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Solvability of partial differential equations of nonlinear totally characteristic type with resonances

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Abstract. Let us consider the following nonlinear singular partial differential equation $(t\partial/\partial t)^m u = F(t,x,\{(t\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+\alpha\le m, j<m})$ in the complex domain. When the equation is of totally characteristic type, the author has proved with H. Chen in [**2**] the existence of the unique holomorphic solution provided that the equation satisfies the Poincaré condition and that no resonances occur. In this paper, he will solve the same equation in the case where some resonances occur.

§1. Introduction.

Notations: $(t,x)\in \boldsymbol{C}_t\times\boldsymbol{C}_x$, $\boldsymbol{N}=\{0,1,2,\dots\}$, and $\boldsymbol{N}^*=\{1,2,\dots\}$. Let $m\in\boldsymbol{N}^*$, set $N=\#\{(j,\alpha)\in\boldsymbol{N}\times\boldsymbol{N}; j+\alpha\le m, j<m\}$ (that is, $N=m(m+3)/2$), and denote the complex variables z as $z=\{z_{j,\alpha}\}_{j+\alpha\le m, j<m}\in\boldsymbol{C}^N$.

In this paper we will consider the following nonlinear singular partial differential equation:

$$\text{(E)}\qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+\alpha\le m\\ j<m}}\right),$$

where $F(t,x,z)$ is a function of the variables (t,x,z) defined in a neighborhood Δ of the origin of $\boldsymbol{C}_t\times\boldsymbol{C}_x\times\boldsymbol{C}_z^N$, and $u=u(t,x)$ is the unknown function. Set $\Delta_0=\Delta\cap\{t=0,z=0\}$. We impose the following conditions on $F(t,x,z)$:

A_1) $F(t,x,z)$ is a holomorphic function on Δ;

A_2) $F(0,x,0)\equiv 0$ on Δ_0.

Set also $I_m=\{(j,\alpha)\in\boldsymbol{N}\times\boldsymbol{N}; j+\alpha\le m, j<m\}$ and $I_m(+)=\{(j,\alpha)\in I_m;\alpha>0\}$. Then the situation is divided into the following three cases:

Case 1: $\dfrac{\partial F}{\partial z_{j,\alpha}}(0,x,0)\equiv 0$ on Δ_0 for all $(j,\alpha)\in I_m(+)$;

Case 2: $\dfrac{\partial F}{\partial z_{j,\alpha}}(0,0,0)\ne 0$ for some $(j,\alpha)\in I_m(+)$;

Case 3: the other case.

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In the case 1, equation (E) is called a *nonlinear Fuchsian type* partial differential equation and it was studied quite well in Gérard-Tahara [**3**] [**4**]. In the case 2, a kind of Grousat problem appears: Gérard-Tahara [**5**] discussed a particular class of the case 2 and proved the existence of holomorphic solutions and also singular solutions of (E).

In the case 3, equation (E) is called a *nonlinear totally characteristic type* partial differential equation. The main thema of this paper is to discuss the case 3 under the following condition:

$$\text{A}_3)\quad \frac{\partial F}{\partial z_{j,\alpha}}(0,x,0) = O(x^\alpha)\ \ (\text{as}\ x \to 0)\ \ \text{for all}\ \ (j,\alpha) \in I_m(+).$$

Under this condition, Chen-Tahara [**2**] has proved the existence of the unique holomorphic solution provided that the equation satisfies both non-resonance condition and the Poincaré condition.

Let us now recall the result in [**2**]. By the condition A_3) we have

$$\frac{\partial F}{\partial z_{j,\alpha}}(0,x,0) = x^\alpha c_{j,\alpha}(x), \quad (j,\alpha) \in I_m \tag{1.1}$$

for some holomorphic functions $c_{j,\alpha}(x)$. Set

$$L(\lambda,\rho) = \lambda^m - \sum_{\substack{j+\alpha\le m\\ j<m}} c_{j,\alpha}(0)\lambda^j \rho(\rho-1)\cdots(\rho-\alpha+1), \tag{1.2}$$

$$L_m(X) = X^m - \sum_{\substack{j+\alpha=m\\ j<m}} c_{j,\alpha}(0)X^j,$$

and denote by $c_1,\dots,c_m$ the roots of the equation $L_m(X)=0$ in X. Then our non-resonance condition and the Poincaré condition are stated as follows:

(N) (non-resonance) $L(k,l) \ne 0$ holds for any $(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}$,

(P) (Poincaré condition) $c_i \in \boldsymbol{C}\setminus[0,\infty)$ for $i=1,\dots,m$.

Note that if we factorize $L(\lambda,l)$ into the form

$$L(\lambda,l) = (\lambda-\xi_1(l))\cdots(\lambda-\xi_m(l)), \quad l \in \boldsymbol{N}, \tag{1.3}$$

by renumbering the subscript i of $\xi_i(l)$ suitably we have

$$\lim_{l\to\infty}\frac{\xi_i(l)}{l} = c_i \quad \text{for}\ i=1,\dots,m. \tag{1.4}$$

THEOREM 1 (Chen-Tahara [**2**]). *Assume* A_1), A_2) *and* A_3). *If the conditions* (N) *and* (P) *are satisfied, equation* (E) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ *satisfying* $u(0,x) \equiv 0$ *near* $x=0$.

The purpose of this paper is to solve equation (E) in the case where the Poincaré condition (P) is satisfied but the non-resonance condition (N) is not satisfied.

§2. Main result.

Let $L(\lambda, \rho)$ be the polynomial in (1.2), and let $\xi_i(l)$ $(i = 1, \dots, m)$ be as in (1.3) and (1.4). Set

$$\mathscr{M} = \{(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}; L(k,l) = 0\},$$

$$\mathscr{M}_{(i)} = \{(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}; k - \xi_i(l) = 0\} \quad (i = 1, \dots, m).$$

We have $\mathscr{M} = \mathscr{M}_{(1)} \cup \cdots \cup \mathscr{M}_{(m)}$. Note that $\mathscr{M} = \varnothing$ is equivalent to the non-resonance condition (N). In the case $\mathscr{M} \neq \varnothing$ we note:

Proposition 1. *If the Poincaré condition* (P) *is satisfied, we have the following properties.*

(1) *$\mathscr{M}$ is a finite set.*

(2) *There is a $\sigma > 0$ such that $|k - \xi_i(l)| \geq \sigma(k+l)$ holds for any $(k,l) \in (\boldsymbol{N}^* \times \boldsymbol{N}) \backslash \mathscr{M}_{(i)}$ $(i = 1, \dots, m)$.*

Proof. It is sufficient to prove the following assertion: $(*)_i$ there are $\sigma_i > 0$, $K_i > 0$ and $L_i > 0$ such that $|k - \xi_i(l)| \geq \sigma_i(k+l)$ holds for any (k,l) satisfying $k \geq K_i$ or $l \geq L_i$. Let us show this now.

Denote by $\ell(1, -c_i)$ the segment in the complex plane joining the two points 1 and $-c_i$. Let d_i be the distance from the origin to $\ell(1, -c_i)$. Since (P) is assumed, we easily see that $d_i > 0$. Since $(k - c_i l)/(k+l)$ is a point on the segment $\ell(1, -c_i)$, we have $|(k - c_i l)/(k+l)| \geq d_i$ for any $(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}$. Moreover, by (1.4) we can take an $L_i > 0$ so that $|(\xi_i(l)/l) - c_i| \leq d_i/2$ holds for any $l \geq L_i$.

If $k \in \boldsymbol{N}^*$ and $l \geq L_i$ we have

$$\begin{aligned} |k - \xi_i(l)| &\geq |k - c_i l| - |\xi_i(l) - c_i l| \\ &\geq (|(k - c_i l)/(k+l)| - |(\xi_i(l)/l) - c_i|)(k+l) \\ &\geq \left(d_i - \frac{d_i}{2}\right)(k+l) = \frac{d_i}{2}(k+l). \end{aligned} \tag{2.1}$$

If $k \in \boldsymbol{N}^*$ and $0 \leq l < L_i$, we have $|k - \xi_i(l)| \geq k - |\xi_i(l)| = (k+l)/3 + (k/3 - |\xi_i(l)|) + (k-l)/3$; therefore if we set $K_i = \max\{3|\xi_i(0)|, \dots, 3|\xi_i(L_i - 1)|, L_i - 1\}$ we obtain

$$|k - \xi_i(l)| \geq \frac{1}{3}(k+l) \quad \text{for } k \geq K_i \text{ and } 0 \leq l < L_i. \tag{2.2}$$

(2.1) and (2.2) complete the proof of the assertion $(*)_i$. □

For $(k,l) \in \mathscr{M}$ we set $\mu(k,l) = \#\{i; \xi_i(l) = k\}$ and we say that $\mu(k,l)$ is the *multiplicity of resonance* of $L(\lambda, \rho)$ at (k,l). We denote by μ the total number of the multiplicities of resonances, that is,

$$\mu = \sum_{(k,l) \in \mathscr{M}} \mu(k,l). \tag{2.3}$$

The following is the main theorem of this paper.

Theorem 2 (when resonances occur). *Assume* A_1), A_2), A_3), (P) *and* $\mathscr{M} \neq \emptyset$. *Then, equation* (E) *has a family of solutions* $u(t,x)$ *of the form*

$$u(t,x) = w(t, t(\log t), t(\log t)^2, \ldots, t(\log t)^\mu, x) \tag{2.4}$$

where $w(t_0, t_1, \ldots, t_\mu, x)$ *is a holomorphic function in a neighborhood of the origin* $(t_0, t_1, \ldots, t_\mu, x) = (0, 0, \ldots, 0, 0)$ *satisfying* $w(0, 0, \ldots, 0, x) \equiv 0$ *near* $x = 0$; *moreover* $w(t_0, t_1, \ldots, t_\mu, x)$ *contains* μ *arbitrary constants.*

Now, in order to look for a solution of the form (2.4) we set

$$t_0 = t, \quad t_1 = t(\log t), \ \ldots, \quad t_\mu = t(\log t)^\mu \tag{2.5}$$

and suppose that $u(t,x)$ is expressed in the form (2.4) for some holomorphic function $w(t_0, t_1, \ldots, t_\mu, x)$. We have

$$t\frac{\partial u}{\partial t} = \tau w$$

where τ is the vector field defined by

$$\tau = \sum_{i=0}^{\mu} t_i \frac{\partial}{\partial t_i} + \sum_{i=1}^{\mu} i t_{i-1} \frac{\partial}{\partial t_i}. \tag{2.6}$$

If we could find a holomorphic solution $w(t_0, t_1, \ldots, t_\mu, x)$ of the equation

$$\tau^m w = F\left(t_0, x, \left\{\left(\tau^j \left(\frac{\partial}{\partial x}\right)^\alpha w\right\}_{(j,\alpha) \in I_m}\right), \tag{2.7}$$

Theorem 2 is straightforward; though, to solve (2.7) is impossible in general. By this reason, we will consider equation (2.7) in a modulo class as follows.

Denote by $\boldsymbol{C}[t_0, t_1, \ldots, t_\mu, x]$ the ring of polynomials in $(t_0, t_1, \ldots, t_\mu, x)$ with coefficients in $\boldsymbol{C}$, and set

$$\mathscr{R} = \sum_{i+j=p+q} \boldsymbol{C}[t_0, t_1, \ldots, t_\mu, x](t_i t_j - t_p t_q).$$

For $f(t_0, t_1, \ldots, t_\mu, x)$ and $g(t_0, t_1, \ldots, t_\mu, x)$, we denote by $f \equiv g \pmod{\mathscr{R}}$ if $f - g \in \mathscr{R}$ holds. It is clear that if $f(t_0, t_1, \ldots, t_\mu, x) \in \mathscr{R}$ we have

$$f(t, t(\log t), t(\log t)^2, \ldots, t(\log t)^\mu, x) \equiv 0$$

as a function of (t,x). Therefore, to get a solution $u(t,x)$ of the form (2.4) it is sufficient to consider the following equation with respect to w:

$$\tau^m w \equiv F\left(t_0, x, \left\{\tau^j \left(\frac{\partial}{\partial x}\right)^\alpha w\right\}_{(j,\alpha) \in I_m}\right) \pmod{\mathscr{R}}. \tag{2.8}$$

Thus our target is to prove the following theorem:

Theorem 3 (result on (2.8)). *Assume* A_1), A_2), A_3), (P) *and* $\mathscr{M} \neq \emptyset$. *Then, equation* (2.8) *has a family of solutions* $w(t_0, t_1, \ldots, t_\mu, x)$ *holomorphic in a neighborhood* *of the origin* $(t_0, t_1, \dots, t_\mu, x) = (0, 0, \dots, 0, 0)$ *satisfying* $w(0, 0, \dots, 0, x) \equiv 0$ *near* $x = 0$ *and with* μ *arbitrary constants.*

The proof of this theorem will be given in sections 4 and 5; in section 4 we will construct a formal solution $w(t_0, t_1, \dots, t_\mu, x)$ and in section 5 we will prove the convergence of the formal solution. In the next section 3 we will present some lemmas which are needed in the proof of theorem 3.

§3. Some lemmas.

For $k \in \boldsymbol{N}$ and $0 \le d \le \mu$, we denote by $H_k[t_0, t_1, \dots, t_d]$ the set of all the homogeneous polynomials of degree k in $(t_0, t_1, \dots, t_d)$. It is easy to see:

Lemma 1. *If* $k \ge 1$, *the set* $H_k[t_0, t_1, \dots, t_d]$ *is decomposed into*

$$H_k[t_0, t_1, \dots, t_d] = \sum_{0 \le i \le d} H_{k-1}[t_0, t_1, \dots, t_i] t_i.$$

For $\vec{k} = (k_0, k_1, \dots, k_d) \in \boldsymbol{N}^{d+1}$ we write $|\vec{k}| = k_0 + k_1 + \cdots + k_d$ and $\langle \vec{k} \rangle = k_1 + 2k_2 + \cdots + dk_d$. Let $c > 0$ and let us define the norm $|w|_c$ of

$$w = \sum_{|\vec{k}|=k} w_{\vec{k}} t_0^{k_0} t_1^{k_1} \cdots t_d^{k_d} \in H_k[t_0, t_1, \dots, t_d] \tag{3.1}$$

by

$$|w|_c = \sum_{|\vec{k}|=k} \frac{|w_{\vec{k}}|}{c^{\langle \vec{k} \rangle}}. \tag{3.2}$$

Lemma 2. *For the vector field* τ *in* (2.6) *and* $w \in H_k[t_0, t_1, \dots, t_d]$ *we have*

$$|\tau w|_c \le (1 + cd) k |w|_c. \tag{3.3}$$

Proof. If w is expressed in the form (3.1) we have

$$\begin{aligned} \tau w = &\sum_{|\vec{k}|=k} (k_0 + k_1 + \cdots + k_d) w_{\vec{k}} t_0^{k_0} t_1^{k_1} \cdots t_d^{k_d} \\ &+ \sum_{|\vec{k}|=k} \left[w_{\vec{k}} \left(\sum_{1 \le i \le d} i k_i t_0^{k_0} \cdots t_{i-1}^{k_{i-1}+1} t_i^{k_i - 1} \cdots t_d^{k_d} \right) \right] \end{aligned} \tag{3.4}$$

and therefore by the definition of the norm we see

$$\begin{aligned} |\tau w|_c &\le k|w|_c + \sum_{|\vec{k}|=k} |w_{\vec{k}}| \sum_{1 \le i \le d} \frac{i k_i}{c^{\langle \vec{k} \rangle - 1}} \\ &\le k|w|_c + \sum_{|\vec{k}|=k} |w_{\vec{k}}| \frac{d(k_1 + \cdots + k_d)}{c^{\langle \vec{k} \rangle - 1}} \le k|w|_c + cdk|w|_c \end{aligned}$$

which completes the proof of Lemma 2. □

Next, let τ be the vector field in (2.6) and let us consider the following equation

$$(\tau - \xi)w = g \in H_k[t_0, t_1, \dots, t_d] \tag{3.5}$$

with $\xi \in \boldsymbol{C}$. The non-resonant case corresponds to the case $\xi \neq k$, and the resonant case corresponds to the case $\xi = k$. We have

Lemma 3 (when $\xi \neq k$). *Let $k \geq 1$, $0 \leq d \leq \mu$ and $g \in H_k[t_0, t_1, \dots, t_d]$. If $\xi \neq k$, equation* (3.5) *has a unique solution $w \in H_k[t_0, t_1, \dots, t_d]$; moreover we have the estimate*

$$|w|_c \leq \frac{(2/\sigma)}{(k+l)}|g|_c$$

for any σ and c with $|k - \xi| \geq \sigma(k+l)$ and $0 < c \leq \sigma/2d$.

Proof. Since $\xi \neq k$ is assumed, the former part is verified by a simple calculation. Moreover by the same argument as in the proof of Lemma 2 we have

$$\begin{aligned} |g|_c &= |\tau w - \xi w| \geq |k - \xi|\,|w|_c - cdk|w|_c \\ &\geq \sigma(k+l)|w|_c - (\sigma/2d)dk|w|_c \geq (\sigma/2)(k+l)|w|_c; \end{aligned}$$

this proves the latter part. □

Let $L(\lambda, \rho)$ be the polynomial in (1.2), let $\xi_i(l)$ $(i = 1, \dots, m)$ be as in (1.3), and let us consider the following equation

$$L(\tau, l)w = g \in H_k[t_0, t_1, \dots, t_d]. \tag{3.6}$$

Since (3.6) is decomposed into

$$(\tau - \xi_1(l)) \cdots (\tau - \xi_m(l))w = g,$$

applying Lemma 3 m-times we obtain

Corollary 1. *Let $(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}$, $0 \leq d \leq \mu$ and $g \in H_k[t_0, t_1, \dots, t_d]$. Assume the Poincaré condition* (P) *and let σ be the constant in Proposition* 1. *If $(k,l) \notin \mathscr{M}$, equation* (3.6) *has a unique solution $w \in H_k[t_0, t_1, \dots, t_d]$; moreover we have the estimate*

$$|w|_c \leq \frac{(2/\sigma)^m}{(k+l)^m}|g|_c$$

for any c with $0 < c \leq \sigma/2d$.

Now let us consider equation (3.5) in the resonant case (that is, $\xi = k$). In this case, to solve (3.5) exactly in $H_k[t_0, t_1, \dots, t_d]$ is impossible in general and so we will employ the following idea: we introduce a new variable t_{d+1} and find a solution $w \in H_k[t_0, t_1, \dots, t_d, t_{d+1}]$ in the following modulo class:

$$(\tau - \xi)w \equiv g \pmod{\mathscr{R}_k^0} \tag{3.7}$$

where $\mathscr{R}_0^0 = \{0\}$, $\mathscr{R}_1^0 = \{0\}$ and $\mathscr{R}_k^0$ (for $k \geq 2$) is defined by

$$\mathscr{R}_k^0 = \sum_{i+j=p+q} H_{k-2}[t_0, t_1, \dots, t_\mu](t_i t_j - t_p t_q).$$

Lemma 4 (when $\xi = k$). *Let* $k \geq 1$, $0 \leq d \leq \mu - 1$ *and* $g \in H_k[t_0, t_1, \dots, t_d]$. *If* $\xi = k$, *equation* (3.7) *has a family of solutions* $w \in H_k[t_0, t_1, \dots, t_d, t_{d+1}]$ *of the form*

$$w = At_0^k + \sum_{0 \leq i \leq d} \left[\sum_{|\vec{h}|=k-1} w_{i,\vec{h}} t_0^{h_0} t_1^{h_1} \cdots t_i^{h_i} \right] t_{i+1} \tag{3.8}$$

with an arbitrary constant $A \in \boldsymbol{C}$ (*where* $\vec{h} = (h_0, h_1, \dots, h_i)$ *and* $|\vec{h}| = h_0 + h_1 + \cdots + h_i$); *moreover we have the following estimate for any* $c > 0$:

$$|w|_c \leq |A| + \frac{1}{c} |g|_c. \tag{3.9}$$

Proof. If w is expressed in the form (3.8) and if $\xi = k$, we have

$$(\tau - \xi)w = \sum_{0 \leq i \leq d} \sum_{|\vec{h}|=k-1} w_{i,\vec{h}} \left(\sum_{1 \leq j \leq i} jh_j t_0^{h_0} \cdots t_{j-1}^{h_{j-1}+1} t_j^{h_j - 1} \cdots t_i^{h_i} t_{i+1} + (i+1) t_0^{h_0} t_1^{h_1} \cdots t_i^{h_i + 1} \right).$$

Since $t_{j-1} t_{i+1} \equiv t_j t_i \pmod{\mathscr{R}_2^0}$, we see that

$$(\tau - \xi)w \equiv \sum_{0 \leq i \leq d} \left[\sum_{|\vec{h}|=k-1} w_{i,\vec{h}} \left(\sum_{1 \leq j \leq i} jh_j + (i+1) \right) t_0^{h_0} t_1^{h_1} \cdots t_i^{h_i} \right] t_i.$$

Hence if we express $g \in H_k[t_0, t_1, \dots, t_d]$ in the form (by Lemma 1)

$$g = \sum_{0 \leq i \leq d} \left[\sum_{|\vec{h}|=k-1} g_{i,\vec{h}} t_0^{h_0} t_1^{h_1} \cdots t_i^{h_i} \right] t_i \tag{3.10}$$

(where $\vec{h} = (h_0, h_1, \dots, h_i)$), by taking A arbitrarily and by setting

$$w_{i,\vec{h}} = \frac{g_{i,\vec{h}}}{\langle \vec{h} \rangle + (i+1)} \quad (0 \leq i \leq d, |\vec{h}| = k-1)$$

we get a solution of the form (3.8). The estimate (3.9) is verified as follows:

$$\begin{aligned} |w|_c &= |A| + \sum_{0 \leq i \leq d} \sum_{|\vec{h}|=k-1} \frac{|w_{i,\vec{h}}|}{c^{\langle \vec{h} \rangle + (i+1)}} \\ &\leq |A| + \sum_{0 \leq i \leq d} \sum_{|\vec{h}|=k-1} \frac{|g_{i,\vec{h}}|}{c^{\langle \vec{h} \rangle + (i+1)}} = |A| + \frac{1}{c} |g|_c. \end{aligned}$$ □

In the resonant case, our equation (3.6) should be considered in the modulo class as follows:

$$L(\tau, l)w \equiv g \pmod{\mathscr{R}_k^0}. \tag{3.11}$$

Corollary 2. *Assume the Poincaré condition* (P) *and* $(k, l) \in \mathscr{M}$. *Let* σ *be the constant in Proposition* 1, $\mu(k, l)$ *be the multiplicity of resonance at* (k, l), $0 \leq d \leq \mu - \mu(k, l)$ *and* $g \in H_k[t_0, t_1, \dots, t_d]$. *Then, equation* (3.11) *has a family of solutions* $w \in H_k[t_0, t_1, \dots, t_d, t_{d+1}, \dots, t_{d+\mu(k,l)}]$ *of the form*

$$w = \sum_{n=0}^{\mu(k,l)-1} A_n t_0^{k-1} t_n + \sum_{0 \le i \le d} \left[\sum_{|\vec{h}|=k-1} w_{i,\vec{h}} t_0^{h_0} t_1^{h_1} \cdots t_i^{h_i} \right] t_{i+\mu(k,l)} \tag{3.12}$$

with $\mu(k,l)$ arbitrary constants $A_0, \dots, A_{\mu(k,l)-1}$. Moreover we have the estimate

$$|w|_c \le \sum_{n=0}^{\mu(k,l)-1} \frac{|A_n|}{c^n} + \frac{1}{c^{\mu(k,l)}\mu(k,l)!} \frac{(2/\sigma)^{m-\mu(k,l)}}{(k+l)^{m-\mu(k,l)}} |g|_c \tag{3.13}$$

for any c with $0 < c \le \sigma/2d$.

Proof. (3.11) is written in the form

$$(\tau - \xi_m(l)) \cdots (\tau - \xi_{\mu(k,l)}(l)) \cdots (\tau - \xi_1(l)) w \equiv g, \tag{3.14}$$

and without loss of generality we may assume that

$$\begin{cases} \xi_i(l) = k, & \text{for } i = 1, \dots, \mu(k,l), \\ \xi_i(l) \ne k, & \text{for } i = \mu(k,l)+1, \dots, m. \end{cases}$$

By Lemma 3 we have a unique solution $W \in H_k[t_0, t_1, \dots, t_d]$ of the equation

$$(\tau - \xi_m(l)) \cdots (\tau - \xi_{\mu(k,l)+1}(l)) W = g$$

and have the estimate

$$|W|_c \le \frac{(2/\sigma)^{m-\mu(k,l)}}{(k+l)^{m-\mu(k,l)}} |g|_c$$

for any c with $0 < c \le \sigma/2d$.

Then, to get a solution w of (3.11) we have to solve the equation

$$(\tau - \xi_{\mu(k,l)}(l)) \cdots (\tau - \xi_1(l)) w \equiv W \pmod{\mathscr{R}_k^0}. \tag{3.15}$$

Since $\xi_i(l) = k$ for $i = 1, \dots, \mu(k,l)$ we can apply Lemma 4 to (3.15): if W is expressed in the form (3.10) with g replaced by W, then by taking $A_0, \dots, A_{\mu(k,l)-1}$ arbitrarily and by setting

$$w_{i,\vec{h}} = \frac{W_{i,\vec{h}}}{(\langle \vec{h} \rangle + i + 1) \cdots (\langle \vec{h} \rangle + i + \mu(k,l))}$$

we get a solution of (3.15) of the form (3.12). Since $|w_{i,\vec{h}}| \le |W_{i,\vec{h}}|/\mu(k,l)!$ holds we have the estimate

$$|w|_c \le \sum_{n=0}^{\mu(k,l)-1} \frac{|A_n|}{c^n} + \frac{1}{c^{\mu(k,l)}\mu(k,l)!} |W|_c$$

for any $c > 0$. This completes the proof of Corollary 2. □

Lastly, let us give a lemma which will play an important role in the proof of Theorem 3. For $k \in \boldsymbol{N}$ we denote by $H_k[t_0, t_1, \dots, t_d][[x]]$ the set of all the formal power series in x with coefficients in $H_k[t_0, t_1, \dots, t_d]$. For $c > 0$, $\rho > 0$ and a function

$$f = \sum_{l \geq 0} f_{k,l}(t_0, t_1, \ldots, t_d) x^l \in H_k[t_0, t_1, \ldots, t_d][[x]]$$

we define the norm $\|f\|_{c,\rho}$ (or the formal norm $\|f\|_{c,\rho}$) by

$$\|f\|_{c,\rho} = \sum_{l \geq 0} |f_{k,l}|_c \rho^l. \tag{3.16}$$

Similarly, for $\rho > 0$ and $f(x) = \sum_{l \geq 0} f_l x^l \in \boldsymbol{C}[[x]]$ (the ring of formal power series in x) we define the norm $\|f\|_\rho$ (or the formal norm $\|f\|_\rho$) by

$$\|f\|_\rho = \sum_{l \geq 0} |f_l| \rho^l. \tag{3.17}$$

Note that we can regard (3.17) as a particular case $k = 0$ of (3.16).

Lemma 5. *Let $c > 0$ and $R > 0$ be fixed. If the estimate*

$$\|f\|_{c,\rho} \leq \frac{C}{(R-\rho)^a} \quad \textit{for any } 0 < \rho < R$$

holds for some $C > 0$ and $a \geq 0$, we have

$$\left\| \frac{\partial f}{\partial x} \right\|_{c,\rho} \leq \frac{(a+1)eC}{(R-\rho)^{a+1}} \quad \textit{for any } 0 < \rho < R.$$

Proof. Since

$$l\rho^{l-1} = \frac{l\rho^{l-1}h}{h} \leq \frac{(\rho+h)^l}{h}$$

holds for any $l \geq 1$, $\rho > 0$ and $h > 0$, we have

$$\left\| \frac{\partial f}{\partial x} \right\|_{c,\rho} = \sum_{l \geq 1} |f_{k,l}|_c l\rho^{l-1} \leq \sum_{l \geq 1} |f_{k,l}|_c \frac{(\rho+h)^l}{h} \leq \frac{1}{h} \|f\|_{c,\rho+h};$$

therefore, by the assumption we have

$$\left\| \frac{\partial f}{\partial x} \right\|_{c,\rho} \leq \frac{1}{h} \frac{C}{(R-\rho-h)^a} \tag{3.18}$$

for any $0 < h < R - \rho$.

When $a = 0$, by (3.18) we have $\|\partial f/\partial x\|_{c,\rho} \leq C/h$ and by letting $h \to R - \rho$ we obtain

$$\left\| \frac{\partial f}{\partial x} \right\|_{c,\rho} \leq \frac{C}{R-\rho} \leq \frac{eC}{R-\rho}.$$

When $a > 0$, we take $h = (R-\rho)/(a+1)$; then by (3.18) we have

$$\left\|\frac{\partial f}{\partial x}\right\|_{c,\rho} \le \frac{1}{h}\frac{C}{(R-\rho-h)^a} = \frac{a+1}{R-\rho} \times \frac{C}{\left((R-\rho) - \dfrac{(R-\rho)}{a+1}\right)^a}$$

$$= \frac{(a+1)C}{(R-\rho)^{a+1}} \times \left(1+\frac{1}{a}\right)^a \le \frac{(a+1)eC}{(R-\rho)^{a+1}}. \qquad \square$$

§4. Construction of a formal solution.

In this section we will construct a formal solution of equation (2.8).

Set $S = \{k \in \boldsymbol{N}^*; L(k,l) = 0 \text{ for some } l \in \boldsymbol{N}\}$, and $S(k) = \{l \in \boldsymbol{N}; L(k,l) = 0\}$ for $k \in S$. Since (P) is assumed, S and $S(k)$ are finite sets and so we can write $S = \{k_1, k_2, \dots, k_q\}$ where $1 \le k_1 < k_2 < \cdots < k_q < \infty$, and for $k = k_i$ $(i = 1, \dots, q)$ we can write $S(k_i) = \{l_{i,1}, l_{i,2}, \dots, l_{i,J(i)}\}$ where $0 \le l_{i,1} < l_{i,2} < \cdots < l_{i,J(i)} < \infty$. For simplicity we denote by $\mu_{i,j}$ the multiplicity of resonance of $L(\lambda,\rho)$ at $(k_i, l_{i,j})$ and set $\mu_i = \mu_{i,1} + \cdots + \mu_{i,J(i)}$ for $i = 1, \dots, q$. Note that the number μ in (2.3) is given by $\mu = \mu_1 + \cdots + \mu_q$.

Let us define a sequence $\{N(k); k \in \boldsymbol{N}\}$ by the following:

$$N(k) = \begin{cases} 0, & \text{if } 0 \le k < k_1, \\ \mu_1, & \text{if } k_1 \le k < k_2, \\ \mu_1 + \mu_2, & \text{if } k_2 \le k < k_3, \\ \cdots & \cdots \\ \mu_1 + \cdots + \mu_{q-1}, & \text{if } k_{q-1} \le k < k_q, \\ \mu_1 + \cdots + \mu_{q-1} + \mu_q, & \text{if } k_q \le k < \infty. \end{cases}$$

Note that $N(k) = \mu$ holds for all $k \ge k_q$. Similarly let us define a sequence $\{d(k,l); (k,l) \in \boldsymbol{N}^* \times (\{-1\} \cup \boldsymbol{N})\}$ by the following (i) and (ii): (i) if $k \ne k_1, \dots, k_q$ we set $d(k,l) = N(k-1)$ for all $l \in \{-1\} \cup \boldsymbol{N}$; (ii) if $k = k_i$ we set

$$d(k,l) = \begin{cases} N(k-1), & \text{if } -1 \le l < l_{i,1}, \\ N(k-1) + \mu_{i,1}, & \text{if } l_{i,1} \le l < l_{i,2}, \\ N(k-1) + \mu_{i,1} + \mu_{i,2}, & \text{if } l_{i,2} \le l < l_{i,3}, \\ \cdots\cdots & \cdots\cdots \\ N(k-1) + \mu_{i,1} + \cdots + \mu_{i,J(i)-1}, & \text{if } l_{i,J(i)-1} \le l < l_{i,J(i)}, \\ N(k-1) + \mu_{i,1} + \cdots + \mu_{i,J(i)-1} + \mu_{i,J(i)}, & \text{if } l_{i,J(i)} \le l < \infty. \end{cases}$$

It is easy to see

Lemma 6. (1) *$d(k,l) \le N(k)$ holds for all $l \in \boldsymbol{N}$.*
(2) *$d(k,l) = N(k)$ holds for sufficiently large $l \in \boldsymbol{N}$.*
(3) *If $(k,l) \notin \mathscr{M}$ we have $d(k,l) = d(k,l-1)$.*
(4) *If $(k,l) \in \mathscr{M}$ we have $d(k,l) = d(k,l-1) + \mu(k,l)$.*

Now, let us prove

Proposition 2. *Equation* (2.8) *has a formal solution w of the form*

$$w = \sum_{k \ge 1, l \ge 0} w_{k,l} x^l \tag{4.1}$$

with

$$w_{k,l} = w_{k,l}(t_0, \dots, t_{d(k,l)}) \in H_k[t_0, \dots, t_{d(k,l)}] \quad ((k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}); \tag{4.2}$$

moreover we may assume that if $(k,l) \in \mathscr{M}$ *the polynomial* $w_{k,l}(t_0, \dots, t_{d(k,l)})$ *contains* $\mu(k,l)$ *arbitrary constants.*

Proof. By the Taylor expansion of $F(t,x,z)$ in (t,z) and by the conditions A_1), A_2), A_3) we see that $F(t,x,z)$ is expressed in the form

$$F(t,x,z) = a(x)t + \sum_{(j,\alpha) \in I_m} x^\alpha c_{j,\alpha}(x) z_{j,\alpha} + \sum_{p+|\nu| \ge 2} g_{p,\nu}(x) t^p z^\nu,$$

where $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha) \in I_m} \in \boldsymbol{N}^N$, $|\nu| = \sum_{(j,\alpha) \in I_m} \nu_{j,\alpha}$, $z^\nu = \prod_{(j,\alpha) \in I_m} (z_{j,\alpha})^{\nu_{j,\alpha}}$ and all the coefficients $a(x), c_{j,\alpha}(x), g_{p,\nu}(x)$ are holomorphic functions on Δ_0.

For simplicity, we write

$$C(x,\lambda,\rho) = \lambda^m - \sum_{(j,\alpha) \in I_m} c_{j,\alpha}(x) \lambda^j \rho(\rho - 1) \cdots (\rho - \alpha + 1),$$

$$Dw = \{D_{j,\alpha} w\}_{(j,\alpha) \in I_m}, \quad \text{and} \quad D_{j,\alpha} w = \tau^j \left(\frac{\partial}{\partial x}\right)^\alpha w.$$

Then equation (2.8) is written in the form

$$C\left(x, \tau, x\frac{\partial}{\partial x}\right) w \equiv a(x) t_0 + \sum_{p+|\nu| \ge 2} g_{p,\nu}(x) t_0^p (Dw)^\nu \pmod{\mathscr{R}}. \tag{4.3}$$

Let w be a formal solution of the form (4.1) with (4.2), and set

$$w_k = \sum_{l \ge 0} w_{k,l}(t_0, \dots, t_{d(k,l)}) x^l \in H_k[t_0, \dots, t_{N(k)}][[x]] \quad (k \ge 1); \tag{4.4}$$

then we have $w = \sum_{k \ge 1} w_k$. By substituting this into (4.3) and by comparing the homogeneous part of degree k with respect to $t_0, t_1, \dots, t_{N(k)}$ in both sides of (4.3) we see that equation (4.3) is decomposed into the following recursive family:

$$C\left(x, \tau, x\frac{\partial}{\partial x}\right) w_k \equiv f_k \pmod{\mathscr{R}} \tag*{$(4.5)_k$}$$

for $k \ge 1$, where $f_1 = a(x) t_0$ and f_k (for $k \ge 2$) is a polynomial of $\{D_{j,\alpha} w_p; 1 \le p \le k-1, (j,\alpha) \in I_m\}$; precisely, it is given by

$$f_k = \sum_{2 \le p+|\nu| \le k} g_{p,\nu}(x) t_0^p \left[\sum_{|k^*|=k-p} \left(\prod_{(j,\alpha) \in I_m} (D_{j,\alpha} w_{k_{j,\alpha}(1)}) \times \cdots \times (D_{j,\alpha} w_{k_{j,\alpha}(\nu_{j,\alpha})})\right)\right] \tag{4.6}$$

where $|k^*| = \sum_{(j,\alpha) \in I_m} (k_{j,\alpha}(1) + \cdots + k_{j,\alpha}(\nu_{j,\alpha}))$.

Moreover, by (4.4) we see also that equation $(4.5)_k$ is decomposed into the following recursive family:

$$(4.7)_{k,l} \qquad L(\tau,l)w_{k,l} \equiv g_{k,l} \pmod{\mathscr{R}_k^0}$$

with

$$(4.8) \qquad g_{k,l} = \sum_{(j,\alpha)\in I_m} \sum_{h=0}^{l-1} c_{j,\alpha,l-h}\tau^j [h]_\alpha w_{k,h} + \phi_{k,l},$$

where $c_{j,\alpha,l}$ are the coefficients of the Taylor expansion $c_{j,\alpha}(x) = \sum_{l\geq 0} c_{j,\alpha,l}x^l$, $[\lambda]_0 = 1$, $[\lambda]_\alpha = \lambda(\lambda-1)\cdots(\lambda-\alpha+1)$ for $\alpha \geq 1$, and $\phi_{k,l}$ are the coefficients of

$$(4.9) \qquad f_k = \sum_{l\geq 0} \phi_{k,l}(t_0,t_1,\ldots,t_{N(k-1)})x^l \in H_k[t_0,t_1,\ldots,t_{N(k-1)}][[x]]$$

which is determined by $w_1,\ldots,w_{k-1}$ provided that $w_1,\ldots,w_{k-1}$ are of the form (4.4).

Note the following proposition.

Proposition 3. *Let σ be the constant in Proposition* 1, *and let μ be as in* (2.3). *Then, in the above context we have*:

(1) *If $w_1,\ldots,w_{k-1}$ are of the form* (4.4), *then f_k is expressed in the form* (4.9).

(2) *If $w_1,\ldots,w_{k-1}$ are of the form* (4.4) *and if $w_{k,q} \in H_k[t_0,\ldots,t_{d(k,l-1)}]$ for all $q < l$, we have $g_{k,l} \in H_k[t_0,\ldots,t_{d(k,l-1)}]$.*

(3) *If $(k,l) \notin \mathscr{M}$ and if $g_{k,l} \in H_k[t_0,\ldots,t_{d(k,l-1)}]$, equation $(4.7)_{k,l}$ has a unique exact solution $w_{k,l} \in H_k[t_0,\ldots,t_{d(k,l)}]$ and it satisfies*

$$(4.10) \qquad |w_{k,l}|_c \leq \frac{(2/\sigma)^m}{(k+l)^m} |g_{k,l}|_c$$

for any c with $0 < c \leq \sigma/2\mu$. Here an "exact solution" means that $w_{k,l}$ satisfies $L(\tau,l)w_{k,l} = g_{k,l}$ exactly.

(4) *If $(k,l) \in \mathscr{M}$ and if $g_{k,l} \in H_k[t_0,\ldots,t_{d(k,l-1)}]$, equation $(4.7)_{k,l}$ has a solution $w_{k,l} \in H_k[t_0,\ldots,t_{d(k,l)}]$ with $\mu(k,l)$ arbitrary constants $A(k,l;n)$ $(n = 0,1,\ldots,\mu(k,l)-1)$. Moreover, we have*

$$(4.11) \qquad |w_{k,l}|_c \leq \sum_{n=0}^{\mu(k,l)-1} \frac{|A(k,l;n)|}{c^n} + \frac{1}{c^{\mu(k,l)}\mu(k,l)!} \frac{(2/\sigma)^{m-\mu(k,l)}}{(k+l)^{m-\mu(k,l)}} |g_{k,l}|_c$$

for any c with $0 < c \leq \sigma/2\mu$.

Proof. (1) and (2) are clear from the definition of the sequences $N(k)$ and $d(k,l)$. By Lemma 6 we see: if $(k,l) \notin \mathscr{M}$ we have $d(k,l) = d(k,l-1)$; if $(k,l) \in \mathscr{M}$ we have $d(k,l) = d(k,l-1) + \mu(k,l)$. Therefore, (3) and (4) are consequences of Corollaries 1 and 2. □

Thus, to get a formal solution w in the form (4.1) with (4.2), we have only to apply Proposition 3 to $(4.7)_{k,l}$ inductively on (k,l) in the following way. i) First we solve $(4.7)_{1,0}$, then we solve $(4.7)_{1,l}$ inductively on l, and by (4.4) we obtain w_1. ii) If $w_1, \dots, w_{k-1}$ are already constructed, we solve $(4.7)_{k,0}$, then we solve $(4.7)_{k,l}$ inductively on l, and we obtain w_k. iii) Repeating the same procedure, we can obtain a formal solution of (2.8) with μ arbitrary constants.

This completes the proof of Proposition 2. □

§5. Convergence of the formal solution.

In this section we will prove the convergence of the formal solution constructed in section 4.

Let

$$w = \sum_{k \geq 1, l \geq 0} w_{k,l} x^l \tag{5.1}$$

be the formal solution in Proposition 2, and let $w_k, f_k, g_{k,l}$ and $\phi_{k,l}$ be as in (4.4), (4.6), (4.8) and (4.9), respectively.

Set $c = \sigma/2\mu$, and take $C > 0$ so that $C \geq (2/\sigma)^m$ and

$$C \geq \frac{(k+l)^{\mu(k,l)} (2/\sigma)^{m-\mu(k,l)}}{c^{\mu(k,l)} \mu(k,l)!} \quad \text{for any } (k,l) \in \mathscr{M} \tag{5.2}$$

(recall that $\mathscr{M}$ is a finite set). From (4.10) and (4.11) we have the estimate

$$|w_{k,l}|_c \leq \begin{cases} \dfrac{C}{(k+l)^m} |g_{k,l}|_c, & \text{when } (k,l) \notin \mathscr{M}, \\ \displaystyle\sum_{n=0}^{\mu(k,l)-1} \frac{|A(k,l;n)|}{c^n} + \frac{C}{(k+l)^m} |g_{k,l}|_c, & \text{when } (k,l) \in \mathscr{M}. \end{cases} \tag{5.3}$$

Take an $R > 0$ sufficiently small so that $0 < R \leq 1$ and

$$C(1+c\mu)^{m-1} R \sum_{(j,\alpha) \in I_m} \|S(c_{j,\alpha})\|_R \leq \frac{1}{2}, \tag{5.4}$$

where S is the shift operator defined by the following: for $a(x) = \sum_{l \geq 0} a_l x^l$ we write $S(a)(x) = \sum_{l \geq 0} a_{l+1} x^l$. Set also

$$A_i = 2k_i^m \sum_{j=1}^{J(i)} \sum_{n=0}^{\mu_{i,j}-1} \frac{|A(k_i, l_{i,j}; n)|}{c^n} R^{l_{i,j}}, \quad i = 1, \dots, q. \tag{5.5}$$

Then we have the following estimates:

Proposition 4. *In the above context, we have*:
(1) *If $k \neq k_1, k_2, \dots, k_q$, we have*

$$\|w_k\|_{c,\rho} \leq \frac{2C}{k^m} \|f_k\|_{c,\rho} \quad \textit{for any } 0 < \rho \leq R.$$

(2) *If $k = k_i$ for some i, we have*

$$\|w_k\|_{c,\rho} \le \frac{A_i}{k^m} + \frac{2C}{k^m}\|f_k\|_{c,\rho} \quad \text{for any } 0 < \rho \le R.$$

PROOF. We first note the following: by (4.8) and Lemma 2 we have

$$|g_{k,l}|_c \le \sum_{(j,\alpha)\in I_m} \sum_{h=0}^{l-1} |c_{j,\alpha,l-h}|(1+c\mu)^j k^j l^\alpha |w_{k,h}|_c + |\phi_{k,l}|_c$$

and therefore

$$\frac{C}{(k+l)^m}|g_{k,l}|_c \le C(1+c\mu)^{m-1} \sum_{(j,\alpha)\in I_m} \sum_{h=0}^{l-1} |c_{j,\alpha,l-h}|\,|w_{k,h}|_c + \frac{C}{k^m}|\phi_{k,l}|_c. \tag{5.6}$$

When $k \ne k_1, k_2, \dots, k_q$, we have $(k,l) \notin \mathscr{M}$ for all $l \ge 0$: then, by combining (5.6) with (5.3) and (5.4) we have

$$\begin{aligned}\|w_k\|_{c,\rho} &= \sum_{l\ge 0} |w_{k,l}|_c \rho^l \le \sum_{l\ge 0} \frac{C}{(k+l)^m}|g_{k,l}|_c \rho^l \\ &\le C(1+c\mu)^{m-1}\rho \sum_{(j,\alpha)\in I_m} \|S(c_{j,\alpha})\|_\rho \|w_k\|_{c,\rho} + \frac{C}{k^m}\|f_k\|_{c,\rho} \\ &\le \frac{1}{2}\|w_k\|_{c,\rho} + \frac{C}{k^m}\|f_k\|_{c,\rho}\end{aligned} \tag{5.7}$$

which yields the result (1).

When $k = k_i$ holds for some i, we have $(k, l_{i,j}) \in \mathscr{M}$ for $j = 1, \dots, J(i)$, and also $(k,l) \notin \mathscr{M}$ for $l \ne l_{i,1}, \dots, l_{i,J(i)}$. Therefore by (5.3)–(5.6) we obtain

$$\begin{aligned}\|w_k\|_{c,\rho} &\le \sum_{j=1}^{J(i)} \sum_{n=0}^{\mu(k_i,l_{i,j})-1} \frac{|A(k_i, l_{i,j}; n)|}{c^n} \rho^{l_{i,j}} \\ &\quad + C(1+c\mu)^{m-1}\rho \sum_{(j,\alpha)\in I_m} \|S(c_{j,\alpha})\|_\rho \|w_k\|_{c,\rho} + \frac{C}{k^m}\|f_k\|_{c,\rho} \\ &\le \frac{1}{2}\frac{A_i}{k^m} + \frac{1}{2}\|w_k\|_{c,\rho} + \frac{C}{k^m}\|f_k\|_{c,\rho}\end{aligned}$$

which yields the result (2). □

Now, set

$$B = \max_{(j,\alpha)\in I_m} ((1+c\mu)^j (em)^\alpha),$$

choose an $H > 0$ sufficiently large so that

$$H \ge \|D_{j,\alpha} w_1\|_{c,R} \quad \text{for any } (j,\alpha) \in I_m \tag{5.8}$$

(where $c = \sigma/2\mu$), and let us consider the following equation with respect to Y:

$$Y = Ht + \sum_{i=1}^{q} \frac{A_i}{(R-\rho)^{m(k_i-1)}} t^{k_i} + \frac{2C}{(R-\rho)^m} \sum_{p+|v|\geq 2} \frac{\|g_{p,v}\|_R}{(R-\rho)^{m(p+|v|-2)}} t^p (BY)^v \tag{5.9}$$

where ρ is a parameter with $0 < \rho < R$.

Since (5.9) is an analytic functional equation, by the implicit function theorem we see that (5.9) has a unique holomorphic solution $Y = Y(t)$ in a neighborhood of $t = 0$ satisfying $Y(0) = 0$. If we expand this into

$$Y = \sum_{k\geq 1} Y_k t^k,$$

the coefficients Y_k $(k = 1, 2, \ldots)$ are determined uniquely by the following recursive family:

$$Y_1 = \begin{cases} H, & \text{when } k_1 > 1, \\ H + A_1, & \text{when } k_1 = 1 \end{cases} \tag{5.10}$$

and for $k \geq 2$

$$\begin{aligned} Y_k = &\sum_{i=1}^{q} \delta_{k_i,k} \frac{A_i}{(R-\rho)^{m(k_i-1)}} \\ &+ \frac{2C}{(R-\rho)^m} \sum_{2\leq p+|v|\leq k} \frac{\|g_{p,v}\|_R}{(R-\rho)^{m(p+|v|-2)}} \\ &\times \left[\sum_{|k^*|=k-p} \left(\prod_{(j,\alpha)\in I_m} (BY_{k_{j,\alpha}(1)}) \times \cdots \times (BY_{k_{j,\alpha}(v_{j,\alpha})}) \right) \right] \end{aligned} \tag{5.11}$$

where $\delta_{k,j}$ denotes the Kronecker's delta. Moreover we can easily see by induction on k that Y_k has the form

$$Y_k = \frac{C_k}{(R-\rho)^{m(k-1)}} \quad \text{for } k \geq 1$$

where $C_k \geq 0$ is a constant independent of the parameter ρ.

We have:

Lemma 7. *The following estimates hold for all $k = 1, 2, \ldots$:*

$$\|D_{j,\alpha} w_k\|_{c,\rho} \leq BY_k \quad \textit{for any } 0 < \rho < R \textit{ and } (j,\alpha) \in I_m. \tag{5.12$_k$}$$

Let us admit this lemma for a while; then the convergence of the formal solution (5.1) is proved as follows. Let $|t_0| \leq \varepsilon, |t_1| \leq \varepsilon/c, |t_2| \leq \varepsilon/c^2, \ldots, |t_\mu| \leq \varepsilon/c^\mu$, and $|x| \leq \rho$. Then we have $|g_k| \leq |g_k|_c \varepsilon^k$ for any $g_k \in H_k[t_0, t_1, \ldots, t_\mu]$. Therefore

$$\sum_{k\geq 1, l\geq 0} |w_{k,l}|\,|x|^l \leq \sum_{k\geq 1, l\geq 0} |w_{k,l}|_c \varepsilon^k \rho^l = \sum_{k\geq 1} \|w_k\|_{c,\rho}\varepsilon^k$$
$$\leq \sum_{k\geq 1} BY_k\varepsilon^k = BY(\varepsilon).$$

This implies that if $\varepsilon > 0$ is sufficiently small, our formal solution (5.1) converges on $\{(t_0, t_1, \dots, t_\mu, x); |t_0| \leq \varepsilon, |t_1| \leq \varepsilon/c, \dots, |t_\mu| \leq \varepsilon/c^\mu, |x| \leq \rho\}$. Summing up, we obtain Theorem 3.

Thus, to complete the proof of Theorem 3 it is suffient to prove Lemma 7 above.

Proof of Lemma 7. Note that the case $k = 1$ is clear from (5.8), (5.10) and the fact $B \geq 1$. The general case is proved by induction on k.

Let $k \geq 2$ and suppose that $(5.12)_p$ is already proved for $p = 1, \dots, k-1$. Then, by Proposition 4, (4.6) and the induction hypothesis we have

$$\begin{aligned}\|w_k\|_{c,\rho} &\leq \sum_{i=1}^{q} \delta_{k_i,k}\frac{A_i}{k^m} + \frac{2C}{k^m}\|f_k\|_{c,\rho} \\ &\leq \frac{1}{k^m}\sum_{i=1}^{q}\delta_{k_i,k}A_i + \frac{2C}{k^m}\sum_{2\leq p+|\nu|\leq k}\|g_{p,\nu}\|_R \\ &\quad \times \left[\sum_{|k^*|=k-p}\left(\prod_{(j,\alpha)\in I_m}(BY_{k_{j,\alpha}(1)}) \times \cdots \times (BY_{k_{j,\alpha}(\nu_{j,\alpha})})\right)\right]\end{aligned}$$

for any $0 < \rho < R$. Since $0 < R \leq 1$ is assumed, by comparing this with (5.11) we obtain

$$\|w_k\|_{c,\rho} \leq \frac{(R-\rho)^m}{k^m} Y_k = \frac{1}{k^m}\frac{C_k}{(R-\rho)^{m(k-2)}} \quad \text{for any } 0 < \rho < R. \tag{5.13}$$

By using this, $(5.12)_k$ is verified in the following way. Let $(j,\alpha) \in I_m$. By Lemma 2 we see that $\|\tau^j w_k\|_{c,\rho} \leq (1+c\mu)^j k^j \|w_k\|_{c,\rho}$ and therefore by (5.13) we have

$$\|\tau^j w_k\|_{c,\rho} \leq \frac{(1+c\mu)^j k^j}{k^m}\frac{C_k}{(R-\rho)^{m(k-2)}} \quad \text{for any } 0 < \rho < R.$$

Applying Lemma 5 α-times we obtain

$$\begin{aligned}\|D_{j,\alpha}w_k\|_{c,\rho} &= \left\|\left(\frac{\partial}{\partial x}\right)^\alpha \tau^j w_k\right\|_{c,\rho} \\ &\leq \frac{(1+c\mu)^j k^j}{k^m}\frac{(m(k-2)+1)\cdots(m(k-2)+\alpha)e^\alpha C_k}{(R-\rho)^{m(k-2)+\alpha}} \\ &\leq (1+c\mu)^j (em)^\alpha \frac{C_k}{(R-\rho)^{m(k-1)}} \leq BY_k\end{aligned}$$

for any $0 < \rho < R$, which proves $(5.12)_k$. □

§6. Supplementary remark.

In this last section, let us show that we can also get a good result in the first order equations with $(n+1)$ independent variables $(t, x_1, \ldots, x_n) \in \boldsymbol{C} \times \boldsymbol{C}^n$.

Consider the following nonlinear partial differential equation:

$$t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \tag{6.1}$$

where $t \in \boldsymbol{C}$, $x = (x_1, \ldots, x_n) \in \boldsymbol{C}^n$, $\partial u/\partial x = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$, $F(t, x, u, v)$ with $v = (v_1, \ldots, v_n)$ is a function of the variables (t, x, u, v) defined in a neighborhood Δ of the origin of $\boldsymbol{C}_t \times \boldsymbol{C}_x^n \times \boldsymbol{C}_u \times \boldsymbol{C}_v^n$, and $u = u(t, x)$ is the unknown function. Set $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$. We assume the following conditions:

B_1) $F(t, x, u, v)$ is a holomorphic function on Δ;

B_2) $F(0, x, 0, 0) \equiv 0$ on Δ_0;

B_3) $\dfrac{\partial F}{\partial v_i}(0, 0, 0, 0) = 0$ for $i = 1, \ldots, n$.

These conditions correspond to the conditions A_1), A_2) and A_3) in section 1, respectively.

Set $a(x) = (\partial F/\partial t)(0, x, 0, 0)$, $\gamma(x) = (\partial F/\partial u)(0, x, 0, 0)$, and $b_i(x) = (\partial F/\partial v_i)(0, x, 0, 0)$ $(i = 1, \ldots, n)$. By B_3) we see that $b_i(0) = 0$ holds for $i = 1, \ldots, n$. We denote by $(\partial b/\partial x)(x) = [\partial b_i/\partial x_j](x)$ the Jacobi matrix of $b(x) = (b_1(x), \ldots, b_n(x))$, and by $c_1, \ldots, c_n$ the eigenvalues of the matrix $(\partial b/\partial x)(0)$. Set

$$L_1(\lambda, \rho) = \lambda - c_1\rho_1 - \cdots - c_n\rho_n - \gamma(0) \quad \text{for } \rho = (\rho_1, \ldots, \rho_n) \in \boldsymbol{R}^n.$$

Then the non-resonance condition and the Poincaré condition for (6.1) are stated as follows:

(N_1) (non-resonance) $L_1(k, l) \neq 0$ holds for any $(k, l) \in \boldsymbol{N}^* \times \boldsymbol{N}^n$,

(P_1) (Poincaré condition) the convex hull of the set $\{1, -c_1, \ldots, -c_n\}$ in $\boldsymbol{C}$ does not contain the origin of $\boldsymbol{C}$.

By Chen-Luo [**1**], Shirai [**6**] we have:

Theorem 4. *Assume* B_1), B_2) *and* B_3). *If the conditions* (N_1) *and* (P_1) *are satisfied, equation* (6.1) *has a unique holomorphic solution* $u(t, x)$ *in a neighborhood of* $(0, 0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n$ *satisfying* $u(0, x) \equiv 0$ *near* $x = 0$.

Now, let us consider the case where the Poincaré condition (P_1) is satisfied but the non-resonance condition (N_1) is not satisfied.

Set $\mathscr{M}_1 = \{(k, l) \in \boldsymbol{N}^* \times \boldsymbol{N}^n; L_1(k, l) = 0\}$ and assume that $\mathscr{M}_1 \neq \emptyset$. Since (P_1) is assumed, it is easy to see that $\mathscr{M}_1$ is a finite set. Denote by μ_1 the cardinal of $\mathscr{M}_1$. Then by the same argument as in Theorem 2 we have:

Theorem 5. *Assume* B_1), B_2), B_3), (P_1), *and* $\mathscr{M}_1 \neq \emptyset$. *Then, equation* (6.1) *has a family of solutions* $u(t, x)$ *of the form*

(6.2) $$u(t,x) = w(t, t(\log t), \dots, t(\log t)^{\mu_1}, x)$$

where $w(t_0, t_1, \dots, t_{\mu_1}, x)$ *is a holomorphic function in a neighborhood of the origin* $(t_0, t_1, \dots, t_{\mu_1}, x) = (0, 0, \dots, 0, 0)$ *satisfying* $w(0, 0, \dots, 0, x) \equiv 0$ *near* $x = 0$; *moreover* $w(t_0, t_1, \dots, t_{\mu_1}, x)$ *contains* μ_1 *arbitrary constants.*

In order to prove this, we expand $F(t,x,u,v)$ into the Taylor series in (t,u,v):

$$F(t,x,u,v) = a(x)t + \gamma(x)u + \sum_{i=1}^{n} b_i(x)v_i + G_2(t,x,u,v)$$

where $G_2(t,x,u,v)$ is the sum of terms whose degree (with respect to (t,u,v)) is greater than or equal to 2. By a suitable linear change of variables $x = (x_1, \dots, x_n)$, we can reduce the equation to the case where the matrix $(\partial b/\partial x)(0)$ is lower triangular; therefore, without loss of generality we may suppose that $b_i(x)$ $(i = 1, \dots, n)$ has the form

$$b_i(x) = c_i x_i + \sum_{1 \le j < i} c_{i,j} x_j + O(|x|^2) \quad (\text{as } x \to 0),$$

where $c_1, \dots, c_n$ are the eigenvalues of the matrix $(\partial b/\partial x)(0)$.

Set

$$t_0 = t, \quad t_1 = t(\log t), \; \dots, \quad t_{\mu_1} = t(\log t)^{\mu_1}$$

and set the vector field τ as

$$\tau = \sum_{i=0}^{\mu_1} t_i \frac{\partial}{\partial t_i} + \sum_{i=1}^{\mu_1} i t_{i-1} \frac{\partial}{\partial t_i}.$$

Under the relation (6.2), equation (6.1) is reduced to the following equation with respect to the unknown function $w(t_0, t_1, \dots, t_{\mu_1}, x)$:

(6.3) $$\tau w \equiv a(x)t + \gamma(x)u + \sum_{i=1}^{n} \left(c_i x_i + \sum_{1 \le j < i} c_{i,j} x_j + O(|x|^2) \right) \frac{\partial w}{\partial x_i} + G_2\left(t_0, x, w, \frac{\partial w}{\partial x} \right) \quad (\operatorname{mod} \mathscr{R}).$$

Now let us set a formal solution w in the form

(6.4) $$w = \sum_{(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}^n} w_{k,l} x^l$$

where

$$w_{k,l} \in H_k[t_0, t_1, \dots, t_{d(k,l)}] \quad ((k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}^n)$$

and $\{d(k,l); (k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}^n\}$ is a suitable sequence in $\{0, 1, \dots, \mu_1\}$. Then, by the argument quite parallel to the one in section 4 we can determine $w_{k,l}$ inductively on

(k,l), if we introduce the following total order relation $\prec$ in $\boldsymbol{N}^* \times \boldsymbol{N}^n$: for $(k,p) = (k,p_1,\dots,p_n)$ and $(h,q) = (h,q_1,\dots,q_n)$ we write $(k,p) \prec (h,q)$ if one of the following conditions is satisfied;

1) $k < h$,
2) $k = h$ and $|p| < |q|$ (where $|p| = p_1 + \cdots + p_n$ and $|q| = q_1 + \cdots + q_n$),
3) $k = h$, $|p| = |q|$ and $p_1 < q_1$,
4) $k = h$, $|p| = |q|$, $p_1 = q_1$ and $p_2 < q_2$,
5) and so on.

Since the discussion of the proof of Theorem 5 is almost the same as in sections 4 and 5, we may omit the details.

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