1)$ such that $f^n \circ \gamma = \gamma \circ \eta$. Since $\mathrm{Im}\ c \cap \mathrm{Per}(f) = \varnothing$, we have $\mathrm{Per}(\eta) = \varnothing$. It implies that η is semi-conjugate to an irrational rotation, and hence, η has no hyperbolic invariant set. In particular, $\mathrm{Im}\ c = \mathrm{Im}\ \gamma$ does not intersect with a hyperbolic invariant set of saddle type. Now, the lemma follows from Theorem 3.12. $\square$

Proof of Lemma 3.15. Put $\Lambda = \bigcup_{i=1}^{k-1} \mathscr{O}(p_i; f)$ and $C = \bigcup_{n=0}^{m-1} \bigcup_{i=1}^{k-1} \mathrm{Im}\ [f^n(p_i), f^{n+n_i}(p_{i+1})]^u$. It is easy to see that $\mathrm{Int}\ [p,q]^u \cap \mathrm{Im}\ [p',q']^u \neq \varnothing$ implies $p = p'$ and $q = q'$. Hence, C is locally diffeomorphic to an interval at any point of $C \setminus \Lambda$.

Fix a point $f^n(p_i)$ of Λ. Recall $f^{n_i}(p_{i+1}) \in D^u_+ p_i$ and $p_{i-1} \in D^u_-(f^{n_i}(p_i))$. By Lemma 3.11, we have $D^u_+(f^n(p_i)) \cap \mathscr{O}(p_{i+1}; f) = \{f^{n+n_i}(p_{i+1})\}$ and $D^u_-(f^n(p_i)) \cap \mathscr{O}(p_{i-1}; f) = \{f^{n-n_{i-1}}(p_{i-1})\}$. It implies that $\mathrm{Im}\ [f^{n-n_{i-1}}(p_{i-1}), f^n(p_i), f^{n+n_i}(p_{i+1})]^u$ is a neighborhood of $f^n(p_i)$ in C. Therefore, C is locally diffeomorphic to an interval at $f^n(p_i)$. $\square$

The following is a consequence of Lemmas 3.14 and 3.15, which we use later.

Lemma 3.16. *For any periodic point $c \in \mathscr{C}(e^u)$ of $\mathscr{C}(f)$ and any $z \in \mathrm{Im}\ c$, there exists $C \in \mathscr{E}(\sqcup_{<\infty} S^1, e^u)$ satisfying $f(C) = C$ and $z \in C \subset \bigcup_{n\geq 0} f^n(\mathrm{Im}\ c)$.*

Proof. If $\mathrm{Im}\ c \cap \mathrm{Per}(f) = \varnothing$, then $\bigcup_{n\geq 0} f^n(\mathrm{Im}\ c)$ is a disjoint union of normally attracting irrational circles by Lemma 3.14.

Suppose that $\mathrm{Im}\ c \cap \mathrm{Per}(f) \neq \varnothing$. Then, $c = \pi_c(\xi)$ for some $\xi = [p_0, p_1, \dots, p_k]^u$ with $p_k = p_0$. By Lemma 3.11, periodic points $p_0, \dots, p_k$ have the same period m. Without loss of generality, we assume that $z \in \mathrm{Im}\ [p_0, p_1]^u$. If $p_1 = f^{n'}(p_0)$ for some $n' \geq 0$, then Lemma 3.15 implies that $C = \bigcup_{n=0}^{m} \mathrm{Im}\ [f^n(p_0), f^{n+n'}(p_0)]^u$ satisfies the required conditions.

Suppose $\mathscr{O}(p_0; f) \neq \mathscr{O}(p_1; f)$. We define a directed graph $G = (E, V)$ so that $V = \{1, \dots, k\}$ and $i \to j$ if and only if $p_{i+1} \in \mathscr{O}(p_j; f)$. Note that $1 \to \cdots \to k$ is a path of G and if $i \to j$ and $p_{j'} \in \mathscr{O}(p_j)$ then $i \to j'$.

Take the shortest path $j_1 \to \cdots \to j_l$ of G satisfying $j_1 = 1$ and $j_l = k$. It is easy to see that $\mathscr{O}(p_{j_i}) \neq \mathscr{O}(p_{j_{i'}})$ if $i \neq i'$. Take n_i so that $p_{j_i+1} = f^{n_i}(p_{j_{(i+1)}})$ for each i. Then, we have $f^{n_i}(p_{j_{(i+1)}}) \in D^u_+ p_{j_i}$ and $\mathrm{Im}\ [p_{j_i}, f^{n_i}(p_{j_{(i+1)}})]^u \subset \mathrm{Im}\ c$ for any $i = 1, \dots, l-1$. Put $j_0 = 0$ and $n_0 = 0$. Then, Lemma 3.15 for $\{p_{j_i}\}_{i=0}^{l-1}$ and $\{n_i\}_{i=0}^{l-1}$ implies that $C = \bigcup_{n=0}^{m-1} \bigcup_{i=0}^{l-1} \mathrm{Im}\ [f^n(p_{j_i}), f^{n+n_i}(p_{j_{(i+1)}})]^u$ is the required one. $\square$

4. Non-degenerate $\boldsymbol{P}$A diffeomorphisms and Proof of Theorem B.

We study the topology of invariant foliations of a non-degenerate $\boldsymbol{P}$A diffeomorphism and give a structure of CW complex structure to $\mathscr{C}_a(e^u)$. In Subsection 4.1, we show that a non-degenerate $\boldsymbol{P}$A diffeomorphism admits a C^1 invariant foliation tangent to E^u on $\boldsymbol{T}^2 \setminus (\Omega^0_*(f) \cup \mathrm{Per}^0_h(f))$. One of the important consequences is Proposition 4.4, which asserts that the non-trivial part of $\mathscr{C}(e^u)$ is $\mathscr{C}(e^u, \mathrm{Per}^0_h(f))$. In Subsection 4.2, we investigate the singularity of the above foliation at attracting periodic points. It allows us to show the continuity of segments and the existence of a decomposition of $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$. We give a structure of CW complex to $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$ in Subsection 4.3, where we see that the combinatorics of connecting segments determines the topology of $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$ completely. We prove Theorem B in Subsection 4.4.

In this section, we fix a framed $\boldsymbol{P}$A diffeomorphism (f, e^u, e^s) and suppose that f is non-degenerate and of class C^2.

4.1. The unstable foliation.

Put $\Omega^k(f) = \mathrm{Per}^k_h(f) \cup \Omega^k_*(f)$ for $k = 0, 2$ and $\Omega^1(f) = \Omega(f) \setminus (\Omega^0(f) \cup \Omega^2(f))$. Theorem 3.12 implies that $\Omega^1(f)$ is a hyperbolic set of saddle-type. It is easy to see that both $\mathrm{Per}^0_h(f)$ and $\mathrm{Per}^2_h(f)$ have no accumulation points. In particular, they are finite sets.

Proposition 4.1. *Let $E^u \oplus E^s$ be the unique $\boldsymbol{P}$A splitting associated to f. Then, the subbundle E^u generates a C^1 f-invariant foliation $\mathscr{F}^u$ on $\boldsymbol{T}^2 \setminus \Omega^0(f)$. Moreover, for any given neighborhood U of $\Omega^0(f)$, there exist $\lambda_U \in]0,1[$ and $K_U > 0$ such that $\|Df^{-n}|_{E^u(z)}\| \leq K_U \lambda_U^n$ for any $n \geq 0$ and $z \in \boldsymbol{T}^2 \setminus U$.*

Proof. Since $\Omega^0(f)$ is an attracting invariant set, we can choose an open neighborhood U_* of $\Omega^0(f)$ so that $f(\overline{U_*}) \subset U_*$ and $\bigcap_{n \geq 0} f^n(U_*) = \Omega^0(f)$.

Since the α-limit set $\bigcap_{n\geq 1} \overline{\{f^{-m}(z) \mid m \geq n\}}$ is contained in $\Omega^1(f) \cup \Omega^2(f)$ for any $z \in \boldsymbol{T}^2 \setminus U_*$, $\|Df^{-n}|_{E^u(z)}\|$ converges to zero. By the compactness of $\boldsymbol{T}^2 \setminus U_*$, there exist $K > 0$ and $\lambda \in (0, 1)$ such that $\|Df^{-n}|_{E^u(z)}\| \leq K\lambda^n$ for any $z \in \boldsymbol{T}^2 \setminus U_*$. It is easy to see that it implies the second assertion of the proposition.

Since $f(\overline{U_*}) \subset U_*$, we obtain the C^1 regularity of E^u on $\boldsymbol{T}^2 \setminus U_*$ by the same argument as in the case of codimension one hyperbolic splittings (see, *e.g.*, Theorem 19.1.8 of [**3**]). By the Df-invariance of E^u, the restriction of E^u to $\boldsymbol{T}^2 \setminus \Omega^0(f)$ is of class C^1. In particular, it generates a C^1 f-invariant foliation $\mathscr{F}^u$. □

For $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$, let $\mathscr{F}^u(z)$ be the leaf of $\mathscr{F}^u$ that contains z, and $\mathscr{F}^u_+(z)$ and $\mathscr{F}^u_-(z)$ denote the positive and the negative half components of $\mathscr{F}^u(z) \setminus \{z\}$. By the same argument as Proposition 4.1, E^s generates a C^1 foliation $\mathscr{F}^s$ on $\boldsymbol{T}^2 \setminus \Omega^2(f)$. We define $\mathscr{F}^s(z)$ and $\mathscr{F}^s_\pm(z)$ similarly.

Lemma 4.2. *The foliation $\mathscr{F}^u$ on $\boldsymbol{T}^2 \setminus \Omega^0(f)$ satisfies the followings.*

1. *$\mathscr{F}^u$ has no compact leaves.*
2. *If $\mathscr{F}^u(z) \cap W^u(p) \neq \varnothing$ for $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$ and $p \in \mathrm{Per}^1_h(f) \cup \mathrm{Per}^2_h(f)$, then $\mathscr{F}^u(z) \subset W^u(p)$.*
3. *If $f(\mathscr{F}^u(z)) = \mathscr{F}^u(z)$ for $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$, then there exists $p \in \mathrm{Fix}^1_h(f) \cup \mathrm{Fix}^2_h(f)$*

such that $\mathscr{F}^u(z) = W^{uu}(p)$.

The similar statements hold for $\mathscr{F}^s$.

Proof. By Proposition 4.1, any $I \in \mathscr{E}(S^1, e^u) \cup \mathscr{E}([0,1], e^u)$ with $I \cap \Omega^0(f) = \varnothing$ satisfies $\sum_{n \geq 0} |f^{-n}(I)| < \infty$.

If $\mathscr{F}^u$ has a compact leaf C, then $|f^{-n}(C)|$ converges to zero as $n \to \infty$ since $C \cap \Omega^0(f) = \varnothing$. It contradicts the Poincaré-Bendixon theorem. Therefore, $\mathscr{F}^u$ has no compact leaves.

Suppose $\mathscr{F}^u(z) \cap W^u(p) \neq \varnothing$ for $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$ and $p \in \mathrm{Per}^1_h(f) \cup \mathrm{Per}^2_h(f)$. Take a point $z_0 \in \mathscr{F}^u(z) \cap W^u(p)$. Any compact subinterval I_1 of $\mathscr{F}^u(z)$ containing z_0 is a subset of $W^u(p)$ since $|f^{-n}(I_1)|$ converges to zero as $n \to \infty$. It implies $\mathscr{F}^u(z) \subset W^u(p)$.

Suppose $f(\mathscr{F}^u(z)) = \mathscr{F}^u(z)$ for $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$. Choose a compact subinterval I_2 of $\mathscr{F}^u(z)$ and a neighborhood U_* of $\Omega^0(f)$ so that $f^{-1}(I_2) \cap I_2 \neq \varnothing$, $f(\overline{U_*}) \subset U_*$, and $\overline{U_*} \cap I_2 = \varnothing$. Put $I_2^n = \bigcup_{m=0}^n f^{-m}(I_2)$ for $n \geq 1$ and $I_2^* = \overline{\bigcup_{n \geq 0} I_2^n}$. Then, $\{I_2^n\}_{n \geq 0}$ is an increasing sequence of compact intervals with $I_2^n \cap U_* = \varnothing$ and $\sup_{n \geq 0} |I_2^n| < \infty$. Hence, the set I_2^* is a compact interval with $I_2^* \cap \Omega^0(f) = \varnothing$ and $f^{-1}(I_2^*) \subset I_2^*$. It implies that I_2^* is a subset of $\mathscr{F}^u(z)$ and contains a point p of $\mathrm{Fix}^1_h(f) \cup \mathrm{Fix}^2_h(f)$. By the second assertion, we have $\mathscr{F}^u(z) \subset W^u(p)$. It implies that $\mathscr{F}^u(z) = W^{uu}(p)$. □

Lemma 4.3. *For any $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$, the positive or the negative boundary point $\partial_\pm \mathscr{F}^u(z)$ is an attracting periodic point if it exists.*

Proof. Proof is done by contradiction. Suppose that $\partial_\tau \mathscr{F}^u(z) \in \boldsymbol{T}^2 \setminus \mathrm{Per}^0_h(f)$ for $z \in \boldsymbol{T}^2 \setminus \Omega^0(f)$ and $\tau \in \{+, -\}$. Without loss of generality, we assume $\tau = -$. Since $\partial_- \mathscr{F}^u(z)$ is contained in $\Omega^0(f) = \mathrm{Per}^0_h(f) \cup \Omega^0_*(f)$, we have $\partial_- \mathscr{F}^u(z) \in \Omega^0_*(f)$. Choose a 1/4-canonical coordinate φ at p such that $\varphi([-1,1] \times \{0\}) = \Omega^0_*(f) \cap \mathrm{Im}\, \varphi$ and $\varphi(\{x\} \times [-1,1]) \subset \mathscr{F}^s(\varphi(x,0))$ for any $x \in [-1,1]$. There exists a C^1 function h on $[0,1]$ such that $h(0) = 0$ and $\varphi(\{(x, h(x)) \mid x \in]0,1]\})$ is a connected component of $\mathscr{F}^u(z) \cap \mathrm{Im}\, \varphi$. Without loss of generality, we assume that $h(x) > 0$ for all $x \in]0,1]$.

Since $\Omega^0_*(f)$ consists of irrational normally attracting circles, there exist sequences $\{n_i \geq 1\}$, $\{x_i \in]-1,0[\}$ and $\{x'_i \in]1/4, 1/2[\}$ such that $f^{n_i} \circ \varphi([0,1/2] \times \{0\}) = \varphi([x_i, x'_i] \times \{0\})$ for any i and $[x_i, x'_i]$ converges to $[0, 1/2]$ as $i \to \infty$. Since $\varphi(\{x\} \times [0, h(x)]) \subset \mathscr{F}^s(\varphi(x,0))$ for any $x \in]0,1[$, the set $f^{n_i} \circ \varphi(\{(x, h(x)) \mid x \in [0,1/2]\})$ converges to $\varphi([0,1/2] \times \{0\})$ as $i \to \infty$. Hence, for any sufficiently large i, there exists a C^1 function h_i on $[x_i, x'_i]$ such that $h_i(x_i) = 0$, $h_i(x) > 0$ for any $x \in]x_i, x'_i]$, $\varphi(\{(x, h_i(x)) \mid x \in [x_i, x'_i]\}) \subset \mathscr{F}^u_-(f^{n_i}(z))$, and $0 < h_i(x'_i) < \inf\{h(x) \mid x \in [1/4, 1/2]\}$. Since $x_i < 0$ and $h_i(x_i) = 0$, we have $\mathscr{F}^u(f^{n_i}(z)) \cap \mathscr{F}^u(z) \neq \varnothing$ for any sufficiently large i. It contradicts that $\mathscr{F}^u$ is a foliation on $\boldsymbol{T}^2 \setminus \Omega^0(f)$. □

The following proposition implies all we need to know is the structure of the image of $\widetilde{\mathscr{C}_a}(e^u, \mathrm{Per}^0_h(f))$ under π_c.

Proposition 4.4. *$\mathscr{C}_a(e^u)$ is the disjoint union of $\mathscr{C}_a(e^u, \mathrm{Per}^0_h(f))$ and $\mathscr{C}_{a,*}(e^u)$.*

Proof. It is trivial that $\mathscr{C}_a(e^u, \mathrm{Per}^0_h(f)) \cap \mathscr{C}_{a,*}(e^u) = \varnothing$. Fix $c \in \mathscr{C}_a(e^u)$. By Lemma 4.2, we have $\mathrm{Im}\, c \cap \Omega^0(f) \neq \varnothing$. Lemma 4.3 implies either $\mathrm{Im}\, c \cap \mathrm{Per}^0_h(f) \neq \varnothing$ or $\mathrm{Im}\, c \subset \Omega^0_*(f)$. The latter implies $c \in \mathscr{C}_{a,*}(e^u)$. □

4.2. A decomposition of $\mathscr{S}(e^u, \mathrm{Fix}_h^0(f))$.

First, we investigate the local structure of $\mathscr{F}^u$ at attracting fixed points. Fix an attracting fixed point r of f and a 1/8-canonical coordinate φ satisfying $\varphi(0,0) = r$, $\mathrm{Im}\,\varphi \subset W^s(r)$, and $\varphi(\{0\}\times[-1,1]) \subset W^{ss}(r) = \mathscr{F}^s(r)$. For $y \in [-1/2, 1/2]$, let I_y denote the connected component of $\mathscr{F}^u(\varphi(1/2, y)) \cap \varphi(]0,1[\times]-1,1[)$ that contains $\varphi(1/2, y)$. Then, there exists a C^1 function h_y on $[0,1]$ such that $\overline{I_y} = \varphi(\{(x, h_y(x)) \mid x \in [0,1]\})$. Remark that the family $\{h_y\}_{y\in[-1/2,1/2]}$ is continuous with respect to the C^1 topology and $|dh_y/dx| \leq 1/8$ for any y. In particular, the set $V = \bigcup_{y\in[-1/2,1/2]} \overline{I_y}$ contains $\varphi([0,1] \times [-1/4, 1/4])$ and the family $\{\overline{I_y}\}_{y\in[-1/2,1/2]}$ defines a foliation on V with a unique possible singularity at r.

Proposition 4.5. *There exist $k \geq 0$, a sequence $\{q_i\}_{i=1}^{2k+1}$ of fixed points of f, and an increasing sequence $\{y_i\}_{i=1}^{2k+1}$ in $[-1/8, 1/8]$ such that*

1. *q_i is of saddle-type if i is odd, and is repelling if i is even,*
2. *$I_{y_i} \subset W_-^{uu}(q_i)$ for any i,*
3. *$\{y \in [-1/2, 1/2] \mid \partial_- I_y = p\} = [y_1, y_{2k+1}]$,*
4. *$\{y \in [-1/2, 1/2] \mid \partial_- I_y \in \varphi(\{0\}\times]-1, 0[)\} = [-1/2, y_1[$,*
5. *$\{y \in [-1/2, 1/2] \mid \partial_- I_y \in \varphi(\{0\}\times]0, 1[)\} =]y_{2k+1}, 1/2]$,*
6. *if $k \geq 1$, then $I_y \subset W^u(q_{2j})$ for any $j = 1, \ldots, k$ and $y \in]y_{2j-1}, y_{2j+1}[$.*

Proof. Fix $N \geq 1$ so that $f^N([0,1] \times [-1/2, 1/2]) \subset \varphi([0,1] \times [-1/8, 1/8])$. The proof is done by investigating the action of f^N on the leaf space of foliation $\{I_y\}$. Take a map $g : [-1/2, 1/2] \to [-1/4, 1/4]$ so that $f^N(I_y) = I_{g(y)}$. The assertion 3 of Lemma 4.2 implies that there exists a map $\alpha : \mathrm{Fix}(g) \to \mathrm{Fix}(f^N)$ such that $I_{y_*} \subset \mathscr{F}^u(\alpha(y_*))$ for any $y_* \in \mathrm{Fix}(g)$. Since $\partial_- I_y = f^N(\partial_- I_y)$ if and only if $\partial_- I_y = r$, we have $\partial_- I_{y_*} = r$ for any $y_* \in \mathrm{Fix}(g)$. In particular, $\mathrm{Im}\,\alpha \subset D_+^u r$. On the other hand, any $q \in D_+^u r$ is a fixed point of f by Lemma 3.11, and hence, there exists a unique fixed point y_* of $\mathrm{Fix}(g)$ with $I_{y_*} \subset \mathscr{F}^u(q)$. Hence, the map α is a one-to-one correspondence between $\mathrm{Fix}(g)$ and $D_+^u r$.

By the finiteness of $\mathrm{Fix}(f)$, there exists an increasing sequence $\{y_i\}_{i=1}^l$ such that $\mathrm{Fix}(g) = \{y_i \mid i = 1, \ldots, l\}$. Put $q_i = \alpha(y_i)$. Since the map g is the composition

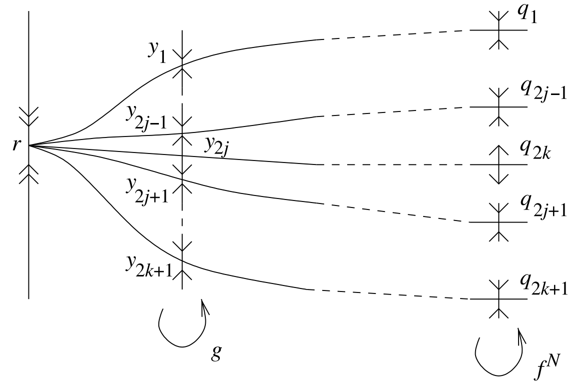

Figure 3. Branching of $\mathscr{F}^u$ at an attracting fixed point.

of f^N and a holonomy map of $\mathscr{F}^u$, it is a C^1 orientation preserving map satisfying $\|Dg(y_i)\| = \|Df^N|_{E^s(q_i)}\|$ for any i. In particular, y_i is attracting if and only if q_i is of saddle-type and y_i is repelling if and only if q_i is repelling.

Recall that $g([-1/2,1/2])$ is contained in $[-1/4,1/4]$. The maximal invariant set $\bigcap_{n\geq 0} g^n([-1/2,1/2])$ must coincide with $[y_1, y_{2k+1}]$ and both y_1, y_{2k+1} are attracting fixed points of g. It implies that $l = 2k+1$ for some $k \geq 0$, y_i is attracting if i is odd, and y_i is repelling if i is even. Hence, $q_i \in \mathrm{Fix}_h^1(f)$ if i is odd, and $q_i \in \mathrm{Fix}_h^2(f)$ if i is even.

The set $J = \{y \in [-1/2,1/2] \mid \partial_- I_y = r\}$ is a g-invariant subinterval of $[-1/2,1/2]$. Since $[y_1, y_{2k+1}]$ is the maximal invariant subinterval and J contains both y_1 and y_{2k+1}, we obtain $J = [y_1, y_{2k+1}]$. Now, it is clear that the assertions 4 and 5 of the proposition hold.

Finally, it remains only to show the last assertion. Suppose that $k \geq 1$. Fix $j = 1,\dots,k$ and $y \in]y_{2j-1}, y_{2j+1}[$. Since $W^u(y_{2j}; g) =]y_{2j-1}, y_{2j+1}[$, $\varphi(1/2, y_{2j}) \in I_{y_{2j}} \subset W^u(q_{2j})$, and q_{2j} is repelling, we have $\varphi(1/2, g^{-n}(y)) \in W^u(q_{2j})$ for some sufficiently large $n \geq 1$, and hence, $\mathscr{F}^u(\varphi(1/2,y)) \cap W^u(q_{2j}) \neq \varnothing$. The second assertion of Lemma 4.2 implies $I_y \subset \mathscr{F}^u(\varphi(1/2,y)) \subset W^u(q_{2j})$. □

One of the consequences is the following.

Corollary 4.6. *$\bigcup_{q \in D_+^u r} W^u(q) = \{z \in \boldsymbol{T}^2 \setminus \Omega^0(f) \mid \partial_- \mathscr{F}^u(z) = r\}$ for any attracting fixed point r of f.* □

Another is the continuity of segments tangent to e^u.

Lemma 4.7. *For any $q \in D_+^u r$ and any compact subinterval J of $\mathscr{F}^s(r)$ with $r \in \mathrm{Int}\, J$, there exists a continuous map $\hat{\xi}$ from a neighborhood U of q to $\mathscr{S}(e^u)$ such that $\hat{\xi}(z)(0) \in J$, $\hat{\xi}(z)(1) = z$, and $\mathrm{Int}\, \hat{\xi}(z) \cap J = \varnothing$ for any $z \in U$. Moreover, if $q \in \mathrm{Fix}_h^2(f)$, then we can choose the above $\hat{\xi}$ so that $\hat{\xi}(z)(0) = r$ for any $z \in U$.*

We remark that the map $\hat{\xi}$ is uniquely determined once U is fixed.

Proof. Suppose that $q = q_i$ and put $I' = \overline{\mathscr{F}_-^u(q_i) \setminus I_{y_i}}$. Since $\overline{\mathscr{F}_-^u(q_i)} = \mathrm{Im}\,[r, q_i]^u$ is an f-invariant compact interval, and $\overline{\mathscr{F}_-^u(q_i)} \cap J$ contains r, we apply Lemma 3.3 to $\overline{\mathscr{F}_-^u(q_i)}$ and $J \subset W^s(r)$, and obtain $I' \cap J = \varnothing$. Since $I' \cap \Omega^0(f) = \varnothing$, there exists an embedding $\psi : [-1,1]^2 \to \boldsymbol{T}^2 \setminus (J \cup \Omega^0(f))$ such that $\psi(0,0) = q_i$, $\psi([-1,0] \times \{0\}) = I'$, $\psi(\{-1\} \times [-1,1]) \subset \varphi(\{1/2\} \times [-1/2,1/2])$, and $\psi([-1,1] \times \{y\}) \subset \mathscr{F}^u(\psi(0,y))$ for any y. Take a function $h : [-1,1] \to [-1/2,1/2]$ so that $\psi(-1,y) = \varphi(1/2, h(y))$. Then, we can define a map $\hat{\xi} : \psi([-1/2,1] \times [-1,1]) \to \mathscr{S}(e^u)$ so that $\mathrm{Im}\,(\hat{\xi}(\psi(x,y)) = \overline{I_{h(y)}} \cup \psi([-1,x] \times \{y\})$. It is easy to see that $\hat{\xi}$ is continuous and satisfies the assertions of the lemma.

The latter part of the lemma is an immediate consequence of the assertion 3 of Proposition 4.5. □

Now, we show that $\mathscr{S}(e^u, \mathrm{Fix}_h^0(f))$ admits a decomposition by the unstable sets of fixed points of $\mathscr{S}(f)$. For $k = 1, 2$, let $\Sigma^k(f)$ denote the set consisting of $p \in \mathrm{Fix}_h^k(f)$ such that $\mathscr{F}^u(p)$ has finite length. Put $\Sigma(f) = \Sigma^1(f) \cup \Sigma^2(f)$. For $p \in \Sigma(f)$, let $(z)^u$

denote the unique element of $\mathscr{S}(e^u)$ with $\mathrm{Int}\,(z)^u = \mathscr{F}^u(z)$. Let $(z_1, \ldots, z_n)^u$ denote the composition $(z_1)^u * \cdots * (z_n)^u$ if it is well-defined.

Lemma 4.8. *Let p be a fixed point in $\Sigma(f)$ with $D^u_\pm p = \{r_\pm\}$. Then, $\partial_\pm \mathscr{F}^u(z) = r_\pm$ for any $z \in W^u(p)$ and the map $z \in W^u(p) \mapsto (z)^u \in \mathscr{S}(e^u)$ is continuous.*

Proof. It is clear when $p \in \mathrm{Fix}^1_h(f)$. When $p \in \mathrm{Fix}^2_h(f)$, Lemma 4.7 implies that there exists a neighborhood U of p such that $\partial_\pm \mathscr{F}^u(z) = r_\pm$ for any $z \in U$ and $(z)^u$ depends continuously on $z \in U$. Since $W^u(p) = \bigcup_{n \geq 0} f^{-n}(U)$, the same holds for any $z \in W^u(p)$. $\square$

Proposition 4.9. *The set $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$ admits a decomposition*

$$\mathscr{S}\big(e^u, \mathrm{Fix}^0_h(f)\big) = \bigcup W^u(\xi_0; \mathscr{S}(f)),$$

where the union runs over all $\xi_0 \in \mathscr{S}(e^u, \mathrm{Fix}^0_h(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$. Moreover, the unstable set $W^u(\xi_0; \mathscr{S}(f))$ coincides with $\{\xi_1 * \cdots * \xi_k \mid \xi_i \in W^u((p_i)^u; \mathscr{S}(f))\}$ for any $\xi_0 = (p_1, \ldots, p_n)^u$ with $p_1, \ldots, p_n \in \Sigma(f)$.

Proof. Put $W(\xi_0) = \{(z_1, \ldots, z_k)^u \mid z_i \in W^u(p_i; f)\}$ for $\xi_0 = (p_1, \ldots, p_n)^u$ with $p_1, \ldots, p_n \in \Sigma(f)$. Lemma 4.8 implies that $\mathscr{S}(f)^{-n}((z_1, \ldots, z_k)^u) = (f^{-n}(z_1), \ldots, f^{-n}(z_k))^u$ converges to ξ_0 as $n \to \infty$ if $z_i \in W^u(p_i; f)$ for any i. Therefore, we have $W(\xi_0) \subset W^u(\xi_0; \mathscr{S}(f))$.

We claim that $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f)) = \bigcup W(\xi_0)$, where the union runs over all $\xi_0 \in \mathscr{S}(e^u, \mathrm{Fix}^0_h(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$. Recall that $\mathrm{Per}^0_h(f)$ is a finite set. Take $\xi \in \mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$ and let $\xi = \xi_1 * \cdots * \xi_k$ be the irreducible decomposition with respect to $\mathrm{Per}^0_h(f)$. Choose $z_i \in \mathrm{Int}\,\xi_i$ for $i = 1, 2, \ldots, k$. Lemma 4.3 implies $\xi_i = (z_i)^u$. Put $r_{-,i} = \partial_- \mathscr{F}^u(z_i) = \xi_i(0)$ and $r_{+,i} = \partial_+ \mathscr{F}^u(z_i) = \xi_i(1)$. By Corollary 4.6, there exist $p_i \in \mathrm{Per}^1_h(f) \cup \mathrm{Per}^2_h(f)$ such that $\partial_\pm \mathscr{F}^u(p_i) = r_{\pm,i}$ and $z_i \in W^u(p_i)$. Since $r_{-,1} = \xi(0) \in \mathrm{Fix}^0_h(f)$, Lemma 3.11 implies all p_i and $r_{+,i}$ are fixed points of f. Hence, $\xi = (z_1)^u * \cdots * (z_k)^u$ is contained in $W((p_1, \ldots, p_k)^u)$ with $p_1, \ldots, p_k \in \Sigma(f)$.

Since the unstable sets are mutually disjoint, the above claim implies that $W^u(\xi_0; \mathscr{S}(f)) = W(\xi_0)$ for any $\xi_0 \in \mathscr{S}(e^u, \mathrm{Fix}^0_h(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$. It also implies the latter assertion of the lemma. $\square$

4.3. The CW structure of $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$.

In this subsection, we show that the decomposition in Proposition 4.9 gives a structure of a CW complex to $\mathscr{S}(e^u, \mathrm{Fix}^0_h(f))$.

We need the following easy lemma, whose proof we omit.

Lemma 4.10. *Let X be a metric space and h a continuous map from a cube $[-1,1]^n$ to X. Suppose that the restriction of h on $]-1,1[^n$ is injective and $h(\partial[-1,1]^n) \cap h(]-1,1[^n) = \emptyset$. Then, the restriction of h on $]-1,1[^n$ is a homeomorphism onto its image.* $\square$

The following lemma shows that the totally ordered sets $(D^u_+ p, \prec^p_+)$ and $(D^u_- p, \prec^p_-)$ determine the boundary of $W^u((p)^u; \mathscr{S}(f))$.

LEMMA 4.11. *For any repelling fixed point p of f, $D^s_+ p$ consists of odd number of elements. Let $D^s_+ p = \{q_1 \prec^p_+ \cdots \prec^p_+ q_{2k+1}\}$. Then, the followings hold*:

1. *q_i is of saddle-type if i is odd, and is attracting if i is even.*
2. *$q_{i+1} \in D^u_+ q_i$ for each i.*
3. *If $D^u_+ p = \{r_+\}$ and $D^u_- p = \{r_-\}$, then $D^u_- q_1 = \{r_-\}$, $D^u_+ q_{2k+1} = \{r_+\}$, and $(z)^u$ converges to $[r_-, q_1, \dots, q_{2k+1}, r_+]^u$ as $z \in \operatorname{Int}[p, q_1]^s$ tends to q_1.*

PROOF. Fix a 1/8-canonical coordinate φ satisfying $\varphi(0,0) = p$, $\operatorname{Im}\varphi \subset W^u(p)$, and $\varphi([-1,1] \times \{y\}) \subset \mathscr{F}^u(\varphi(0,y))$ for any $y \in [-1,1]$. For $x \in [-1/2, 1/2]$, let I_x denote the connected component of $\mathscr{F}^s(\varphi(x,1/2)) \cap \varphi(]-1,1[\times]0,1[)$ that contains $\varphi(x,1/2)$. Applying the same argument as in Proposition 4.5 to $\mathscr{F}^s$, we obtain $k \geq 0$, a sequence $\{q_i\}_{i=1}^{2k+1}$ of fixed points of f, and an increasing sequence $\{x_i\}_{i=1}^{2k+1}$ in $[-1/8, 1/8]$ such that

1. q_i is of saddle-type if i is odd, and is attracting if i is even,
2. $I_{x_i} \subset W^{ss}_-(q_i)$ for any i,
3. $\{x \in [-1/2, 1/2] \mid \partial_- I_x = p\} = [x_1, x_{2k+1}]$,
4. $\{x \in [-1/2, 1/2] \mid \partial_- I_x \in \varphi(]-1,0[\times\{0\})\} = [-1/2, x_1[$,
5. $\{x \in [-1/2, 1/2] \mid \partial_- I_x \in \varphi(]0,1[\times\{0\})\} =]x_{2k+1}, 1/2]$,
6. if $k \geq 1$, $I_x \subset W^s(q_{2j})$ for any $j = 1, \dots, k$ and $x \in]x_{2j-1}, x_{2j+1}[$.

In particular, $D^s_+ p = \{q_1 \prec^p_+ \cdots \prec^p_+ q_{2k+1}\}$.

First, we show that $D^u_+ q_{2j-1} = \{q_{2j}\}$ for any j. Take an interval $J \subset W^{uu}_+(q_{2j-1})$ so that $\partial_- J = q_{2j-1}$ and $\mathscr{F}^s(z) \cap \varphi(]x_{2j-1}, x_{2j}[\times\{1/2\}) \neq \emptyset$ for any $z \in J$. Since $\varphi(]x_{2j-1}, x_{2j}[\times\{1/2\}) \subset W^s(q_{2j})$, we have $J \subset W^s(q_{2j})$ by the assertion 2 of Lemma 4.2 for $\mathscr{F}^s$. It implies $W^{uu}_+(q_{2j-1}) \subset W^s(q_{2j})$, and hence, $D^u_+ q_{2j-1} = \{q_{2j}\}$ by Lemma 3.6. The same argument shows $D^u_- q_{2j+1} = \{q_{2j}\}$ for any j.

Suppose that $D^u_+ p = \{r_+\}$ and $D^u_- p = \{r_-\}$. Then, the set $\varphi(]-1,0[\times\{0\}) \subset \mathscr{F}^u_-(p)$ is contained in $W^s(r_-)$. Since $\mathscr{F}^s(\varphi(x,1/2)) \cap \varphi(]-1,0[\times\{0\}) \neq \emptyset$ for any $x \in [-1/2, x_1[$, we have $\varphi([-1/2, x_1[\times\{1/2\}) \subset W^s(r_-)$. The same argument as above implies that $D^u_- q_1 = \{r_-\}$. Similarly, we obtain $D^u_+ q_{2k+1} = \{r_+\}$.

Finally, we show the convergence of $(z)^u$. For any $z \in \operatorname{Int}[p, q_1]^s = W^{ss}_-(q_1)$ and any compact subinterval I_z of $\mathscr{F}^u(z)$, there exist $n \geq 1$ and $y \in]0, 1/2[$ such that $f^{-n}(I_z) \subset \varphi([-1/2, 1/2] \times \{y\})$. Hence, there exists a decomposition $(z)^u = \xi_{z,0} * \cdots * \xi_{z,2k+1}$ such that $\xi_{z,i-1}(1) = \xi_{z,i}(0) \in W^{ss}_-(q_i)$ for any i. Lemma 3.2 implies that $\xi_{z,i}(0)$ is the unique intersection point of $\mathscr{F}^u(z)$ and $\mathscr{F}^s_-(q_i)$. By Lemma 3.3, we have $\operatorname{Im}[r_-, p, r_+]^u \cap \mathscr{F}^s_+(q_i) = \emptyset$, and hence, $\varphi([-1/2, 1/2] \times \{y\}) \cap \mathscr{F}^s_+(q_i) = \emptyset$ for any $y \in]0, 1/2[$. The same argument as above implies $\mathscr{F}^u(z) \cap \mathscr{F}^s_+(q_i) = \emptyset$ for any $z \in \operatorname{Int}[p, q_1]^s$. Therefore, $\xi_{z,i}(0)$ is the unique intersection point of $\mathscr{F}^u(z)$ and $\mathscr{F}^s(q_i)$ for $z \in \operatorname{Int}[p, q_1]^s$. It is easy to see that it depends continuously on $z \in \operatorname{Int}[p, q_1]^s$.

For any $t \in]0, 1[$, there exists $n \geq 0$ such that $\mathscr{F}^s(f^{-n}(\xi_{z,0}(t))) \cap \operatorname{Int}[r_-, p]^u \neq \emptyset$. Applying Lemma 3.3 to $\operatorname{Im}[r_-, p]^u$ and $\mathscr{F}^s(r_-) = W^{ss}(r_-)$, we obtain $\mathscr{F}^s(\varphi(x, 0)) \neq \mathscr{F}^s(r_-)$ for any $x \in]-1, 0[$. It implies that $\operatorname{Int}\xi_{z,0} \cap \mathscr{F}^s(r_-) = \emptyset$. Similarly, we have $\operatorname{Int}\xi_{z,2k+1} \cap \mathscr{F}^s(r_+) = \emptyset$.

Put $q_0 = r_-$ and $q_{2k+2} = r_+$. Fix a compact interval $I^s_j \subset \mathscr{F}^s(q_{2j})$ for every $j = 0, \dots, k+1$. We have shown $\operatorname{Int}\xi_{z,2j} \cap \mathscr{F}^s(q_{2j}) = \emptyset$ and $\operatorname{Int}\xi_{z,2j+1} \cap \mathscr{F}^s(q_{2j+2}) = \emptyset$

for any $z \in \operatorname{Int}\,[p,q_1]^s$ and $j = 0,\dots,k$. By Lemma 4.7, there exists a continuous function $\hat{\xi}_j$ from a neighborhood U_j of q_{2j+1} to $\mathscr{S}(e^u)$ such that $\hat{\xi}_j(z)(0) \in I^s_j$, $\hat{\xi}_j(z)(1) \in U_j$, and $\operatorname{Int}\hat{\xi}_j(z) \cap I^s_j = \varnothing$ for any $z \in U_j$. It is clear that $\hat{\xi}_j(q_{2j+1}) = [q_{2j}, q_{2j+1}]^u$ and $\xi_{z,2j} = \hat{\xi}_j(\xi_{z,2j}(1))$ if $\xi_{z,2j}(1) \in U_j$. Therefore, $\xi_{z,2j}$ converges to $[q_{2j}, q_{2j+1}]^u$ as $z \in \operatorname{Int}\,[p,q_1]^s$ tends to q_1. Similarly, we obtain that $\xi_{z,2j+1}$ converges to $[q_{2j+1}, q_{2j+2}]^u$ as $z \in \operatorname{Int}\,[p,q_1]^s$ tends to q_1. □

For $p \in \Sigma^2(f)$ with $D^u_\pm p = \{r_\pm\}$, $D^s_+ p = \{q_1 \prec^p_+ \cdots \prec^p_+ q_{2k+1}\}$, and $D^s_- p = \{q'_1 \prec^p_- \cdots \prec^p_- q'_{2l+1}\}$, we define two elements d_+p and d_-p of $\mathscr{S}(e^u)$ by

$$d_+p = \left[r_-, q_1, \dots, q_{2k+1}, r_+\right]^u$$
$$d_-p = \left[r_-, q'_1, \dots, q'_{2l+1}, r_+\right]^u.$$

Proposition 4.12. *For $p \in \Sigma^2(f)$, there exists a continuous map μ_p from $[-1,1]$ to $\mathscr{S}(e^u)$ such that $\mu_p(0) = (p)^u$, $\mu_p(1) = d_+p$, $\mu_p(-1) = d_-p$, and $\mu_p|_{]-1,1[}$ is a homeomorphism onto $W^u((p)^u; \mathscr{S}(f))$.*

Proof. Put $I = \operatorname{Im}\,[q'_1, p, q_1]^s$. Lemmas 4.8 and 4.11 imply that the map $z \in \operatorname{Int} I \mapsto (z)^u$ is continuous and converges to d_+p and d_-p as z tends to q_1 and q'_1 respectively.

By Lemma 4.10, we have only to show that $\mathscr{F}^u(z)$ and I have a unique intersection point for any $z \in W^u(p)$. We have $\mathscr{F}^u(z) \cap I \neq \varnothing$ since $f^{-n}(z)$ tends to p as $n \to \infty$ and I is an invariant curve transverse to $\mathscr{F}^u(p)$. Applying Lemma 3.2 to I and $\mathscr{F}^u(z)$, we obtain that the intersection point is unique. □

For $p \in \Sigma(f)$, we define a subset I_p of $[-1,1]$ by

$$I_p = \begin{cases} \{0\} & \text{if } p \in \Sigma^1(f) \\ ]-1,1[& \text{if } p \in \Sigma^2(f). \end{cases}$$

We call a continuous map μ_p from $\overline{I_p}$ to $\mathscr{S}(e^u)$ *a characteristic map* of p if $\mu_p(0) = (p)^u$, the restriction of μ_p to I_p is a homeomorphism onto $W^u((p)^u; \mathscr{S}(f))$, and $\mu_p(\pm 1) = d_\pm p$ when $p \in \Sigma^2(f)$. Proposition 4.12 implies any $p \in \Sigma^2(f)$ has a characteristic map. Proposition 4.9 also implies that a characteristic map μ_q of $q \in \Sigma^1(f)$ is given by $\mu_q(0) = (q)^u$.

For $\xi = (p_1, \dots, p_n)^u$ with $p_1, \dots, p_n \in \Sigma(f)$, we put $I_\xi = \prod_{i=1}^n I_{p_i} \subset [-1,1]^n$ and define a map λ_ξ from $\overline{I_\xi}$ to $\mathscr{S}(e^u)$ by $\lambda_\xi(s_1, \dots, s_n) = \mu_{p_1}(s_1) * \cdots * \mu_{p_n}(s_n)$. We define *the index* by $\#\{i \mid p_i \in \operatorname{Fix}^2_h(f)\}$ and denote it by $\operatorname{ind}\xi$.

Proposition 4.13. *The decomposition*

$$\mathscr{S}\big(e^u, \operatorname{Fix}^0_h(f)\big) = \bigcup W^u(\xi; \mathscr{S}(f)) \tag{1}$$

gives a structure of locally finite CW complex to $\mathscr{S}(e^u, \operatorname{Fix}^0_h(f))$ with $\dim W^u(\xi; \mathscr{S}(f)) =$

ind ξ, *where the union runs over all* ξ *in* $\mathscr{S}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$.

PROOF. The uniqueness of irreducible decomposition with respect to $\mathrm{Fix}_h^0(f)$ implies that the restriction of λ_ξ on I_ξ is a homeomorphism onto $W^u(\xi; \mathscr{S}(f))$. In particular, $W^u(\xi; \mathscr{S}(f))$ is homeomorphic to an open disk of dimension ind ξ.

It is easy to see that $\partial W^u(\xi; \mathscr{S}(f))$ is contained in a union of finitely many $W^u(\xi'; \mathscr{S}(f))$ with ind $\xi' <$ ind ξ. Hence, Proposition 4.9 implies that the decomposition (1) gives a structure of cellular complex to $\mathscr{S}(e^u, \mathrm{Fix}_h^0(f))$.

Fix $\xi_0 \in \mathscr{S}(e^u, \mathrm{Fix}_h^0(f))$. Since $\Sigma(f)$ is a finite set, there exists $K > 0$ such that $|(p)^u| \leq K|\mu_p(t)|$ for any $p \in \Sigma(f)$ and $t \in \overline{I_p}$. It implies that $|\xi_1| \leq K|\xi|$ for any $\xi_1 \in \mathscr{S}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$ and $\xi \in \overline{W^u(\xi_1; \mathscr{S}(f))}$. Let A be the set consisting of curves $\xi_1 \in \mathscr{S}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$ with $|\xi_1| \leq K(|\xi_0| + 1)$. Corollary 3.8 implies that A is a finite set. Hence, the set $\bigcup_{\xi_1 \in A} \overline{W^u(\xi_1; \mathscr{S}(f))}$ is a finite complex which contains a neighborhood $\{\xi \in \mathscr{S}(e^u, \mathrm{Fix}_h^0(f)) \mid |\xi| < |\xi_0| + 1\}$ of ξ_0. Therefore, the decomposition (1) gives a structure of locally finite CW complex to $\mathscr{S}(e^u, \mathrm{Fix}_h^0(f))$. □

4.4. Proof of Theorem B.

Fix a prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$. By replacing f with its iteration f^n, we may assume that $\mathrm{Per}_h^0(f) = \mathrm{Fix}_h^0(f)$. It is easy to see that $\pi_c(W^u(\gamma_0; \mathscr{S}(f))) \subset W^u(\pi_c(\gamma_0); \mathscr{C}(f))$ for any $\gamma_0 \in \widetilde{\mathscr{C}_a}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$. Hence, Proposition 4.9 implies that

$$\mathscr{C}_a\big(e^u, \mathrm{Fix}_h^0(f)\big) = \bigcup W^u(c; \mathscr{C}(f)), \tag{2}$$

where the union runs over all $c \in \mathscr{C}_a(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{C}(f))$. Since the unstable sets are mutually disjoint, it implies that $W^u(\pi_c(\gamma_0); \mathscr{C}(f)) = \pi_c(W^u(\gamma_0; \mathscr{S}(f)))$ for any $\gamma_0 \in \widetilde{\mathscr{C}_a}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$.

LEMMA 4.14. *For* $\gamma_0 \in \widetilde{\mathscr{C}_a}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$, *the restriction* π_c *to* $W^u(\gamma_0; \mathscr{S}(f))$ *is a homeomorphism onto* $W^u(\pi_c(\gamma_0); \mathscr{C}(f))$.

PROOF. By Lemma 4.10, it is sufficient to show that the restriction of π_c to $W^u(\gamma_0; \mathscr{S}(f))$ is injective. Suppose that $\pi_c(\gamma) = \pi_c(\gamma')$ for some $\gamma, \gamma' \in W^u(\gamma_0; \mathscr{S}(f))$. Then, there exist $\xi_1, \xi_2 \in \mathscr{S}(e^u, \mathrm{Fix}_h^0(f)) \cap \mathrm{Fix}(\mathscr{S}(f))$, $s_1 \in I_{\xi_1}$, and $s_2 \in I_{\xi_2}$ such that $|\xi_i| \neq 0$ for $i = 1, 2$, $\gamma = \lambda_{\xi_1}(s_1) * \lambda_{\xi_2}(s_2)$, and $\gamma' = \lambda_{\xi_2}(s_2) * \lambda_{\xi_1}(s_1)$. Since $\gamma, \gamma' \in W^u(\gamma_0; \mathscr{S}(f))$, we have $\xi_1 * \xi_2 = \xi_2 * \xi_1 = \gamma_0$. Put $\xi_1 = (p_1, \ldots, p_m)^u$ and $\xi_2 = (p_{m+1}, \ldots, p_n)^u$ with $p_1, \ldots, p_n \in \mathrm{Fix}(f)$. Let k be the greatest common divisor of m and n. Then, we can see that $p_{ik+j} = p_j$ for any $i = 0, \ldots, (n/k) - 1$ and $j = 1, \ldots, k$. In particular, we have $\gamma_0 = (p_1, \ldots, p_k)^u * \cdots * (p_1, \ldots, p_k)^u$. It contradicts that γ and γ_0 represent the prime homology class a. □

By Proposition 4.13, the decomposition (1) gives a structure of locally finite CW complex to $\mathscr{S}(e^u, \mathrm{Fix}_h^0(f))$. It is easy to see that $\widetilde{\mathscr{C}_a}(e^u, \mathrm{Fix}_h^0(f))$ is its subcomplex. Since the restriction of π_c on $\widetilde{\mathscr{C}}(e^u, \mathrm{Fix}_h^0(f))$ is finite to one, Lemma 4.10 implies that π_c induces the corresponding structure of CW complex to $\pi_c(\widetilde{\mathscr{C}_a}(e^u, \mathrm{Fix}_h^0(f))) = \mathscr{C}_a(e^u, \mathrm{Fix}_h^0(f))$, which is given by the decomposition (2). It is trivial that $\dim W^u(\pi_c(\gamma); \mathscr{C}(f)) =$

ind $\gamma_0 = $ ind $\pi_c(\gamma_0)$.

It is clear that ind c is zero for any $c \in \mathscr{C}_{a,*}(e^u)$. Since $\Omega^0_*(f)$ consists of finitely many embedded circles, Proposition 4.4 implies Theorem B.

5. Reduction to a simple $\boldsymbol{P}$A homotopy.

We begin the study of $\boldsymbol{P}$A homotopy. The main aim of this subsection is the reduction of the proof of Theorem A to the case of a simple $\boldsymbol{P}$A homotopy, which exhibits an essential bifurcation only once and it is a saddle-node bifurcation of a non-hyperbolic periodic orbit. In Subsection 5.1, we observe the local persistence of combinatorics of connecting segments. In Subsection 5.2, we show that any $\boldsymbol{P}$A homotopy can be approximated by a regular $\boldsymbol{P}$A homotopy and see some basic properties of saddle-node bifurcations. In Subsection 5.3, we give the compactness of families of invariant one-dimensional submanifolds. It is a keystone to show the discreteness of essential bifurcations and the continuity of connecting segments under a $\boldsymbol{P}$A homotopy. As an application, we show the existence of global continuations of connecting segments for a regular $\boldsymbol{P}$A homotopy of special type. In Subsection 5.4, we give a definition of a simple $\boldsymbol{P}$A homotopy and show that any regular $\boldsymbol{P}$A homotopy is equivalent to a concatenation of simple $\boldsymbol{P}$A homotopy locally.

When we fix a $\boldsymbol{P}$A homotopy $\{f_\lambda\}$, let the symbols $D^{u(s)}_{\pm,\lambda}p$, $\prec^p_{\pm,\lambda}$, $[p_1, \dots, p_k]^{u(s)}_\lambda$, and $(q_1, \dots, q_k)^{u(s)}_\lambda$ denote $D^{u(s)}_{\pm}p$, $\prec^p_{\pm}$, $[p_1, \dots, p_k]^{u(s)}$, and $(q_1, \dots, q_k)^{u(s)}$ for f_λ.

5.1. Local persistence of combinatorics.

Take a C^1 framed $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda \in I_0}$. For a subinterval I of I_0, $\lambda_0 \in I$, and $p \in \mathrm{Per}(f_{\lambda_0})$, we call a continuous map $\hat{p} : I \to \boldsymbol{T}^2$ *a continuation* of p on I if $\hat{p}(\lambda_0) = p$ and $\hat{p}(\lambda) \in \mathrm{Per}(f_\lambda)$ for any $\lambda \in I$.

The following lemma summarizes the facts on the local persistence of the combinatorics of connecting segments.

Lemma 5.1. *Let p, q_1, and q_2 be hyperbolic periodic points of f_{λ_0} for $\lambda_0 \in I_0$. Then, there exists a neighborhood I of λ_0 such that*

1. *p, q_1, and q_2 admit unique continuations $\hat{p}$, $\hat{q}_1$, and $\hat{q}_2$ on I,*
2. *if $q_i \in D^\sigma_{\tau,\lambda_0}p$ for $i = 1, 2$, $\sigma = u, s$, and $\tau = \pm$, then $\hat{q}_i(\lambda) \in D^\sigma_{\tau,\lambda}\hat{p}(\lambda)$ for any $\lambda \in I$, and*
3. *if $q_1 \prec^p_{\tau,\lambda_0} q_2$, then $\hat{q}_1(\lambda) \prec^{\hat{p}(\lambda)}_{\tau,\lambda} \hat{q}_2(\lambda)$ for any $\lambda \in I$.*

Proof. Without loss of generality, we can assume that $\sigma = u$, $\tau = +$, and p is attracting. The first is a consequence of the implicit function theorem. The second is a consequence of the continuity of $W^{uu}_\delta(\hat{q}_i(\lambda))$, the lower semi-continuity of $W^s(\hat{p}(\lambda))$ with respect to λ, and Lemma 3.6. The last follows from Proposition 3.9 since the condition given there is persistent under perturbation. $\square$

5.2. Saddle-node bifurcations and regular $\boldsymbol{P}$A homotopy.

For a $\boldsymbol{P}$A homotopy $\{f_\lambda\}_{\lambda \in [-1,1]}$, $\lambda_* \in]-1, 1[$, and $p_* \in \mathrm{Fix}(f^n_{\lambda_*})$ with some $n \geq 1$, we say that p_* *exhibits a generating saddle-node bifurcation* when there exist a neighborhood $[\lambda_-, \lambda_+]$ of λ_*, a neighborhood U of p_*, and continuations $\hat{p}_s$ and $\hat{p}_n$ of p_* on $[\lambda_*, \lambda_+]$ such that

1. $\mathrm{Fix}(f^n_{\lambda_*}) \cap U = \{p_*\}$,
2. $\mathrm{Fix}(f^n_\lambda) \cap U = \varnothing$ for any $\lambda \in [\lambda_-, \lambda_*[$,
3. $\mathrm{Fix}(f^n_\lambda) \cap U = \{\hat{p}_s(\lambda), \hat{p}_n(\lambda)\}$, $\hat{p}_s(\lambda) \in \mathrm{Fix}^1_h(f^n_\lambda)$, and $\hat{p}_n(\lambda) \in \mathrm{Fix}^0_h(f^n_\lambda) \cup \mathrm{Fix}^2_h(f^n_\lambda)$ for any $\lambda \in]\lambda_*, \lambda_+]$.

We call the pair $(\hat{p}_s, \hat{p}_n)$ *a saddle-node continuation* of p_* and the set $U \times [\lambda_-, \lambda_+]$ *an isolating region* of p_*. Remark that p_* has a unique saddle-node continuation on $[\lambda_*, \lambda_+]$. We say a periodic point exhibits an *annihilating* saddle-node bifurcation when it exhibits a generating saddle-node bifurcation for the $\boldsymbol{P}$A homotopy $\{f_{-\lambda}\}_{\lambda \in [-1,1]}$.

For a $\boldsymbol{P}$A diffeomorphism f, let $\mathrm{Fix}_*(f)$ and $\mathrm{Per}_*(f)$ denote the set of non-hyperbolic fixed points and the set of non-hyperbolic periodic points, respectively. A $\boldsymbol{P}$A homotopy $\{f_\lambda\}$ is called *regular* if f_λ is of class C^2, $\mathrm{Per}_*(f_\lambda)$ consists of at most one periodic orbit, and it exhibits a saddle-node bifurcation for any λ. For a regular $\boldsymbol{P}$A homotopy $\{f_\lambda\}_{\lambda \in I}$, let $\mathscr{N}\mathscr{D}(\{f_\lambda\})$ be the set of λ such that f_λ is non-degenerate. The set $\{\lambda \mid \mathrm{Fix}_*(f^n_\lambda) \neq \varnothing\}$ is discrete and $\mathrm{Fix}(f^n_\lambda)$ is a finite set for any λ once we fix $n \geq 1$. In particular, $\mathscr{N}\mathscr{D}(\{f_\lambda\})$ is a dense subset of I.

Lemma 5.2. *Any $\boldsymbol{P}$A homotopy between two C^2 non-degenerate orientable $\boldsymbol{P}$A diffeomorphisms of $\boldsymbol{T}^2$ is approximated by a regular $\boldsymbol{P}$A homotopy between them in the space of continuous family of C^1-diffeomorphisms.*

Proof. Since the set of $\boldsymbol{P}$A diffeomorphisms is an open subset of $\mathrm{Diff}^1(\boldsymbol{T}^2)$, we may take a $\boldsymbol{P}$A homotopy $\{f_\lambda\}_{\lambda \in [0,1]}$ between given two diffeomorphisms so that the map $(z, \lambda) \mapsto f_\lambda(z)$ is of class C^2. By the standard bifurcation theory, we can approximate $\{f_\lambda\}$ by $\{g_\lambda\}$ so that $f_0 = g_0$, $f_1 = g_1$, the set $\mathrm{Per}_*(g_\lambda)$ consists of at most one periodic orbit, and it exhibits a generic bifurcation, namely, one of a period doubling bifurcation, a saddle-node bifurcation or a Hopf bifurcation for any $\lambda \in]-1, 1[$ (see *e.g.* Section 3.2 of [**5**]). Since g_λ is an orientable $\boldsymbol{P}$A diffeomorphism, no periodic point exhibits a period doubling or a Hopf bifurcation. □

In the rest of the subsection, we show some basic properties of saddle-node bifurcations. Fix a regular framed $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda \in [-1,1]}$ and suppose that a fixed point p_* of f_0 exhibits a generating saddle-node bifurcation. Let $(\hat{p}_s, \hat{p}_n)$ be a saddle-node continuation and $U_* \times [\lambda_*, \lambda_*]$ an isolating region of p_*.

For $\sigma \in \{u, s\}$ and $\tau = \pm$, we say p_* is of *type* (σ, τ) when $\|Df(e^\sigma_0(p_*))\| = 1$ and there exists a compact embedded interval I with $\partial_\tau I = p_*$, $e^\sigma_0(p_*) \in T_{p_*} I$, $\mathrm{Fix}(f_0) \cap I = \{p_*\}$, and $f_0(I) \subset I$. Remark that $\hat{p}_n(\lambda)$ is attracting if $\sigma = u$ and is repelling if $\sigma = s$.

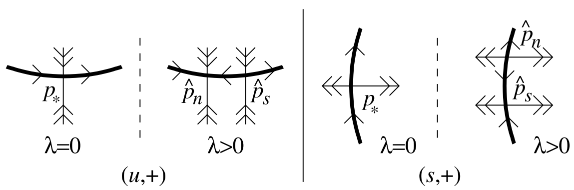

Figure 4. Types of non-hyperbolic periodic points.

Lemma 5.3. *A non-hyperbolic periodic point p_* admits a unique type. p_* is of type $(\sigma, +)$ (resp. $(\sigma, -)$) if and only if $\|Df_0(e_0^\sigma(p_*))\| = 1$ and there exists a compact embedded interval I satisfying $\partial_- I = p_*$ (resp. $\partial_+ I = p_*$), $e_0^\sigma(p_*) \in T_{p_*} I$, $\mathrm{Fix}(f_0) \cap I = \{p_*\}$, and $f_0^{-1}(I) \subset I$.*

Proof. Let $I(p, f)$ denote the Lefschetz fixed point index of an isolated fixed point p of a map f. It is easy to see that $I(p_*, f_0) = 0$.

We assume $\|Df_0(e_0^u(p_*))\| = 1$ since the proof for the other case is similar. We call a compact embedded interval I *a local center-unstable manifold* of p_* if $p_* \in \mathrm{Int}\, I$, $e_0^u(p_*) \in T_{p_*} I$, $I \cap \mathrm{Fix}(f_0) = \{p_*\}$, and each connected component J of $I \setminus \{p_*\}$ satisfies $f_0(J) \subset J$ or $f_0^{-1}(J) \subset J$. We claim that for such an interval I, one of the connected component J of $I \setminus \{p_*\}$ satisfies $f_0(J) \subset J$ and the other J' satisfies $f_0^{-1}(J') \subset J'$. In fact, fix a 1/8-canonical coordinate φ so that $\varphi(0, 0) = p_*$, $D\varphi^{-1}(e^s(p_*))$ is parallel to the y-axis, and $\varphi([-1, 1] \times \{0\}) \subset I$. Since $\|Df_0(e_0^s(p_*))\| < 1$, there exists $\epsilon > 0$ such that $f \circ \varphi([-\epsilon, \epsilon] \times [-y, y]) \subset \varphi(]-1, 1[\times]-y, y[)$ for any $y \in]0, \epsilon[$. Therefore, we have $I(p_*, f_0|_I) = I(p_*, f_0) = 0$. It is easy to see that it implies the claim.

Since p_* is an isolated fixed point of f_0, the center manifold theorem (see *e.g.* [**8**]) implies that p_* admits a local center-unstable manifold. Hence, p_* admits a unique type by the claim. The latter assertion of the lemma is also a consequence of the existence of a local center-unstable manifold and the above claim. $\square$

Lemma 5.4. *Suppose that $p, q \in \mathrm{Per}_h(f_0)$ and $\xi \in \mathscr{S}(e_0^u) \cap \mathrm{Per}(\mathscr{S}(f_0))$ satisfy $\xi(0) = p$, $\xi(1) = q$, and $\mathrm{Int}\, \xi \cap \mathrm{Per}_h(f_0) = \varnothing$. Then, either one of the followings holds*:

1. $\xi = [p, q]_0^u$,
2. $\xi = [p, p_*, q]_0^u$ *for some $p_* \in \mathrm{Per}_*(f_0)$ of type $(u, \pm)$, or*
3. $\xi = [p, p_*, q]_0^u$ *for some $p_* \in \mathrm{Per}_*(f_0)$ of type $(s, \pm)$.*

In the former two cases, one of p and q is attracting and the other is not. In the last case, both p and q are attracting.

Proof. If $\mathrm{Int}\, \xi \cap \mathrm{Per}_*(f_0) = \varnothing$, then the lemma is trivial.

Suppose that $\mathrm{Im}\, \xi$ contains a non-hyperbolic periodic point p_*. Since $\{f_\lambda\}$ is regular, we have $\mathrm{Per}_*(f_0) = \mathscr{O}(p_*; f_0)$. Hence, there exist $n_1, \ldots, n_k \geq 0$ such that $\xi = [p, f_0^{n_1}(p_*), \ldots, f_0^{n_k}(p_*), q]_0^u$. If p_* is of type $(s, \pm)$, then $k = 1$ and $p, q \in \mathrm{Per}_h^0(f_0)$. If p_* is of type $(u, +)$, then q is an attracting periodic point and $D_{+,0}^u f_0^{n_k}(p_*) = \{q\}$ since $[f_0^{n_k}(p_*), q]_0^u$ is ascending. It implies $D_{+,0}^u f_0^{n_1}(p_*) = \{f_0^{n_1 - n_k}(q)\} \subset \mathrm{Per}_h(f_0)$, and hence, $k = 1$. The proof for the case where p_* is of type $(u, -)$ is similar. $\square$

Lemma 5.5. *If p_* is of type $(u, +)$, then $D_{+,\lambda}^u \hat{p}_n(\lambda) = \{\hat{p}_s(\lambda)\}$ for any λ sufficiently close to zero.*

Proof. Take $\delta_0 > 0$ so that any immersed curve L tangent to e_λ^u with $|L| \leq \delta_0$ is an embedded interval. Let δ be a number less than both δ_0 and the distance between p_* and $\boldsymbol{T}^2 \setminus U_*$.

First, we claim that $|W_-^{uu}(\hat{p}_s(\lambda); f_\lambda)| < \delta$ for any sufficiently small $\lambda > 0$. Once it is shown, we have $\partial_- W^{uu}(\hat{p}_s(\lambda)) = \hat{p}_n(\lambda)$, and hence, $\hat{p}_s(\lambda) \in D_{+,\lambda}^u \hat{p}_n(\lambda)$ since $U_* \cap \mathrm{Fix}_h^0(f_\lambda) = \{\hat{p}_n(\lambda)\}$. Assume that the claim does not hold. Then, we can take a sequence

$\{\lambda_i > 0\}_{i=1}^{\infty}$ converging to 0 and a sequence $\{I_i\}_{i=1}^{\infty}$ of embedded compact intervals satisfying $|I_i| = \delta$, $\partial_+ I_i = \hat{p}_s(\lambda_i)$, and $\mathrm{Int}\, I_i \subset W_-^{uu}(\hat{p}_s(\lambda_i); f_\lambda)$ for any i. Notice that I_i is tangent to $e^u_{\lambda_i}$ and $f^{-1}(I_i) \subset I_i$. By Lemma 2.1, we may assume that I_i converges to an embedded interval I_* tangent to e^u_0. Since $|I_*| = \delta$ and $\partial_+ I_* = p_*$, we have $I_* \subset U_*$, and hence, $I_* \cap \mathrm{Fix}(f_0) = \{p_*\}$. Since $f^{-1}(I_*) \subset I_*$, Lemma 5.3 implies that p_* must be of type $(u, -)$. It contradicts that p_* is of type $(u, +)$.

Next, we show that $\hat{p}_s(\lambda)$ is the unique element of $D^u_{+,\lambda}\hat{p}_n(\lambda)$ for any sufficiently small $\lambda > 0$. Suppose that it does not hold. Take a sequence $\{\lambda'_i > 0\}_{i=1}^{\infty}$ which converges to 0 and a sequence $\{q_i \in D^u_{+,\lambda'_i}\hat{p}_n(\lambda'_i)\}_{i=1}^{\infty}$ with $q_i \neq \hat{p}_s(\lambda'_i)$. Then, there exist $\delta' \in]0, \delta[$ and a sequence $\{I'_i\}$ of embedded intervals with $|I'_i| = \delta'$, $\partial_- I'_i = \hat{p}_n(\lambda'_i)$, and $I'_i \subset W_-^{uu}(q_i; f_{\lambda'_i})$. Notice that I'_i is tangent to $e^u_{\lambda'_i}$ and $f(I'_i) \subset I'_i$. By Lemma 2.1, we may assume that I'_i converges to an embedded interval I'_* tangent to e^u_0. Since $|I'_*| = \delta'$ and $\partial_- I'_* = p_*$, we have $I'_* \subset U_*$, and hence, $I'_* \cap \mathrm{Fix}(f_0) = \{p_*\}$. Since $f(I'_*) \subset I'_*$, p_* is of type $(u, -)$. It contradicts that p_* is of type $(u, +)$. □

Finally, we consider bifurcations of connecting segments between p_* and hyperbolic fixed points. Suppose that $D^u_{-,0}p_*$ and $D^u_{+,0}p_*$ contain hyperbolic fixed points p and q respectively. We assume $p, q \in \mathrm{Fix}(f_\lambda)$ for any $\lambda \in [-1, 1]$, that is, p and q have constant continuations on $[-1, 1]$. This assumption holds for *a simple* $\boldsymbol{P}$A *homotopy*, which we define and investigate later.

Lemma 5.6. *If p_* is of type $(s, \pm)$, then $D^u_{-,\lambda}\hat{p}_s(\lambda) = D^u_{-,\lambda}\hat{p}_n(\lambda) = \{p\}$ and $D^u_{+,\lambda}\hat{p}_s(\lambda) = D^u_{+,\lambda}\hat{p}_n(\lambda) = \{q\}$ for any $\lambda > 0$ sufficiently close to zero.*

Remark that both p and q are attracting since $[p, q_*]^u_0$ is descending and $[p_*, q]^u_0$ is ascending.

Proof. It is a consequence of the continuity of $W^{uu}(\hat{p}_s(\lambda))$ and $W^{uu}(\hat{p}_n(\lambda))$, the persistence of attracting fixed points, and Lemma 3.6. □

Lemma 5.7. *If p_* is of type $(u, +)$, then the followings hold*:

1. $D^u_{+,\lambda}p = \{q\}$ *for any $\lambda < 0$ sufficiently close to zero.*
2. $D^u_{+,\lambda}p = \{\hat{p}_n(\lambda)\}$ *and* $D^u_{+,\lambda}\hat{p}_s(\lambda) = \{q\}$ *for any $\lambda > 0$ sufficiently close to zero.*

Remark that p is of saddle-type or repelling and q is attracting since both $[p, p_*]^u_0$ and $[p_*, q]^u_0$ are ascending.

Proof. Take $a_0 < 0 < a_1$ and a coordinate $\psi : [a_0 - 1, a_1 + 1] \times [-1, 1] \to \boldsymbol{T}^2$ so that

1. $\psi(a_0, 0) = p$, $\psi(0, 0) = p_*$, $\psi(a_1, 0) = q$, $\mathrm{Im}\, \psi \cap \mathrm{Fix}(f_0) = \{p, q, p_*\}$,
2. $\mathrm{Im}\, [p, p_*, q]^u_0 = \psi([a_0, a_1] \times \{0\})$, and
3. the map $\varphi_a : (x, y) \mapsto \psi(x + a, y)$ is a 1/8-canonical coordinate associated to (e^u_0, e^s_0) for any $a \in [a_0, a_1]$,

Since $[p, p_*]^u_0$ is ascending, q is attracting, and $\|Df^n(e^s_0(z))\|$ tends to zero as $n \to \infty$ for any $z \in \mathrm{Im}\, [p_*, q]^u_0$, there exists a neighborhood B of $\varphi([0, a_1] \times \{0\})$ such that $f_0(\overline{B}) \subset \mathrm{Int}\, B$ and $\overline{B} \subset \mathrm{Im}\, \psi \setminus \{p\}$.

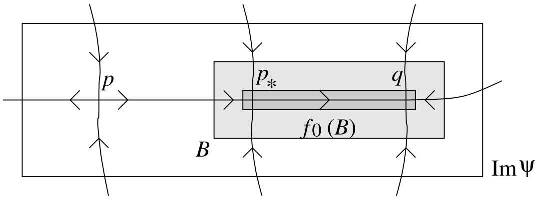

Figure 5. Proof of Lemma 5.7.

Let $W^{uu}_{\delta,+}(p; f_\lambda)$ be the positive half component of $W^{uu}_\delta(p; f_\lambda)$ for $\delta > 0$. For $m \geq 1$, let $I(m, \lambda)$ be the closure of $f^{m+1}_\lambda(W^{uu}_{\delta,+}(p; f_\lambda)) \setminus f^m_\lambda(W^{uu}_{\delta,+}(p; f_\lambda))$. Since $\partial_+ W^{uu}(p; f_0) = p_*$, there exists $m_0 \geq 0$ such that $I(m_0, 0) \subset \mathrm{Int}\ B$. Take $\lambda_1 > 0$ so that the followings hold for any $\lambda \in [-\lambda_1, \lambda_1]$:

1. $\mathrm{Fix}(f_\lambda) \cap \mathrm{Im}\ \psi = \{p, q\}$ if $\lambda < 0$, and $\mathrm{Fix}(f_\lambda) \cap \mathrm{Im}\ \psi = \{p, q, \hat{p}_s(\lambda), \hat{p}_n(\lambda)\}$ if $\lambda > 0$.
2. φ_a is a 1/8-canonical coordinate associated to $(e^u_\lambda, e^s_\lambda)$ for any $a \in [a_0, a_1]$.
3. $f_\lambda(\overline{B}) \subset \mathrm{Int}\ B$.
4. $I(m_0, \lambda) \subset \mathrm{Int}\ B$, and $\bigcup_{m \leq m_0} I(m, \lambda) \subset \mathrm{Im}\ \psi$.

Remark that if a curve $J \subset B$ is tangent to e^u_λ for $\lambda \in [-\lambda_1, \lambda_1]$, then $\overline{J}$ is an embedded interval with finite length. In particular, it satisfies $\partial_\pm J \in \overline{B}$.

Suppose $\lambda \in]-\lambda_1, 0[$. Since $\bigcup_{n \geq m_0} I(n, \lambda)$ is a curve which is contained in B and is tangent to e^u_λ, we have $\partial_+ W^{uu}(p; f_\lambda) \in \overline{B}$. It implies that $D^u_{+,\lambda} p = \{q\}$ since $\mathrm{Fix}(f_\lambda) \cap \overline{B} = \{q\}$.

Next, suppose $\lambda \in]0, \lambda_1[$. Since $W^{uu}(\hat{p}_s(\lambda))$ is contained in $\mathrm{Int}\ B$ and is tangent to e^u_λ, $W^{uu}(\hat{p}_s(\lambda))$ is an embedded interval with $\partial W^{uu}(\hat{p}_s(\lambda)) = \{\hat{p}_n(\lambda), q\}$. By Lemma 5.5, we have $\partial_- W^{uu}(\hat{p}_s(\lambda)) = \hat{p}_n(\lambda)$, and hence, $\partial_+ W^{uu}(\hat{p}_s(\lambda)) = q$. It implies that $D^u_{+,\lambda}\hat{p}_s(\lambda) = \{q\}$. Since $\bigcup_{n \geq m_0} I(n, \lambda) \subset \mathrm{Int}\ B$, we also have $\partial_+ W^{uu}(p; f_\lambda) \in \{\hat{p}_n(\lambda), q\}$. If $\partial_+ W^{uu}(p; f_\lambda) = q$, then $W^{uu}_+(p; f_\lambda)$ must intersect with $W^{ss}(\hat{p}_n(\lambda))$, which separates p and q in $\mathrm{Im}\ \psi$. It contradicts Lemma 3.2 for $\mathrm{Im}\ [p, q]^u_\lambda$ and $W^{ss}(\hat{p}_n(\lambda))$. Therefore, we obtain $\partial_+ W^{uu}(p; f_\lambda) = \hat{p}_n(\lambda)$, and hence, $D^u_{+,\lambda} p = \{\hat{p}_n(\lambda)\}$. □

5.3. The compactness of families of invariant embedded curves.

The following Lemma is a keystone to show the finiteness and the compactness of the family of compact invariant one-dimensional submanifolds.

Lemma 5.8. *Let (f, e^u, e^s) be a C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphism of $\boldsymbol{T}^2$, and L be an element of $\mathscr{E}(\sqcup_{<\infty}[0,1], e^u)$ or $\mathscr{E}(\sqcup_{<\infty}S^1, e^u)$ with $f(L) = L$. If φ is a 1/80-canonical coordinate associated to (e^u, e^s) such that $L \cap \varphi([-1/3, 1/3]^2) \neq \varnothing$ and $\partial L \cap \mathrm{Im}\ \varphi = \varnothing$, then $L \cap \varphi([-1/2, 1/2]^2)$ is connected.*

Proof. Replacing f with its iteration f^n if it is necessary, we assume that any attracting or repelling periodic point of f is a fixed point. Put $\epsilon = 1/80$. We say a triple $(\sigma_*, L_*, \varphi_*)$ of $\sigma_* \in \{u, s\}$, $L_* \in \mathscr{E}(\sqcup_{<\infty}S^1, e^{\sigma_*}) \cup \mathscr{E}(\sqcup_{<\infty}[0,1], e^{\sigma_*})$, and an ϵ-canonical coordinate φ associated to (e^u, e^s), is *bad* when $f(L_*) = L_*$, $L_* \cap \varphi_*([-1/3, 1/3]^2) \neq \varnothing$, $\partial L_* \cap \mathrm{Im}\ \varphi_* = \varnothing$, and $L_* \cap \varphi_*([-1/2, 1/2]^2)$ is not connected. We show that if $(\sigma_0, L_0, \varphi_0)$ is a bad triple then there exists another bad triple $(\sigma_1, L_1, \varphi_1)$ such that

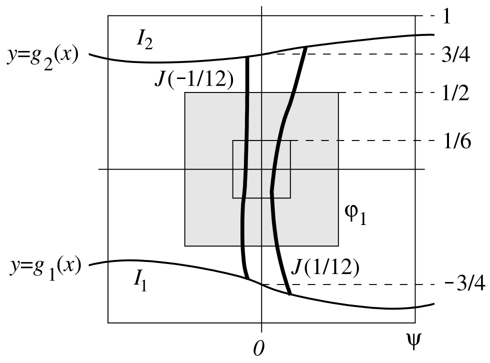

Figure 6. Proof of Lemma 5.8.

Im $\varphi_1 \subset$ Im φ_0 and (Im $\varphi_0 \setminus$ Im $\varphi_1) \cap \mathrm{Fix}(f) \neq \emptyset$. Notice that it completes the proof of the lemma. In fact, if the lemma does not hold, then the above claim implies that there exists a sequence $\{(\sigma_n, L_n, \varphi_n)\}_{n\geq 0}$ of bad triples such that Im $\varphi_{n+1} \subset$ Im φ_n and (Im $\varphi_n \setminus$ Im $\varphi_{n+1}) \cap \mathrm{Fix}(f) \neq \emptyset$. However, it contradicts the finiteness of $\mathrm{Fix}(f)$.

Fix a bad triple $(\sigma_0, L_0, \varphi_0)$. Without loss of generality, we can assume $\sigma_0 = u$. First, we claim that there exist two distinct connected components I_1 and I_2 of $L_0 \cap$ Im φ_0 which intersects with $\varphi_0([-1/2, 1/2]^2)$. Suppose it does not hold. Then, there exists a connected component I_0 of $L_0 \cap$ Im φ_0 such that $I_0 \cap \varphi_0([-1/3, 1/3]^2) \neq \emptyset$ and $I_0 \cap \varphi_0([-1/2, 1/2]^2) = L_0 \cap \varphi_0([-1/2, 1/2]^2)$. We can take a C^1-function h_0 on $[-1, 1]$ so that $I_0 = \varphi_0(\{(x, h_0(x)) \mid x \in [-1, 1]\})$. Since $|h_0'(x)| \leq \epsilon$, we have $|h_0(x)| \leq 1/3 + 2\epsilon < 1/2$. In particular, $I_0 \cap \varphi_0([-1/2, 1/2]^2)$ is connected. However, it contradicts the badness of $(\sigma_0, L_0, \varphi_0)$.

Let I_1 and I_2 be two distinct connected components of $L_0 \cap$ Im φ_0 which intersects with $\varphi_0([-1/2, 1/2]^2)$. Take C^1-functions h_1 and h_2 on $[-1, 1]$ so that $I_i = \varphi_0(\{(x, h_i(x)) \mid x \in [-1, 1]\})$ for $i = 1, 2$. Without loss of generality, we may assume that $h_1(x) < h_2(x)$ for any $x \in [-1, 1]$. Since $|h_i'(x)| \leq \epsilon$, we have $|h_i(x)| \leq 1/2 + 2\epsilon$. Put $a = (2/3)(h_2(0) - h_1(0))$ and $b = (1/2)(h_2(0) + h_1(0))$. Since

$$|a| + |b| \leq \max_{i,j=1,2} \left| \frac{7}{6} h_i(0) - \frac{1}{6} h_j(0) \right| < \frac{4}{3}\left(\frac{1}{2} + 2\epsilon\right) < 1,$$

we can define an ϵ-canonical coordinate ψ, and functions g_1 and g_2 on $[-1, 1]$ by $\psi(x, y) = \varphi_0(ax, ay + b)$ and $g_i(x) = a^{-1}(h_i(ax) - b)$. It is easy to see that $g_1(0) = -3/4$, $g_2(0) = 3/4$, $|g_i(x) - g_i(0)| \leq \epsilon|x|$ for $x \in [-1, 1]$, and $I_i \cap$ Im $\psi = \psi(\{(x, g_i(x)) \mid x \in [-1, 1]\})$ for each $i = 1, 2$. In particular, we have $(I_1 \cup I_2) \cap \psi([-1/2, 1/2]^2) = \emptyset$.

Put $B(x_0, \delta) = \psi(\{(x, y) \mid |x - x_0| \leq \delta, g_1(x) \leq y \leq g_2(x)\})$ for any $x_0 \in [-1/2, 1/2]$ and $\delta \in]0, 1[$. We claim that for any $x_0 \in [-1/4, 1/4]$, there exists $J(x_0) \in \mathscr{E}([0, 1], e^s)$ such that $J(x_0) \subset B(x_0, 4\epsilon)$, $f(J(x_0)) = J(x_0)$, $\partial_- J(x_0) \in I_1$ and $\partial_+ J(x_0) \in I_2$. Once it is shown, we can take a bad triple $(\sigma_1, \varphi_1, L_1)$ by $\sigma_1 = s$, $\varphi_1(x, y) = \psi(x/2, y/2)$, and $L_1 = J(1/12) \cup J(-1/12)$. Since (Im $\varphi_0 \setminus$ Im $\varphi_1) \cap \mathrm{Fix}(f)$ contains ∂L_1, the triple is the required one.

Fix $x_0 \in [-1/4, 1/4]$. The proof of the claim is divided into three steps. First, we show Int $B(x_0, 2\epsilon) \cap \mathrm{Fix}_h^2(f) \neq \varnothing$. Suppose it does not hold. Then, $\Omega^2(f) \cap B(x_0, 2\epsilon)$ is empty or a union of finitely many circles tangent to e^s. Hence, we can take $x_1 \in]x_0 - \epsilon, x_0 + \epsilon[$ so that $\psi(x_1, 0) \notin \Omega^2(f)$. Since $\partial_\pm \mathscr{F}^s(\psi(x_1, 0))$ is not contained in Int $B(x_0, 2\epsilon)$ by Lemma 4.3, there exists $J_* \in \mathscr{E}([0,1], e^s)$ such that $J_* \subset \mathscr{F}^s(\psi(x_1, 0))$, $\partial_- J_* \in I_1$, and $\partial_+ J_* \in I_2$. Proposition 4.1 implies that $|f^n(J_*)|$ tends to zero as $n \to \infty$. However, it contradicts $L \in \mathscr{E}(\sqcup_{<\infty} S^1, e^u) \cup \mathscr{E}(\sqcup_{<\infty}[0,1], e^u)$ and $f(L) = L$.

Second, we show that for any given $p \in \mathrm{Fix}(f) \cap B(x_0, 4\epsilon) \setminus I_2$, there exists $J_+(p) \in \mathscr{E}([0,1], e^s)$ such that $J_+(p) \subset B(x_0, 6\epsilon)$, $f(J_+(p)) = J_+(p)$, and $\partial_- J_+(p) = p$. If $p \in \mathrm{Fix}_h^2(f)$, then $D_+^s p$ is non-empty by Lemma 4.11. We put $L = W_-^{ss}(q)$ with some $q \in D_+^s p$ in this case. If $p \in \mathrm{Fix}_h^0(f) \cup \mathrm{Fix}_h^1(f)$, then we put $L = W_+^{ss}(p)$. The set L is an element of $\mathscr{E}([0,1], e^s)$ with $f(L) = L$ and $\partial_- L = p$. If L is not contained in $B(x_0, 6\epsilon)$, then it must intersects with I_2. However, it contradicts Lemma 3.3 for L_0 and L since $L \subset W^s(q) \setminus \{q\}$ or $L \subset W^{ss}(p) \setminus \{p\}$. Therefore, L is a subset of $B(x_0, 6\epsilon)$. In particular, L has finite length, and hence, $J_+(p) = \overline{L}$ is the required one. By the similar way, we can show that for any given $p \in \mathrm{Fix}(f) \cap B(x_0, 4\epsilon) \setminus I_1$, there exists $J_-(p) \in \mathscr{E}([0,1], e^s)$ such that $J_-(p) \subset B(x_0, 6\epsilon)$, $f(J_-(p)) = J_-(p)$, and $\partial_+ J_-(p) = p$.

Finally, we prove the claim. By the first step, we can take $p \in$ Int $B(x_0, 2\epsilon) \cap \mathrm{Fix}_h^2(f)$. Let $J_+(p)$ be the interval obtained in the second step. Since $p \in B(x_0, 2\epsilon)$ and $J_+(p)$ is tangent to e^s, we have $J_+(p) \in B(x_0, 4\epsilon)$. If $p' = \partial_+ J_+(p)$ is not contained I_2, then let $J_+(p')$ be the interval obtained in the second step for p'. We replace $J_+(p)$ by $J_+(p) \cup J_+(p')$ and repeat the same procedure until $\partial_+ J_+(p) \in I_2$. It must stop in finite times since the number of fixed point of f is finite. Hence, we obtain $J_+^* \in B(x_0, 4\epsilon)$ such that $f(J_+^*) = J_+^*$, $\partial_- J_+^* = p$, and $\partial_+ J_+^* \in I_2$. Similarly, we can also obtain $J_-^* \in B(x_0, 4\epsilon)$ such that $f(J_-^*) = J_-^*$, $\partial_+ J_-^* = p$, and $\partial_- J_-^* \in I_1$. Now, the interval $J_+^* \cup J_-^*$ satisfies the condition required in the claim. As we see before, the claim completes the proof. □

Proposition 5.9. *Let (f, e^u, e^s) be a C^2 framed $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$ such that f has at most one non-hyperbolic periodic orbit. Then, there exists $N \geq 1$ such that $\mathscr{C}(e^u) \cap \mathrm{Per}(\mathscr{C}(f))$ is the disjoint union of $\mathscr{C}(e^u, \mathrm{Fix}_h(f^N)) \cap \mathrm{Per}(\mathscr{C}(f))$ and $\mathscr{C}_*(e^u)$.*

Remark that it is trivial that $\mathscr{C}(e^u) \cap \mathrm{Per}(\mathscr{C}(f))$ is the disjoint union of $\mathscr{C}(e^u, \mathrm{Per}_h(f)) \cap \mathrm{Per}(\mathscr{C}(f))$ and $\mathscr{C}_*(e^u)$. However, $\mathrm{Per}_h(f)$ may contain periodic points of arbitrary large period in general.

Proof. If the proposition does not hold, then there exist a sequence $\{n_i\}_{i \geq 1}$ of integers and a sequence $\{c_i\}_{i \geq 1}$ in $\mathscr{C}(e^u) \cap \mathrm{Per}(\mathscr{C}(f))$ such that n_i tends to infinity as $i \to \infty$ and Im c_i contains a periodic point of f of period n_i for each $i \geq 1$. By Lemma 3.16, we can also assume that an f-invariant set $C_i = \bigcup_{n=0}^{n_i - 1} f^n(\mathrm{Im}\, c_i)$ is an element of $\mathscr{E}(\sqcup_{<\infty} S^1, e^u)$. Lemma 3.11 implies $C_i \cap C_j = \varnothing$ if $n_i \neq n_j$.

Since $\mathrm{Per}_*(f)$ is a finite set, we can assume that $C_i \cap \mathrm{Per}_*(f) = \varnothing$ for all i. Take a 1/80-canonical coordinate φ and $i_1, i_2 \geq 1$ so that $n_{i_1} \neq n_{i_2}$ and $C_{i_k} \cap \varphi(]1/3, 1/3[^2) \neq \varnothing$ for $k = 1, 2$. By the persistence of connecting segments of hyperbolic periodic points, both C_{i_1} and C_{i_2} are persistent under perturbation. Hence, if a C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphism (f_1, e_1^u, e_1^s) is sufficiently close to (f, e^u, e^s), then φ is a 1/80-canonical coordinate for (e_1^u, e_1^s) and there exist $C_1', C_2' \in \mathscr{E}(\sqcup_{<\infty} S^1, e_1^u)$ such that $C_1' \cap C_2' = \varnothing$, $f(C_k') = C_k'$ and $C_k' \cap \varphi(]1/3, 1/3[^2) \neq \varnothing$ for each $k = 1, 2$. It contradicts Lemma 5.8. □

We also obtain the compactness of the family of invariant sets consisting of embedded circles.

PROPOSITION 5.10. *Let $\{(f_i, e_i^u, e_i^s)\}_{i\geq 1}$ be a sequence of non-degenerate C^2 framed $\boldsymbol{P}$A diffeomorphisms which converges to a C^1 framed $\boldsymbol{P}$A diffeomorphism (f_*, e_*^u, e_*^s) in the C^1-topology. Then, any sequence $\{C_i \in \mathscr{E}(\sqcup_{<\infty} S^1, e_i^u)\}_{i\geq 1}$ satisfying $f_i(C_i) = C_i$ for each i admits a subsequence which converges to an element C_* of $\mathscr{E}(\sqcup_{<\infty} S^1, e_*^u)$ with $f_*(C_*) = C_*$.*

PROOF. Choose a sequence $\{i_k\}$ so that C_{i_k} converges to a compact set C_* with respect to the Hausdorff metric. Remark that C_* is f_*-invariant.

Fix $z \in C_*$. Take a 1/80-canonical coordinate φ associated to (f_*, e_*^u, e_*^s) with $\varphi(0,0) = z$. Lemma 5.8 implies that there exists a C^1 function h_k on $[-1/2, 1/2]$ such that $\varphi([-1/2,1/2]^2) \cap C_{i_k} = \varphi(\{(x, h_k(x)) \mid x \in [-1/2, 1/2]\})$ for any sufficiently large k. Since C_{i_k} converges to C_*, h_k converges to a C^1-function h_* satisfying $\varphi([-1/2,1/2]^2) \cap C_* = \varphi(\{(x, h_*(x)) \mid x \in [-1/2,1/2]\})$. It implies $C_* \in \mathscr{E}(\sqcup_{<\infty} S^1, e_*^u)$. □

Finally, we show the existence of global continuation of connecting segments. Let $\{(f_\lambda, e_\lambda^u, e_\lambda^s)\}_{\lambda\in I}$ be a framed $\boldsymbol{P}$A homotopy. For $\sigma \in \{u, s\}$, $\lambda_0 \in I$, a subinterval I_1 of I containing λ_0, and $\xi \in \mathscr{S}(e_{\lambda_0}^\sigma) \cap \mathrm{Per}(\mathscr{S}(f_{\lambda_0}))$, we call a continuous map $\hat{\xi}$ from I_1 to $C^1([0,1], \boldsymbol{T}^2)$ *a continuation* of ξ on I_1 if $\hat{\xi}(\lambda_0) = \xi$ and $\hat{\xi}(\lambda) \in \mathscr{S}(e_\lambda^\sigma) \cap \mathrm{Per}(\mathscr{S}(f_\lambda))$ for any $\lambda \in I_1$.

PROPOSITION 5.11. *Let $\{(f_\lambda, e_\lambda^u, e_\lambda^s)\}_{\lambda\in[0,1]}$ be a regular framed $\boldsymbol{P}$A homotopy such that $\mathrm{Fix}_*(f_\lambda) = \varnothing$ for any $\lambda \in]0,1[$. Suppose that hyperbolic fixed points p and q of f_0 satisfy $q \in D_{+,0}^\sigma q$ for $\sigma = u$ or s, and admit continuations $\hat{p}$ and $\hat{q}$ on $[0,1]$. Then, there exists a continuation $\hat{\xi}$ of $[p,q]_0^\sigma$ on $[0,1]$ such that $\hat{\xi}(\lambda) = [\hat{p}(\lambda), \hat{q}(\lambda)]_\lambda^\sigma$ for any $\lambda \in [0,1[$ and $\mathrm{Int}\, \hat{\xi}(1) \cap \mathrm{Per}_h^0(f_1) = \varnothing$.*

PROOF. To simplify the proof, we assume $\sigma = u$ and $q \in \mathrm{Fix}_h^0(f_0)$. Then, we have $\hat{p}(\lambda) \in \mathrm{Fix}_h^1(f_\lambda) \cup \mathrm{Fix}_h^2(f_\lambda)$ and $\hat{q}(\lambda) \in \mathrm{Fix}_h^0(f_\lambda)$ for any $\lambda \in [0,1[$. Put $J = \{\lambda \in [0,1[\mid |W_+^{uu}(\hat{p}(\lambda))| < \infty\}$. Since $\partial_+ W_+^{uu}(\hat{p}(\lambda))$ must be an attracting fixed point, Lemma 5.1 implies that J is an open subset of $[0,1[$.

First, we claim that there exists $K > 0$ such that $|W_+^{uu}(\hat{p}(\lambda))| \leq K$ for any $\lambda \in J \cap \mathscr{N}\mathscr{D}(\{f_\lambda\})$. Suppose that it does not hold. Then, it is easy to see that we can take $\lambda_0 \in J \cap \mathscr{N}\mathscr{D}(\{f_\lambda\})$ and a 1/80-canonical coordinate φ associated to $(e_{\lambda_0}^u, e_{\lambda_0}^s)$ so that $W_+^{uu}(\hat{p}(\lambda_0)) \cap \varphi([-1/3, /1/3]^2) \neq \varnothing$, $W_+^{uu}(\hat{p}(\lambda_0)) \cap \varphi([-1/2,1/2]^2)$ is not connected, and $\mathrm{Im}\, \varphi \cap \{\hat{p}(\lambda_0), \hat{q}(\lambda_0)\} = \varnothing$. However, it contradicts Lemma 5.8.

By the continuity of $W_\delta^{uu}(\hat{p}(\lambda))$ with respect to λ, the function $\lambda \mapsto |W_+^{uu}(p; f_\lambda)| \in]0,\infty]$ is lower semi-continuous. Since $J \cap \mathscr{N}\mathscr{D}(\{f_\lambda\})$ is a dense subset of J, the above claim implies that J is a closed subset of $[0,1[$ and $|W_+^{uu}(\hat{p}(\lambda))| \leq K$ for any $\lambda \in J$. Since J is an open subset of $[0,1[$, we obtain $J = [0,1[$.

Put $J_1 = \{\lambda \in [0,1[\mid D_{+,\lambda}^u \hat{p}(\lambda) = \{\hat{q}(\lambda)\}\}$. Recall that $\partial_+ W^{uu}(\hat{p}(\lambda))$ is an attracting fixed point for any $\lambda \in [0,1[$. By lemma 5.1, both J_1 and $[0,1[\setminus J_1$ are open subset of $[0,1[$. Since $J_1 \neq \varnothing$, we obtain $J_1 = [0,1[$.

Put $\hat{\xi}(\lambda) = [\hat{p}(\lambda), \hat{q}(\lambda)]_\lambda^u$ for $\lambda \in [0,1[$. We claim that if $\hat{\xi}(\lambda_n)$ converges to an element

ξ_* of $\mathscr{S}(e^u_{\lambda_*})$ for a sequence $\{\lambda_n\}_{n\geq 1}$ which tends to $\lambda_* \in [0,1]$, then Int $\xi_* \cap \mathrm{Per}^0_h(f_{\lambda_*}) = \varnothing$. Suppose that there exists an attracting periodic point q_* of f_{λ_*} in Int ξ_*. Take a continuation $\hat{q}_*$ of q_* on a neighborhood I_* of λ_*. By the persistence of $W^{ss}_\delta(\hat{q}_*(\lambda))$, we have Int $\hat{\xi}(\lambda_n)) \cap W^{ss}(\hat{q}(\lambda_n)) \neq \varnothing$ for any sufficiently large n. It contradicts Lemma 3.3 for Im $\hat{\xi}(\lambda_n)$ and $W^{ss}(\hat{q}_*(\lambda_n))$ since Int $\hat{\xi}(\lambda_n) \cap \mathrm{Per}(f_{\lambda_n}) = \varnothing$. Hence, we obtain the claim. Remark that the claim implies that $\xi_* = [\hat{p}(\lambda_*), \hat{q}(\lambda_*)]^u_{\lambda_*} = \hat{\xi}(\lambda_*)$ for $\lambda_* \in [0,1[$.

Since $|\hat{\xi}(\lambda)| \leq K$ for any $\lambda \in [0,1[$, Lemma 2.1 implies the family $\{\hat{\xi}(\lambda)\}_{\lambda\in[0,1[}$ is relatively compact in $\mathscr{S}(\boldsymbol{T}^2)$. By the above claim, we obtain the continuity of $\hat{\xi}$ at any $\lambda_* \in [0,1[$.

The set $\bigcap_{\lambda_0<1} \overline{\{\hat{\xi}(\lambda) \mid \lambda \in [\lambda_0, 1[\}}$ is a connected subset of $\mathscr{S}(e^u_1) \cap \mathrm{Fix}(\mathscr{S}(f_1))$. Since the set $\mathscr{S}(e^u_1) \cap \mathrm{Fix}(\mathscr{S}(f_1))$ is discrete by Corollary 3.8, $\hat{\xi}(\lambda)$ converges to an element $\hat{\xi}(1)$ of $\mathscr{S}(e^u_1) \cap \mathrm{Fix}(\mathscr{S}(f_1))$ as $\lambda \to 1$. By the above claim, we have Im $\hat{\xi}(1) \cap \mathrm{Per}^0_h(f_1) = \varnothing$. □

Corollary 5.12. *Let $\{f_\lambda\}_{\lambda\in[0,1]}$ be a regular framed $\boldsymbol{P}$A homotopy such that $\mathrm{Fix}_*(f_\lambda) = \varnothing$ for any $\lambda \in]0,1[$. Suppose that hyperbolic fixed points p, q_1, and q_2 of f_0^N satisfy $q_1, q_2 \in D^s_{+,0}p$ and admit continuations $\hat{p}$, $\hat{q}_1$, and $\hat{q}_2$ on $[0,1]$. If $q_1 \prec^p_{+,0} q_2$, then $\hat{q}_1(\lambda), \hat{q}_2(\lambda) \in D^s_{+,\lambda}\hat{p}(\lambda)$ and $\hat{q}_1(\lambda) \prec^{\hat{p}(\lambda)}_{+,\lambda} \hat{q}_2(\lambda)$ for any $\lambda \in [0,1[$.*

Proof. It is a consequence of the above proposition for $\sigma = s$ and the persistence of the order $\prec^p_+$ which is observed in Lemma 5.1. □

5.4. A Simple $\boldsymbol{P}$A homotopy.

We say a framed $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u, e^s)\}_{\lambda\in[-1,1]}$ is *simple* if it is regular and there exist an integer $N \geq 1$, a finite subset Δ_h of $\boldsymbol{T}^2$, and a disjoint union Δ^0_* of finitely many embedded circles in $\boldsymbol{T}^2$ which satisfy the following conditions:

1. Any non-hyperbolic periodic point of f_0 exhibits a generating saddle-node bifurcation.
2. $\Delta_h = \mathrm{Fix}_h(f_0^N)$ and $\Delta^0_* = \Omega^0_*(f)$.
3. For any $\lambda \in [-1,1]$, $\Delta_h \subset \mathrm{Fix}_h(f_\lambda^N)$ and each connected component of Δ^0_* is a 2-normally attracting circle for f_λ.
4. $\mathrm{Fix}_*(f_\lambda^N) = \varnothing$ for any $\lambda \neq 0$.
5. For any $\lambda \in \mathscr{N}\mathscr{D}(\{f_\lambda\})$ and $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, $\mathscr{C}_a(e^u_\lambda)$ is the disjoint union of $\mathscr{C}_a(e^u_\lambda, \Delta^0_h)$ and $\{c \in \mathscr{C}_a(e^u_\lambda) \mid \mathrm{Im}\, c \subset \Delta^0_*\}$, where $\Delta^0_h = \Delta_h \cap \mathrm{Fix}^0_h(f_0^N)$.

It is worth to remark that a simple $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}$ may exhibit a bifurcation of periodic points in Δ^0_*. In general, hyperbolic periodic points with very large period may appear in Δ^0_* for f_λ with $\lambda \neq 0$. However, it makes no change of the topology of the trivial part $\{c \in \mathscr{C}_a(e^u_\lambda) \mid \mathrm{Im}\, c \subset \Delta^0_*\}$ of $\mathscr{C}_a(e^u_\lambda)$. We also remark that Δ_h may not coincide with $\mathrm{Fix}_h(f_\lambda^N)$ for $\lambda \neq 0$ in general. In fact, as we see later, the set $\mathrm{Fix}_h(f_\lambda^N)$ coincide with Δ_h for $\lambda \in [-1,0[$, but it is the union of Δ_h and periodic orbits generated by the saddle-node bifurcation at $\lambda = 0$ for $\lambda > 0$.

We say two framed $\boldsymbol{P}$A homotopies $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda\in I}$ and $\{(g_\lambda, e'^u_\lambda, e'^s_\lambda)\}_{\lambda\in I'}$ are C^r-*equivalent* if there exist a continuous family $\{h_\lambda\}_{\lambda\in I}$ of C^r diffeomorphisms of $\boldsymbol{T}^2$ and a homeomorphism $\rho : I \to I'$ such that each h_λ is isotopic to identity, $g_{\rho(\lambda)} = h_\lambda \circ f_\lambda \circ h_\lambda^{-1}$, $Dh_\lambda(e^\sigma_\lambda(z)) \in \{a \cdot e'^\sigma_{\rho(\lambda)}(h_\lambda(z)) \mid a > 0\}$ for any $\lambda \in I$, $\sigma \in \{u,s\}$, and $z \in \boldsymbol{T}^2$.

Proposition 5.13. *Let $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda \in I_0}$ be a regular framed $\boldsymbol{P}$A homotopy. For any $\lambda_0 \in I_0$, there exists a neighborhood I of λ_0 such that $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda \in I}$ is C^2-equivalent to a simple $\boldsymbol{P}$A homotopy.*

Proof. First of all, we assume that $\lambda_0 = 0$ and any non-hyperbolic periodic point of f_0 exhibits a generating saddle-node bifurcation without loss of generality.

By Proposition 3.13, $\Omega^0_*(f)$ consists of finitely many mutually disjoint 2-normally attracting circles. By Proposition 5.9 there exists $N \geq 1$ such that any $c \in \mathscr{C}(e^u_0) \cap \mathrm{Per}(\mathscr{C}(f_0))$ satisfies either $\mathrm{Im}\ c \subset \Omega^0_*(f_0)$ or $\mathrm{Im}\ c \cap \mathrm{Fix}_h(f_0^N) \neq \varnothing$. Put $\Delta_h = \mathrm{Fix}_h(f_0^N)$, $\Delta^0_h = \mathrm{Fix}^0_h(f_0^N)$, and $\Delta^0_* = \Omega^0_*(f_0)$. Recall that any hyperbolic fixed points and 2-normally attracting circles have local continuations. By replacing $\{f_\lambda\}$ in the C^2-equivalence class if it is necessary, we assume that $\Delta_h \subset \mathrm{Fix}_h(f_\lambda^N)$ and any connected component of Δ^0_* is 2-normally attracting circle associated to f_λ for any $\lambda \in I_0$. Since $\{f_\lambda\}$ is regular, the set $\{\lambda \mid \mathrm{Fix}_*(f_\lambda^N) \neq \varnothing\}$ is discrete. Hence, there exists a neighborhood I_1 of 0 such that $I_1 \subset I_0$ and $\mathrm{Fix}_*(f_\lambda^N) = \varnothing$ for any $\lambda \in I_1 \setminus \{0\}$.

It is sufficient to show that there exists a neighborhood $I_2 \subset I_1$ of 0 such that any $c \in \mathscr{C}(e^u_\lambda)$ satisfies either $\mathrm{Im}\ c \subset \Delta^0_*$ or $\mathrm{Im}\ c \cap \Delta^0_h \neq \varnothing$ for any $\lambda \in I_2 \cap \mathscr{N}\mathscr{D}(\{f_\lambda\})$. Suppose it does not hold. Then, there exist a sequence $\{\lambda_i \in \mathscr{N}\mathscr{D}(\{f_\lambda\})\}_{i \geq 1}$ and $\{c_i \in \mathscr{C}(e^u_{\lambda_i})\}_{i \geq 1}$ such that λ_i tends to 0 as $i \to \infty$, $\mathrm{Im}\ c_i \not\subset \Delta^0_*$ and $\mathrm{Im}\ c_i \cap \Delta^0_h = \varnothing$ for any i. By Theorem B, there exists $c'_i \in \mathscr{C}(e^u_{\lambda_i}) \cap \mathrm{Per}(\mathscr{C}(f_{\lambda_i}))$ such that $c_i \in W^u(c'_i; \mathscr{C}(f_{\lambda_i}))$ for each i. We can see that $\mathrm{Im}\ c'_i \cap \Delta^0_h = \mathrm{Im}\ c_i \cap \Delta^0_h$, and if $\mathrm{Im}\ c'_i$ is contained in Δ^0_* then $\mathrm{Im}\ c_i$ also is. In particular, the choice of c_i implies $\mathrm{Im}\ c'_i \not\subset \Delta^0_*$ and $\mathrm{Im}\ c'_i \cap \Delta^0_h = \varnothing$ for any i. By Lemma 3.16, we may assume $C_i = \bigcup_{n \geq 1} f^n_{\lambda_i}(\mathrm{Im}\ c'_i)$ is an element of $\mathscr{E}(\sqcup_{<\infty} S^1, e^u_{\lambda_i})$. Then, Proposition 5.10 implies that there exists a subsequence $\{C_{i_k}\}$ which converges to an f_0-invariant set $C_* \in \mathscr{E}(\sqcup_{<\infty} S^1, e^u_0)$.

If $C_* \subset \Delta^0_*$, then C_{i_k} is contained in $W^s(\Delta^0_*; f_{\lambda_{i_k}})$ for any sufficiently large k. Since C_{i_k} is $f_{\lambda_{i_k}}$-invariant, it is contained in Δ^0_*. However, it contradicts $\mathrm{Im}\ c'_{i_k} \not\subset \Delta^0_*$. Therefore, we have $C_* \not\subset \Delta^0_*$.

The choice of the integer N implies $C_* \cap \mathrm{Fix}_h(f_0^N) \neq \varnothing$. Hence, we can take $\xi \in \mathscr{S}(e^u_0, \mathrm{Fix}_h(f_0^N)) \cap \mathrm{Per}(\mathscr{S}(f))$ with $\mathrm{Im}\ \xi \subset C_*$ and $\mathrm{Int}\ \xi \cap \mathrm{Fix}_h(f_0^N) = \varnothing$. By Lemma 5.4, $\xi(0)$ or $\xi(1)$ is an element of $\Delta^0_h = \mathrm{Fix}^0_h(f_0^N)$. The persistence of the stable set of an attracting fixed point implies that $C_{i_k} \cap \Delta^0_h \neq \varnothing$ for any sufficiently large k. It contradicts that $\mathrm{Im}\ c'_i \cap \Delta^0_h = \varnothing$. □

Let $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda \in [-1,1]}$ be a simple $\boldsymbol{P}$A homotopy. Let N and Δ_h be the ones given in the definition. The persistence of hyperbolic fixed points observed in Lemma 5.1 and the finiteness of fixed points of f_λ^N imply that all fixed point of f_λ^N with $\lambda \in [-1, 0[$ (*resp.* $\lambda \in]0, 1]$) admits a unique continuation on $[-1, 0]$ (*resp.* $[0, 1]$). In particular, there exists a unique pair $(\hat{\theta}_s, \hat{\theta}_n)$ of continuous maps from $\mathrm{Fix}_*(f_0^N) \times [0, 1]$ to $\boldsymbol{T}^2$ such that $\hat{\theta}_\sigma(p_*, \cdot)$ is a continuation of p_* on $[0, 1]$ for any $p_* \in \mathrm{Fix}_*(f_0^N)$ and $\sigma = s, n$, and $(\hat{\theta}_s(p_*, \cdot), \hat{\theta}_n(p_*, \cdot))$ is a saddle-node continuation of p_* on a small interval containing $\lambda = 0$. Put $\theta_\sigma = \hat{\theta}_\sigma(\cdot, 1)$ for $\sigma \in \{s, n\}$. Then, the existence and the uniqueness of a continuation of a hyperbolic fixed point imply that $\mathrm{Fix}(f_{-1}^N) = \Delta_h$, $\mathrm{Fix}(f_1^N) = \Delta_h \cup \mathrm{Im}\ \theta_s \cup \mathrm{Im}\ \theta_n$, and $f_1 \circ \theta_\sigma = \theta_\sigma \circ f_0$ for $\sigma \in \{s, n\}$. In particular, both $\mathrm{Im}\ \theta_s$ and $\mathrm{Im}\ \theta_n$ are periodic orbits of f_1.

6. Bifurcations in a simple $\boldsymbol{P}$A homotopy.

In this section, we investigate the bifurcation of the combinatorial description of $\mathscr{S}_a(e^u_\lambda, \Delta^0_h) \cap \mathrm{Fix}(\mathscr{S}(f^N_\lambda))$ for a simple $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}$. It allows us to describe the bifurcation of $\mathscr{C}_a(e^u_\lambda, \Delta^0_h)$ completely. In Subsection 6.1, we study the bifurcation of connecting segments and define a natural correspondence Θ between $\mathscr{S}_a(e^u_{-1}, \Delta^0_h)$ and $\mathscr{S}_a(e^u_1, \Delta^0_h)$. In Subsection 6.2, we consider the bifurcation of $D^s_\pm p$ for a repelling periodic point p.

We fix a simple framed $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda\in[-1,1]}$. Let N, Δ_h,Δ^0_h, and Δ^0_* be the ones in the definition, and $(\hat{\theta}_s, \hat{\theta}_n)$ and (θ_s, θ_n) be the pairs of maps defined in the last paragraph of Subsection 5.4. To simplify the proofs, we assume that any point of $\mathrm{Per}_*(f_0)$ is of type $(u,+)$ or $(s,+)$.

6.1. Correspondence of segments.

In this subsection, we construct a natural correspondence Θ between $\mathscr{S}(e^u_{-1}, \Delta^0_h)$ and $\mathscr{S}(e^u_1, \Delta^0_h)$.

Lemma 6.1. *For periodic points $p, q \in \Delta_h$ with $q \in D^u_{+,-1}p$, there exists a continuation $\hat{\xi}$ of $[p,q]^u_{-1}$ on $[-1,1]$ such that one of the followings holds*:

1. $\hat{\xi}(\lambda) = [p,q]^u_\lambda$ *for any* $\lambda \in [-1,1]$.
2. *There exists* $p_* \in \mathrm{Fix}_*(f^N_0)$ *of type* $(u,+)$ *such that*

$$\hat{\xi}(\lambda) = \begin{cases} [p,q]^u_\lambda & \text{if } \lambda \in [-1,0[, \\ [p,p_*,q]^u_0 & \text{if } \lambda = 0, \\ \left[p, \hat{\theta}_n(p_*,\lambda), \hat{\theta}_s(p_*,\lambda), q\right]^u_\lambda & \text{if } \lambda \in]0,1]. \end{cases}$$

In the latter case, we have $p \in \mathrm{Per}^1_h(f^N_\lambda) \cup \mathrm{Per}^2_h(f^N_\lambda)$ and $q \in \mathrm{Per}^0_h(f^N_\lambda)$ for any $\lambda \in [-1,1]$.

Proof. By Proposition 5.11, there exists a continuation $\hat{\xi}_-$ of $[p,q]^u_{-1}$ on $[-1,0]$ satisfying $\hat{\xi}_-(\lambda) = [p,q]^u_\lambda$ for any $\lambda \in [-1,0[$ and $\mathrm{Int}\ (\hat{\xi}_-(0)) \cap \mathrm{Fix}^0_h(f^N_0) = \varnothing$. Lemma 5.4 implies either $\hat{\xi}_-(0) = [p,q]^u_0$ or $\hat{\xi}_-(0) = [p,p_*,q]^u_0$ for some $p_* \in \mathrm{Fix}_*(f^N_0)$ of type $(u,+)$ since one of p or q is not attracting. In the former case, $[p,q]^u_0$ has a continuation $\hat{\xi}$ on $[0,1]$ with $\hat{\xi}(\lambda) = [p,q]^u_\lambda$ by Proposition 5.11 again.

Notice that p is repelling or of saddle-type, and q is attracting in the latter case. By Lemmas 5.5 and 5.7, we obtain $D^u_{+,\lambda}p = \{\hat{\theta}_n(p_*,\lambda)\}$, $D^u_{+,\lambda}\hat{\theta}_n(p_*,\lambda) = \{\hat{\theta}_s(p_*,\lambda)\}$, and $D^u_{+,\lambda}\hat{\theta}_s(p_*,\lambda) = \{q\}$ for some sufficiently small $\lambda_+ > 0$. Proposition 5.11 implies that there exists a continuation $\hat{\xi}_+$ of a segment such that $\hat{\xi}_+(\lambda) = [p, \hat{\theta}_n(p_*,\lambda), \hat{\theta}_s(p_*,\lambda), q]^u_\lambda$ for any $\lambda \in]0,1]$ and $\mathrm{Int}\ (\hat{\xi}_+(0)) \cap \Delta^0_h = \varnothing$. Since $D^u_{+,0}p = \{p_*\}$ and $D^u_{+,0}p_* = \{q\}$, we obtain $\hat{\xi}_+(0) = [p,p_*,q]^u_0$. □

Lemma 6.2. *Suppose $p, q \in \Delta_h$ and $\xi \in \mathscr{S}(e^u_1) \cap \mathrm{Fix}(\mathscr{S}(f_1)^N)$ satisfy $\xi(0) = p$, $\xi(1) = q$, and $\mathrm{Int}\ \xi \cap \Delta_h = \varnothing$. Then, one of the followings holds*:

1. $q \in D^u_{+,-1}p$ *and* $\xi = [p,q]^u_1$.
2. $q \in D^u_{+,-1}p$ *and there exists* $p_* \in \mathrm{Fix}_*(f^N_0)$ *of type* $(u,+)$ *such that* $\xi =$

$[p, \theta_n(p_*), \theta_s(p_*), q]^u_1$.

3. *There exist* $p_* \in \mathrm{Fix}_*(f_0^N)$ *of type* $(s,+)$ *and* $\sigma \in \{s, n\}$ *such that* $D^u_{-,0}p_* = \{p\}$, $D^u_{+,0}p_* = \{q\}$, *and* $\xi = [p, \theta_\sigma(p_*), q]^u_1$.

Proof. Suppose $\xi = [p_1, \dots, p_k]^u_1$ for some $k \geq 2$ and $p_1, \dots, p_k \in \mathrm{Fix}(f_1^N)$, where $p_1 = p$ and $p_k = q$. By Proposition 5.11, $[p_i, p_{i+1}]^u_1$ has a continuation $\hat{\xi}_i$ on $[0, 1]$ with $\mathrm{Int}\,(\hat{\xi}_i(0)) \cap \Delta^0_h = \varnothing$. Notice that $\hat{\xi}_i(\lambda)(0)$ is not contained in Δ^0_h for $i = 2, \dots, k-1$.

Put $\xi_0 = \hat{\xi}_1(0) * \cdots * \hat{\xi}_{k-1}(0)$. Then, we have $\xi_0(0) = p$, $\xi_0(1) = q$, and $\mathrm{Int}\,\xi_0 \cap \mathrm{Per}^0_h(f_0) = \varnothing$. Since $\{p, q\} \subset \Delta_h = \mathrm{Fix}_h(f_0^N)$, Lemma 5.4 implies that either $\xi_0 = [p, q]^u_0$ or $\xi_0 = [p, p_*, q]^u_0$ for some $p_* \in \mathrm{Fix}_*(f_0)$. For the former case, we have $k = 2$, and hence, $\xi_0 = [p, q]^u_0$ and $\xi = [p, q]^u_1$. Proposition 5.11 implies $q \in D^u_{+,-1}p$.

For the latter case, we see that $k \geq 3$ and $(\hat{\xi}_{i-1}(0))(1) = (\hat{\xi}_i(0))(0) = p_*$ for any $i = 2, \dots, k-1$. By the uniqueness of continuations, each p_i is either $\theta_s(p_*)$ or $\theta_n(p_*)$. If p_* is of type $(u,+)$, then $q \in D^u_{+,-1}p$ and $\xi = [p, \theta_n(p_*), \theta_s(p_*), q]^u_1$ by Lemmas 5.5 and 5.7. If p_* is of type $(s,+)$, then $\|Df_1^N(e^u_1(p_i))\| > 1$ for any $i = 2, \dots, k-1$. It implies that $k = 3$. Hence, we obtain $\xi = [p, \theta_\sigma(p_*), q]^u_1$ for some $\sigma \in \{s, n\}$. $\square$

Let Σ_h be the subset of $\Delta_h \setminus \Delta^0_h$ consisting of p satisfying $|W^{uu}(p; f_{-1})| < \infty$. Put $\Sigma^k_h = \Sigma_h \cap \mathrm{Fix}^k_h(f_0^N)$ for $k = 1, 2$. For $p \in \Sigma_h$, we define a subset I^-_p of $[-1, 1]$ by

$$I^-_p = \begin{cases} \{0\} & \text{if } p \in \Sigma^1_h \\ ]-1, 1[& \text{if } p \in \Sigma^2_h. \end{cases}$$

If $\xi_0 = (p_1, \dots, p_k)^u_{-1}$ is well-defined, then we define a subset $I^-_{\xi_0}$ of $[-1, 1]^k$ by $I^-_{\xi_0} = \prod_{i=1}^k I^-_{p_i}$.

Let μ^-_p be a characteristic map from $\overline{I^-_p}$ to $\overline{W^u((p)^u_{-1}; \mathscr{S}(f_{-1}))}$ associated to f_{-1}. By Lemmas 6.1 and 6.2, a point $p \in \Delta_h$ is contained in Σ_h if and only if $W^{uu}(p; f_1)$ has finite length and one of the followings holds:

1. $(p)^u_1 \in \mathscr{S}(e^u_1, \Delta^0_h)$.
2. There exists $p_* \in \mathrm{Fix}_*(f_0^N)$ of type $(u,+)$ with $\theta_n(p_*) \in D^u_{+,1}p$ and $(p, \theta_s(p_*))^u_1 \in \mathscr{S}(e^u_1, \Delta^h_0)$.

We define the map μ^+_p by $\mu^+_p = \mu_{p,1}$ in the former case and $\mu^+_p = \mu_{p,1}(\cdot) * (\theta_s(p_*))^u_1$ in the latter case, where $\mu_{p,1}$ is a characteristic map from $\overline{I^-_p}$ to $\overline{W^u((p)^u_1; \mathscr{S}(f_1))}$ associated to f_1. Remark that the restriction of μ^+_p to I^-_p is a homeomorphism onto $W^u(\mu^+_p(0); \mathscr{S}(f_1))$.

By Proposition 4.9, there exists a unique map Θ from $\mathscr{S}(e^u_{-1}, \Delta^0_h)$ to $\mathscr{S}(e^u_1, \Delta^0_h)$ such that

1. $\Theta \circ \mu^-_p(s) = \mu^+_p(s)$ for any $p \in \Sigma_h$ and $s \in I_p$, and
2. $\Theta(\xi_1 * \xi_2) = \Theta(\xi_1) * \Theta(\xi_2)$ for any $\xi_1, \xi_2 \in \mathscr{S}(e^u_{-1}, \Delta^0_h)$ with $\xi_1(1) = \xi_2(0)$.

It is important to remark that the map Θ is not continuous in general. However, the restriction to each $W^u(\xi_0; \mathscr{S}(f_{-1}))$ is continuous.

Proposition 6.3. *The map* Θ *is injective. An element* $\xi \in \mathscr{S}(e^u_1, \Delta^0_h)$ *with* $|\xi| \neq 0$ *is contained in the image of* Θ *if and only if there exists a sequence* $\{p_1, \dots, p_k\}$ *such* *that* $\xi \in W^u((p_1,\dots,p_k)_1^u;\mathscr{S}(f_1))$ *and each* p_i *satisfies either* $p_i \in \Sigma_h$ *or* $p_i = \theta_s(p_{*,i})$ *for some* $p_{*,i} \in \mathrm{Fix}_*(f_0^N)$ *of type* $(u,+)$.

Proof. Observe that the restriction of Θ on $W^u(\xi_0;\mathscr{S}(f_{-1}))$ is a bijective map onto $W^u(\Theta(\xi_0);\mathscr{S}(f_1))$ for any $\xi_0 \in \mathscr{S}(e_{-1}^u,\Delta_h^0)\cap \mathrm{Fix}(\mathscr{S}(f_{-1}^N))$.

Take $\xi_+ = (p_1,\dots,p_k)_1^u$ with $p_1,\dots,p_k \in \mathrm{Fix}(f_1^N)$. By Lemma 6.2, $\xi_+ \in \mathrm{Im}\ \Theta$ if and only if each p_i satisfies $p_i \in \Sigma_h$ or $p_i = \theta_s(p_{*,i})$ for some $p_{*,i} \in \mathrm{Fix}_*(f_0^N)$ of type $(u,+)$. Hence, Proposition 4.9 implies the latter half of the assertion.

An element $\xi_+ = (p_1,\dots,p_k)_1^u$ of Im Θ with $p_1,\dots,p_k \in \mathrm{Fix}(f_1^N)$ have a unique irreducible decomposition $\xi_- = \mu_{q_1}^+(0) * \cdots * \mu_{q_m}^+(0)$ associated to Δ_h^0. Since $\mu_p^+(0) \neq \mu_{p'}^+(0)$ for any $p,p' \in \Sigma_h$ with $p \neq p'$, the restriction of Θ to $\mathscr{S}(e_{-1}^u,\Delta_h^0)\cap\mathrm{Fix}(\mathscr{S}(f_{-1}^N))$ is injective. Hence, the observation at the beginning of the proof implies that Θ is injective. □

6.2. Correspondence at the boundary of cells.

Let Θ, Σ_h, Σ_h^1, and Σ_h^2 be the ones defined in the previous subsection. We denote by Σ_* the set of non-hyperbolic fixed points p_* of f_0^N that are of type $(s,+)$ and satisfy $|W^{uu}(p_*;f_0)| < \infty$.

Lemma 6.4. *If* $p_* \in \mathrm{Fix}_*(f_0^N)$ *is of type* $(s,+)$, *then* $D_{-,1}^s\theta_n(p_*) = \{\theta_s(p_*)\}$. *Moreover, the following three conditions on* p_* *are equivalent*:

- $p_* \in \Sigma_*$.
- $|W^{uu}(\theta_s(p_*);f_1)| < \infty$.
- $|W^{uu}(\theta_n(p_*);f_1)| < \infty$.

Proof. The former is a consequence of Lemma 5.6 and Proposition 5.11. The latter follows from Lemma 5.5 since $p_* \in \Sigma_*$ is a non-hyperbolic fixed point of type $(u,+)$ for $(f_0^{-N},-e_0^s,e_0^u)$. □

For $p_* \in \Sigma_*$, let $\mu_{p_*}^* : [-1,1] \to \overline{W^u((\theta_n(p_*);\mathscr{S}(f_1)))}$ be a characteristic map of $\theta_n(p_*)$ for f_1.

Lemma 6.5. *Suppose that* $p_* \in \Sigma_*$ *satisfies* $D_{+,1}^s\theta_n(p_*)\cap \mathrm{Im}\ \theta_s \neq \varnothing$. *Then, the followings hold*:

1. *For any* $q_* \in \Sigma_*$, *there exist* $\xi_-,\xi_+ \in \mathrm{Im}\ \Theta$ *and* $q_*' \in \Sigma_*$ *such that* $|\xi_-| \neq 0$, $|\xi_+| \neq 0$, *and* $\mu_{q_*}^*(1) = \xi_- * (\theta_s(q_*'))_1^u * \xi_+$.
2. $\mu_q(1) \in \mathrm{Im}\ \Theta$ *for any* $q \in \Sigma_h^2$.

Proof. Put $\mu_{p_*}^*(1) = (q_1,\dots,q_k)_1^u$ and $r_- = \partial_- W^{uu}(\theta_n(p_*);f_1)$. Since Im $\theta_s = \mathscr{O}(\theta_s(p_*);f_1)$, Lemma 3.11 implies that there exists a unique $l = 1,\dots,k$ such that $q_l \in \mathrm{Im}\ \theta_s$. Since $D_{+,1}^u r_-$ contains $\theta_n(p_*)$, $\theta_s(p_*)$, and q_1, we can show $\mathscr{O}(q_1;f_1) \neq \mathscr{O}(\theta_s(p_*);f_1)$ by the same way as the proof of Lemma 3.11. In particular, we have $l \geq 2$. The same argument also implies $l \leq k-1$. Therefore, $\xi_- = (q_1,\dots,q_{l-1})_1^u$, $\xi_+ = (q_{l+1},\dots,q_k)_1^u$, and $q_*' = \theta_s^{-1}(q_l)$ satisfy the former assertion for p_*.

Since Σ_* is the orbit of p_*, the former assertion also holds for any $q_* \in \Sigma_*$. Since $D_{-,1}^s q_l = \{\theta_n(p_*)\}$ and $\mathscr{O}(q_l;f_1) = \mathrm{Im}\ \theta_s = \theta_s(\Sigma_*)$, we also obtain the second assertion. □

Lemma 6.6. *$D^s_{-,1}p \cap \mathrm{Im}\,\theta_s = \varnothing$ and $D^s_{+,1}p \cap \mathrm{Im}\,\theta_s$ contains at most one point for any $p \in \Sigma_h^2$.*

Proof. Fix $p \in \Sigma_h^2$. By Lemma 6.4, we have $D^s_{+,1}\theta_s(p_*) = \{\theta_n(p_*)\}$ for any $p_* \in \mathrm{Fix}_*(F_0^N)$. It implies $D^s_{-1}p \cap \mathrm{Im}\ \theta_s = \varnothing$. If $\mathrm{Fix}_*(f_0^N)$ is not empty, then we have $\mathrm{Im}\ \theta_s = \mathscr{O}(\theta_s(q_*); f_1)$ for any $q_* \in \mathrm{Fix}_*(f_0^N)$. Hence, Lemma 3.11 implies that $D^s_{+,1}p \cap \mathrm{Im}\,\theta_s$ contains at most one point. □

Now, let us consider the correspondence between $\mu_p^+(s)$ and $\mu_p^-(s)$ at $s = \pm 1$.

Lemma 6.7. *A pair (p, q) of elements of Δ_h satisfies $q \in D^s_{+,-1}p$ if and only if either one of the followings holds:*

1. *$q \in D^s_{+,1}p$.*
2. *There exists $p_* \in \mathrm{Fix}_*(f_0^N)$ of type $(s, +)$ such that $q \in D^s_{+,1}\theta_n(p_*)$, $D^s_{-,1}\theta_n(p_*) = \{\theta_s(p_*)\}$, and $\theta_s(p_*) \in D^s_{+,1}p$.*

In the former case, the map $\lambda \mapsto [p, q]^s_\lambda$ is a continuation of $[p, q]^s_{-1}$ on $[-1, 1]$. In the latter case, the map $\hat{\xi}$ given by

$$\hat{\xi}(\lambda) = \begin{cases} [p, q]^s_\lambda & \text{if } \lambda \in [-1, 0[, \\ [p, p_*, q]^s_0 & \text{if } \lambda = 0, \\ \left[p, \hat{\theta}_s(p_*, \lambda), \hat{\theta}_n(p_*, \lambda), q\right]^s_\lambda & \text{if } \lambda \in]0, 1] \end{cases}$$

is a continuation of $[p, q]^s_{-1}$ on $[-1, 1]$.

Proof. Notice that a non-hyperbolic fixed point of type $(s, +)$ for (f_0^N, e_0^u, e_0^s) is of type $(u, +)$ for $(f_0^{-N}, -e^s, e^u)$. Hence, the lemma follows from Lemmas 6.1 and 6.2 for the $\boldsymbol{P}$A homotopy $\{(f_\lambda^{-N}, -e^s_\lambda, e^u_\lambda)\}$. □

Lemma 6.8. *The equation $\Theta \circ \mu_p^-(-1) = \mu_p^+(-1)$ holds for any $p \in \Sigma_h^2$. If $D^s_{+,1}p \cap \theta_s(\Sigma_*) = \varnothing$, then the equation $\Theta \circ \mu_p^-(1) = \mu_p^+(1)$ also holds, and hence, $\Theta \circ \mu_p^-$ is a continuous map on $\overline{I_p^-} = [-1, 1]$.*

Proof. We show that the latter assertion holds. The proof for the former is same since $D^s_{-,1}p \cap \theta_s(\Sigma_*) = \varnothing$ by Lemma 6.6.

Suppose that $D^s_{+,1}p \cap \Delta_h^1 = \{q_1, \ldots, q_k\}$ and $q_i \prec^p_{+,1} q_j$ if $i < j$. Since $D^s_{+,1}p \cap \theta_s(\Sigma_*) = \varnothing$, we can obtain $\mu_p^+(1) = \mu_{q_1}^+(0) * \cdots * \mu_{q_k}^+(0)$. By Lemma 6.7, we have $D^s_{+,-}p \cap \mathrm{Fix}_h^1(f_{-1}^N) = \{q_1, \ldots, q_k\}$ and $[p, q_i]^s_\lambda$ is a continuation of $[p, q_i]^s_{-1}$ for any $i = 1, \ldots, k$. Corollary 5.12 implies $q_i \prec^p_{+,-1} q_j$ if $i < j$. It implies $\mu_p^-(1) = (q_1, \ldots, q_k)^u_{-1}$, and hence, $\mu_p^+(1) = \Theta \circ \mu_p^-(1)$. In particular, the map $\Theta \circ \mu_p^-(s) = \mu_p^+(s)$ is continuous at $s = 1$. □

Lemma 6.9. *For $p \in \Sigma_h^2$ and $p_* \in \Sigma_*$ with $\theta_s(p_*) \in D^s_{+,1}p$, there exist $\xi_-, \xi_*, \xi_+ \in \mathscr{S}(e^u_{-1}, \Delta_h^0)$ such that $\mu_p^-(1) = \xi_- * \xi_* * \xi_+$, $\mu_p^+(1) = \Theta(\xi_-) * (\theta_s(p_*))^u_1 * \Theta(\xi_+)$, and $\mu^*_{p_*}(1) = \Theta(\xi_*)$.*

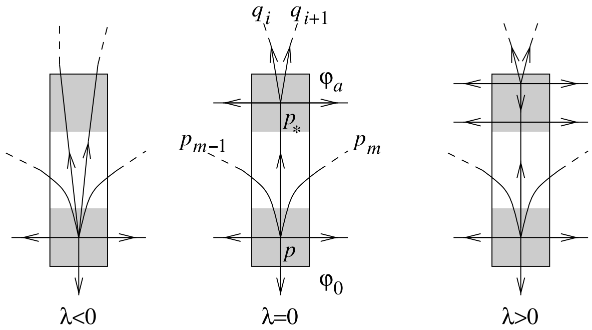

Figure 7. Proof of Lemma 6.9.

Proof. Lemmas 6.6 and 6.5 imply that $D^s_{+,1}p \cap \mathrm{Im}\ \theta_s = \{\theta_s(p_*)\}$ and $D^s_{+,1}\theta_n(p_*) \subset \Delta_h$. There exist $r_-, r_+ \in \Delta^0_h$, $p_1, \dots, p_k, q_1, \dots, q_l \in \Delta_h$, and $m \geq 1$ such that $\mu^+_p(1) = [r_-, p_1, \dots, p_{m-1}, \theta_s(p_*), p_m, \dots, p_k, r_+]^u_1$ and $\mu^*_{p_*}(1) = [p_{m-1}, q_1, \dots, q_l, p_m]^u_1$. By Lemma 6.4, we have $D^s_{-,1}\theta_n(p_*) = \{\theta_s(p_*)\}$. Hence, Lemma 6.7 implies $D^s_{+,-1}p = \{p_1, \dots, p_k, q_1, \dots, q_l\}$, $D^s_{+,0}p = \{p_1, \dots, p_k, p_*\}$, and $D^s_{+,0}p_* = \{q_1, \dots, q_l\}$.

We have $p_i \prec^p_{+,-1} p_j$ if $i < j$ by Corollary 5.12. We claim that $q_i \prec^p_{+,-1} q_j$ if $i < j$ and $p_{m-1} \prec^p_{+,-1} q_i \prec^p_{+,-1} p_m$ for any i. Take $a_0 > 1$ and an embedding $\psi : [-1, a_0+1] \times [-1,1] \to \boldsymbol{T}^2$ so that $\psi(0,0) = p$, $\psi(0,a_0) = p_*$, $\psi(\{0\}\times[0,a_0]) = \mathrm{Im}\ [p,p_*]^s_0$, and the map $\varphi_a : (x,y) \to \psi(x, y+a)$ on $[-1,1]^2$ is a 1/8-canonical coordinate associated to (e^u_0, e^s_0) for any $a \in [0,a_0]$. We fix $\lambda_0 > 0$ so that φ_a is a 1/8-canonical coordinate associated to $(e^u_\lambda, e^s_\lambda)$ for any $a \in [0,a_0]$ and $\lambda \in [-\lambda_0, \lambda_0]$, and $\hat{\theta}_s(p_*,\lambda), \hat{\theta}_n(p_*,\lambda) \in \varphi_{a_0}([-1/2,1/2]^2)$ for any $\lambda \in [0,\lambda_1]$. By the continuity of connecting segments with respect to λ, there exist $\lambda_1 \in]0,\lambda_0]$ and a family $\{\hat{v}_{q_j}\}^l_{j=1}$ of functions on $[0,a_0+1] \times [-\lambda_1, \lambda_1]$ such that

1. $\hat{v}_{q_j}(0,\lambda) = 0$ for any $\lambda \in [-\lambda_1, \lambda_1]$, and
2. $\psi(\{(\hat{v}_{q_j}(y), y) \mid y \in [0, a_0+1]\})$ is a subset of $\mathrm{Im}\ [p, q_j]^s_\lambda$ for any $\lambda \in [-\lambda_1, 0[$, and of $\mathrm{Im}\ [p, \hat{\theta}_s(p_*,\lambda), \hat{\theta}_n(p_*,\lambda), q_j]^s_\lambda$ for any $\lambda \in]0,\lambda_1]$.

By Corollary 5.12, we have $q_i \prec^{\hat{\theta}_n(p_*,\lambda)}_{+,\lambda} q_{i+1}$ for any $\lambda \in]0,1]$. Since φ_{a_0} is a 1/8-canonical coordinate, Proposition 3.9 implies that $\hat{v}_{q_i}(a_0+1,\lambda) < \hat{v}_{q_{i+1}}(a_0+1,\lambda)$ for any $\lambda \in]0,\lambda_1]$. Since $W^{ss}_-(q_i; f_\lambda) \cap W^{ss}_-(q_{i+1}; f_\lambda) = \varnothing$, and $\hat{v}_{q_i}$ and $\hat{v}_{q_{i+1}}$ are continuous functions, we obtain $\hat{v}_{q_i}(1,\lambda) < \hat{v}_{q_{i+1}}(1,\lambda)$ for any $\lambda \in [-\lambda_1, 0[$. Since φ_0 is a 1/8-canonical coordinate, Proposition 3.9 implies that $q_i \prec^p_{+,\lambda} q_{i+1}$ for any $\lambda \in [-\lambda_1, 0[$. By Corollary 5.12, we obtain $q_i \prec^p_{+,-1} q_{i+1}$.

Take a family $\{\hat{v}_{p_i}\}^k_{i=1}$ of continuous functions on $[0,1] \times [-\lambda_1, \lambda_1]$ such that $\hat{v}_{p_i}(0,\lambda) = 0$ and $\psi(\{(\hat{v}_{p_i}(y), y) \mid y \in [0,1]\}) \subset \mathrm{Im}\ [p, p_i]^s_\lambda$ for any $\lambda \in [-\lambda_1, \lambda_1]$. The same argument as above implies $\hat{v}_{p_{m-1}}(1,\lambda) < \hat{v}_{q_1}(1,\lambda)$ and $\hat{v}_{q_l}(1,\lambda) < \hat{v}_{p_m}(1,\lambda)$ for any $\lambda \in [-\lambda_1, \lambda_1]$. It implies $p_{m-1} \prec^p_{+,-1} q_1$ and $q_l \prec^p_{+,-1} p_m$. It completes the proof of the claim.

Put $\xi_- = [r_-, p_1, \dots, p_{m-1}]^u_{-1}$, $\xi_* = [p_{m-1}, q_1, \dots, q_l, p_m]^u_{-1}$, and $\xi_+ = [p_m, \dots, p_k, r_+]^u_{-1}$. Then, the above claim implies $\mu^-_p(1) = \xi_- * \xi_* * \xi_+$, $\mu^+_p(1) = \Theta(\xi_-) * (\theta_s(p_*))^u_1 * \Theta(\xi_+)$, and $\mu^*_{p_*}(1) = \Theta(\xi_*)$. □

7. Proof of Theorem A.

In this section, we show the following proposition, which completes the proof of Theorem A. Recall that $X \cup \{\infty\}$ is the one point compactification of a topological space X.

PROPOSITION 7.1. *For any prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, pointed spaces $(\mathscr{C}_a(e^u_1, \Delta^0_h) \cup \{\infty\}, \infty)$ and $(\mathscr{C}_a(e^u_{-1}, \Delta^0_h) \cup \{\infty\}, \infty)$ have the same homotopy type.*

In fact, Lemma 5.2 and Proposition 5.13 allow us to reduce the proof to the case of a simple $\boldsymbol{P}$A homotopy. For a simple $\boldsymbol{P}$A homotopy $\{(f_\lambda, e^u_\lambda, e^s_\lambda)\}_{\lambda \in [-1,1]}$, the set $\{c \in \mathscr{C}_a(e^u_\lambda) \mid \operatorname{Im} c \subset \Delta^0_*\}$ does not depend on the choice of $\lambda \in [-1,1]$ and it consists of finitely many isolated elements of index 0. Hence, the proposition implies that $(\mathscr{C}_a(e^u_1) \cup \{\infty\}, \infty)$ and $(\mathscr{C}_a(e^u_{-1}) \cup \{\infty\}, \infty)$ have the same homotopy type for any prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$.

7.1. Compatible maps.

Before the proof of Proposition 7.1, we introduce the concept of *compatible maps* and consider their continuity.

Let e_1 and e_2 be continuous unit vector fields on $\boldsymbol{T}^2$, and Λ_1 and Λ_2 be finite subsets of $\boldsymbol{T}^2$ with $\Lambda_1 \subset \Lambda_2$. We call a map Ψ from $\mathscr{S}(e_1, \Lambda_1)$ to $\mathscr{S}(e_2, \Lambda_2)$ *compatible* if the following conditions hold:

1. $\Psi(\xi)(0) = \xi(0)$, $\Psi(\xi)(1) = \xi(1)$, and $\Psi(\xi)$ is homotopic to ξ relative to end points for any $\xi \in \mathscr{S}(e_1, \Lambda)$.
2. $\Psi(\xi_1 * \xi_2) = \Psi(\xi_1) * \Psi(\xi_2)$ for any $\xi_1, \xi_2 \in \mathscr{S}(e_1, \Lambda)$ with $\xi_1(1) = \xi_2(0)$.

Remark that the map Θ defined in Subsection 6.1 is a compatible map.

Let (f, e^u_f, e^s_f) and (g, e^u_g, e^s_g) be C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphisms and suppose $\operatorname{Fix}^0_h(f^n) \subset \operatorname{Fix}^0_h(g^n)$ for some $n \geq 1$. Recall that $\Sigma(f^n)$ is the set of points p of $\operatorname{Fix}^1_h(f^n) \cup \operatorname{Fix}^2_h(f^n)$ satisfying $|W^{uu}(p; f)| < \infty$.

LEMMA 7.2. *Let Ψ be a compatible map from $\mathscr{S}(e^u_f, \operatorname{Fix}^0_h(f^n))$ to $\mathscr{S}(e^u_g, \operatorname{Fix}^0_h(g^n))$. Suppose that $\Psi \circ \mu_p$ is continuous for each $p \in \Sigma(f^n)$, where $\mu_p : \overline{I}_p \to \mathscr{S}(e^u_f)$ is a characteristic map of p. Then, the map Ψ is continuous and proper. In particular, it induces a proper continuous map from $\mathscr{C}_a(e^u_f, \operatorname{Fix}^0_h(f^n))$ to $\mathscr{C}_a(e^u_g, \operatorname{Fix}^0_h(g^n))$ for any $a \in H_1(\boldsymbol{T}^2; \boldsymbol{Z})$.*

PROOF. By Propositions 4.9 and 4.13, the map Ψ is continuous. Since $\Sigma(f^n)$ is a finite set, there exists $K > 0$ such that $|\Psi \circ \mu_p(s)| \geq K|\mu_p(s)|$ for any $p \in \Sigma(f^n)$ and $s \in \overline{I_p}$. It implies that $|\Psi(\xi)| \geq K|\xi|$ for any $\xi \in \mathscr{S}(e^u_f, \operatorname{Fix}^0_h(f^n))$. Hence, the Ψ is proper by Lemma 2.1.

For any $a \in H_1(\boldsymbol{T}^2; \boldsymbol{Z})$, we can define a map Ψ_a from $\mathscr{C}_a(e^u_f, \operatorname{Fix}^0_h(f^n))$ to $\mathscr{C}_a(e^u_g, \operatorname{Fix}^0_h(g^n))$ by $\pi_c \circ \Psi = \Psi_a \circ \pi_c$ on $\widetilde{\mathscr{C}_a}(e^u_f, \operatorname{Fix}^0_h(f^n))$. Since π_c and Ψ are continuous and proper, the map Ψ_a also is. $\square$

We also call a map $\hat{\Psi}$ from $\mathscr{S}(e^u_f, \operatorname{Fix}^0_h(f^n)) \times [0,1]$ to $\mathscr{S}(e^u_g, \operatorname{Fix}^0_h(g^n))$ *compatible* if $\hat{\Psi}(\cdot, t)$ is compatible for any $t \in [0,1]$.

Lemma 7.3. *Let $\hat{\Psi}$ be a compatible map from $\mathscr{S}(e_f^u, \mathrm{Fix}_h^0(f^n)) \times [0,1]$ to $\mathscr{S}(e_g^u, \mathrm{Fix}_h^0(g^n))$. Suppose that the map $(s,t) \mapsto \hat{\Psi}(\mu_p(s), t)$ is continuous on $\overline{I_p} \times [0,1]$ for each $p \in \Sigma(f)$. Then, $\hat{\Psi}$ is continuous and proper. In particular, it induce a proper continuous map from $\mathscr{C}_a(e_f^u, \mathrm{Fix}_h^0(f^n)) \times [0,1]$ to $\mathscr{C}_a(e_g^u, \mathrm{Fix}_h^0(g^n))$ for any $a \in H_1(\boldsymbol{T}^2; \boldsymbol{Z})$.*

Proof. The proof is the same as the previous lemma. $\square$

7.2. Proof of Proposition 7.1.

Now, we prove Proposition 7.1. Fix a framed simple $\boldsymbol{P}$A homotopy $\{(f_\lambda, e_\lambda^u, e_\lambda^s)\}_{\lambda \in [-1,1]}$. Let Δ_h, Δ_h^0, and $N \geq 1$ be the ones in the definition of a simple $\boldsymbol{P}$A homotopy, and Σ_h, Σ_h^1, Σ_h^2, Σ_*, Θ, $\mu_p^{\pm}$, $\mu_{p_*}^*$, and I_p^- be the ones given in Section 6. Without loss of generality, we assume that all non-hyperbolic periodic point of f_0 is of type $(u,+)$ or $(s,+)$.

Lemma 7.4. *Suppose $D_{+,1}^s p \cap \theta_s(\Sigma_*) = \varnothing$ for any $p \in \Sigma_h^2$. Then, Θ is a homeomorphism onto its image.*

Proof. Recall that the map Θ is compatible. Lemmas 6.8 and 7.2 imply that it is continuous and proper. Since Θ is injective and both $\mathscr{S}(e_{-1}^u, \Delta_h^0)$ and $\mathscr{S}(e_1^u, \Delta_h^0)$ are locally compact, Θ is a homeomorphism onto Im Θ. $\square$

We consider three cases:

1. $\Sigma_* = \varnothing$
2. $\Sigma_* \neq \varnothing$ and $D_{+,1}^s \theta_n(p_*) \cap \theta_s(\Sigma_*) = \varnothing$ for any $p_* \in \Sigma_*$.
3. $\Sigma_* \neq \varnothing$ and $D_{+,1}^s \theta_n(p_*) \cap \theta_s(\Sigma_*) \neq \varnothing$ for some $p_* \in \Sigma_*$.

Remark that all non-hyperbolic periodic points of f_0 are of type $(s,+)$ in the latter two cases. In the first case, we show that the spaces $\mathscr{C}_a(e_{-1}^u, \Delta_h^0)$ and $\mathscr{C}_a(e_{-1}^u, \Delta_h^0)$ are homeomorphic. In the second case, they have the same proper homotopy type. The last is the worst case since the set $\mathscr{C}_a(e_{-1}^u)$ may be empty while $\mathscr{C}_a(e_1^u)$ is not. In this case, we need to consider the homotopy type of the compactifications of $\mathscr{C}_a(e_{-1}^u)$ and $\mathscr{C}_a(e_1^u)$. See Figure 8.

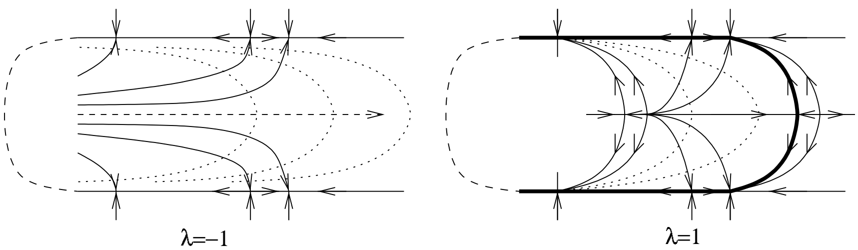

Figure 8. The worst case.

The first case. We consider the case $\Sigma_* = \varnothing$.

Lemma 7.5. *If $\Sigma_* = \varnothing$, then $\mathscr{C}_a(e_{-1}^u, \Delta_h^0)$ is homeomorphic to $\mathscr{C}_a(e_1^u, \Delta_h^0)$. In particular, their one point compactifications are homeomorphic as pointed spaces.*

PROOF. Proposition 6.3 implies Im $\Theta = \mathscr{S}(e_1^u, \Delta_h^0)$. By Lemma 7.4, Θ is a homeomorphism between $\mathscr{S}(e_{-1}^u, \Delta_h^0)$ to $\mathscr{S}(e_1^u, \Delta_h^0)$. Fix $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$. Since the restriction of Θ to $\widetilde{\mathscr{C}}_a(e_{-1}^u, \Delta_h^0)$ commutes with π_c, it induce a homeomorphism between $\mathscr{C}_a(e_{-1}^u, \Delta_h^0)$ and $\mathscr{C}_a(e_1^u, \Delta_h^0)$. □

THE SECOND CASE. Next, we consider the case that $\Sigma_* \neq \varnothing$ and $D_{+,1}^s \theta_n(p_*) \cap \theta_s(\Sigma_*) = \varnothing$ for any $p_* \in \Sigma_*$. Remark that $\Delta_h^0 = \mathrm{Fix}_h^0(f_{-1}^N) = \mathrm{Fix}_h^0(f_1^N)$, Im $\theta_\sigma = \theta_\sigma(\Sigma_*) = \mathscr{O}(\theta_\sigma(p_*); f_0)$, and $\mu_{p_*}^*(1) \in$ Im Θ for any $p_* \in \Sigma_*$ and $\sigma \in \{s, n\}$ in this case.

LEMMA 7.6. *Suppose that $\Sigma_* \neq \varnothing$ and $D_{+,1}^s \theta_n(p_*) \cap \theta_s(\Sigma_*) = \varnothing$ for any $p_* \in \Sigma_*$. Then, there exists a proper homotopy equivalence between $\mathscr{C}_a(e_1^u, \Delta_h^0)$ and $\mathscr{C}_a(e_{-1}^u, \Delta_h^0)$ for any prime homology class $a \in H_1(\boldsymbol{T}^2; \boldsymbol{Z})$. In particular, pointed spaces $(\mathscr{C}_a(e_1^u, \Delta_h^0) \cup \{\infty\}, \infty)$ and $(\mathscr{C}_a(e_{-1}^u, \Delta_h^0) \cup \{\infty\}, \infty)$ have the same homotopy type.*

PROOF. For $p \in \Sigma_h^2$ and $p_* \in \Sigma_*$ with $\theta_s(p_*) \in D_{+,1}^s p$, take a decomposition $\mu_p^-(1) = \xi_- * \xi_* * \xi_+$ in Lemma 6.9. We define maps ν_{p_*} and ν_p from $[-1,1] \times [0,1]$ to $\mathscr{S}(e_1^u, \Delta_h^0)$ by

$$\nu_{p_*}(s,t) = \mu_{p_*}^*((1-t)s + t)$$

and

$$\nu_p(s,t) = \begin{cases} \mu_p^+(s) & \text{if } s \leq 0, \\ \mu_p^+((1+2t)s) & \text{if } s > 0 \text{ and } (1+2t)s \leq 1, \\ \Theta(\xi_-) * \mu_{p_*}^*((1+2t)s - 2) * \Theta(\xi_+) & \text{if } s > 0 \text{ and } (1+2t)s > 1. \end{cases}$$

See Figure 9.

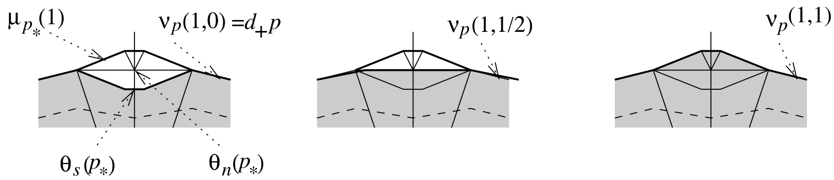

Figure 9. The map ν_p.

We define a compatible map Φ from $\mathscr{S}(e_1^u, \Delta_h^0) \times [0,1]$ to $\mathscr{S}(e_1^u, \Delta_h^0)$ as follows:

1. If $p \in \Sigma_h^1$, or $p \in \Sigma_h^2$ and $D_{+,1}^s p \cap \theta_s(\Sigma_*) = \varnothing$, then $\Phi(\mu_p^+(s), t) = \mu_p^+(s)$ for any $(s,t) \in I_p^- \times [0,1]$.
2. If $p \in \Sigma_h^2$ and $D_{+,1}^s p \cap \theta_s(\Sigma_*) \neq \varnothing$, then $\Phi(\mu_p^+(s), t) = \nu_p(s,t)$ for any $(s,t) \in I_p^- \times [0,1]$.
3. $\Phi((\theta_s(p_*))_1^s, t) = \nu_{p_*}(-1, t)$ and $\Phi(\mu_{p_*}^*(s), t) = \nu_{p_*}(s,t)$ for $p_* \in \Sigma_*$ and $(s,t) \in]-1,1[\times[0,1]$.

Put $\Theta_+ = \Phi(\cdot,1)\circ\Theta$. Notice that Φ and Θ_+ are compatible. Since $\nu_p(1,t) = \xi_- * \nu_{p_*}(-1,t) * \xi_+$ for any $t \in [-1,1]$, we can check that Φ is continuous and proper by Lemma 7.3. By Lemmas 6.9 and 7.2, the map Θ_+ is continuous and proper.

We define a compatible map Θ_- from $\mathscr{S}(e^u_1,\Delta^0_h)$ to $\mathscr{S}(e^u_{-1},\Delta^0_h)$ by

1. $\Theta_-\circ\mu^+_p(s) = \mu^-_p(s)$ for $p \in \Sigma_h$ and $s \in I^-_p$, and
2. $\Theta_-((\theta_s(p_*))^u_1) = \Theta_-(\mu^*_{p_*}(s)) = \Theta^{-1}(\mu^*_{p_*}(1))$ for $p_* \in \Sigma_*$ and $s \in]-1,1[$.

Since $\mu^*_{p_*}(1) \in \operatorname{Im}\Theta$, the map Θ_- is well-defined. It is clear that $\Theta_-\circ\mu^*_{p_*}$ is continuous on $[-1,1]$ for any $p_* \in \Sigma_*$. Lemma 6.9 implies that $\Theta_-\circ\mu^+_p$ is continuous for any $p \in \Sigma^2_h$. Hence, the map Θ_- is continuous and proper by Lemma 7.2. We can also verify that the map $(\xi,t) \mapsto \Theta_-\circ\Phi(\Theta(\xi),t)$ from $\mathscr{S}(e^u_{-1},\Delta^0_h)\times[0,1]$ to $\mathscr{S}(e^u_{-1},\Delta^0_h)$ is continuous and proper.

By their constructions, $\Theta_-\circ\Theta$ is the identity map and is homotopic to $\Theta_-\circ\Theta_+$. We can check that $\Theta_+\circ\Theta_-$ coincides with $\Phi(\cdot,1)$, and hence, is homotopic to the identity map. Therefore, Θ_+ and Θ_- are proper homotopy equivalences between $\mathscr{S}(e^u_{-1},\Delta^0_h)$ and $\mathscr{S}(e^u_1,\Delta^0_h)$. Since all maps constructed above are compatible, Θ_+ and Θ_+ induce proper homotopy equivalences between $\mathscr{C}_a(e^u_{-1},\Delta^0_h)$ and $\mathscr{C}_a(e^u_1,\Delta^0_h)$ for any prime homology class $a \in H_1(\boldsymbol{T}^2;\boldsymbol{Z})$. □

The last case. Finally, we consider the case that $D^s_{+,1}\theta_n(p_0)\cap\theta_s(\Sigma_*) \neq \varnothing$ for some $p_0 \in \Sigma_0$. Put $\mathscr{S}_*(e^u_1,\Delta^0_h) = \mathscr{S}(e^u_1,\Delta^0_h)\setminus\operatorname{Im}\Theta$.

Lemma 7.7. *$\operatorname{Im}\Theta$ and $\mathscr{S}_*(e^u_1,\Delta^0_h)$ are mutually disjoint subcomplexes of $\mathscr{S}(e^u_1,\Delta^0_h)$.*

Proof. Lemmas 6.4 and 6.5 imply $D^s_{\pm,1}\theta_n(p_*)\cap\theta_s(\Sigma_*) \neq \varnothing$ for any $p_* \in \Sigma_*$. Hence, we have $\mu^*_{p_*}(\pm1) \in \mathscr{S}_*(e^u_1,\Delta^0_h)$. It implies that $\mathscr{S}_*(e^u_1,\Delta^0_h)$ is a subcomplex of $\mathscr{S}(e^u_1,\Delta^0_h)$.

Lemmas 6.6 and 6.5 also imply $D^s_{\pm,1}\theta_n(q)\cap\theta_s(\Sigma_*) = \varnothing$ for any $q \in \Sigma^2_h$. It implies $\mu^+_p(\pm1) \in \operatorname{Im}\Theta$ for any $q \in \Sigma^2_h$, and hence, $\operatorname{Im}\Theta$ is a subcomplex of $\mathscr{S}(e^u_1,\Delta^0_h)$. □

Lemma 7.8. *There exists a continuous compatible map Ψ from $\mathscr{S}(e^u_1,\Delta^0_h)\times[0,\infty[$ to $\mathscr{S}(e^u_1,\Delta^0_h)$ such that*

- *$\Psi(\xi,t) = \xi$ for any $\xi \in \operatorname{Im}\Theta$ and $t \in [0,1]$,*
- *$\Psi(\xi_*,0) = \xi_*$ for any $\xi_* \in \mathscr{S}_*(e^u_1,\Delta^0_h)$,*
- *$\inf\{|\Psi(\xi_*,t)| \mid \xi_* \in \mathscr{S}_*(e^u_1,\Delta^0_h)\}$ tends to infinity as $t \to \infty$.*

Proof. For any $p_* \in \Sigma_*$, there exists $\xi_-,\xi_+ \in \operatorname{Im}\Theta$ and $q_* \in \Sigma_*$ such that $|\xi_-| \neq 0$, $|\xi_+| \neq 0$, and $\mu^*_{p_*}(1) = \xi_- * (\theta_s(q_*))^u_1 * \xi_+$ by Lemma 6.5. First, we define a compatible map Ψ_1 from $\mathscr{S}(e^u_1,\Delta^0_h)\times[0,1]$ to $\mathscr{S}(e^u_1,\Delta^0_h)$ by

1. $\Psi_1(\mu^+_p(s),t) = \mu^+_p(s)$ for $p \in \Sigma_h$ and $s \in I^-_p$,
2. $\Psi_1((\theta_s(p_*))^u_1,t) = \mu^*_{p_*}(2t-1)$ and

$$\Psi_1(\mu^*_{p_*}(s),t) = \begin{cases} \mu^*_{p_*}(s+2t) & (s+2t \leq 1) \\ \xi_- * \mu^*_{q_*}(s+2t-2) * \xi_+ & (s+2t > 1) \end{cases}$$

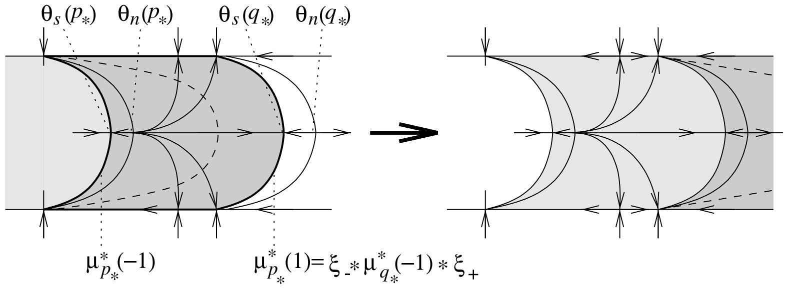

Figure 10. The map Ψ_1.

for $p_* \in \Sigma_*$, where $\mu^*_{p_*}(1) = \xi_- * (\theta_s(q_*))^u_1 * \xi_+$.

See Figure 10. It is easy to check that Ψ_1 satisfies the assumption of Lemma 7.3. Hence, it is continuous and proper. Remark that $\Psi_1(\cdot, t)$ is the identity on $\mathrm{Im}\,\Theta$ and $\Psi_1(\mathscr{S}_*(e^u_1, \Delta^0_h) \times \{t\})$ is a subset of $\mathscr{S}_*(e^u_1, \Delta^0_h)$ for any $t \in [0,1]$.

Define an integer valued function l on $\mathscr{S}(e^u_1, \Delta^0_h)$ by $l(\xi) = 0$ if $|\xi| = 0$ and $l(\xi) = k$ if ξ has the irreducible decomposition $\xi = \xi_1 * \cdots * \xi_k$ with respect to Δ^0_h. It is easy to see $l(\xi_1 * \xi_2) = l(\xi_1) + l(\xi_2)$ for any ξ_1, ξ_2 with $\xi_1(1) = \xi_2(0) \in \Delta^0_h$, and $l(\Psi_1(\xi, t)) \geq l(\xi)$ for any ξ and t. Recall that $\mu^*_{p_*}(1) = \xi_- * (\theta_s(q_*))^u_1 * \xi_+$ where $\xi_-, \xi_+ \in \mathrm{Im}\,\Theta$, $|\xi_\pm| \neq 0$, and $q_* \in \Sigma_*$. It implies $l(\Psi_1(\mu^*_{p_*}(s), 1)) = l(\xi_- * \mu^*_{q_*}(s) * \xi_+) \geq 3$ for any $p_* \in \Sigma_*$ and $s \in [-1, 1]$. In particular, we have $l(\Psi(\xi_*, 1)) \geq l(\xi_*) + 2$ for any $\xi_* \in \mathscr{S}_*(e^u_1, \Delta^0_h)$.

Put $\psi_1 = \Psi_1(\cdot, 1)$. We define a map Ψ from $\mathscr{S}(e^u_1, \Delta^0_h) \times [0, \infty[$ to $\mathscr{S}(e^u_1, \Delta^0_h)$ by $\Psi(\xi, t) = \Psi(\psi_1^{[t]}(\xi), t - [t])$, where $[t]$ is the largest integer with $[t] \leq t$. It is clear that Ψ is compatible and continuous. We also see that $\Psi(\xi, t) = \xi$ for any $\xi \in \mathrm{Im}\,\Theta$ and $t \geq 0$, and $\Psi(\xi_*, 0) = \xi_*$ and $l(\Psi(\xi_*, t)) \geq l(\xi_*) + 2[t]$ for any $\xi_* \in \mathscr{S}_*(e^u_1, \Delta^0_h)$ and $t \geq 0$.

By the finiteness of Σ_h and Σ_*, we can take $K > 0$ such that $|\xi| \geq K \cdot l(\xi)$ for any $\xi \in \mathscr{S}(e^u_1, \Delta^0_h)$. Hence, we obtain that $\inf\{|\Psi(\xi, t)| \mid \xi \in \mathscr{S}_*(e^u_1, \Delta^0_h)\}$ tends to infinity as $t \to \infty$. □

The following is the last piece of the proof of Proposition 7.1. As remarked at the beginning of this section, it completes the proof of Theorem A.

Lemma 7.9. *Suppose that $D^s_{+,1}\theta_n(p_0) \cap \theta_s(\Sigma_*) \neq \emptyset$ for some $p_0 \in \Sigma_*$. Then, pointed spaces $(\mathscr{C}_a(e^u_1, \Delta^0_h) \cup \{\infty\}, \infty)$ and $(\mathscr{C}_a(e^u_{-1}, \Delta^0_h) \cup \{\infty\}, \infty)$ have the same homotopy type.*

Proof. The map Ψ in Lemma 7.8 induces a homotopy between pointed spaces $(\mathscr{C}_a(e^u_1, \Delta^0_h) \cup \{\infty\}, \infty)$ and $(\pi_c(\mathrm{Im}\,\Theta \cap \widetilde{\mathscr{C}}_a(e^u)) \cup \{\infty\}, \infty)$ for any prime homology class $a \in H_1(\boldsymbol{T}^2; \boldsymbol{Z})$. Since Θ is a homeomorphism between $\mathscr{S}(e^u_{-1}, \Delta^0_h)$ and $\mathrm{Im}\,\Theta$ by Lemma 7.4, $\pi_c(\mathrm{Im}\,\Theta \cap \widetilde{\mathscr{C}}_a(e^u))$ and $\mathscr{C}_a(e^u_{-1}, \Delta^0_h)$ have the same proper homotopy type. □

8. Proof of Theorems C and D.

8.1. Proof of Theorem C.

Before the proof of Theorem C, we need some preparations. We say an element $c \in \mathscr{C}(e^u)$ is *injective* if any $\gamma \in \pi_c^{-1}(c)$ is injective as a map from S^1 to $\boldsymbol{T}^2$. Remark that if $\mathscr{C}_a(e^u)$ contains an injective element for $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, then the class a is prime.

Lemma 8.1. *Let (f, e^u, e^s) be a C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$. If $\mathscr{C}_a(e^u) \neq \varnothing$ for a prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, then there exist a decomposition $a = \sum_{i=1}^{l} a_i$ and a sequence $\{c_i \in \mathscr{C}_{a_i}(e^u) \cap \mathrm{Per}(\mathscr{C}(f))\}_{i=1}^{l}$ such that c_i is injective for any i.*

Proof. By Theorem B, we can take $c \in \mathscr{C}_a(e^u) \cap \mathrm{Per}(\mathscr{C}(f))$. If $\mathrm{Im}\, c \cap \mathrm{Per}(f) = \varnothing$, then $\mathrm{Im}\, c$ is an irrational normally attracting circle by Proposition 3.13. Hence, the lemma is clear in this case.

We claim that if $c \in \mathscr{C}_a(e^u, \mathrm{Fix}(f^n)) \cap \mathrm{Per}(\mathscr{C}(f))$ is not injective, then there exist $a_1, a_2 \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, $c_1 \in \mathscr{C}_{a_1}(e^u, \mathrm{Fix}(f^n)) \cap \mathrm{Per}(\mathscr{C}(f))$, and $c_2 \in \mathscr{C}_{a_2}(e^u, \mathrm{Fix}(f^n)) \cap \mathrm{Per}(\mathscr{C}(f))$ such that $a_1 + a_2 = a$ and $|c_1| + |c_2| = |c|$. Notice that the claim completes the proof. In fact, since any element of $\mathscr{C}(e^u)$ has the length larger than the injectivity radius of $\boldsymbol{T}^2$, we can obtain the required sequences by applying the claim several times for c.

Now, we prove the claim. Suppose that $c \in \mathscr{C}_a(e^u, \mathrm{Fix}(f^n)) \cap \mathrm{Per}(\mathscr{C}(f))$ is not injective. Take $\gamma = [p_0, \ldots, p_{k-1}, p_0]^u \in \pi_c^{-1}(c) \cap \widetilde{\mathscr{C}}(e^u, \mathrm{Fix}(f^n))$ and put $\Lambda = \{z \in \mathrm{Im}\, c = \mathrm{Im}\, \gamma \mid \#(\gamma^{-1}(z) \cap [0,1[) \geq 2\}$. Notice that $\mathrm{Fix}(f^n) \cap \mathrm{Im}\, \gamma = \{p_0, \ldots, p_{k-1}\}$ and Λ is a non-empty f^n-invariant set. Without loss of generality, we may assume that Λ intersects with $[p_0, p_1]^u([0,1[)$. Since either the positive or negative f^n-orbit of a point in $[p_0, p_1]^u([0,1[)$ converges to p_0, the set Λ contains p_0. In particular, we have $p_l = p_0$ for some $l \neq 0$. Put $c_1 = \pi_c([p_0, \ldots, p_l]^u)$ and $c_2 = \pi_c([p_l, \ldots, p_{k_1}, p_0]^u)$, and let a_i be the homology class represented by c_i for $i = 1, 2$. Then, a_1, a_2, c_1, and c_2 satisfy the required conditions. □

Lemma 8.2. *Let (f, e^u, e^s) be a C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$. Suppose that there exist prime homology classes $a_1, a_2 \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$ such that $a_1 \neq \pm a_2$ and $H_c^*(\mathscr{C}_{a_i}(e^u)) \neq \{0\}$ for $i = 1, 2$. Then, there exist $a_1', a_2' \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, $c_1 \in \mathscr{C}_{a_1'}(e^u) \cap \mathrm{Per}(\mathscr{C}(f))$, and $c_2 \in \mathscr{C}_{a_2'}(e^u) \cap \mathrm{Per}(\mathscr{C}(f))$ such that $a_1' \neq \pm a_2'$, and c_1 and c_2 are injective.*

Proof. For each $i = 1, 2$ take the decomposition $a_i = \sum_{j=1}^{l} a_{i,j}$ and the sequence $\{c_{i,j} \in \mathscr{C}_{a_{i,j}}(e^u)\}_{j=1}^{l}$ in Lemma 8.1. Notice that all $a_{i,j}$ are prime since all $c_{i,j}$ are injective. If $a_{i,j} \neq \pm a_{i,1}$ for some $i = 1, 2$ and $j = 2, \ldots, l$, then, $a_1' = a_{i,1}$, $a_2' = a_{i,j}$, $c_1 = c_{i,1}$, and $c_2 = c_{i,j}$ satisfy the required conditions.

Suppose $a_{i,j} = \pm a_{i,1}$ for any $i = 1, 2$ and $j = 2, \ldots, l$. Since all a_i and $a_{i,j}$ are prime, we have $a_{i,j} = \pm a_i$ for any $i = 1, 2$ and $j = 1, \ldots, l$. In particular, the assumption implies $a_{1,1} \neq \pm a_{2,1}$. Hence, $a_1' = a_{1,1}$, $a_2' = a_{2,1}$, $c_1 = c_{1,1}$, and $c_2 = c_{2,1}$ satisfy the required conditions. □

Let us prove Theorem C. Let (f, e^u, e^s) be a C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphism on $\boldsymbol{T}^2$. Suppose that there exist prime homology classes $a_1, a_2 \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$ such that $a_1 \neq \pm a_2$ and $H_c^*(\mathscr{C}_{a_i}(e^u)) \neq \{0\}$ for $i = 1, 2$.

Fix a C^1 framed $\boldsymbol{P}$A diffeomorphism (g, e_*^u, e_*^s) which is $\boldsymbol{P}$A homotopic to (f, e^u, e^s). Since the set of $\boldsymbol{P}$A diffeomorphisms is an open subset of $\mathrm{Diff}^1(\boldsymbol{T}^2)$, there exists a sequence $\{(g_n, e_n^u, e_n^s)\}$ of C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphisms which converges to (g, e^u_*, e^s_*) in the C^1-topology. By Theorem A, we may assume that $\mathscr{C}_{a_i}(e^u_n) \neq \varnothing$ for any $i = 1, 2$ and $n \geq 1$.

By Lemma 8.2, there exist sequences $\{a_{1,n}\}_{n\geq 1}$ and $\{a_{2,n}\}_{n\geq 1}$ in $H_1(\boldsymbol{T}^2, \boldsymbol{Z})$, $\{c_{1,n}\}_{n\geq 1}$ and $\{c_{2,n}\}_{n\geq 1}$ in $\mathscr{C}(e^u_n)$, and $\{m_n\}_{n\geq 1}$ of positive integers such that $a_{1,n}$ and $a_{2,n}$ are prime homology classes with $a_{1,n} \neq \pm a_{2,n}$, $c_{i,n} \in \mathscr{C}_{a_{i,n}}(e^u_n)$, and $\mathrm{Im}\, c_{i,n}$ is an $g_n^{m_n}$-invariant embedded circle for any $n \geq 1$ and $i = 1, 2$. By Lemma 5.8, if φ is a 1/80-canonical coordinate for (e^u_n, e^s_n) with $\varphi([-1/3, 1/3]^2) \cap \mathrm{Im}\, c_{i,n} \neq \varnothing$, then there exists a C^1-function h on $[-1/2, 1/2]$ such that $\varphi([-1/2, 1/2]^2) \cap \mathrm{Im}\, c_{i,n} = \varphi(\{(x, h(x)) \mid x \in [-1/2, 1/2]\})$. Hence, the same argument as the proof of Proposition 5.10 implies that a subsequence of $\{\mathrm{Im}\, c_{i,n}\}$ converges to an element C_i of $\mathscr{E}(S^1, e^u_*)$ for each $i = 1, 2$. Since $a_{1,n} \neq \pm a_{2,n}$ for any n, we obtain that C_1 and C_2 are mutually distinct but intersect with each other. It implies that E^u_g is not uniquely integrable.

8.2. Proof of Theorem D.

We prepare two lemmas to prove Theorem D.

Lemma 8.3 (A variant of the Pliss lemma, Corollary 3.1 of [**6**]). *For $\lambda, \lambda_1 \in]0,1[$ with $\lambda < \lambda_1$, suppose that a sequence $\{a_n\}_{n\geq 0}$ of positive numbers satisfies $\prod_{k=0}^n a_k \leq \lambda^n$ for any sufficiently large $n \geq 1$. Then, there exists a sequence $\{n_i\}_{i\geq 1}$ such that n_i tends to infinity as $i \to \infty$ and $\prod_{k=n_i}^n a_k \leq \lambda_1^{n-n_i}$ for any $n \geq n_i$.* □

Lemma 8.4. *Let h be a C^1 orientation preserving diffeomorphism on S^1 with $\mathrm{Per}(h) = \varnothing$. Then, $\limsup_{n\to\infty}(1/n)\log Dh^n(z) \geq 0$ for any $z \in S^1$.*

Proof. Suppose that the lemma does not hold. Then, there exist $z_0 \in S^1$, $n_0 > 0$ and $\lambda \in]0,1[$ satisfying $Dh^n(z_0) < \lambda^n$ for any $n \geq n_0$. Fix two constants λ_1, λ_2 such that $\lambda < \lambda_1 < \lambda_2 < 1$. By Lemma 8.3, there exists a sequence $\{n_i\}_{i\geq 1}$ such that n_i tends to infinity and $Dh^n(h^{n_i}(z_0)) < \lambda_1^n$ for any i and n.

Let $I(z, \delta)$ denote the interval $]z - \delta, z + \delta[$ for $z \in S^1$ and $\delta > 0$. Choose $\delta > 0$ such that $Dh(z')/Dh(z) < \lambda_2/\lambda_1$ for any $z \in S^1$ and $z' \in I(z, \delta)$. We claim that $h^n(I(h^{n_i}(z_0), \delta)) \subset I(h^{n_i+n}(z_0), \lambda_2^n \delta)$ for any $n \geq 0$ and $i \geq 1$. Proof is by induction of n. It is trivial for the case $n = 0$. Assume that $h^m(I(h^{n_i}(z_0), \delta)) \subset I(h^{n_i+m}(z_0), \lambda_2^n \delta)$ for any $m = 0, \ldots, n$. Then, we have

$$Dh^{n+1}(z) = \prod_{l=0}^{n} \frac{Dh(h^l(z))}{Dh(h^{l+n_i}(z_0))} \cdot Dh^{n+1}(h^{n_i}(z_0)) < \left(\frac{\lambda_2}{\lambda_1}\right)^{n+1} \cdot \lambda_1^{n+1} < \lambda_2^{n+1}$$

for any $z \in I(h^{n_i}(z_0), \delta)$. It implies $h^{n+1}(I(h^{n_i}(z_0), \delta)) \subset I(h^{n_i+n+1}(z_0), \lambda_2^{n+1}\delta)$.

It is known that the positive orbit of any point of S^1 converges to a Cantor set Λ which is a minimal h-invariant set (see *e.g.* Theorem 8.4 of [**7**]). Choose $i_0 \geq 1$ so that $I(h^{n_{i_0}}(z_0), \delta) \cap \Lambda \neq \varnothing$. Then, $I(h^{n_{i_0}}(z_0), \delta)$ contains infinitely many points of the positive orbit of $h^{n_{i_0}}(z_0)$. Hence, $h^m(I(h^{n_{i_0}}(z_0), \delta))$ is a subset of $I(h^{n_{i_0}}(z_0), \delta)$ for some large m. It implies the existence of periodic points, and hence, contradicts the assumption. □

Now, we prove Theorem D. Let (f, e^u, e^s) be a C^2 non-degenerate framed $\boldsymbol{P}$A diffeomorphism and suppose that there exist $k \geq 1$ and a prime homology class $a \in H_1(\boldsymbol{T}^2, \boldsymbol{Z})$

such that $H^k_c(\mathscr{C}_a(e^u)) \neq \{0\}$. Fix a C^1 framed $\boldsymbol{P}$A diffeomorphism (g, e^u_*, e^s_*) which is $\boldsymbol{P}$A homotopic to (f, e^u, e^s). Choose a sequence $\{(g_i, e^u_i, e^s_i)\}$ of C^2 framed non-degenerate $\boldsymbol{P}$A diffeomorphisms which converges to (g, e^u_*, e^s_*) in the C^1-topology. We may assume that all (g_i, e^u_i, e^s_i) are $\boldsymbol{P}$A homotopic to (f, e^u, e^s). By the continuity of the $\boldsymbol{P}$A splittings, there exists $\lambda \in]0,1[$ such that $\|Dg_i^{-n}(e^u_i(p))\|\lambda^{-n} < \|Dg_i^{-n}(e^s_i(p))\| < 1$ for any $i \geq 1$, $n \geq 1$, and $p \in \mathrm{Fix}^2_h(g_i^n)$. Fix $\lambda_1 \in]\lambda, 1[$. By Theorems A and B, there exist $c_i \in \mathscr{C}(e^u_i) \cap \mathrm{Per}(\mathscr{C}(g_i))$ and $p_i \in \mathrm{Per}^2_h(g_i)$ such that $p_i \in \mathrm{Im}\, c_i$ for every i. Lemma 8.3 implies that we may assume $\|Dg_i^{-n}(e^u_i(p_i))\| \leq \lambda_1^n$ for any i and $n \geq 1$ by replacing p_i by another point in its orbit. By Lemma 3.16 there exists a sequence $\{C_i\}$ of compact subsets of $\boldsymbol{T}^2$ such that $C_i \in \mathscr{E}(\sqcup_{<\infty} S^1, e^u_i)$, $p_i \in C_i$, $g_i(C_i) = C_i$ for any i. Proposition 5.10 implies that there exists an increasing sequence $\{i_k\}$ such that p_{i_k} converges to a point p_∞ and C_{i_k} converges to an element C_∞ of $\mathscr{E}(\sqcup_{<\infty} S^1, e^u_*)$ with $g(C_\infty) = C_\infty$. Remark that $p_\infty \in C_\infty$ and $\|Dg^{-n}(e^u_*(p_\infty))\| \leq \lambda_1^n$ for all $n \geq 1$. If C_∞ contains no periodic points, then it contradicts Lemma 8.4. If C_∞ contains exactly one periodic orbit, then C_∞ is a union of 1-normally attracting circles for g by the same argument as in Proposition 3.13. It contradicts that C_{i_k} contains a repelling periodic point p_{i_k} and converges to C_∞. Therefore, C_∞ contains at least two distinct periodic orbits.

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Masayuki Asaoka

Department of Mathematics
Kyoto University
Kyoto, 606-8502
Japan
E-mail: asaoka@math.kyoto-u.ac.jp