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A certain Galois action on nearly holomorphic modular forms with respect to a unitary group in three variables

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Abstract. In this paper we will consider Fourier-Jacobi expansions of nearly holomorphic modular forms on $SU(2,1)$, whose coefficients are nearly holomorphic theta functions, and will construct a certain Galois action on them.

0. Introduction.

The classical Eisenstein series

$$E_k(z) = \left[\sum_{\substack{c,d \in \boldsymbol{Z} \\ (c,d) \neq (0,0)}} \mathrm{Im}(z)^s (cz+d)^{-k} |cz+d|^{-2s} \right]_{s=0}$$

is a holomorphic modular form if k is an even integer and $k \geq 4$. In the case $k = 2$, it is well known that

$$E_2(z) = a \cdot \mathrm{Im}(z)^{-1} + \sum_{m=0}^{\infty} b_m \exp(2\pi\sqrt{-1}mz)$$

with $a, b_m \in \boldsymbol{C}$, $N \in \boldsymbol{N}$, where $\mathrm{Im}(z)$ denotes the imaginary part of z. This is the most famous example of a nearly holomorphic modular form. In general, a nearly holomorphic modular form f in elliptic modular case can be expressed as

$$f(z) = \sum_{j=0}^{p} \left\{ (\pi\,\mathrm{Im}(z))^{-j} \sum_{m=0}^{\infty} c_{(f,j,m)} \exp(2\pi\sqrt{-1}mz/N) \right\} \tag{0.1}$$

with $c_{(f,j,m)} \in \boldsymbol{C}$, $p \in \boldsymbol{N}$ and $N \in \boldsymbol{N}$. Then for any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ (the group of all automorphisms of the complex number field), it is proved in [**12**] that there exists another holomorphic modular form f^σ such that

$$f^\sigma(z) = \sum_{j=0}^{p} \left\{ (\pi\,\mathrm{Im}(z))^{-j} \sum_{m=0}^{\infty} c^\sigma_{(f,j,m)} \exp(2\pi\sqrt{-1}mz/N) \right\}. \tag{0.2}$$

In that book, this Galois action of $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ was constructed in case of any symplectic groups.

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In [9], Shimura defined and investigated nearly holomorphic modular forms with respect to unitary and orthogonal groups precisely. On the other hand, the author constructed a Galois action on modular forms for certain unitary groups in [13]. The main theme of this paper is to construct an extension of that Galois action on nearly holomorphic modular forms for three dimensional unitary groups.

To be more concrete, let F be a totally real algebraic number field of finite degree and K be its CM-extension, *i.e.* totally imaginary quadratic extension. We denote by $\boldsymbol{a}$ the set of all archimedean primes of F. Consider a non-degenerate skew-hermitian matrix R of degree 3 with coefficients in K given by

$$R = \begin{pmatrix} & & 1 \\ & s & \\ -1 & & \end{pmatrix},$$

where $s \in K^{\times}$ is pure imaginary at each $v \in \boldsymbol{a}$. Then $\sqrt{-1}R$ is a hermitian matrix of signature $(1,2)$ or $(2,1)$ for any embedding of K into $\boldsymbol{C}$. Define the algebraic group G of unitary similitudes with respect to R by

$$G(\boldsymbol{Q}) = \{\gamma \in GL(3,K) \mid {}^t\gamma^{\rho} R\gamma = \nu(\gamma)R, \ \ \nu(\gamma) \in F^{\times}\},$$

where ρ stands for the non-trivial element of $\mathrm{Gal}(K/F)$; ρ is the complex conjugation for any embedding of K into $\boldsymbol{C}$. We can define the symmetric domain $\mathfrak{D}$ corresponding to G as

$$\mathfrak{D} = \left\{ \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v \in \boldsymbol{a}} \in (\boldsymbol{C}^2)^{\boldsymbol{a}} \,\middle|\, \begin{array}{l} \sqrt{-1}(s^{\Psi_v} w_v \overline{w_v} + \overline{z_v} - z_v) > 0 \\ \text{for each } v \in \boldsymbol{a} \end{array} \right\},$$

where Ψ_v is the embedding of K into $\boldsymbol{C}$ that lies above v such that $\mathrm{Im}(s^{\Psi_v}) > 0$. Take CM-type $\Psi = (\Psi_v)_{v \in \boldsymbol{a}}$ and call it "the CM-type determined by s." Put

$$r_{1,v}(\mathfrak{z}) = (s^{\Psi_v} w_v \overline{w_v} + \overline{z_v} - z_v)^{-1},$$

$$r_{2,v}(\mathfrak{z}) = -s^{\Psi_v} \overline{w_v} (s^{\Psi_v} w_v \overline{w_v} + \overline{z_v} - z_v)^{-1},$$

for each $v \in \boldsymbol{a}$. Then a nearly holomorphic modular form f on $\mathfrak{D}$ is written as a polynomial of $\bigcup_{v \in \boldsymbol{a}} \{r_{1,v}, r_{2,v}\}$ with coefficients in holomorphic functions:

$$f(\mathfrak{z}) = \sum_{\substack{j_1, j_2 \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ j_1 + j_2 \leq p}} c_{(f;j_1,j_2)}(\mathfrak{z}) \left(\prod_{v \in \boldsymbol{a}} r_{1,v}(\mathfrak{z})^{j_{1,v}} \right) \left(\prod_{v \in \boldsymbol{a}} r_{2,v}(\mathfrak{z})^{j_{2,v}} \right),$$

with $j_1 = (j_{1,v})_{v \in \boldsymbol{a}}$, $j_2 = (j_{2,v})_{v \in \boldsymbol{a}}$, $p = (p_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ and holomorphic functions $c_{(f;j_1,j_2)}$. If f is $\boldsymbol{C}$-valued, then f can be rewritten as

$$f(\mathfrak{z}) = \sum_{\substack{j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ 0 \leq j \leq p}} \sum_{0 \leq m \in F} \left\{ \prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1} r_{1,v}(\mathfrak{z}))^{j_v} \right\} g_{(f,j,m)}(w) \times \exp\left(2\pi\sqrt{-1} \sum_{v \in \boldsymbol{a}} m_v z_v \right). \tag{0.3}$$

We call this expression the Fourier-Jacobi expansion of f. Each "coefficient" $g_{(f,j,m)}$ is a nearly holomorphic function (*i.e.* a polynomial of $\{\overline{w_v} \mid v \in \boldsymbol{a}\}$ with coefficients in holomorphic functions) on $\prod_{v\in\boldsymbol{a}} \boldsymbol{C}$ which satisfies the same property of classical theta function. We will call such a function as "nearly holomorphic theta function". Then $g_{(f,j,m)}$ is a nearly holomorphic theta function with respect to L^{Ψ}, where L is a certain $\boldsymbol{Z}$-lattice in K. (We regard Ψ as an embedding of K into $\prod_{v\in\boldsymbol{a}} \boldsymbol{C}$ by $b^{\Psi} = (b^{\Psi_v})_{v\in\boldsymbol{a}}$.) Put

$$(g_{(f,j,m)})_*(w) = \exp\left(\pi\sqrt{-1}\sum_{v\in\boldsymbol{a}}(ms)^{\Psi_v}\overline{w_v}w_v\right)g_{(f,j,m)}(w).$$

Then for any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, there exists a theta function $g_{(f,j,m)}^{(\sigma,\Psi,a)}$ with respect to the lattice $(aL)^{\Psi\sigma}$ of $\prod_{v\in\boldsymbol{a}} \boldsymbol{C}$ which satisfies

$$(g_{(f,j,m)}^{(\sigma,\Psi,a)})_*((au)^{\Psi\sigma}) = \{(g_{(f,j,m)})_*(u^{\Psi})\}^{\sigma}$$

for any $u \in K$, where $\Psi\sigma$ means the CM-type defined by $\Psi\sigma = \{\Psi_v\sigma \mid v \in \boldsymbol{a}\}$ and the symbol a denotes an element of $K_A^{\times}$, the idele group of K, depending on σ and Ψ (*cf.* (1.5)). This is a generalization of the theorem which is proven in the case of holomorphic theta functions (Theorem 3.1 of [**13**]).

The main theorem (Theorem 4.1) is as follows.

Let f be a nearly holomorphic modular form given by (0.3). *Let $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, Ψ be CM-type of K determined by s, and choose $a \in K_A^{\times}$ suitably* (*cf.* (1.5)). *Then there exists $b \in F^{\times}$ and another nearly holomorphic modular form $f^{(\sigma,\Psi,a)}$ with respect to another group $\tilde{G}$ of unitary similitudes and symmetric domain $\tilde{\mathfrak{D}}$ corresponding to a skew-hermitian form* $\begin{pmatrix} & & 1 \\ & bs & \\ -1 & & \end{pmatrix}$ ($b \in F^{\times}$ *is determined by σ, Ψ and a*) *whose Fourier-Jacobi expansion is*

$$f^{(\sigma,\Psi,a)}(\mathfrak{z}) = \sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\ 0\le j\le p}}\sum_{0\le m\in F}\left\{\prod_{v\in\boldsymbol{a}}((\pi\sqrt{-1})^{-1}r_{1,v\sigma}(\mathfrak{z}))^{j_v}\right\}$$
$$\times\, g_{(f,j,m)}^{(\sigma,\Psi,a)}(\tilde{w})\exp\left(2\pi\sqrt{-1}\sum_{v\in\boldsymbol{a}}m_v\tilde{z}_v\right),\quad \left(\tilde{\mathfrak{z}} = \begin{pmatrix}\tilde{z}\\ \tilde{w}\end{pmatrix} = \begin{pmatrix}\tilde{z}_v\\ \tilde{w}_v\end{pmatrix}_{v\in\boldsymbol{a}} \in \tilde{\mathfrak{D}}\right).$$

We shall review the results of holomorphic cases constructed mainly in [**13**] in section 1. In section 2, we examine the near holomorphy of modular forms explicitly for three dimensional unitary groups. The Galois action to nearly holomorphic theta functions will be constructed in section 3. Using these results, we can prove the main theorem in section 4. And in section 5, the last section, we study an orthogonal projection map of nearly holomorphic modular forms to holomorphic ones which was stated in [**12**], and prove that it is compatible with the Galois action in this paper.

Notation. For a ring A, we denote by A_n^m the set of all $m \times n$-matrices with entries in A, and denote A_1^n simply by A^n. We express the identity matrix of degree n as 1_n. The transpose of a matrix X is denoted by ${}^tX$. We denote as usual by $\boldsymbol{Z}, \boldsymbol{N}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the set of all positive rational integers, the field of rational numbers, real numbers, and complex numbers, respectively. For $z \in \boldsymbol{C}$ we denote by $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $\bar{z}$ the real part, the imaginary part, and the complex conjugation of z, respectively. Moreover, when $z = x + \sqrt{-1}y$ with $x, y \in \boldsymbol{R}$, we put $\partial/\partial z = (1/2)((\partial/\partial x) - \sqrt{-1}(\partial/\partial y))$, $\partial/\partial \bar{z} = (1/2)((\partial/\partial x) + \sqrt{-1}(\partial/\partial y))$ as differential operators on C^∞-functions on $\boldsymbol{C}$. If K is an algebraic number field, K_{ab} denotes the maximal abelian extension of K, and we denote by K_A (resp. $K_A^\times$) the adele ring (resp. the idele group) of K. By class field theory, every element x of $K_A^\times$ defines an element of $\mathrm{Gal}(K_{ab}/K)$. We denote this by $[x, K]$. We denote by $\mathcal{O}_K$ and $\mathcal{O}_K^\times$ the ring of algebraic integers of K and its unit group. For each finite prime $\mathfrak{p}$ of K, we denote the $\mathfrak{p}$-completion of K and its maximal compact subring by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$. In the same way, $\boldsymbol{Q}_p$ and $\boldsymbol{Z}_p$ denote the p-completion of $\boldsymbol{Q}$ and $\boldsymbol{Z}$ for each rational prime number p. For an algebraic group G defined over a field k, we denote by $G(K)$ the group of K-rational elements of G if K is an extension field of k. We denote by G_A, G_∞, and $G_\mathfrak{f}$ the adelization of G, the archimedean component of G_A, and the non-archimedean component of G_A. By a variety, we understand a Zariski open subset of an absolutely irreducible projective variety.

1. Unitary groups in three variables and corresponding modular forms.

In this section we shall recall the results of [**13**]. In that paper we constructed the Galois action on holomorphic modular forms with respect to a unitary group. We restrict ourselves to the case of 3-dimensional unitary groups in this paper.

Let F be a totally real algebraic number field of finite degree and K be its CM-extension (namely, a totally imaginary quadratic extension of F). Such a field K is called a CM-field. As is well known, the non-trivial element of $\mathrm{Gal}(K/F)$ is the complex conjugation for any embedding of K into $\boldsymbol{C}$. We denote this by ρ. Let $\boldsymbol{a}$ be the set of all archimedean primes of F, which can be identified with those of K. We denote by $|\boldsymbol{a}|$, the number of elements in $\boldsymbol{a}$, which is equal to $[F : \boldsymbol{Q}]$. For each $v \in \boldsymbol{a}$, there are two embeddings of K into $\boldsymbol{C}$ which lie above v. By a CM-type of K, we denote a set $\Psi = (\Psi_v)_{v \in \boldsymbol{a}}$ where each Ψ_v is an embedding of K into $\boldsymbol{C}$ which lies above v.

Define a non-degenerate skew-hermitian matrix $R \in K_3^3$ by

$$R = \begin{pmatrix} & & 1 \\ & s & \\ -1 & & \end{pmatrix}, \quad \text{where } s \in K^\times \text{ and } s^\rho = -s.$$

Determine the CM-type $\Psi = (\Psi_v)_{v \in \boldsymbol{a}}$ of K so that $\mathrm{Im}(s^{\Psi_v}) > 0$ for each $v \in \boldsymbol{a}$. Let $G^{(1,1)}(s, \Psi)$ be the group of unitary similitudes with respect to R, and we view $G^{(1,1)}(s, \Psi)$ as an algebraic group defined over $\boldsymbol{Q}$. Then

$$G^{(1,1)}(s, \Psi)(\boldsymbol{Q}) = \{\gamma \in GL(3, K) \mid {}^t\gamma^\rho R \gamma = \nu(\gamma) R \text{ with } \nu(\gamma) \in F^\times\}.$$

We have $\nu(\gamma) \gg 0$ ($\gg 0$ means totally positive from now on) for any $\gamma \in G^{(1,1)}(s, \Psi)(\boldsymbol{Q})$, since the hermitian matrix $-\sqrt{-1}{}^t\gamma^{\rho\Psi_v} R^{\Psi_v} \gamma^{\Psi_v}$ must have the same signature as $-\sqrt{-1}R^{\Psi_v}$, that is, $(2,1)$ for each $v \in \boldsymbol{a}$. Note that for any $\gamma \in G^{(1,1)}(s, \Psi)(\boldsymbol{Q})$, it holds $\det(\gamma)\det(\gamma)^\rho = \nu(\gamma)^3$. Next, we define an algebraic subgroup $G_1^{(1,1)}(s, \Psi)$ of $G^{(1,1)}(s, \Psi)$ as

$$G_1^{(1,1)}(s, \Psi)(\boldsymbol{Q}) = \{\gamma \in G^{(1,1)}(s, \Psi)(\boldsymbol{Q}) \mid \nu(\gamma) = \det(\gamma) = 1\}.$$

Then $G_1^{(1,1)}(s, \Psi)$ has the strong approximation property. Hereafter we write $G^{(1,1)}(s, \Psi)$ (resp. $G_1^{(1,1)}(s, \Psi)$) as G (resp. G_1) if there is no fear of confusion.

For $v \in \boldsymbol{a}$ and $b \in F$, we denote by b_v the image of b by the embedding $v : F \hookrightarrow \boldsymbol{R}$. For any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ and $v \in \boldsymbol{a}$, we denote by $v\sigma$ an element of $\boldsymbol{a}$ so that $b_{v\sigma} = (b_v)^\sigma$. Given a set X, we denote by $X^{\boldsymbol{a}}$ the set of all indexed elements $(x_v)_{v\in\boldsymbol{a}}$ with $x_v \in X$. For $x = (x_v)_{v\in\boldsymbol{a}} \in X^{\boldsymbol{a}}$ and $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, we denote by x^σ the element $y = (y_v)_{v\in\boldsymbol{a}} \in X^{\boldsymbol{a}}$ such that $y_{v\sigma} = x_v$.

We can view a CM-type Ψ as an embedding of K into $\boldsymbol{C}^{\boldsymbol{a}}$ such that $b^\Psi = (b^{\Psi_v})_{v\in\boldsymbol{a}}$ for any $b \in K$. Through Ψ, we can view K as a dense subset of $\boldsymbol{C}^{\boldsymbol{a}}$. When $b \in F$, we drop the symbol Ψ (since b^Ψ does not depend on Ψ) and regard b as the element $(b_v)_{v\in\boldsymbol{a}}$ in $\boldsymbol{R}^{\boldsymbol{a}}$. For $x = (x_v)_{v\in\boldsymbol{a}} \in \boldsymbol{C}^{\boldsymbol{a}}$, we write $\boldsymbol{e}_{\boldsymbol{a}}(x) = \exp(2\pi\sqrt{-1}\sum_{v\in\boldsymbol{a}} x_v)$.

For each $v \in \boldsymbol{a}$, we can define the v-component $G_v = G^{(1,1)}(s, \Psi)_v$ of the algebraic group G as follows.

$$G_v = \{\gamma \in GL(3, \boldsymbol{C}) \mid {}^t\bar{\gamma} R^{\Psi_v}\gamma = \nu(\gamma) R^{\Psi_v} \text{ with } \nu(\gamma) \in \boldsymbol{R}^\times\}.$$

Note that for any $\gamma \in G_v$, $\nu(\gamma) > 0$. We can define the corresponding symmetric domain $\mathfrak{D}_v = \mathfrak{D}^{(1,1)}(s, \Psi)_v$ as

$$\mathfrak{D}_v = \left\{ \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in \boldsymbol{C}_1^2 \,\middle|\, \sqrt{-1}(s^{\Psi_v} w\bar{w} + \bar{z} - z) > 0 \right\}.$$

Now we can define the action of G_v on $\mathfrak{D}_v$. For any $\mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathfrak{D}_v$ and $\alpha = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in G_v$, put

$$\alpha\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} (a_1 z + b_1 w + c_1)(a_3 z + b_3 w + c_3)^{-1} \\ (a_2 z + b_2 w + c_2)(a_3 z + b_3 w + c_3)^{-1} \end{pmatrix}.$$

Then the group G_v acts on $\mathfrak{D}_v$ as a group of holomorphic automorphism by $\mathfrak{z} \to \alpha(\mathfrak{z})$. The automorphic factors are

$$\mu_v(\alpha, \mathfrak{z}) = a_3 z + b_3 w + c_3,$$

$$\lambda_v(\alpha, \mathfrak{z}) = \begin{pmatrix} \overline{a_3} z + \overline{c_3} & \overline{a_3} w - \overline{b_3}(s^{\Psi_v})^{-1} \\ -s^{\Psi_v}\overline{a_2} z - s^{\Psi_v}\overline{c_2} & -s^{\Psi_v}\overline{a_2} w + \overline{b_2} \end{pmatrix}.$$

Then for $\alpha, \beta \in G_v$, we have

$$\begin{aligned}
&\mu_v(\alpha\beta, \mathfrak{z}) = \mu_v(\alpha, \beta(\mathfrak{z}))\mu_v(\beta, \mathfrak{z}),\\
&\lambda_v(\alpha\beta, \mathfrak{z}) = \lambda_v(\alpha, \beta(\mathfrak{z}))\lambda_v(\beta, \mathfrak{z}),\\
&\begin{pmatrix} dz \\ dw \end{pmatrix} \circ \alpha = \nu(\alpha)\mu_v(\alpha, \mathfrak{z})^{-1}\,{}^t\lambda_v(\alpha, \mathfrak{z})^{-1}\begin{pmatrix} dz \\ dw \end{pmatrix},\\
&\det(\lambda_v(\alpha, \mathfrak{z})) = \det(\alpha)^{-1}\nu(\alpha)^2\mu_v(\alpha, \mathfrak{z}).
\end{aligned} \tag{1.1}$$

Put

$$\begin{aligned}
\eta_v(\mathfrak{z}) &= \eta_v^{(1,1)}(s, \Psi)(\mathfrak{z}) = \sqrt{-1}(s^{\Psi_v}w\bar{w} + \bar{z} - z),\\
\xi_v(\mathfrak{z}) &= \xi_v^{(1,1)}(s, \Psi)(\mathfrak{z}) = \sqrt{-1}\begin{pmatrix} \bar{z} - z & \bar{w} \\ -w & (s^{\Psi_v})^{-1} \end{pmatrix}.
\end{aligned} \tag{1.2}$$

Then the following assertions hold.

$$\begin{aligned}
\nu(\alpha)\xi_v(\mathfrak{z}) &= {}^t\lambda_v(\alpha, \mathfrak{z})\xi_v(\alpha(\mathfrak{z}))\overline{\lambda_v(\alpha, \mathfrak{z})},\\
\nu(\alpha)\eta_v(\mathfrak{z}) &= \overline{\mu_v(\alpha, \mathfrak{z})}\eta_v(\alpha(\mathfrak{z}))\mu_v(\alpha, \mathfrak{z}),
\end{aligned}$$

for $\alpha \in G_v$ and $\mathfrak{z} \in \mathfrak{D}_v$. Put

$$\begin{aligned}
G_\infty &= G^{(1,1)}(s, \Psi)_\infty = \prod_{v \in \boldsymbol{a}} G^{(1,1)}(s, \Psi)_v,\\
\mathfrak{D} &= \mathfrak{D}^{(1,1)}(s, \Psi) = \prod_{v \in \boldsymbol{a}} \mathfrak{D}^{(1,1)}(s, \Psi)_v,
\end{aligned}$$

and G_∞ acts on $\mathfrak{D}$ componentwise. We can define an embedding of $G(\boldsymbol{Q}) = G^{(1,1)} \cdot (s, \Psi)(\boldsymbol{Q})$ into G_∞ by $\alpha \to (\alpha^{\Psi_v})_{v \in \boldsymbol{a}}$ and also define an action of $G(\boldsymbol{Q})$ onto $\mathfrak{D}$, $\alpha(\mathfrak{z}) = (\alpha^{\Psi_v}(\mathfrak{z}_v))_{v \in \boldsymbol{a}}$ where $\alpha \in G(\boldsymbol{Q})$, $\mathfrak{z} = (\mathfrak{z}_v)_{v \in \boldsymbol{a}} \in \mathfrak{D}$. Set

$$\begin{aligned}
\mu_v(\alpha, \mathfrak{z}) &= \mu_v(\alpha^{\Psi_v}, \mathfrak{z}_v), \quad & \lambda_v(\alpha, \mathfrak{z}) &= \lambda_v(\alpha^{\Psi_v}, \mathfrak{z}_v),\\
\xi_v(\mathfrak{z}) &= \xi_v(\mathfrak{z}_v), & \eta_v(\mathfrak{z}) &= \eta_v(\mathfrak{z}_v).
\end{aligned}$$

Now let us consider a congruence subgroup of $G(\boldsymbol{Q})$. For any integral ideal $\mathfrak{a}$ of K, set

$$\Gamma_\mathfrak{a} = \{\gamma \in G_1^{(1,1)}(s, \Psi)(\boldsymbol{Q}) \cap GL(3, \mathcal{O}_K) \mid \gamma \equiv 1_3 \mod \mathfrak{a}_3^3\},$$

where $\mathcal{O}_K$ denotes the ring of integers of K. By a congruence subgroup of $G(\boldsymbol{Q})$, we understand a subgroup Γ of $G(\boldsymbol{Q})$ which contains $\Gamma_\mathfrak{a}$ for some integral ideal $\mathfrak{a}$ of K and $K^\times\Gamma_\mathfrak{a}$ is a subgroup of $K^\times\Gamma$ of finite index.

Next let us define modular forms on $\mathfrak{D}$. Take a rational representation ω of $GL(2, \boldsymbol{C})^{\boldsymbol{a}} \times (\boldsymbol{C}^\times)^{\boldsymbol{a}}$ on V_ω, where V_ω is a finite dimensional vector space over $\boldsymbol{C}$. Then for any V_ω-valued function f on $\mathfrak{D}$ and any $\alpha \in G(\boldsymbol{Q})$, we define another V_ω-valued function $f|_\omega\alpha$ on $\mathfrak{D}$ by

$$(f|_\omega\alpha)(\mathfrak{z}) = \omega((\lambda_v(\alpha, \mathfrak{z}))_{v \in \boldsymbol{a}}, (\mu_v(\alpha, \mathfrak{z}))_{v \in \boldsymbol{a}})^{-1}f(\alpha(\mathfrak{z})).$$

Clearly $f|_{\omega}(\alpha\beta) = (f|_{\omega}\alpha)|_{\omega}\beta$ for any $\alpha, \beta \in G(\boldsymbol{Q})$. If f is holomorphic (or C^{∞}) on $\mathfrak{D}$, so is $f|_{\omega}\alpha$.

For any congruence subgroup Γ of $G^{(1,1)}(s,\Psi)(\boldsymbol{Q})$, we denote by $\mathscr{M}_{\omega}^{(1,1)}(s,\Psi)(\Gamma)$, the set of all holomorphic V_{ω}-valued functions f on $\mathfrak{D}^{(1,1)}(s,\Psi)$ such that $f|_{\omega}\gamma = f$ for any $\gamma \in \Gamma$. We denote by $\mathscr{M}_{\omega}^{(1,1)}(s,\Psi)$, the union of $\mathscr{M}_{\omega}^{(1,1)}(s,\Psi)(\Gamma)$ for all congruence subgroups Γ of $G^{(1,1)}(s,\Psi)(\boldsymbol{Q})$. We write simply $\mathscr{M}_{\omega}^{(1,1)}(s,\Psi), \mathscr{M}_{\omega}^{(1,1)}(s,\Psi)(\Gamma)$ by $\mathscr{M}_{\omega}, \mathscr{M}_{\omega}(\Gamma)$ respectively if there is no fear of confusion.

Take $k = (k_v)_{v\in\boldsymbol{a}} \in \boldsymbol{Z}^{\boldsymbol{a}}$ and consider the rational representation ω_k of $GL(2,\boldsymbol{C})^{\boldsymbol{a}} \times (\boldsymbol{C}^{\times})^{\boldsymbol{a}}$ as

$$\omega_k(a,b) = \prod_{v\in\boldsymbol{a}} (b_v)^{k_v},$$

where $a = (a_v)_{v\in\boldsymbol{a}} \in GL(2,\boldsymbol{C})^{\boldsymbol{a}}$ and $b = (b_v)_{v\in\boldsymbol{a}} \in (\boldsymbol{C}^{\times})^{\boldsymbol{a}}$. In this case we write simply $|_{\omega_k}$ and $\mathscr{M}_{\omega_k}$ by $|_k$ and $\mathscr{M}_k$. From now on till the end of this section we treat the case when $f \in \mathscr{M}_k$ with some $k \in \boldsymbol{Z}^{\boldsymbol{a}}$.

Hereafter we identify $\boldsymbol{Z}^{\boldsymbol{a}}$ with the free module $\sum_{v\in\boldsymbol{a}} \boldsymbol{Z}\cdot v$ by putting $(k_v)_{v\in\boldsymbol{a}} = \sum_{v\in\boldsymbol{a}} k_v v$. Also put $\mathbf{1} = (1)_{v\in\boldsymbol{a}} = \sum_{v\in\boldsymbol{a}} v$. We can define the action of $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ on $\boldsymbol{Z}^{\boldsymbol{a}}$ by $(\sum_{v\in\boldsymbol{a}} k_v v)^{\sigma} = \sum_{v\in\boldsymbol{a}} k_v(v\sigma)$. For any $k \in \boldsymbol{Z}^{\boldsymbol{a}}$, we denote by $F(k)$ the algebraic number field corresponding to $\{\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \mid k^{\sigma} = k\}$. Then $F(k)$ is contained in the Galois closure of F over $\boldsymbol{Q}$. For $p = (p_v)_{v\in\boldsymbol{a}}$, $q = (q_v)_{v\in\boldsymbol{a}} \in \boldsymbol{Z}^{\boldsymbol{a}}$, we write $p \le q$ if $p_v \le q_v$ for each $v \in \boldsymbol{a}$.

We can define a certain parabolic subgroup of $G = G^{(1,1)}(s,\Psi)$ and consider corresponding Fourier-Jacobi expansions of holomorphic modular forms. Set

$$\mathsf{N}^{(1,1)}(s,\Psi)(\boldsymbol{Q}) = \left\{ \mathsf{h}_{y,b} = \begin{pmatrix} 1 & sy^{\rho} & b + (1/2)syy^{\rho} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\, \begin{matrix} y \in K \\ b \in F \end{matrix} \right\},$$

$$\mathsf{H}^{(1,1)}(s,\Psi)(\boldsymbol{Q}) = \mathsf{N}^{(1,1)}(s,\Psi)(\boldsymbol{Q}) \cdot \left\{ \begin{pmatrix} \alpha & & \\ & 1 & \\ & & (\alpha^{\rho})^{-1} \end{pmatrix} \middle|\, \alpha \in K^{\times} \right\},$$

$$\mathsf{P}^{(1,1)}(s,\Psi)(\boldsymbol{Q}) = \mathsf{N}^{(1,1)}(s,\Psi)(\boldsymbol{Q}) \cdot \left\{ \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \beta\beta^{\rho}(\alpha^{\rho})^{-1} \end{pmatrix} \middle|\, \begin{matrix} \beta \in K^{\times} \\ \alpha \in K^{\times} \end{matrix} \right\}.$$

Then $\mathsf{P}^{(1,1)}(s,\Psi)$ is a parabolic subgroup of G and $\mathsf{N}^{(1,1)}(s,\Psi)$ is its unipotent radical. We write simply $\mathsf{N}^{(1,1)}(s,\Psi), \mathsf{H}^{(1,1)}(s,\Psi), \mathsf{P}^{(1,1)}(s,\Psi)$ by $\mathsf{N}, \mathsf{H}, \mathsf{P}$ respectively if there is no fear of confusion.

For any $f \in \mathscr{M}_k$, we can write f as

$$\begin{aligned} f(\mathfrak{z}) &= \sum_{m\in F} g_{(f,m)}(w) \exp\left(2\pi\sqrt{-1}\sum_{v\in\boldsymbol{a}} m_v z_v\right) \\ &= \sum_{m\in F} g_{(f,m)}(w) \boldsymbol{e}_{\boldsymbol{a}}(mz) \quad \left(\mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v\in\boldsymbol{a}} \in \mathfrak{D}\right), \end{aligned} \tag{1.3}$$

where $g_{(f,m)} \not\equiv 0$ only if m belongs to a certain $\boldsymbol{Z}$-lattice of F. Considering the action of $\mathsf{N}(\boldsymbol{Q})$, the function $g_{(f,m)}$ must be a certain theta function.

For a lattice Λ in $\boldsymbol{C}^{\boldsymbol{a}}$ and a hermitian form H on $\boldsymbol{C}^{\boldsymbol{a}}$ so that $\mathrm{Im}(H(l_1, l_2)) \in \boldsymbol{Z}$ for any $l_1, l_2 \in \Lambda$, we denote by $\mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, \Lambda, H)$, the space of holomorphic functions g on $\boldsymbol{C}^{\boldsymbol{a}}$ satisfying

$$g(u+l) = g(u)\exp\left(\pi H\left(l, u + \frac{1}{2}l\right)\right) \quad \text{for any } l \in \Lambda,\ u = (u_v)_{v \in \boldsymbol{a}} \in \boldsymbol{C}^{\boldsymbol{a}}. \tag{1.4}$$

As is well known, $\mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, \Lambda, H) \neq \{0\}$ only if H is semi-positive definite.

Then in (1.3), the function $g_{(f,m)}$ belongs to $\mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{ms,\Psi})$, where L is a certain $\boldsymbol{Z}$-lattice in K and

$$H_{t,\Psi}(u_1, u_2) = -2\sqrt{-1}\sum_{v \in \boldsymbol{a}} t^{\Psi_v}\overline{u_{1,v}}u_{2,v},$$

for $t \in K^{\times}$ so that $t^{\rho} = -t$. Since $H_{t,\Psi}$ is semi-positive definite if and only if $\mathrm{Im}(t^{\Psi_v}) \geq 0$ for each $v \in \boldsymbol{a}$, the function $g_{(f,m)}$ in (1.3) is non-zero only if m is totally positive or 0. We rewrite (1.3) as

$$f(\mathfrak{z}) = \sum_{0 \leq m \in F} g_{(f,m)}(w)\boldsymbol{e}_{\boldsymbol{a}}(mz),$$

where $0 \leq m$ means m is totally positive or 0.

We denote by $\mathfrak{T}_{t,\Psi}$ the union of $\mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$ for all $\boldsymbol{Z}$-lattices L in K. The Galois action on $\mathfrak{T}_{t,\Psi}$ was constructed in [**13**]. To explain that, we must review the reflex of CM-type. For a CM-field K, its CM-type Ψ, and any $\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, we can define another CM-type $\Psi\sigma = \{\Psi_v\sigma \mid v \in \boldsymbol{a}\}$ of K. Note that $(\Psi\sigma)_{(v\sigma)} = \Psi_v\sigma$. We denote by K^*_{Ψ} (or simply K^* if there is no fear of confusion), the algebraic number field corresponding to $\{\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \mid \Psi\sigma = \Psi\}$ which is a finite index subgroup of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. As is well known, K^*_{Ψ} is a CM-field contained in the Galois closure of K. Viewing Ψ as a union of $[F : \boldsymbol{Q}]$ different $\mathrm{Gal}(\bar{\boldsymbol{Q}}/K)$-cosets in $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, we can define a CM-type Ψ^* of K^*_{Ψ} as

$$\mathrm{Gal}(\bar{\boldsymbol{Q}}/K^*_{\Psi})\Psi^* = (\mathrm{Gal}(\bar{\boldsymbol{Q}}/K)\Psi)^{-1}.$$

We call Ψ^* by "the reflex of Ψ" and the couple (K^*_{Ψ}, Ψ^*) by "the reflex of (K, Ψ)". From the definition, we have $(K^*_{\Psi})^{\sigma} = K^*_{\Psi\sigma}$ for any $\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. By N'_{Ψ}, we denote the group homomorphism $x \to \prod_{\psi^* \in \Psi^*} x^{\psi^*}$ from $K^{*\times}_{\Psi}$ to $K^{\times}$. It is a morphism of algebraic groups if we view $K^{*\times}_{\Psi}$ and $K^{\times}$ as algebraic groups defined over $\boldsymbol{Q}$, and so it can naturally be extended to the homomorphism of $(K^*_{\Psi})^{\times}_A$ to $K^{\times}_A$.

For any $g \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, \Lambda, H)$, we define a function g_* on $\boldsymbol{C}^{\boldsymbol{a}}$ (which may be non-holomorphic) by

$$g_*(u) = \exp\left(-\frac{\pi}{2}H(u,u)\right)g(u) \quad (u \in \boldsymbol{C}^{\boldsymbol{a}}).$$

Now for any $\boldsymbol{Z}$-lattice L in K and $g \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$, consider the restriction of g_* to

K^{Ψ}. Take any $\boldsymbol{Z}$-lattice L_1 in K. Then there exists a certain sublattice L_2 of L_1 so that

$$g_*(y^{\Psi}+x^{\Psi}) = g_*(y^{\Psi}) \quad \text{for any } y \in L_1,\ x \in L_2.$$

(This does not mean g_* is periodic on whole K^{Ψ}.) Hence we can define $g_*(y^{\Psi})$ for $y \in K_A$. (See, section 3 of [**13**].)

For a CM-type Ψ and any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, a certain idele class $g_{\Psi}(\sigma) \in K_A^{\times}/K^{\times}K_{\infty}^{\times}$ is defined in [**2**] (or essentially in [**1**]). Take an abelian variety (A, ι) of type (K, Ψ) with a $\boldsymbol{Z}$-lattice L in K and a complex analytic isomorphism Θ of $\boldsymbol{C}^{\boldsymbol{a}}/L^{\Psi}$ onto A. (See, [**11**].) We denote by A_{tor} the subgroup of all torsion elements of A, which coincides with the image of K/L by $\Theta \circ \Psi$. Next take $(A, \iota)^{\sigma}$. Then it is an abelian variety of type $(K, \Psi\sigma)$ and we have the following commutative diagram

$$\begin{array}{ccc} K/L & \xrightarrow{\Theta\circ\Psi} & A_{\mathrm{tor}} \\ {\scriptstyle \times a}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ K/aL & \xrightarrow{\Theta_a\circ(\Psi\sigma)} & A_{\mathrm{tor}}^{\sigma} \end{array}$$

with some $a \in K_A^{\times}$ and complex analytic isomorphism Θ_a of $\boldsymbol{C}^{\boldsymbol{a}}/(aL)^{\Psi\sigma}$ onto A^{σ}. The coset $aK^{\times}K_{\infty}^{\times}$ is uniquely determined only by (K, Ψ) and σ (not depending on A or L). We denote this coset by $g_{\Psi}(\sigma)$. For $a \in g_{\Psi}(\sigma)$, we have $aa^{\rho} \in \chi(\sigma)F^{\times}F_{\infty}^{\times}$, where $\chi(\sigma) \in \prod_p \boldsymbol{Z}_p^{\times} \subset \boldsymbol{Q}_A^{\times}$ which satisfies $[\chi(\sigma)^{-1}, \boldsymbol{Q}] = \sigma|_{\boldsymbol{Q}_{ab}}$. We define $\iota(\sigma, a) \in F^{\times}$ by $\chi(\sigma)/(aa^{\rho}) \in \iota(\sigma, a)F_{\infty}^{\times}$. If σ is trivial on K_{Ψ}^{*}, we have $g_{\Psi}(\sigma) = N_{\Psi}'(b)K^{\times}K_{\infty}^{\times}$ with $b \in (K_{\Psi}^{*})_A^{\times}$ such that $[b^{-1}, K_{\Psi}^{*}] = \sigma|_{K_{\Psi ab}^{*}}$; this fact is a main theorem of complex multiplication theory of [**11**]. Note that $g_{\Psi}(\sigma_1)g_{\Psi\sigma_1}(\sigma_2) = g_{\Psi}(\sigma_1\sigma_2)$. In [**13**], we defined

$$C_{\Psi}(\boldsymbol{C}) = \{(\sigma, \Psi, a) \mid \sigma \in \mathrm{Aut}(\boldsymbol{C}), a \in g_{\Psi}(\sigma)\}. \tag{1.5}$$

We have shown the following theorems in [**13**]. (See, Theorems 3.1 and 6.1 of [**13**].)

THEOREM 1.1. *For any* $g \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$ *and* $(\sigma, \Psi, a) \in C_{\Psi}(\boldsymbol{C})$, *there exists* $g^{(\sigma,\Psi,a)} \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, (aL)^{\Psi\sigma}, H_{\iota(\sigma,a)t,\Psi\sigma})$ *which satisfies*

$$(g^{(\sigma,\Psi,a)})_*((ay)^{\Psi\sigma}) = \{g_*(y^{\Psi})\}^{\sigma} \quad \textit{for any } y \in K.$$

THEOREM 1.2. *For* $k \in \boldsymbol{Z}^{\boldsymbol{a}}$, *take* $f \in \mathscr{M}_k^{(1,1)}(s, \Psi)$ *and let*

$$f(\mathfrak{z}) = \sum_{0 \le m \in F} g_{(f,m)}(w)\boldsymbol{e}_{\boldsymbol{a}}(mz) \quad \left(\mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathfrak{D}^{(1,1)}(s, \Psi)\right),$$

be its Fourier-Jacobi expansion. For any $(\sigma, \Psi, a) \in C_{\Psi}(\boldsymbol{C})$, *there exists* $f^{(\sigma,\Psi,a)} \in \mathscr{M}_{k^{\sigma}}^{(1,1)}(\iota(\sigma,a)s, \Psi\sigma)$ *whose Fourier-Jacobi expansion is*

$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}) = \sum_{0 \le m \in F} g_{(f,m)}^{(\sigma,\Psi,a)}(\tilde{w})\boldsymbol{e}_{\boldsymbol{a}}(m\tilde{z}),$$

where $\tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} \in \mathfrak{D}^{(1,1)}(\iota(\sigma,a)s, \Psi\sigma)$.

We also have to review modular forms with respect to symplectic groups. Let $F, \boldsymbol{a}$ be as above. Put $G^{(l)}(\boldsymbol{Q}) = GSp(l,F)$, $G_1^{(l)}(\boldsymbol{Q}) = Sp(l,F)$, that is,

$$G^{(l)}(\boldsymbol{Q}) = \left\{ \gamma \in GL(2l,F) \mid {}^t\gamma \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \gamma = \nu(\gamma) \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \ \text{ with } \nu(\gamma) \in F^\times \right\},$$

$$G_1^{(l)}(\boldsymbol{Q}) = \left\{ \gamma \in GL(2l,F) \mid {}^t\gamma \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \right\}.$$

We view $G^{(l)}, G_1^{(l)}$ as algebraic groups defined over $\boldsymbol{Q}$. Then $G_1^{(l)}$ has the strong approximation property. As is well known, we have $\det(\gamma) = 1$ for any $\gamma \in G_1^{(l)}(\boldsymbol{Q})$. Set

$$G^{(l)}(\boldsymbol{Q})_+ = \{\gamma \in G^{(l)}(\boldsymbol{Q}) \mid \nu(\gamma) \gg 0\},$$

where $\gg 0$ means totally positive, and set

$$\mathfrak{H}_l^{\boldsymbol{a}} = \{ z = (z_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{C}_l^l)^{\boldsymbol{a}} \mid {}^t z_v = z_v, \ \operatorname{Im}(z_v) > 0 \text{ for any } v \in \boldsymbol{a}\},$$

where >0 means positive definite. In case $F = \boldsymbol{Q}$, $\mathfrak{H}_l^{\boldsymbol{a}}$ is the Siegel upper half space, which is written as $\mathfrak{H}_l$.

Now $G^{(l)}(\boldsymbol{Q})_+$ acts on $\mathfrak{H}_l^{\boldsymbol{a}}$ as $\alpha((z_v)_{v\in\boldsymbol{a}}) = ((a_v z_v + b_v)(c_v z_v + d_v)^{-1})_{v \in \boldsymbol{a}}$ with $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^{(l)}(\boldsymbol{Q})_+$ and $a, b, c, d \in F_l^l$. The automorphic factor is defined by

$$j_v^{(l)}(\alpha, (z_v)_{v\in\boldsymbol{a}}) = c_v z_v + d_v$$

for each $v \in \boldsymbol{a}$. We define congruence subgroups of $G^{(l)}(\boldsymbol{Q})_+$ and modular forms on $\mathfrak{H}_l^{\boldsymbol{a}}$ with respect to them as in [**11**]. For a congruence subgroup $\Gamma^{(l)}$ and $k \in \boldsymbol{Z}^{\boldsymbol{a}}$, we denote by $\mathcal{M}_k^{(l)}(\Gamma^{(l)})$, the space of holomorphic modular forms on $\mathfrak{H}_l^{\boldsymbol{a}}$ of weight k with respect to $\Gamma^{(l)}$, that is, the set of all holomorphic functions f on $\mathfrak{H}_l^{\boldsymbol{a}}$ which satisfy $f(\gamma(z)) \prod_{v\in\boldsymbol{a}} \det(j_v(\gamma,z))^{-k_v} = f(z)$ for any $\gamma \in \Gamma^{(l)}$ (and are holomorphic at any cusp if $F = \boldsymbol{Q}$ and $l = 1$). The union of $\mathcal{M}_k^{(l)}(\Gamma^{(l)})$ for all congruence subgroups $\Gamma^{(l)}$ is denoted by $\mathcal{M}_k^{(l)}$. Set

$$\mathcal{A}_k^{(l)} = \bigcup_{e \in \boldsymbol{Z}^{\boldsymbol{a}}} \{ f_1 f_2^{-1} \mid f_1 \in \mathcal{M}_{k+e}^{(l)}, \ 0 \not\equiv f_2 \in \mathcal{M}_e^{(l)} \}.$$

As is well known, any $f \in \mathcal{M}_k^{(l)}$ has a Fourier expansion as

$$f(z) = \sum_{r \in L} c_r \boldsymbol{e}_{\boldsymbol{a}}(\operatorname{tr}(rz)),$$

where L is a certain lattice in the space of symmetric matrices of degree l with coefficients in F. (Note that $c_r \neq 0$ only if r_v is semi-positive definite for each $v \in \boldsymbol{a}$.) Then for any $\sigma \in \operatorname{Aut}(\boldsymbol{C})$, there exists $f^\sigma \in \mathcal{M}_{k^\sigma}^{(l)}$ whose Fourier expansion is

$$f^\sigma(z) = \sum_{r \in L} c_r^\sigma \boldsymbol{e}_{\boldsymbol{a}}(\operatorname{tr}(rz)). \tag{1.6}$$

This fact is proved in [**4**] (*cf.* also in section 26 of [**11**]). We can also define f^σ in case $f = f_1/f_2 \in \mathscr{A}_k^{(l)}$ ($f_1 \in \mathscr{M}_{k+e}^{(l)}, 0 \neq f_2 \in \mathscr{M}_e^{(l)}$) by $f^\sigma = f_1^\sigma/f_2^\sigma$. (It is well defined.)

In [**3**], the canonical models for symplectic cases are constructed. Take $G_A^{(l)} = GSp(l, F_A)$ and set

$$\mathscr{G}_+^{(l)} = \{x \in G_A^{(l)} \mid \nu(x) \in \overline{F^\times F_{\infty+}^\times} \boldsymbol{Q}_A^\times,\ \ \nu(x)_v > 0 \text{ for each } v \in \boldsymbol{a}\}.$$

We denote by $\mathscr{Z}^{(l)}$ the $\mathscr{Z}$ defined in [**3**], which is a certain family of subgroups X of $G_A^{(l)}$ such that $\Gamma_X^{(l)} = X \cap G^{(l)}(\boldsymbol{Q})$ is a congruence subgroup of $G^{(l)}(\boldsymbol{Q})_+$. For $X \in \mathscr{Z}^{(l)}$, the canonical model of $\Gamma_X^{(l)} \backslash \mathfrak{H}_l^{\boldsymbol{a}}$ (which is written V_X in [**3**]) is denoted by $V_X^{(l)}$, and the projection map of $\mathfrak{H}_l^{\boldsymbol{a}}$ to $V_X^{(l)}$ is written $\varphi_X^{(l)}$ (which is φ_X in [**3**]). For $X, Y \in \mathscr{Z}^{(l)}$ and $x \in \mathscr{G}_+^{(l)}$ so that $X \supset xYx^{-1}$, we can take the morphism $J_{XY}^{(l)}(x)$ (called $J_{XY}(x)$ in [**3**]) of $V_Y^{(l)}$ to $(V_X^{(l)})^{\sigma(x)}$, where $\sigma(x) \in \mathrm{Aut}(\boldsymbol{C})$ is determined by x, as in [**3**].

Take $0 \neq f_1, f_2 \in \mathscr{M}_k^{(l)}$ and f_1^σ, f_2^σ in the sense of (1.6). Then for any $Y \in \mathscr{Z}^{(l)}$ such that $(f_1/f_2) \circ (\varphi_Y^{(l)})^{-1}$ is defined as a rational function on $V_Y^{(l)}$, we have

$$(f_1^\sigma/f_2^\sigma) = [(f_1/f_2) \circ (\varphi_Y^{(l)})^{-1}]^\sigma \circ J_{Y\tilde{Y}}^{(l)}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}\right) \circ \varphi_{\tilde{Y}}^{(l)}, \tag{1.7}$$

where $\chi(\sigma) \in \prod_p \boldsymbol{Z}_p^\times$ so that $[\chi(\sigma)^{-1}, \boldsymbol{Q}] = \sigma|_{\boldsymbol{Q}_{ab}}$ and

$$\tilde{Y} = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} Y \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}.$$

For any CM-extension K of F and its CM-type Ψ, put

$$W^{(l)}(\Psi) = \{z \in \mathfrak{H}_l^{\boldsymbol{a}} \mid z = \tau^\Psi \text{ for some } \tau \in K_l^l\}.$$

Then $\varphi_X^{(l)}(z)$ is $K_{\Psi ab}^*$-rational for any $z \in W^{(l)}(\Psi)$ and $X \in \mathscr{Z}^{(l)}$. For any $z = \tau^\Psi \in W^{(l)}(\Psi)$, we define the group injection $\Phi_z^{(l)} : K_A^\times \to G_A^{(l)}$ as

$$\Phi_z^{(l)}(a) = \begin{pmatrix} (a\tau - a^\rho\tau^\rho)(\tau - \tau^\rho)^{-1} & -(a - a^\rho)\tau^\rho(\tau - \tau^\rho)^{-1}\tau \\ (a - a^\rho)(\tau - \tau^\rho)^{-1} & (\tau - \tau^\rho)^{-1}(a^\rho\tau - a\tau^\rho) \end{pmatrix},$$

namely the $h(\overbrace{a, \ldots, a}^{l \text{ times}})$ in section 24.10 of [**11**] with h corresponding to z. Then it satisfies $\Phi_z^{(l)}(a)\begin{pmatrix} \tau \\ 1_l \end{pmatrix} = \begin{pmatrix} a \cdot \tau \\ a \cdot 1_l \end{pmatrix}$ and $\nu(\Phi_z^{(l)}(a)) = aa^\rho$. If $a \in K^\times$, then $\Phi_z^{(l)}(a) \in G^{(l)}(\boldsymbol{Q})_+$ and $\Phi_z^{(l)}(a)(z) = z$. We have the relation between canonical models and CM-points as follows (Proposition 3.2 of [**13**]).

Proposition 1.3. *Take any* $z = \tau^\Psi \in W^{(l)}(\Psi)$ $(\tau \in K_l^l, {}^t\tau = \tau)$ *and* $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. *For any* $X \in \mathscr{Z}^{(l)}$, *put*

$$\tilde{X} = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} X \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix} \in \mathscr{Z}^{(l)}$$

and

$$\left[J_{\tilde{X}X}^{(l)}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1}\right)(\varphi_X^{(l)}(z))\right]^{\sigma} = \varphi_{\tilde{X}}^{(l)}(\tilde{z}).$$

Then we have $\tilde{z} \in W^{(l)}(\Psi\sigma)$ *and*

$$\varphi_{\tilde{X}}^{(l)}(\tilde{z}) = \varphi_{\tilde{X}}^{(l)}(((\alpha_1\tau + \alpha_2)(\alpha_3\tau + \alpha_4)^{-1})^{\Psi\sigma}),$$

where $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in G^{(l)}(\boldsymbol{Q})$ $(\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F_l^l)$ *such that*

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1}(X \cap G_{1A}^{(l)})G_{\infty}^{(l)}\Phi_z^{(l)}(a) \cap G^{(l)}(\boldsymbol{Q}),$$

with $a \in g_{\Psi}(\sigma)$.

To use Shimura's many results in the symplectic cases, we must define some embeddings of algebraic groups and corresponding symmetric domains.

For $z = (z_v)_{v\in\boldsymbol{a}} \in \mathfrak{H}_1^{\boldsymbol{a}}$, define an embedding into $\mathfrak{D}^{(1,1)}(s,\Psi)$ by

$$\varepsilon_0^{(1,1)}(s,\Psi)(z) = \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} z_v \\ 0 \end{pmatrix}_{v\in\boldsymbol{a}}.$$

This is compatible with the injection $I_0^{(1,1)}(s,\Psi)$ of $G_1^{(1)}(\boldsymbol{Q}) = SL(2,F)$ into $G_1^{(1,1)}\cdot(s,\Psi)(\boldsymbol{Q})$ defined by

$$I_0^{(1,1)}(s,\Psi)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}, \quad \text{where } a,b,c,d \in F.$$

As $I_0^{(1,1)}(s,\Psi)$ can be viewed as a homomorphism of algebraic groups, we can extend $I_0^{(1,1)}(s,\Psi)$ to the map $G_{1A}^{(1)} \hookrightarrow G_1^{(1,1)}(s,\Psi)_A$. We denote $I_0^{(1,1)}(s,\Psi), \varepsilon_0^{(1,1)}(s,\Psi)$ by I_0, ε_0 if there is no fear of confusion. We have

$$I_0(\alpha)(\varepsilon_0(z)) = \varepsilon_0(\alpha(z)),$$

$$\mu_v(I_0(\alpha), \varepsilon_0(z)) = j_v^{(1)}(\alpha, z) \quad \text{for any } \alpha \in G_1^{(1)}(\boldsymbol{Q}),\ z \in \mathfrak{H}_1^{\boldsymbol{a}}.$$

Hence we can consider the pull-back of modular forms on $\mathfrak{D}$.

LEMMA 1.4. *For any* $f \in \mathscr{M}_k$, *we have* $f \circ \varepsilon_0 \in \mathscr{M}_k^{(1)}$ $(k \in \boldsymbol{Z}^{\boldsymbol{a}})$.

Next we will define the embedding of $\mathfrak{D} = \mathfrak{D}^{(1,1)}(s,\Psi)$ into $\mathfrak{H}_3^{\boldsymbol{a}}$. Take $\delta \in K^\times$ such that $\delta^\rho = -\delta$. Put

$$\varepsilon_\delta^{(1,1)}(s,\Psi)(\mathfrak{z}) = \begin{pmatrix} z_v - (1/2)s^{\Psi_v}w_v^2 & w_v & -(1/2)\delta^{\Psi_v}s^{\Psi_v}w_v^2 \\ w_v & (-s^{-1})^{\Psi_v} & \delta^{\Psi_v}w_v \\ -(1/2)\delta^{\Psi_v}s^{\Psi_v}w_v^2 & \delta^{\Psi_v}w_v & -(\delta^2)_v z_v - ((\delta^2)_v/2)s^{\Psi_v}w_v^2 \end{pmatrix}_{v\in\boldsymbol{a}},$$

where $\mathfrak{z} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v\in\boldsymbol{a}} \in \mathfrak{D}^{(1,1)}(s,\Psi)$.

This is compatible with the injection $I_\delta = I_\delta^{(1,1)}(s,\Psi)$ of $G(\boldsymbol{Q}) = G^{(1,1)}(s,\Psi)(\boldsymbol{Q})$ into $G^{(3)}(\boldsymbol{Q}) = GSp(3,F)$ defined by

$$I_\delta^{(1,1)}(s,\Psi)(\alpha) = C(s,\delta)\begin{pmatrix} \alpha^\rho & 0 \\ 0 & \alpha \end{pmatrix} C(s,\delta)^{-1},$$

$$\text{where } C(s,\delta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \delta & 0 & 0 & -\delta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -s & 0 & 0 & s & 0 \\ 0 & 0 & -\delta^{-1} & 0 & 0 & \delta^{-1} \end{pmatrix}.$$

Then we have

$$I_\delta(\alpha)(\varepsilon_\delta(\mathfrak{z})) = \varepsilon_\delta(\alpha(\mathfrak{z})),$$
$$\nu(I_\delta(\alpha)) = \nu(\alpha), \quad \det(I_\delta(\alpha)) = \det(\alpha)\det(\alpha)^\rho.$$

Put, for each $v \in \boldsymbol{a}$

$$\Omega_v(\mathfrak{z}) = \begin{pmatrix} 1 & 0 & (1/2)\delta^{\Psi_v} \\ 0 & 1 & (1/2)\delta^{\Psi_v}s^{\Psi_v}w_v \\ (-\delta^{-1})^{\Psi_v} & 0 & (1/2) \end{pmatrix} \quad \text{for } \mathfrak{z} = \begin{pmatrix} z_v \\ w_v \end{pmatrix}_{v \in \boldsymbol{a}} \in \mathfrak{D}.$$

Then we have

$$j_v^{(3)}(I_\delta(\alpha), \varepsilon_\delta(\mathfrak{z})) = \Omega_v(\alpha(\mathfrak{z}))\begin{pmatrix} \lambda_v(\alpha,\mathfrak{z}) & 0 \\ 0 & \mu_v(\alpha,\mathfrak{z}) \end{pmatrix}\Omega_v(\mathfrak{z})^{-1} \tag{1.8}$$

where $\alpha \in G(\boldsymbol{Q})$, $\mathfrak{z} \in \mathfrak{D}$. From (1.1) and (1.8), we obtain

$$\det(j_v^{(3)}(I_\delta(\alpha), \varepsilon_\delta(\mathfrak{z}))) = \det(\mu_v(\alpha,\mathfrak{z}))^2$$

for each $v \in \boldsymbol{a}$ if $\alpha \in G_1(\boldsymbol{Q})$ (since $\det(\Omega_v(\mathfrak{z})) = 1$ for any $\mathfrak{z} \in \mathfrak{D}$). Therefore for any $f \in \mathscr{M}_k^{(3)}$, we have $f \circ \varepsilon_\delta^{(1,1)}(s,\Psi) \in \mathscr{M}_{2k}^{(1,1)}(s,\Psi)$ $(k \in \boldsymbol{Z}^{\boldsymbol{a}})$.

The last embedding is that of $Sp(l,F)$ into $Sp(l|\boldsymbol{a}|,\boldsymbol{Q})$, stated in section 1 of [**8**]. Write $\boldsymbol{a}$ as $\{v_1, v_2, \ldots, v_{|\boldsymbol{a}|}\}$ and take a basis $\{\beta_1, \ldots, \beta_{|\boldsymbol{a}|}\}$ of F over $\boldsymbol{Q}$. Put

$$B = \begin{pmatrix} (\beta_1)_{v_1} & \cdots & (\beta_{|\boldsymbol{a}|})_{v_1} \\ \cdots & \cdots & \cdots \\ (\beta_1)_{v_{|\boldsymbol{a}|}} & \cdots & (\beta_{|\boldsymbol{a}|})_{v_{|\boldsymbol{a}|}} \end{pmatrix}, \quad B^{(l)} = \begin{pmatrix} (\beta_1)_{v_1}1_l & \cdots & (\beta_{|\boldsymbol{a}|})_{v_1}1_l \\ \cdots & \cdots & \cdots \\ (\beta_1)_{v_{|\boldsymbol{a}|}}1_l & \cdots & (\beta_{|\boldsymbol{a}|})_{v_{|\boldsymbol{a}|}}1_l \end{pmatrix}.$$

Let $\{\beta_1', \ldots, \beta_{|\boldsymbol{a}|}'\}$ be the dual basis of $\{\beta_1, \ldots, \beta_{|\boldsymbol{a}|}\}$ with respect to $\mathrm{Tr}_{F/\boldsymbol{Q}}$, that is,

$$({}^tB)^{-1} = \begin{pmatrix} (\beta_1')_{v_1} & \cdots & (\beta_{|\boldsymbol{a}|}')_{v_1} \\ \cdots & \cdots & \cdots \\ (\beta_1')_{v_{|\boldsymbol{a}|}} & \cdots & (\beta_{|\boldsymbol{a}|}')_{v_{|\boldsymbol{a}|}} \end{pmatrix}.$$

We define the embedding $I_B^{(l)}$ of $Sp(l,F)$ into $Sp(l|\boldsymbol{a}|,\boldsymbol{Q})$ as

$$I_B^{(l)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) =$$

$$\begin{pmatrix} {}^tB^{(l)} & 0 \\ 0 & (B^{(l)})^{-1} \end{pmatrix} \begin{pmatrix} a_{v_1} & & & b_{v_1} & & \\ & \ddots & & & \ddots & \\ & & a_{v_{|\boldsymbol{a}|}} & & & b_{v_{|\boldsymbol{a}|}} \\ c_{v_1} & & & d_{v_1} & & \\ & \ddots & & & \ddots & \\ & & c_{v_{|\boldsymbol{a}|}} & & & d_{v_{|\boldsymbol{a}|}} \end{pmatrix} \begin{pmatrix} ({}^tB^{(l)})^{-1} & 0 \\ 0 & B^{(l)} \end{pmatrix},$$

where $a, b, c, d \in F_l^l$. As is well known, $Sp(l|\boldsymbol{a}|, \boldsymbol{Q})$ acts on

$$\mathfrak{H}_{l|\boldsymbol{a}|} = \{Z \in \boldsymbol{C}_{l|\boldsymbol{a}|}^{l|\boldsymbol{a}|} \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$$

as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(Z) = (aZ+b)(cZ+d)^{-1} \quad (a, b, c, d \in \boldsymbol{Q}_{l|\boldsymbol{a}|}^{l|\boldsymbol{a}|}).$$

The corresponding embedding $\varepsilon_B^{(l)}$ of $\mathfrak{H}_l^{\boldsymbol{a}}$ into $\mathfrak{H}_{l|\boldsymbol{a}|}$ is defined by

$$\varepsilon_B^{(l)}((z_v)_{v \in \boldsymbol{a}}) = {}^tB^{(l)} \begin{pmatrix} z_{v_1} & & \\ & \ddots & \\ & & z_{v_{|\boldsymbol{a}|}} \end{pmatrix} B^{(l)}.$$

This embedding is compatible with $I_B^{(l)}$. For any $\alpha \in Sp(l, F)$, put $\begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} = I_B^{(l)}(\alpha)$ with $a_\alpha, b_\alpha, c_\alpha, d_\alpha \in \boldsymbol{Q}_{l|\boldsymbol{a}|}^{l|\boldsymbol{a}|}$. Then we have

$$c_\alpha \varepsilon_B^{(l)}(z) + d_\alpha = (B^{(l)})^{-1} \begin{pmatrix} j_{v_1}^{(l)}(\alpha, z) & & \\ & \ddots & \\ & & j_{v_{|\boldsymbol{a}|}}^{(l)}(\alpha, z) \end{pmatrix} B^{(l)} \tag{1.9}$$

for any $z \in \mathfrak{H}_l^{\boldsymbol{a}}$. Hence we can consider the pull-back of modular forms again in this case. For any holomorphic modular form f on $\mathfrak{H}_{l|\boldsymbol{a}|}$ of weight κ, we have $f \circ \varepsilon_B^{(l)} \in \mathscr{M}_{\kappa\boldsymbol{1}}^{(l)}$ $(\kappa \in \boldsymbol{Z})$.

2. Nearly holomorphic modular forms.

We are going to define nearly holomorphic modular forms, which are basically same as those in [**9**] or [**12**].

For any rational representation (ω, V_ω) of $GL(2, \boldsymbol{C})^{\boldsymbol{a}} \times (\boldsymbol{C}^\times)^{\boldsymbol{a}}$ and each $v \in \boldsymbol{a}$, define new representations $\omega \otimes \tau_v$ and $\omega \otimes \pi_v$ on $\mathrm{Hom}(\boldsymbol{C}_1^2, V_\omega)$ as

$$[(\omega \otimes \tau_v)(a, b)h](u) = \omega(a, b)h({}^ta_v u b_v),$$

$$[(\omega \otimes \pi_v)(a, b)h](u) = \omega(a, b)h(a_v^{-1} u b_v^{-1}),$$

where $a = (a_v)_{v \in \boldsymbol{a}} \in GL(2, \boldsymbol{C})^{\boldsymbol{a}}$ and $b = (b_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{C}^\times)^{\boldsymbol{a}}$.

We also define $\mathrm{Hom}(\boldsymbol{C}_1^2, V_\omega)$-valued functions on $\mathfrak{D}^{(1,1)}(s, \Psi)$. For any $f \in C^\infty(\mathfrak{D}^{(1,1)}(s, \Psi), V_\omega)$, we take

$$(D_v f)(u) = \left(\frac{\partial f}{\partial z_v}, \frac{\partial f}{\partial w_v}\right) \cdot u,$$

$$(\overline{D_v} f)(u) = \left(\frac{\partial f}{\partial \overline{z_v}}, \frac{\partial f}{\partial \overline{w_v}}\right) \cdot u \quad (u \in \boldsymbol{C}_1^2),$$

for each $v \in \boldsymbol{a}$. Further put

$$(D_{\omega, v} f)(u) = \omega(\bar{\xi}, \eta)^{-1} D_v[\omega(\bar{\xi}, \eta) f](u),$$

$$(E_v f)(u) = (\overline{D_v} f)(\overline{\xi_v} u \eta_v)$$

with $\eta = (\eta_v)_{v \in \boldsymbol{a}}$, $\bar{\xi} = (\overline{\xi_v})_{v \in \boldsymbol{a}}$. Here η_v and ξ_v are as in (1.2), and $\overline{\xi_v}$ denotes the complex conjugation of ξ_v. By a formal computation, we have

$$D_{\omega, v}(f|_\omega \alpha) = (D_{\omega, v} f)\,|_{\omega \otimes \tau_v}\, \alpha,$$

$$E_v(f|_\omega \alpha) = (E_v f)\,|_{\omega \otimes \pi_v}\, \alpha,$$

for $\alpha \in G(\boldsymbol{Q})$ such that $\nu(\alpha) = 1$. We have $E_v(fg) = f \cdot (E_v g)$ for any $\boldsymbol{C}$-valued holomorphic function f. Especially $E_v f = 0$ if f is holomorphic.

For a representation (ω, V_ω) of $GL(2, \boldsymbol{C})^{\boldsymbol{a}} \times (\boldsymbol{C}^\times)^{\boldsymbol{a}}$, we define a contraction

$$\Theta_\omega : \mathrm{Hom}(\boldsymbol{C}_1^2, \mathrm{Hom}(\boldsymbol{C}_1^2, V_\omega)) \to V_\omega$$

as follows. Write an element $h \in \mathrm{Hom}(\boldsymbol{C}_1^2, \mathrm{Hom}(\boldsymbol{C}_1^2, V_\omega))$ as $h(u_1, u_2)$ with $u_1, u_2 \in \boldsymbol{C}_1^2$. Then we put

$$\Theta_\omega \circ h = h\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + h\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

We can easily verify that

$$\Theta_\omega \circ (\omega \otimes \tau_v \otimes \pi_v)(a, b) = \Theta_\omega \circ (\omega \otimes \pi_v \otimes \tau_v)(a, b) = \omega(a, b) \Theta_\omega, \tag{2.1}$$

for any $(a, b) \in GL(2, \boldsymbol{C})^{\boldsymbol{a}} \times (\boldsymbol{C}^\times)^{\boldsymbol{a}}$ and each $v \in \boldsymbol{a}$.

For each $v \in \boldsymbol{a}$, put

$$r_{1,v}(\mathfrak{z}) = r_{1,v}^{(1,1)}(s, \Psi)(\mathfrak{z}) = \sqrt{-1}\eta_v^{(1,1)}(s, \Psi)(\mathfrak{z})^{-1},$$

$$r_{2,v}(\mathfrak{z}) = r_{2,v}^{(1,1)}(s, \Psi)(\mathfrak{z}) = -\sqrt{-1} s^{\Psi_v} \overline{w_v} \eta_v^{(1,1)}(s, \Psi)(\mathfrak{z})^{-1}.$$

Then by a calculation, we can get

$$\frac{\partial}{\partial z_v} r_{1,v} = r_{1,v}^2 \qquad \frac{\partial}{\partial w_v} r_{1,v} = r_{1,v} r_{2,v},$$

$$\frac{\partial}{\partial z_v} r_{2,v} = r_{1,v} r_{2,v} \qquad \frac{\partial}{\partial w_v} r_{2,v} = r_{2,v}^2,$$

$$(E_v r_{1,v})(u) = (1, 0)(u),$$

$$(E_v r_{2,v})(u) = (0, 1)(u) \quad (u \in \boldsymbol{C}_1^2).$$

Obviously, we have

$$E_v r_{1,v'} = 0,$$

$$E_v r_{2,v'} = 0,$$

$$\frac{\partial}{\partial z_v} r_{1,v'} = \frac{\partial}{\partial w_v} r_{1,v'} = \frac{\partial}{\partial z_v} r_{2,v'} = \frac{\partial}{\partial w_v} r_{2,v'} = 0,$$

if $v \neq v'$.

From now on we use the notion of nearly holomorphic functions stated in section 2 of [**9**]. Consider $\bigcup_{v\in\boldsymbol{a}}\{r_{1,v}, r_{2,v}\}$, which is a set of C^∞-functions on $\mathfrak{D}^{(1,1)}(s,\Psi)$. We have $\det\begin{pmatrix}(\partial/\partial\overline{z_v})r_{1,v} & (\partial/\partial\overline{w_v})r_{1,v} \\ (\partial/\partial\overline{z_v})r_{2,v} & (\partial/\partial\overline{w_v})r_{2,v}\end{pmatrix} \neq 0$. Hence we can define $\bigcup_{v\in\boldsymbol{a}}\{\partial/\partial r_{1,v}, \partial/\partial r_{2,v}\}$ as (2.1) of [**9**]. For a polynomial $P((x_v)_{v\in\boldsymbol{a}}, (y_v)_{v\in\boldsymbol{a}})$ of $2|\boldsymbol{a}|$-variables with coefficients in holomorphic functions on $\mathfrak{D}^{(1,1)}(s,\Psi)$ and $f = P((r_{1,v})_{v\in\boldsymbol{a}}, (r_{2,v})_{v\in\boldsymbol{a}})$, we have

$$\frac{\partial}{\partial r_{1,v}} f = \left(\frac{\partial}{\partial x_v} P\right)((r_{1,v'})_{v'\in\boldsymbol{a}}, (r_{2,v'})_{v'\in\boldsymbol{a}}),$$

$$\frac{\partial}{\partial r_{2,v}} f = \left(\frac{\partial}{\partial y_v} P\right)((r_{1,v'})_{v'\in\boldsymbol{a}}, (r_{2,v'})_{v'\in\boldsymbol{a}}).$$

Note that they are mutually commutative, that is, $(\partial/\partial r_{i,v_1})(\partial/\partial r_{j,v_2}) = (\partial/\partial r_{j,v_2})(\partial/\partial r_{i,v_1})$ for any $i,j \in \{1,2\}$ and $v_1, v_2 \in \boldsymbol{a}$.

Then for any $f \in C^\infty(\mathfrak{D}^{(1,1)}(s,\Psi), V_\omega)$, we have

$$(E_v f)(u) = \left(\frac{\partial f}{\partial r_{1,v}}, \frac{\partial f}{\partial r_{2,v}}\right)(u) \quad (u \in \boldsymbol{C}_1^2). \tag{2.2}$$

This implies $E_{v_1}E_{v_2} = E_{v_2}E_{v_1}$ for any $v_1, v_2 \in \boldsymbol{a}$.

Now we can define nearly holomorphic modular forms on $\mathfrak{D}^{(1,1)}(s,\Psi)$. For any congruence subgroup Γ of $G(\boldsymbol{Q})$, any $p = (p_v)_{v\in\boldsymbol{a}} \in (\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}$, and any rational representation (ω, V_ω) of $GL(2,\boldsymbol{C})^{\boldsymbol{a}} \times (\boldsymbol{C}^\times)^{\boldsymbol{a}}$, put

$$\begin{aligned}\mathcal{N}_\omega^p(\Gamma) &= \mathcal{N}_\omega^{p,(1,1)}(s,\Psi)(\Gamma) \\ &= \left\{ f \in C^\infty(\mathfrak{D}^{(1,1)}(s,\Psi), V_\omega) \;\middle|\; \begin{array}{l} f|_\omega\gamma = f \text{ for any } \gamma\in\Gamma, \\ E_v^{p_v+1} f = 0 \text{ for each } v \in \boldsymbol{a}. \end{array}\right\}.\end{aligned}$$

We denote by $\mathcal{N}_\omega^{p,(1,1)}(s,\Psi)$ (or $\mathcal{N}_\omega^p$ if there is no fear of confusion) the union of $\mathcal{N}_\omega^{p,(1,1)}(s,\Psi)(\Gamma)$ for all congruence subgroups Γ of $G^{(1,1)}(s,\Psi)(\boldsymbol{Q})$.

From (2.2) and Lemma 2.1 of [**9**], any element f of $\mathcal{N}_\omega^p(\Gamma)$ can be written as

$$f(\mathfrak{z}) = \sum_{\substack{j_1, j_2 \in (\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}} \\ j_1+j_2\leq p}} c_{(f;j_1,j_2)}(\mathfrak{z}) \left(\prod_{v\in\boldsymbol{a}} r_{1,v}(\mathfrak{z})^{j_{1,v}}\right)\left(\prod_{v\in\boldsymbol{a}} r_{2,v}(\mathfrak{z})^{j_{2,v}}\right) \tag{2.3}$$

where $c_{(f;j_1,j_2)}$ are V_ω-valued holomorphic functions on $\mathfrak{D}$ and $j_1 = (j_{1,v})_{v\in\boldsymbol{a}}$, $j_2 = (j_{2,v})_{v\in\boldsymbol{a}}$. We can easily verify the following lemma from (2.2).

LEMMA 2.1. *For* $f \in \mathcal{N}_\omega^p$, *assume* $(\prod_{v \in \boldsymbol{a}} E_v^{q_v}) f = 0$ *with some* $q = (q_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$. *Then* $c_{(f; j_1, j_2)} \equiv 0$ *for any* $j_1, j_2 \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ *such that* $q \le j_1 + j_2$.

From now on till the end of this section, we restrict ourselves in the case when $\omega = \omega_k$ $(\omega_k(a, b) = \prod_{v \in \boldsymbol{a}} b_v^{k_v})$ with some $k = (k_v)_{v \in \boldsymbol{a}} \in \boldsymbol{Z}^{\boldsymbol{a}}$ and $V_\omega = \boldsymbol{C}$. We write simply $\mathcal{N}_{\omega_k}^p, D_{\omega_k, v}$ and Θ_{ω_k} by $\mathcal{N}_k^p, D_{k,v}$ and Θ_k. We can identify $\mathrm{Hom}(\boldsymbol{C}_1^2, \boldsymbol{C})$ with $\boldsymbol{C}_2^1$. For any $q = (q_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ and $\boldsymbol{C}$-valued function f, the function $(\prod_{v \in \boldsymbol{a}} E_v^{q_v}) f$ can be viewed as a $\bigotimes_{v \in \boldsymbol{a}} (\boldsymbol{C}_2^1)^{\otimes q_v}$-valued function. But (2.2) implies that $(\prod_{v \in \boldsymbol{a}} E_v^{q_v}) f$ is $\bigotimes_{v \in \boldsymbol{a}} \mathrm{Sym}_{q_v}(\boldsymbol{C}_2^1)$-valued, where $\mathrm{Sym}_{q_v}(\boldsymbol{C}_2^1)$ denotes the q_v-th symmetric tensor product of $\boldsymbol{C}_2^1$. This is because $(\partial/\partial r_{1,v})(\partial/\partial r_{2,v}) = (\partial/\partial r_{2,v})(\partial/\partial r_{1,v})$.

For $f \in \mathcal{N}_k^p(\Gamma)$, let us consider its Fourier-Jacobi expansion. As $r_{1,v}, r_{2,v}$ are invariant under the action of $\mathsf{h} = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{N}(\boldsymbol{Q}) \cap \Gamma$, holomorphic functions $c_{(f; j_1, j_2)}$ are also. Hence we can express $c_{(f; j_1, j_2)}$ as

$$c_{(f; j_1, j_2)}(\mathfrak{z}) = \left\{ \prod_{v \in \boldsymbol{a}} (\pi\sqrt{-1})^{-j_{1,v} - j_{2,v}} \right\} \sum_{m \in F} g_{(f; j_1, j_2; m)}(w) \boldsymbol{e}_{\boldsymbol{a}}(mz),$$

with $\boldsymbol{C}$-valued holomorphic functions $g_{(f; j_1, j_2; m)}$. Of course $g_{(f; j_1, j_2; m)}$ is non-zero only if m belongs to a certain $\boldsymbol{Z}$-lattice of F. Since $r_{2,v}(\mathfrak{z}) = -s^{\Psi_v} \overline{w_v} r_{1,v}(\mathfrak{z})$, we can rewrite (2.3) as

$$f(\mathfrak{z}) = \sum_{\substack{j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{m \in F} \left[\sum_{\substack{j_1 \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ 0 \le j_1 \le j}} \left\{ \prod_{v \in \boldsymbol{a}} (-s^{\Psi_v} \overline{w_v})^{j_v - j_{1,v}} \right\} g_{(f; j_1, j - j_1; m)}(w) \right]$$
$$\times \left\{ \prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1} r_{1,v}(\mathfrak{z}))^{j_v} \right\} \boldsymbol{e}_{\boldsymbol{a}}(mz).$$

Further, rewriting the inside of the big square bracket by $g_{(f,j,m)}(w)$, we can get

$$f(\mathfrak{z}) = \sum_{\substack{j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{m \in F} \left\{ \prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1} r_{1,v}(\mathfrak{z}))^{j_v} \right\} g_{(f,j,m)}(w) \boldsymbol{e}_{\boldsymbol{a}}(mz). \tag{2.4}$$

We call this expression "the Fourier-Jacobi expansion" of f. Hereafter we will frequently use this type of expansion since it is more convenient to describe the Galois action on f than the usual form of (2.3).

Now we must define nearly holomorphic theta functions, which are essentially same as those defined in [**10**]. For $j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$, we denote by $\mathfrak{T}^j(\boldsymbol{C}^{\boldsymbol{a}}, \Lambda, H)$, the space of functions g on $\boldsymbol{C}^{\boldsymbol{a}}$ which satisfy (1.4) and $(\partial/\partial \overline{u_v})^{j_v + 1} g = 0$ for each $v \in \boldsymbol{a}$. Note that $g \in \mathfrak{T}^j(\boldsymbol{C}^{\boldsymbol{a}}, \Lambda, H)$ can be expressed as a polynomial of $\{\overline{u_v} \mid v \in \boldsymbol{a}\}$ with holomorphic functions as coefficients, and the "degree" of $\overline{u_v}$ is not higher than j_v for each $v \in \boldsymbol{a}$. We also express as $\mathfrak{T}_{t,\Psi}^j$ the union of $\mathfrak{T}^j(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})$ for all $\boldsymbol{Z}$-lattices L in K.

Then in the same way as the Fourier-Jacobi expansions of holomorphic modular forms, each function $g_{(f,j,m)}$ belongs to $\mathfrak{T}^{j}_{ms,\Psi}$. From Lemma 3.2 below, we can rewrite (2.4) as

$$f(\mathfrak{z}) = \sum_{\substack{j\in(\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}\\ 0\le j\le p}} \sum_{0\le m\in F} \left\{\prod_{v\in\boldsymbol{a}}((\pi\sqrt{-1})^{-1}r_{1,v}(\mathfrak{z}))^{j_v}\right\} g_{(f,j,m)}(w)\boldsymbol{e}_{\boldsymbol{a}}(mz), \tag{2.5}$$

where $0\le m$ means that m is totally positive or 0.

Next let us review nearly holomorphic Hilbert modular forms. For $f\in C^{\infty}(\mathfrak{H}_1^{\boldsymbol{a}}, \boldsymbol{C})$ and $k=(k_v)_{v\in\boldsymbol{a}}\in\boldsymbol{Z}^{\boldsymbol{a}}$, put

$$\begin{aligned} E_v f &= -\mathrm{Im}(z_v)^2\frac{\partial}{\partial\overline{z_v}}f,\\ D_{k,v}f &= \frac{\partial f}{\partial z_v}+\frac{k_v}{2\sqrt{-1}\,\mathrm{Im}(z_v)}f. \end{aligned} \tag{2.6}$$

Then we have

$$(E_v f)\,|_{k-2v}\,\alpha = E_v(f|_k\alpha),$$
$$(D_{k,v}f)\,|_{k+2v}\,\alpha = D_{k,v}(f|_k\alpha),$$

for $\alpha\in SL(2,F)$. Note that $E_{v_1}E_{v_2}=E_{v_2}E_{v_1}$ for any $v_1,v_2\in\boldsymbol{a}$. For any $p=(p_v)_{v\in\boldsymbol{a}}\in(\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}$, any $k\in\boldsymbol{Z}^{\boldsymbol{a}}$, and any congruence subgroup $\Gamma^{(1)}$ of $G^{(1)}(\boldsymbol{Q})_+$, we denote by $\mathcal{N}_k^{p,(1)}(\Gamma^{(1)})$ the space of all $f\in C^{\infty}(\mathfrak{H}_1^{\boldsymbol{a}},\boldsymbol{C})$ which satisfy $f|_k\gamma=f$ for any $\gamma\in\Gamma^{(1)}$ and $E_v^{p_v+1}f=0$ for each $v\in\boldsymbol{a}$. We write $\mathcal{N}_k^{p,(1)}$ the union of $\mathcal{N}_k^{p,(1)}(\Gamma^{(1)})$ for all congruence subgroups $\Gamma^{(1)}$ of $G^{(1)}(\boldsymbol{Q})_+$.

Any $f\in\mathcal{N}_k^{p,(1)}$ has a Fourier expansion as

$$f(z) = \sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\ 0\le j\le p}} \sum_{0\le m\in F} \left\{\prod_{v\in\boldsymbol{a}}(\pi\,\mathrm{Im}(z_v))^{-j_v}\right\} c_{(f,j,m)}\boldsymbol{e}_{\boldsymbol{a}}(mz), \tag{2.7}$$

where $c_{(f,j,m)}\in\boldsymbol{C}$. These facts are stated in section 14 of [**12**]. For any $v_0\in\boldsymbol{a}$, the Fourier expansion of $E_{v_0}f$ is as

$$(E_{v_0}f)(z) = \sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\ 0\le j\le p}} \sum_{0\le m\in F} \frac{\sqrt{-1}}{2}j_{v_0}\,\mathrm{Im}(z_{v_0})\left\{\prod_{v\in\boldsymbol{a}}(\pi\,\mathrm{Im}(z_v))^{-j_v}\right\} c_{(f,j,m)}\boldsymbol{e}_{\boldsymbol{a}}(mz).$$

We can also define a Galois action on nearly holomorphic Hilbert modular forms as the following lemma.

LEMMA 2.2. *Let* $f\in\mathcal{N}_k^{p,(1)}$ *whose Fourier expansion is as* (2.7). *For any* $\sigma\in\mathrm{Aut}(\boldsymbol{C})$, *there exists* $f^{\sigma}\in\mathcal{N}_{k^{\sigma}}^{p^{\sigma},(1)}$ *whose Fourier expansion is*

$$f^{\sigma}(z) = \sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\ 0\le j\le p}} \sum_{0\le m\in F} \left\{\prod_{v\in\boldsymbol{a}}(\pi\,\mathrm{Im}(z_{v\sigma}))^{-j_v}\right\} c^{\sigma}_{(f,j,m)}\boldsymbol{e}_{\boldsymbol{a}}(mz).$$

This lemma is also proved in section 14 of [**12**].

Let us consider the relation between this Galois action and the values of nearly holomorphic Hilbert modular forms at so-called CM-points. For a CM-extension K of F, take $\tau \in K$ and a CM-type $\Psi = (\Psi_v)_{v \in \boldsymbol{a}}$ of K so that $\mathrm{Im}(\tau^{\Psi_v}) > 0$ for each $v \in \boldsymbol{a}$. Then $z_0 = \tau^{\Psi} = (\tau^{\Psi_v})_{v \in \boldsymbol{a}} \in W^{(1)}(\Psi)$.

Theorem 2.3. *Let $f \in \mathcal{N}_k^{p,(1)}(\Gamma^{(1)})$ and $g \in \mathcal{M}_k^{(1)}(\Gamma^{(1)})$ with some congruence subgroup $\Gamma^{(1)}$ of $G^{(1)}(\boldsymbol{Q})_+$. Take $X \in \mathcal{X}^{(1)}$ so that $\mathcal{O}_F^\times \Gamma^{(1)} \supset \Gamma_X^{(1)}$. For any $z_0 \in W^{(1)}(\Psi)$ and any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, take $\tilde{z}_0 \in W^{(1)}(\Psi\sigma)$ as*

$$\left[J_{\tilde{X}X}^{(1)}\left(\begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}^{-1} \right)(\varphi_X^{(1)}(z_0)) \right]^\sigma = \varphi_{\tilde{X}}^{(1)}(\tilde{z}_0),$$

where

$$\tilde{X} = \begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}^{-1} X \begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}.$$

Assume that $g(z_0) \neq 0$. Then we have

$$(*) \qquad [g(z_0)^{-1} f(z_0)]^\sigma = g^\sigma(\tilde{z}_0)^{-1} f^\sigma(\tilde{z}_0).$$

Proof. In case when $\sigma \in \mathrm{Aut}(\boldsymbol{C}/K_\Psi^*)$, this theorem is proved in [**12**] (or essentially in [**4**]). For an arbitrary $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, we can prove it in the same way combining Proposition 1.3. We have only to prove this theorem for a sufficiently small X since the left hand side of $(*)$ is invariant when we substitute z_0 by $\gamma(z_0)$ with $\gamma \in \Gamma^{(1)}$.

For an arbitrary $b \in K^\times$ such that $bb^\rho = 1$ and $b^2 \neq 1$, put $\beta = \Phi_{z_0}^{(1)}(b)$. Then $\beta \in SL(2, F)$ and $\beta(z_0) = z_0$. For $l = (l_v)_{v \in \boldsymbol{a}} \in \boldsymbol{N}^{\boldsymbol{a}}$, take $f_0 \in \mathcal{M}_l^{(1)}$ so that $f_0(z_0) \neq 0$. Set $h = (f_0|_l \beta)/f_0 \in \mathcal{A}_0^{(1)}$. For $D_{l,v}$ in (2.6), consider

$$(D_{l,v} f_0)(z) = \frac{\partial f_0}{\partial z_v}(z) + \frac{l_v}{2\sqrt{-1}\,\mathrm{Im}(z_v)} f_0(z).$$

Then we have

$$\begin{aligned} (D_{l,v} f_0)\,|_{l+2v}\, \beta &= D_{l,v}(f_0|_l \beta) \\ &= D_{l,v}(h f_0) \\ &= h(D_{l,v} f_0) + f_0(D_{0,v} h). \end{aligned}$$

Considering the value at z_0, we have

$$(D_{l,v} f_0)(z_0) \cdot ((b^{\Psi_v})^{-2} - 1) \prod_{v' \in \boldsymbol{a}} (b^{\Psi_{v'}})^{-l_{v'}} = f_0(z_0)(D_{0,v} h)(z_0), \tag{2.8}$$

since $j_{v'}^{(1)}(\beta, z_0) = b^{\Psi_{v'}}$ and $h(z_0) = \prod_{v' \in \boldsymbol{a}} (b^{\Psi_{v'}})^{-l_{v'}}$ for each $v' \in \boldsymbol{a}$. Note that $D_{0,v} h \in \mathcal{A}_{2v}^{(1)}$ and $((\pi\sqrt{-1})^{-1} D_{0,v} h)^\sigma = ((\pi\sqrt{-1})^{-1} D_{0,v\sigma} h^\sigma)$. Clearly we can get $D_{l,v} f_0 \in \mathcal{N}_{l+2v}^{v,(1)}$ and

$$(\pi\sqrt{-1})^{-1} D_{l^\sigma, v\sigma} f_0^\sigma = ((\pi\sqrt{-1})^{-1} D_{l,v} f_0)^\sigma.$$

Take $Y \in \mathscr{Y}^{(1)}$ (sufficiently small) so that f_0, $f_0|_l\beta \in \mathscr{M}_l^{(1)}(\Gamma_Y^{(1)} \cap SL(2,F))$ and take $\tilde{z_0} \in W^{(1)}(\Psi\sigma)$ so that

$$\left[J_{\tilde{Y}Y}^{(1)}\left(\begin{pmatrix}1 & 0\\ 0 & \chi(\sigma)\end{pmatrix}^{-1}\right)(\varphi_Y^{(1)}(z_0))\right]^{\sigma} = \varphi_{\tilde{Y}}^{(1)}(\tilde{z_0}),$$

where $\tilde{Y} = \begin{pmatrix}1 & 0\\ 0 & \chi(\sigma)\end{pmatrix}^{-1} Y \begin{pmatrix}1 & 0\\ 0 & \chi(\sigma)\end{pmatrix}$. Put $\tilde{\beta} = \Phi_{\tilde{z}_0}^{(1)}(b)$. Then we have

$$\tilde{\beta} = \Phi_{\tilde{z}_0}^{(1)}(b) \in \begin{pmatrix}1 & 0\\ 0 & \chi(\sigma)\end{pmatrix}^{-1}(Y \cap G_{1A}^{(1)})\beta(Y \cap G_{1A}^{(1)})\begin{pmatrix}1 & 0\\ 0 & \chi(\sigma)\end{pmatrix}.$$

This implies $f_0^{\sigma}|_{l^{\sigma}}\tilde{\beta} = (f_0|_l\beta)^{\sigma}$ from Theorem 1.5 of [**8**]. In the same way as (2.8), we can get

$$(D_{l^{\sigma},v\sigma}f_0^{\sigma})(\tilde{z_0}) \cdot ((b^{\Psi_v\sigma})^{-2} - 1)\prod_{v' \in \boldsymbol{a}}(b^{\Psi_{v'}\sigma})^{-l_{v'}} = f_0^{\sigma}(\tilde{z_0})(D_{0,v}h^{\sigma})(\tilde{z_0}).$$

Combining (1.7) and Proposition 1.3, we can show that $(\pi\sqrt{-1})^{-1}D_{l,v}f_0$ satisfies the property of this theorem. By the next lemma, we can write f as a polynomial of $\{(\pi\sqrt{-1})^{-1}D_{l,v}f_0 \mid v \in \boldsymbol{a}\}$ with meromorphic modular forms as coefficients. Hence the proof of our theorem is completed (admitting Lemma 2.4). $\square$

LEMMA 2.4. *Take any $l \in \boldsymbol{N}^{\boldsymbol{a}}$ and $k \in \boldsymbol{Z}^{\boldsymbol{a}}$. For any $0 \not\equiv f_0 \in \mathscr{M}_l^{(1)}$ and $f \in \mathscr{N}_k^{p,(1)}$, we can express f as*

$$f(z) = f_0(z)^{-N}\sum_{\substack{j \in (\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}\\ 0 \le j \le p}} g_j(z)\prod_{v \in \boldsymbol{a}}\{(\pi\sqrt{-1})^{-1}(D_{l,v}f_0)(z)\}^{j_v}, \tag{2.9}$$

with $N \in \boldsymbol{N}$ and $g_j \in \mathscr{M}_{k-2j+Nl-(\sum_{v\in\boldsymbol{a}} j_v)l}^{(1)}$.

PROOF. Put

$$f(z) = \sum_{0 \le j \le p}\left\{\prod_{v \in \boldsymbol{a}}(\pi\,\mathrm{Im}(z_v))^{-j_v}\right\}h_{(f,j)}(z),$$

with holomorphic functions $h_{(f,j)}$ on $\mathfrak{H}_1^{\boldsymbol{a}}$. Take a finite subset P_f of $(\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}$ as

$$P_f = \left\{ j \in (\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}} \,\middle|\, \begin{array}{l}\text{There exists } j' \in (\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}\\ \text{such that } j \le j' \text{ and } h_{(f,j')} \not\equiv 0\end{array}\right\}.$$

Choose $q = (q_v)_{v\in\boldsymbol{a}} \in P_f$ so that $\sum_{v\in\boldsymbol{a}} q_v$ is maximum. Then we have $\{\prod_{v\in\boldsymbol{a}}((-\pi\sqrt{-1})E_v)^{q_v}\}f = (\prod_{v\in\boldsymbol{a}}(2^{-q_v}q_v!))h_{(f,q)}$ and hence $h_{(f,q)} \in \mathscr{M}_{k-2q}^{(1)}$. Define $\hat{f} \in \mathscr{M}_{k+(\sum_{v\in\boldsymbol{a}} q_v)l}^{(1)}$ as

$$\hat{f}(z) = f_0(z)^{(\sum_{v\in\boldsymbol{a}} q_v)}f(z) - h_{(f,q)}(z)\prod_{v \in \boldsymbol{a}}\left\{\frac{2\sqrt{-1}}{\pi l_v}(D_{l,v}f_0)(z)\right\}^{q_v}.$$

Take $P_{\hat{f}}$ in the same way as P_f. Then $q \notin P_{\hat{f}}$ and $P_{\hat{f}} \subset P_f$. Hence we can prove this lemma inductively. $\square$

3. Galois action on nearly holomorphic theta functions.

In the previous section, we defined a space of nearly holomorphic theta functions. In this section we will construct a certain Galois action on $\mathfrak{T}^p_{t,\Psi}$ which is an extension of Theorem 1.1 constructed in [**13**].

For $g \in \mathfrak{T}^p_{t,\Psi}$, in the same way as holomorphic theta functions, we define

$$g_*(u) = \exp\left(-\frac{\pi}{2}H_{t,\Psi}(u,u)\right)g(u) \quad (u \in \boldsymbol{C}^{\boldsymbol{a}}).$$

We can also define $g_*(y^\Psi)$ for $y \in K_A$ in the same way as the holomorphic case. Then the action of $\mathsf{N}^{(1,1)}(s,\Psi)(\boldsymbol{Q})$ on $\mathfrak{T}^p_{t,\Psi}$ can be defined as

$$(g|\mathsf{h}_{y,b})(u) = g(u+y^\Psi)\exp\left(-\pi H_{t,\Psi}\left(y^\Psi, u+\frac{1}{2}y^\Psi\right) + 2\pi\sqrt{-1}\sum_{v\in\boldsymbol{a}}(s^{-1}t)_v b_v\right)$$

for any $g \in \mathfrak{T}^p_{t,\Psi}$ and $\mathsf{h}_{y,b} = \begin{pmatrix} 1 & sy^\rho & b+(1/2)syy^\rho \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{N}^{(1,1)}(s,\Psi)(\boldsymbol{Q})$. Then $g|\mathsf{h}_{y,b} \in \mathfrak{T}^p_{t,\Psi}$. Note that $(g|\mathsf{h}_{y,0})(0) = g_*(y^\Psi)$ for $y \in K$ and

$$\mathfrak{T}^p(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi}) = \{g \in \mathfrak{T}^p_{t,\Psi} \mid g|\mathsf{h}_{y,0} = g \text{ for any } y \in L\}. \tag{3.1}$$

To know the space of nearly holomorphic theta functions, we must define a differential operator κ_v on $\mathfrak{T}^p_{t,\Psi}$ as

$$(\kappa_v g)(u) = \frac{\partial g}{\partial u_v}(u) + 2\pi\sqrt{-1}t^{\Psi_v}\overline{u_v}g(u)$$

for each $v \in \boldsymbol{a}$. Then we have $\kappa_v g \in \mathfrak{T}^{p+v}_{t,\Psi}$ and by a computation,

$$(\kappa_v g) \mid \mathsf{h} = \kappa_v(g|\mathsf{h}) \tag{3.2}$$

for any $g \in \mathfrak{T}^p_{t,\Psi}$ and any $\mathsf{h} \in \mathsf{N}(\boldsymbol{Q})$. We can easily verify $\kappa_{v_1}\kappa_{v_2} = \kappa_{v_2}\kappa_{v_1}$ for any $v_1, v_2 \in \boldsymbol{a}$. We have the following lemma.

Lemma 3.1. *For $p \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$, we have*

$$\mathfrak{T}^p(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi}) = \sum_{\substack{l \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le l \le p}} \left(\prod_{v\in\boldsymbol{a}}\kappa_v^{l_v}\right)\mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi}).$$

Proof. From (3.1) and (3.2), we can easily show

$$\kappa_v(\mathfrak{T}^j(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})) \subset \mathfrak{T}^{j+v}(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi}).$$

For any $g \in \mathfrak{T}^p(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})$, put

$$g(u) = \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le j \le p}} g_j(u)\prod_{v\in\boldsymbol{a}}(2\pi\sqrt{-1}t^{\Psi_v}\overline{u_v})^{j_v}, \tag{3.3}$$

with holomorphic functions g_j. Take the finite subset P of $(\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ as

$$P = \left\{ j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \,\middle|\, \begin{array}{l} \text{There exists } l \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ \text{such that } l \geq j \text{ and } g_l \not\equiv 0. \end{array} \right\}.$$

Choose $j_0 \in P$ so that $\sum_{v \in \boldsymbol{a}} j_{0,v}$ is maximum. (In general, it is not uniquely determined.) Then, by computing precisely, we get $g_{j_0} \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$. Note that $g_{j_0} \not\equiv 0$. This implies $H_{t,\Psi}$ is semi-positive definite, that is, $\mathrm{Im}(t^{\Psi_v}) \geq 0$ for any $v \in \boldsymbol{a}$. Put

$$\hat{g}(u) = g(u) - \left\{ \left(\prod_{v \in \boldsymbol{a}} \kappa_v^{j_{0,v}} \right) g_{j_0} \right\}(u),$$

and set

$$\hat{g}(u) = \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \leq j \leq p}} \hat{g}_j(u) \prod_{v \in \boldsymbol{a}} (2\pi\sqrt{-1}\, t^{\Psi_v} \overline{u_v})^{j_v}.$$

In the same way, take

$$\hat{P} = \left\{ j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \,\middle|\, \begin{array}{l} \text{There exists } l \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ \text{such that } l \geq j \text{ and } \hat{g}_l \not\equiv 0. \end{array} \right\}.$$

Then we have

$$\hat{P} \subset P - \{j_0\}.$$

Hence we can prove this lemma by an induction. □

In this proof we showed the following lemma.

LEMMA 3.2. *The space* $\mathfrak{T}_{t,\Psi}^{p} \neq \{0\}$ *only if* $\mathrm{Im}(t^{\Psi_v}) > 0$ *for each* $v \in \boldsymbol{a}$ *or* $t = 0$.

The purpose of this section is the following theorem.

THEOREM 3.3. *For* $g \in \mathfrak{T}^p(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$ *and* $(\sigma, \Psi, a) \in C_{\Psi}(\boldsymbol{C})$, *there exists* $g^{(\sigma,\Psi,a)} \in \mathfrak{T}^{p^\sigma}(\boldsymbol{C}^{\boldsymbol{a}}, (aL)^{\Psi\sigma}, H_{\iota(\sigma,a)t,\Psi\sigma})$ *such that*

$$(g^{(\sigma,\Psi,a)})_*((ay)^{\Psi\sigma}) = \{g_*(y^{\Psi})\}^{\sigma}$$

for each $y \in K$.

To prove this, we have only to prove the following lemma.

LEMMA 3.4. *Take any* $g \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$, $(\sigma, \Psi, a) \in C_{\Psi}(\boldsymbol{C})$ *and* $p = (p_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$. *Then there exists a non-zero constant* c *which satisfies*

$$\left\{ \left(\left(\prod_{v \in \boldsymbol{a}} \kappa_v^{p_v} \right) g \right)_* (y^{\Psi}) \right\}^{\sigma} = c \left(\left(\prod_{v \in \boldsymbol{a}} (\kappa_{v\sigma})^{p_v} \right) g^{(\sigma,\Psi,a)} \right)_* ((ay)^{\Psi\sigma}), \tag{3.4}$$

for any $y \in K$.

Combining this lemma and Lemma 3.1, we can easily get Theorem 3.3.

Before the proof, we must review basic facts about theta functions. For $l \in \boldsymbol{N}$, let $\mathfrak{H}_l$ be the Siegel upper half space of degree l. Put

$$\theta^{(l)}(u, Z; p_1, p_2) = \sum_{x \in \boldsymbol{Z}^l} \exp(\pi\sqrt{-1}({}^t(x+p_1)Z(x+p_1) + 2^t(x+p_1)(u+p_2))),$$

$$\varphi^{(l)}(u, Z; p_1, p_2) = \exp(\pi\sqrt{-1}{}^t u(Z-\bar{Z})^{-1}u)\theta^{(l)}(u, Z; p_1, p_2),$$

with $u \in \boldsymbol{C}^l$, $Z \in \mathfrak{H}_l$ and $p_1, p_2 \in \boldsymbol{Q}^l$. As stated in [**5**], take $(\omega_1\ \ \omega_2) \in \boldsymbol{C}^l_{2l}$ such that $Z = \omega_2^{-1}\omega_1 \in \mathfrak{H}_l$ $(\omega_1, \omega_2 \in GL(l, \boldsymbol{C}))$ and set

$$\varphi^{(l)}(u, (\omega_1\ \ \omega_2); p_1, p_2; p_1') = \theta^{(l)}(0, Z; p_1', 0)^{-1}\varphi^{(l)}(\omega_2^{-1}u, Z; p_1, p_2)$$

where $p_1' \in \boldsymbol{Q}^l$ so that $\theta^{(l)}(0, Z; p_1', 0) \neq 0$ (it is possible from [**5**]). Next we consider theta functions on $(\boldsymbol{C}^l)^{\boldsymbol{a}}$. Take B and $B^{(l)}$ as in section 1 and put

$$\theta^{(l)}_{F,B}(u, z; p_1, p_2) = \theta^{(l|\boldsymbol{a}|)}\left({}^t\boldsymbol{B}^{(l)}\begin{pmatrix} u_{v_1} \\ \vdots \\ u_{v_{|\boldsymbol{a}|}}\end{pmatrix}, \varepsilon^{(l)}_B(z); p_1, p_2\right),$$

$$\varphi^{(l)}_{F,B}(u, z; p_1, p_2) = \varphi^{(l|\boldsymbol{a}|)}\left({}^t\boldsymbol{B}^{(l)}\begin{pmatrix} u_{v_1} \\ \vdots \\ u_{v_{|\boldsymbol{a}|}}\end{pmatrix}, \varepsilon^{(l)}_B(z); p_1, p_2\right),$$

for $u \in (\boldsymbol{C}^l)^{\boldsymbol{a}}$, $z \in \mathfrak{H}_l^{\boldsymbol{a}}$ and $p_1, p_2 \in \boldsymbol{Q}^{l|\boldsymbol{a}|}$. Take $(\omega_1\ \ \omega_2) \in (\boldsymbol{C}^l_{2l})^{\boldsymbol{a}}$ such that $\omega_2^{-1}\omega_1 \in \mathfrak{H}_l^{\boldsymbol{a}}$ $(\omega_1, \omega_2 \in GL(l, \boldsymbol{C})^{\boldsymbol{a}})$ and set

$$\varphi^{(l)}_{F,B}(u, (\omega_1\ \ \omega_2); p_1, p_2; p_1') = \theta^{(l)}_{F,B}(0, z; p_1', 0)^{-1}\varphi^{(l)}_{F,B}(\omega_2^{-1}u, z; p_1, p_2)$$

for $u \in (\boldsymbol{C}^l)^{\boldsymbol{a}}$, $z = \omega_2^{-1}\omega_1 \in \mathfrak{H}_l^{\boldsymbol{a}}$ and $p_1, p_2, p_1' \in \boldsymbol{Q}^{l|\boldsymbol{a}|}$ so that $\theta^{(l)}_{F,B}(0, z; p_1', 0) \neq 0$. For fixed p_1, p_2, p_1' (by the same reason as in [**5**]), there exists a congruence subgroup $\Gamma^{(l)}$ of $G^{(l)}(\boldsymbol{Q})_+$ which satisfies

$$\varphi^{(l)}_{F,B}(u, (\omega_1\ \ \omega_2){}^t\gamma; p_1, p_2; p_1') = \varphi^{(l)}_{F,B}(u, (\omega_1\ \ \omega_2); p_1, p_2; p_1')$$

for any $\gamma \in \Gamma^{(l)}$ and $(\omega_1\ \ \omega_2)$.

Proof of Lemma 3.4.

This lemma can be proved by combining Theorem 2.3 and the proof of the main theorem of [**5**] (or Theorem 1.1 proved in [**13**]).

As stated in [**13**], the space $\mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^{\Psi}, H_{t,\Psi})$ is span by functions

$$\begin{aligned} g(u) &= \varphi^{(1)}_{F,B}(u, (\tau_1^{\Psi}, \tau_2^{\Psi}); j, 0; j') \\ &= \theta^{(1)}_{F,B}(0, (\tau_1\tau_2^{-1})^{\Psi}; j', 0)^{-1}\varphi^{(1)}_{F,B}((\tau_2^{-1})^{\Psi}u, (\tau_1\tau_2^{-1})^{\Psi}; j, 0) \end{aligned} \tag{3.5}$$

with some $\tau_1, \tau_2 \in K^{\times}$, $j, j' \in \boldsymbol{Q}^{|\boldsymbol{a}|}$ and $z_0 = (\tau_1\tau_2^{-1})^{\Psi} \in \mathfrak{H}_1^{\boldsymbol{a}}$. Fix this g. Then from the proof of Theorem 3.1 of [**13**], we have

$$g^{(\sigma,\Psi,a)}(u) = \varphi^{(1)}_{F,B}(u, (\alpha_1\tau_1 + \alpha_2\tau_2, \alpha_3\tau_1 + \alpha_4\tau_2)^{\Psi\sigma}; j, 0; j'). \tag{3.6}$$

Here

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$
$$\in GL(2,F) \cap (\tilde{X} \cap SL(2,F_A)) \begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}^{-1} GL(2,F_\infty)\Phi^{(1)}_{z_0}(a)$$

with (sufficiently small) $\tilde{X} \in \mathscr{X}^{(1)}$ so that $\varphi^{(1)}_{F,B}(u, (\omega_1\ \ \omega_2)^t\gamma; j, 0; j') = \varphi^{(1)}_{F,B}(u, (\omega_1\ \ \omega_2); j, 0; j')$ for any $(\omega_1\ \ \omega_2)$ and any $\gamma \in \Gamma^{(1)}_{\tilde{X}} \cap SL(2,F)$. Note that the right hand side of (3.6) is independent of the choice of α. Put

$$\widetilde{z_0} = [(\alpha_1\tau_1 + \alpha_2\tau_2)(\alpha_3\tau_1 + \alpha_4\tau_2)^{-1}]^{\Psi\sigma},$$
$$X = \begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix} \tilde{X} \begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}^{-1} \in \mathscr{X}^{(1)}.$$

Then

$$\left[J^{(1)}_{\tilde{X}X}\left(\begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}^{-1} \right)(\varphi^{(1)}_X(z_0)) \right]^\sigma = \varphi^{(1)}_{\tilde{X}}(\widetilde{z_0})$$

can be verified by Proposition 1.3. Take any $h \in \mathscr{A}^{(1)}_p(\Gamma^{(1)}_X)$ so that h and $1/h$ are holomorphic at z_0. (If it is impossible, take $\tilde{X}$, and hence $X, \widetilde{z_0}$ again.) By Theorem 1.5 of [**8**], we get $h^\sigma \in \mathscr{A}^{(1)}_{p^\sigma}(\Gamma^{(1)}_{\tilde{X}})$ and both of h^σ and $1/h^\sigma$ are holomorphic at $\widetilde{z_0}$.

Hereafter let us prove

$$\left[h(z_0)^{-1} \left\{ \left(\prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1}\tau_2^{\Psi_v}\kappa_v)^{p_v} \right) g \right\}_* (y^\Psi) \right]^\sigma$$
$$= h^\sigma(\widetilde{z_0})^{-1} \left\{ \left(\prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1}(\alpha_3\tau_1 + \alpha_4\tau_2)^{\Psi_v\sigma}\kappa_{v\sigma})^{p_v} \right) g^{(\sigma,\Psi,a)} \right\}_* ((ay)^{\Psi\sigma}), \tag{3.7}$$

for any $y \in K$.

We can rewrite

$$\left\{ \left(\prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1}\kappa_v)^{p_v} \right) g \right\}_* (y^\Psi) = \left[\left\{ \prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1}\kappa_v)^{p_v} \right\} (g|\mathsf{h}_{y,0}) \right](0). \tag{3.8}$$

And by a calculation, we can get

$$(g|\mathsf{h}_{y,0})(u) = \exp(-\pi\sqrt{-1}{}^t q_{1,y} q_{2,y})\varphi^{(1)}_{F,B}(u, (\tau_1^\Psi\ \ \tau_2^\Psi); j + q_{1,y}, q_{2,y}; j'), \tag{3.9}$$

where

$$y = (\tau_1\ \ \tau_2) \begin{pmatrix} \beta_1 & \cdots & \beta_{|\boldsymbol{a}|} & & 0 & \\ & 0 & & \beta'_1 & \cdots & \beta'_{|\boldsymbol{a}|} \end{pmatrix} \begin{pmatrix} q_{1,y} \\ q_{2,y} \end{pmatrix} \quad \text{with } q_{1,y}, q_{2,y} \in \boldsymbol{Q}^{|\boldsymbol{a}|}.$$

Hence (3.8) is equal to

$$\exp(-\pi\sqrt{-1}\,{}^t q_{1,y} q_{2,y})\left\{\prod_{v\in\boldsymbol{a}}\left((\pi\sqrt{-1})^{-1}\left(\frac{\partial}{\partial u_v}\right)\right)^{p_v}\right\}$$
$$\circ\,\varphi^{(1)}_{F,B}((\tau_2^{\Psi})^{-1}u, z_0; j+q_{1,y}, q_{2,y})|_{u=0}\theta^{(1)}_{F,B}(0, z_0; j', 0)^{-1}. \tag{3.10}$$

As is well known, $\theta^{(1)}_{F,B}(0, z; j', 0)$ is a so-called modular form of half integral weight. We can rewrite (3.10) as

$$\exp(-\pi\sqrt{-1}\,{}^t q_{1,y} q_{2,y})\left\{\prod_{v\in\boldsymbol{a}}\left((\pi\sqrt{-1})^{-1}\left(\frac{\partial}{\partial u_v}\right)\right)^{p_v}\right\}$$
$$\circ\,\varphi^{(1)}_{F,B}((\tau_2^{\Psi})^{-1}u, z_0; j+q_{1,y}, q_{2,y})|_{u=0}\theta^{(1)}_{F,B}(0, z_0; j', 0)/\theta^{(1)}_{F,B}(0, z_0; j', 0)^2. \tag{3.11}$$

Put

$$f_y(z) = \exp(-\pi\sqrt{-1}\,{}^t q_{1,y} q_{2,y})\left\{\prod_{v\in\boldsymbol{a}}\left((\pi\sqrt{-1})^{-1}\tau_2^{\Psi_v}\left(\frac{\partial}{\partial u_v}\right)\right)^{p_v}\right\}$$
$$\circ\,\varphi^{(1)}_{F,B}((\tau_2^{\Psi})^{-1}u, z; j+q_{1,y}, q_{2,y})|_{u=0}\theta^{(1)}_{F,B}(0, z; j', 0). \tag{3.12}$$

Then $f_y \in \mathscr{N}^{p,(1)}_{p+\mathbf{1}}$ can be verified and clearly $\theta^{(1)}_{F,B}(0, z; j', 0)^2 \in \mathscr{M}^{(1)}_{\mathbf{1}}$. The left hand side of (3.7) is equal to $[f_y(z_0)/\theta^{(1)}_{F,B}(0, z_0; j', 0)^2 h(z_0)]^{\sigma}$. Note that $\theta^{(1)}_{F,B}(0, z; j', 0)^2$ has a Fourier expansion with $\boldsymbol{Q}$-coefficients, hence it is stable under the action of $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ in the sense of (1.6).

In the same way, the right hand side of (3.7) can also be expressed as a value of nearly holomorphic modular form at a CM-point. Take $\tilde{q}_{2,y} \in \boldsymbol{Q}^{|\boldsymbol{a}|}$ so that $\tilde{q}_{2,y} \equiv \chi(\sigma)_l q_{2,y} \bmod (\boldsymbol{Z}_l)^{|\boldsymbol{a}|}$ and ${}^t q_{1,y}(\tilde{q}_{2,y} - \chi(\sigma)_l q_{2,y}) \in 2(\boldsymbol{Z}_l)^{|\boldsymbol{a}|}$ for each finite prime l, where $\chi(\sigma)_l$ denotes the l-component of $\chi(\sigma)$. Next take $\tilde{Y} \in \mathscr{Z}^{(1)}$ such that $\tilde{Y} \subset \tilde{X}$ and

$$\varphi^{(1)}_{F,B}(u, (\omega_1\ \ \omega_2)\,{}^t\gamma; j+q_{1,y}, \tilde{q}_{2,y}; j') = \varphi^{(1)}_{F,B}(u, (\omega_1\ \ \omega_2); j+q_{1,y}, \tilde{q}_{2,y}; j')$$

for any $\gamma \in \Gamma^{(1)}_{\tilde{Y}} \cap SL(2, F)$ and $(\omega_1\ \ \omega_2)$. Since $g^{(\sigma,\Psi,a)}$ is independent of the choice of α, we can take α' instead of α so that

$$\alpha' = \begin{pmatrix} \alpha_1' & \alpha_2' \\ \alpha_3' & \alpha_4' \end{pmatrix}$$
$$\in GL(2, F) \cap (\tilde{Y} \cap SL(2, F_{\boldsymbol{A}}))\begin{pmatrix} 1 & 0 \\ 0 & \chi(\sigma) \end{pmatrix}^{-1} GL(2, F_\infty)\Phi^{(1)}_{z_0}(a),$$

and take

$$\widetilde{z_0}' = [(\alpha_1'\tau_1 + \alpha_2'\tau_2)(\alpha_3'\tau_1 + \alpha_4'\tau_2)^{-1}]^{\Psi\sigma}.$$

Note that $\alpha(\alpha')^{-1} \in SL(2, F) \cap \Gamma^{(1)}_{\tilde{X}}$ and $\alpha(\alpha')^{-1}(\widetilde{z_0}') = \widetilde{z_0}$. Then we have

$$\left\{\left(\prod_{v\in\boldsymbol{a}}((\pi\sqrt{-1})^{-1}\kappa_{v\sigma})^{p_v}\right)g^{(\sigma,\Psi,a)}\right\}_*((ay)^{\Psi\sigma})$$
$$= \exp(-\pi\sqrt{-1}{}^t q_{1,y}\tilde{q}_{2,y})\left\{\prod_{v\in\boldsymbol{a}}\left((\pi\sqrt{-1})^{-1}\left(\frac{\partial}{\partial u_{v\sigma}}\right)\right)^{p_v}\right\}$$
$$\circ\,\varphi^{(1)}_{F,B}((\alpha_3'\tau_1^{\Psi\sigma}+\alpha_4'\tau_2^{\Psi\sigma})^{-1}u,\tilde{z_0}';j+q_{1,y},\tilde{q}_{2,y})|_{u=0}$$
$$\times\,\theta^{(1)}_{F,B}(0,\tilde{z_0}';j',0)^{-1}. \tag{3.13}$$

Since $j^{(1)}_{v\sigma}(\alpha(\alpha')^{-1},\tilde{z_0}')=(\alpha_3\tau_1+\alpha_4\tau_2)^{\Psi_v\sigma}\{(\alpha_3'\tau_1+\alpha_4'\tau_2)^{\Psi_v\sigma}\}^{-1}$, we can rewrite the right hand side of (3.7) as

$$h^\sigma(\tilde{z_0}')^{-1}\exp(-\pi\sqrt{-1}{}^t q_{1,y}\tilde{q}_{2,y})\left\{\prod_{v\in\boldsymbol{a}}\left((\pi\sqrt{-1})^{-1}(\alpha_3'\tau_1+\alpha_4'\tau_2)^{\Psi_v\sigma}\left(\frac{\partial}{\partial u_{v\sigma}}\right)\right)^{p_v}\right\}$$
$$\circ\,\varphi^{(1)}_{F,B}((\alpha_3'\tau_1^{\Psi\sigma}+\alpha_4'\tau_2^{\Psi\sigma})^{-1}u,\tilde{z_0}';j+q_{1,y},\tilde{q}_{2,y})|_{u=0}\theta^{(1)}_{F,B}(0,\tilde{z_0}';j',0)^{-1}.$$

Put

$$\tilde{f}_y(z)=\exp(-\pi\sqrt{-1}{}^t q_{1,y}\tilde{q}_{2,y})\left\{\prod_{v\in\boldsymbol{a}}\left((\pi\sqrt{-1})^{-1}(\alpha_3'\tau_1+\alpha_4'\tau_2)^{\Psi_v\sigma}\left(\frac{\partial}{\partial u_{v\sigma}}\right)\right)^{p_v}\right\}$$
$$\circ\,\varphi^{(1)}_{F,B}((\alpha_3'\tau_1^{\Psi\sigma}+\alpha_4'\tau_2^{\Psi\sigma})^{-1}u,z;j+q_{1,y},\tilde{q}_{2,y})|_{u=0}\theta^{(1)}_{F,B}(0,z;j',0).$$

Then $\tilde{f}_y\in\mathscr{N}^{p^\sigma,(1)}_{p^\sigma+\mathbf{1}}$, and by a computation, we obtain $\tilde{f}_y=f_y^\sigma$. We can prove (3.7) by using Theorem 2.3 since the right hand side of (3.7) is equal to $\tilde{f}_y(\tilde{z_0}')/\theta^{(1)}_{F,B}(0,\tilde{z_0}';j',0)^2h^\sigma(\tilde{z_0}')$. □

In the rest of this section, we will make important examples of nearly holomorphic modular forms which satisfy Theorem 4.1 (the main theorem of this paper) and will be used in order to prove that theorem. To construct nearly holomorphic modular forms, we use the embeddings defined in section 1.

As stated in [**6**], we can define a $\boldsymbol{C}^{3|\boldsymbol{a}|}_{3|\boldsymbol{a}|}$-valued holomorphic function T on $\mathfrak{H}_{3|\boldsymbol{a}|}$ by

$$T(Z)=\frac{\theta^{(3|\boldsymbol{a}|)}(0,Z;q_1^{(0)},0)}{2\pi\sqrt{-1}}\left.\begin{pmatrix}\dfrac{\partial}{\partial u_1}\theta^{(3|\boldsymbol{a}|)}(u,Z;q_1^{(1)},0) & \cdots & \dfrac{\partial}{\partial u_1}\theta^{(3|\boldsymbol{a}|)}(u,Z;q_1^{(3|\boldsymbol{a}|)},0)\\ \cdots & \cdots & \cdots\\ \dfrac{\partial}{\partial u_{3|\boldsymbol{a}|}}\theta^{(3|\boldsymbol{a}|)}(u,Z;q_1^{(1)},0) & \cdots & \dfrac{\partial}{\partial u_{3|\boldsymbol{a}|}}\theta^{(3|\boldsymbol{a}|)}(u,Z;q_1^{(3|\boldsymbol{a}|)},0)\end{pmatrix}\right|_{u=0} \tag{3.14}$$

where $u=\begin{pmatrix}u_1\\ \vdots\\ u_{3|\boldsymbol{a}|}\end{pmatrix}\in\boldsymbol{C}^{3|\boldsymbol{a}|}$ and $q_1^{(0)},\dots,q_1^{(3|\boldsymbol{a}|)}\in\boldsymbol{Q}^{3|\boldsymbol{a}|}$. For each $Z\in\mathfrak{H}_{3|\boldsymbol{a}|}$, we can take suitable $q_1^{(0)},\dots,q_1^{(3|\boldsymbol{a}|)}\in\boldsymbol{Q}^{3|\boldsymbol{a}|}$ such that $\det(T(Z))\neq0$. As stated in [**6**], T is vector-valued modular form which satisfies

$$T(\gamma(Z)) = \det(\gamma_3 Z + \gamma_4) \cdot (\gamma_3 Z + \gamma_4) T(Z)$$

for $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}$ contained in a certain congruence subgroup of $Sp(3|\boldsymbol{a}|, \boldsymbol{Q})$ ($\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \boldsymbol{Q}^{3|\boldsymbol{a}|}_{3|\boldsymbol{a}|}$). Note that $\det(T(Z))$ is a holomorphic modular form of weight $3|\boldsymbol{a}| + 1$. Using this T and the embedding $\varepsilon_B^{(3)}$ (defined in section 1), we can define a $\boldsymbol{C}^{3|\boldsymbol{a}|}_{3|\boldsymbol{a}|}$-valued holomorphic function $\varDelta$ on $\mathfrak{H}_3^{\boldsymbol{a}}$ by

$$\varDelta(z) = B^{(3)} T(\varepsilon_B^{(3)}(z)) \quad (z \in \mathfrak{H}_3^{\boldsymbol{a}}).$$

By (1.9), there exists a congruence subgroup $\varGamma^{(3)}$ of $G^{(3)}(\boldsymbol{Q})_+$ such that

$$\varDelta(\gamma(z)) = \left(\prod_{v \in \boldsymbol{a}} \det(j_v^{(3)}(\gamma, z))\right) \begin{pmatrix} j_{v_1}^{(3)}(\gamma, z) & & \\ & \ddots & \\ & & j_{v_{|\boldsymbol{a}|}}^{(3)}(\gamma, z) \end{pmatrix} \varDelta(z),$$

for any $\gamma \in \varGamma^{(3)}$. Next define a $\boldsymbol{C}^{3|\boldsymbol{a}|}_{3|\boldsymbol{a}|}$-valued holomorphic function $\hat{\varDelta}$ on $\mathfrak{D}^{(1,1)}(s, \varPsi)$ by

$$\hat{\varDelta}(\mathfrak{z}) = \theta^{(1)}_{F,B}(0, (-s^{-1})^{\varPsi}; q_1', 0)^{-2} \begin{pmatrix} \Omega_{v_1}(\mathfrak{z})^{-1} & & \\ & \ddots & \\ & & \Omega_{v_{|\boldsymbol{a}|}}(\mathfrak{z})^{-1} \end{pmatrix} \varDelta(\varepsilon_\delta(\mathfrak{z}))$$

where $q_1' \in \boldsymbol{Q}^{|\boldsymbol{a}|}$ so that $\theta^{(1)}_{F,B}(0, (-s^{-1})^{\varPsi}; q_1', 0) \neq 0$. Then by (1.8), there exists a congruence subgroup $\varGamma$ of $G(\boldsymbol{Q}) = G^{(1,1)}(s, \varPsi)(\boldsymbol{Q})$ such that

$$\hat{\varDelta}(\alpha(\mathfrak{z})) = \left(\prod_{v \in \boldsymbol{a}} \mu_v(\alpha, \mathfrak{z})^2\right) \begin{pmatrix} \lambda_{v_1}(\alpha, \mathfrak{z}) & & & & \\ & \mu_{v_1}(\alpha, \mathfrak{z}) & & & \\ & & \ddots & & \\ & & & \lambda_{v_{|\boldsymbol{a}|}}(\alpha, \mathfrak{z}) & \\ & & & & \mu_{v_{|\boldsymbol{a}|}}(\alpha, \mathfrak{z}) \end{pmatrix} \hat{\varDelta}(\mathfrak{z})$$

for any $\alpha \in \varGamma$. Note that $\det(\hat{\varDelta}(\mathfrak{z}))$ is non-zero constant times $\det(T(\varepsilon_B \circ \varepsilon_\delta(\mathfrak{z})))$ and hence contained in $\mathscr{M}^{(1,1)}_{(6|\boldsymbol{a}|+2)\cdot\mathbf{1}}(s, \varPsi)$. Put

$$\varLambda(\mathfrak{z}) = \det(\hat{\varDelta}(\mathfrak{z}))\, {}^t\hat{\varDelta}(\mathfrak{z})^{-1}.$$

Then $\varLambda$ is holomorphic on the whole $\mathfrak{D}$ and satisfies

$$\varLambda(\alpha(\mathfrak{z})) = \left(\prod_{v \in \boldsymbol{a}} \mu_v(\alpha, \mathfrak{z})^{6|\boldsymbol{a}|}\right) \begin{pmatrix} {}^t\lambda_{v_1}(\alpha, \mathfrak{z})^{-1} & & & & \\ & \mu_{v_1}(\alpha, \mathfrak{z})^{-1} & & & \\ & & \ddots & & \\ & & & {}^t\lambda_{v_{|\boldsymbol{a}|}}(\alpha, \mathfrak{z})^{-1} & \\ & & & & \mu_{v_{|\boldsymbol{a}|}}(\alpha, \mathfrak{z})^{-1} \end{pmatrix} \varLambda(\mathfrak{z})$$

for α contained in a certain congruence subgroup of $G^{(1,1)}(s, \varPsi)(\boldsymbol{Q})$.

For each $v = v_k \in \boldsymbol{a}$ $(1 \le k \le |\boldsymbol{a}|)$, take $Q \in \boldsymbol{Q}_2^{3|\boldsymbol{a}|}$ and put

$$\begin{pmatrix} \psi_{1,v}(\mathfrak{z}) & \psi_{3,v}(\mathfrak{z}) \\ \psi_{2,v}(\mathfrak{z}) & \psi_{4,v}(\mathfrak{z}) \end{pmatrix} = (0^2_{3k-3} \ \ 1_2 \ \ 0^2_{3|\boldsymbol{a}|-3k+1}) \Lambda(\mathfrak{z}) Q \tag{3.15}$$

where 0^m_n denotes the $m \times n$ matrix whose entries are all 0. Then $\begin{pmatrix} \psi_{1,v} \\ \psi_{2,v} \end{pmatrix}, \begin{pmatrix} \psi_{3,v} \\ \psi_{4,v} \end{pmatrix} \in \mathscr{M}_\omega$ with $\omega(a,b) = (\prod_{v' \in \boldsymbol{a}} b_{v'}^{6|\boldsymbol{a}|})^t a_v^{-1}$. At each $\mathfrak{z}_0 \in \mathfrak{D}^{(1,1)}(s,\boldsymbol{\Psi})$, we can take T so that $\det(\Lambda(\mathfrak{z}_0)) \neq 0$. For such Λ, we can choose Q such that $\det\begin{pmatrix} \psi_{1,v}(\mathfrak{z}_0) & \psi_{3,v}(\mathfrak{z}_0) \\ \psi_{2,v}(\mathfrak{z}_0) & \psi_{4,v}(\mathfrak{z}_0) \end{pmatrix} \neq 0$. Note that ω is equivalent to $\omega_{6|\boldsymbol{a}|\cdot\mathbf{1}+v} \otimes \pi_v$. For some $h \in \mathscr{M}^{(1)}_{\mathbf{1}}$ so that $h((-s^{-1})^{\boldsymbol{\Psi}}) \neq 0$, we can take $\phi_{1,v}, \phi_{2,v} \in \mathscr{N}^v_{6|\boldsymbol{a}|\cdot\mathbf{1}+v}(s,\boldsymbol{\Psi})$ as

$$\phi_{1,v} = (\pi\sqrt{-1})^{-1} h((-s^{-1})^{\boldsymbol{\Psi}})^{-1} \Theta_{6|\boldsymbol{a}|\cdot\mathbf{1}+v} \circ D_{\omega,v}\begin{pmatrix} \psi_{1,v} \\ \psi_{2,v} \end{pmatrix},$$

$$\phi_{2,v} = (\pi\sqrt{-1})^{-1} h((-s^{-1})^{\boldsymbol{\Psi}})^{-1} \Theta_{6|\boldsymbol{a}|\cdot\mathbf{1}+v} \circ D_{\omega,v}\begin{pmatrix} \psi_{3,v} \\ \psi_{4,v} \end{pmatrix}.$$

By a computation, we have

$$\phi_{1,v} = (\pi\sqrt{-1})^{-1} h((-s^{-1})^{\boldsymbol{\Psi}})^{-1} \left\{ \frac{\partial \psi_{1,v}}{\partial z_v} + \frac{\partial \psi_{2,v}}{\partial w_v} - (6|\boldsymbol{a}| - 2)(r_{1,v}\psi_{1,v} + r_{2,v}\psi_{2,v}) \right\},$$

$$\phi_{2,v} = (\pi\sqrt{-1})^{-1} h((-s^{-1})^{\boldsymbol{\Psi}})^{-1} \left\{ \frac{\partial \psi_{3,v}}{\partial z_v} + \frac{\partial \psi_{4,v}}{\partial w_v} - (6|\boldsymbol{a}| - 2)(r_{1,v}\psi_{3,v} + r_{2,v}\psi_{4,v}) \right\}.$$

This implies

$$E_v\phi_{1,v} = -(\pi\sqrt{-1})^{-1} h((-s^{-1})^{\boldsymbol{\Psi}})^{-1}((6|\boldsymbol{a}|-2)\psi_{1,v}, (6|\boldsymbol{a}|-2)\psi_{2,v}),$$

$$E_v\phi_{2,v} = -(\pi\sqrt{-1})^{-1} h((-s^{-1})^{\boldsymbol{\Psi}})^{-1}((6|\boldsymbol{a}|-2)\psi_{3,v}, (6|\boldsymbol{a}|-2)\psi_{4,v}),$$

and so $\det\begin{pmatrix} E_v\phi_{1,v} \\ E_v\phi_{2,v} \end{pmatrix} \neq 0$ at $\mathfrak{z}_0$ if we take Λ and Q so that $\det\begin{pmatrix} \psi_{1,v}(\mathfrak{z}_0) & \psi_{3,v}(\mathfrak{z}_0) \\ \psi_{2,v}(\mathfrak{z}_0) & \psi_{4,v}(\mathfrak{z}_0) \end{pmatrix} \neq 0$. We take $\phi_{1,v}, \phi_{2,v}$ so that $\det\begin{pmatrix} E_v\phi_{1,v} \\ E_v\phi_{2,v} \end{pmatrix} \not\equiv 0$.

LEMMA 3.5. *Take $\phi_{1,v}, \phi_{2,v}$ as above. Let their Fourier-Jacobi expansions be*

$$\phi_{i,v}(\mathfrak{z}) = \sum_{0 \le m \in F} \left(g_{(\phi_{i,v},0,m)}(w) + (\pi\sqrt{-1})^{-1} r^{(1,1)}_{1,v}(s,\boldsymbol{\Psi})(\mathfrak{z}) g_{(\phi_{i,v},v,m)}(w) \right) \boldsymbol{e_a}(mz),$$

$$\left(i = 1,2, \ \ \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathfrak{D}^{(1,1)}(s,\boldsymbol{\Psi}) \right),$$

with $g_{(\phi_{i,v},0,m)} \in \mathfrak{T}_{ms,\Psi}$ and $g_{(\phi_{i,v},v,m)} \in \mathfrak{T}^v_{ms,\Psi}$. For any $(\sigma,\boldsymbol{\Psi},a) \in C_{\Psi}(\boldsymbol{C})$, there exist $\phi^{(\sigma,\Psi,a)}_{1,v}, \phi^{(\sigma,\Psi,a)}_{2,v} \in \mathscr{N}^{v\sigma,(1,1)}_{6|\boldsymbol{a}|\cdot\mathbf{1}+v\sigma}(\iota(\sigma,a)s, \boldsymbol{\Psi}\sigma)$ whose Fourier-Jacobi expansions are

$$\phi_{i,v}^{(\sigma,\Psi,a)}(\mathfrak{z}) = \sum_{0\le m\in F}\Big(g_{(\phi_{i,v},0,m)}^{(\sigma,\Psi,a)}(\tilde{w}) + (\pi\sqrt{-1})^{-1} r_{1,v\sigma}^{(1,1)}(\iota(\sigma,a)s,\Psi\sigma)(\mathfrak{z}) g_{(\phi_{i,v},v,m)}^{(\sigma,\Psi,a)}(\tilde{w})\Big)\boldsymbol{e_a}(m\tilde{z}),$$

where $i=1,2$ *and* $\mathfrak{z} = \begin{pmatrix}\tilde{z}\\ \tilde{w}\end{pmatrix} \in \mathfrak{D}^{(1,1)}(\iota(\sigma,a)s,\Psi\sigma)$.

Proof. For $(\sigma,\Psi,a)\in C_{\Psi}(\boldsymbol{C})$, take $\alpha = \begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix} \in G^{(1)}(\boldsymbol{Q}) = GL(2,F)$ so that

$$\alpha \in \left((\tilde{X}\cap G_{1A}^{(1)})\begin{pmatrix}1 & 0\\ 0 & \chi(\sigma)\end{pmatrix}^{-1} G_{\infty}^{(1)}\Phi_{(-s^{-1})^{\Psi}}^{(1)}(a)\right)\cap G^{(1)}(\boldsymbol{Q}),$$

where $\tilde{X}\in\mathscr{Z}^{(1)}$ satisfies the following conditions (1) and (2).

(1) For any $\gamma = \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix} \in \Gamma_{\tilde{X}}^{(1)}\cap G_1^{(1)}(\boldsymbol{Q})$, put

$$\hat{\gamma} = \begin{pmatrix}1 &&&&& \\ & \gamma_1 &&& \gamma_2 & \\ && 1 &&& \\ &&& 1 && \\ & \gamma_3 &&& \gamma_4 & \\ &&&&& 1\end{pmatrix} \in G_1^{(3)}(\boldsymbol{Q}).$$

Then

$$\left(\prod_{v\in\boldsymbol{a}}\det(j_v^{(3)}(\hat{\gamma},z))\right)^{-1}\begin{pmatrix} j_{v_1}^{(3)}(\hat{\gamma},z) && \\ & \ddots & \\ && j_{v_{|\boldsymbol{a}|}}^{(3)}(\hat{\gamma},z)\end{pmatrix}^{-1}\Delta(\hat{\gamma}(z)) = \Delta(z),$$

for $z\in\mathfrak{H}_3^{\boldsymbol{a}}$.

(2) $\theta_{F,B}^{(1)}(0,z;q_1',0)^2,\ h^{\sigma}\in\mathscr{M}_{\boldsymbol{1}}^{(1)}(\Gamma_{\tilde{X}}^{(1)}\cap G_1^{(1)}(\boldsymbol{Q}))$.

Note that $\nu(\alpha) = \iota(\sigma,a)^{-1}$. For such α, take $\alpha'\in G_1^{(3)}(\boldsymbol{Q})$ by

$$\alpha' = \begin{pmatrix}1 &&&&& \\ & \iota(\sigma,a)\alpha_1 &&& \alpha_2 & \\ && 1 &&& \\ &&& 1 && \\ & \iota(\sigma,a)\alpha_3 &&& \alpha_4 & \\ &&&&& 1\end{pmatrix},$$

and consider the $\boldsymbol{C}_{3|\boldsymbol{a}|}^{3|\boldsymbol{a}|}$-valued function $\tilde{\tilde{\Delta}}$ on $\mathfrak{D}^{(1,1)}(\iota(\sigma,a)s,\Psi\sigma)$ by

$$\begin{aligned}\tilde{\tilde{\Delta}}(\mathfrak{z}) &= \theta_{F,B}^{(1)}(0,[(\alpha_1(-s^{-1})+\alpha_2)(\alpha_3(-s^{-1})+\alpha_4)^{-1}]^{\Psi\sigma};q_1',0)^{-2}\\ &\times\begin{pmatrix}\Omega_{v_1}(\mathfrak{z})^{-1}j_{v_1}^{(3)}(\alpha',\varepsilon_\delta(\mathfrak{z}))^{-1} && \\ & \ddots & \\ && \Omega_{v_{|\boldsymbol{a}|}}(\mathfrak{z})^{-1}j_{v_{|\boldsymbol{a}|}}^{(3)}(\alpha',\varepsilon_\delta(\mathfrak{z}))^{-1}\end{pmatrix}\\ &\times\Delta(\alpha'(\varepsilon_\delta(\mathfrak{z}))),\end{aligned}$$

where $\tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{z}_v \\ \tilde{w}_v \end{pmatrix}_{v \in \boldsymbol{a}} \in \mathfrak{D}^{(1,1)}(\iota(\sigma, a)s, \Psi\sigma)$. In the same way as above, put

$$\tilde{\Lambda}(\tilde{\mathfrak{z}}) = \det(\tilde{\tilde{\Delta}}(\tilde{\mathfrak{z}}))\,{}^t\tilde{\tilde{\Delta}}(\tilde{\mathfrak{z}})^{-1},$$

and

$$\begin{pmatrix} \tilde{\psi}_{1,v}(\tilde{\mathfrak{z}}) & \tilde{\psi}_{3,v}(\tilde{\mathfrak{z}}) \\ \tilde{\psi}_{2,v}(\tilde{\mathfrak{z}}) & \tilde{\psi}_{4,v}(\tilde{\mathfrak{z}}) \end{pmatrix} = (0^2_{3k-3} \ \ 1_2 \ \ 0^2_{3|\boldsymbol{a}|-3k+1})\tilde{\Lambda}(\tilde{\mathfrak{z}})Q,$$

where $v = v_k$ and Q is same as that which was used in the construction of $\phi_{1,v}$ and $\phi_{2,v}$. For each $v \in \boldsymbol{a}$, put

$$\begin{aligned}
\tilde{\phi}_{1,v} = {} & (\pi\sqrt{-1})^{-1} h^{\sigma}([(\alpha_1(-s^{-1}) + \alpha_2)(\alpha_3(-s^{-1}) + \alpha_4)^{-1}]^{\Psi\sigma})^{-1} \\
& \times \left\{ \prod_{v' \in \boldsymbol{a}} (\alpha_3(-s^{-1}) + \alpha_4)^{(\Psi\sigma)_{v'}} \right\} \\
& \times \left\{ \frac{\partial \tilde{\psi}_{1,v}}{\partial \tilde{z}_v} + \frac{\tilde{\psi}_{2,v}}{\partial \tilde{w}_v} - (6|\boldsymbol{a}| - 2)(r_{1,v}\tilde{\psi}_{1,v} + r_{2,v}\tilde{\psi}_{2,v}) \right\}, \\
\tilde{\phi}_{2,v} = {} & (\pi\sqrt{-1})^{-1} h^{\sigma}([(\alpha_1(-s^{-1}) + \alpha_2)(\alpha_3(-s^{-1}) + \alpha_4)^{-1}]^{\Psi\sigma})^{-1} \\
& \times \left\{ \prod_{v' \in \boldsymbol{a}} (\alpha_3(-s^{-1}) + \alpha_4)^{(\Psi\sigma)_{v'}} \right\} \\
& \times \left\{ \frac{\partial \tilde{\psi}_{3,v}}{\partial \tilde{z}_v} + \frac{\tilde{\psi}_{4,v}}{\partial \tilde{w}_v} - (6|\boldsymbol{a}| - 2)(r_{1,v}\tilde{\psi}_{3,v} + r_{2,v}\tilde{\psi}_{4,v}) \right\}.
\end{aligned}$$

Then $\tilde{\phi}_{1,v\sigma}$ and $\tilde{\phi}_{2,v\sigma}$ satisfy the properties of $\phi_{1,v}^{(\sigma,\Psi,a)}$ and $\phi_{2,v}^{(\sigma,\Psi,a)}$. $\square$

The non-zero holomorphic function $\det\begin{pmatrix} E_v\phi_{1,v} \\ E_v\phi_{2,v} \end{pmatrix}$ is contained in $\mathscr{M}_{12|\boldsymbol{a}|\cdot\mathbf{1}-v}$. We can prove the following lemma by looking at the constructions of $\bigcup_{v \in \boldsymbol{a}}\{\phi_{1,v}, \phi_{2,v}\}$ and $\bigcup_{v \in \boldsymbol{a}}\{\phi_{1,v}^{(\sigma,\Psi,a)}, \phi_{2,v}^{(\sigma,\Psi,a)}\}$ carefully.

LEMMA 3.6. *There exists a non-zero constant $c \in \boldsymbol{C}^{\times}$ so that*

$$\left\{\det\begin{pmatrix} E_v\phi_{1,v} \\ E_v\phi_{2,v} \end{pmatrix}\right\}^{(\sigma,\Psi,a)} = c \cdot \det\begin{pmatrix} E_{v\sigma}\phi_{1,v}^{(\sigma,\Psi,a)} \\ E_{v\sigma}\phi_{2,v}^{(\sigma,\Psi,a)} \end{pmatrix}.$$

4. Galois action on modular forms.

In this section we will construct a certain Galois action on the space of nearly holomorphic modular forms. The purpose of this section is to prove the following theorem.

THEOREM 4.1. *Let* $f \in \mathcal{N}_k^{p,(1,1)}(s, \Psi)$ *and let*

$$f(\mathfrak{z}) = \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{0 \le m \in F} \left\{ \prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1} r_{1,v}^{(1,1)}(s, \Psi)(\mathfrak{z}))^{j_v} \right\} g_{(f,j,m)}(w) \boldsymbol{e}_{\boldsymbol{a}}(mz),$$

$$\left(g_{(f,j,m)} \in \mathfrak{T}^j_{ms,\Psi},\ \mathfrak{z} = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathfrak{D}^{(1,1)}(s, \Psi) \right),$$

be its Fourier-Jacobi expansion. Then for any $(\sigma, \Psi, a) \in C_\Psi(\boldsymbol{C})$, *there exists* $f^{(\sigma,\Psi,a)} \in \mathcal{N}_{k^\sigma}^{p^\sigma,(1,1)}(\iota(\sigma,a)s, \Psi\sigma)$ *whose Fourier-Jacobi expansion is*

$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}) = \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{0 \le m \in F} \left\{ \prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1} r_{1,v\sigma}^{(1,1)}(\iota(\sigma,a)s, \Psi\sigma)(\tilde{\mathfrak{z}}))^{j_v} \right\}$$

$$\times\, g^{(\sigma,\Psi,a)}_{(f,j,m)}(\tilde{w}) \boldsymbol{e}_{\boldsymbol{a}}(m\tilde{z}), \quad \left(\tilde{\mathfrak{z}} = \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} \in \mathfrak{D}^{(1,1)}(\iota(\sigma,a)s, \Psi\sigma) \right).$$

To prove this theorem, we have to show the following lemma.

LEMMA 4.2. *Take any* $f \in \mathcal{N}_k^{p,(1,1)}(s, \Psi)$. *Let* $\bigcup_{v \in \boldsymbol{a}} \{\phi_{1,v}, \phi_{2,v}\}$ *be as in section* 3. *Then there exists* $l = (l_v)_{v \in \boldsymbol{a}} \in \boldsymbol{N}^{\boldsymbol{a}}$ *and* $\varphi = \prod_{v \in \boldsymbol{a}} \det \begin{pmatrix} E_v \phi_{1,v} \\ E_v \phi_{2,v} \end{pmatrix}^{l_v} \in \mathcal{M}^{(1,1)}_{-l+12|\boldsymbol{a}|(\sum_{v \in \boldsymbol{a}} l_v)\cdot \boldsymbol{1}}(s, \Psi)$ *so that*

$$f(\mathfrak{z}) = \varphi(\mathfrak{z})^{-1} \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{\substack{q \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le q \le j}} h_{q,j}(\mathfrak{z}) \prod_{v \in \boldsymbol{a}} (\phi_{1,v}(\mathfrak{z})^{q_v} \phi_{2,v}(\mathfrak{z})^{j_v - q_v}), \tag{4.1}$$

with some $h_{q,j} \in \mathcal{M}^{(1,1)}_{\{6|\boldsymbol{a}|(\sum_{v \in \boldsymbol{a}} 2l_v - j_v)\}\cdot\boldsymbol{1} - j - l + k}(s, \Psi)$.

PROOF. As stated in section 2, f can be written as

$$f(\mathfrak{z}) = \sum_{\substack{j_1, j_2 \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ j_1 + j_2 \le p}} c_{(f;j_1,j_2)}(\mathfrak{z}) \left(\prod_{v \in \boldsymbol{a}} r_{1,v}(\mathfrak{z})^{j_{1,v}} \right) \left(\prod_{v \in \boldsymbol{a}} r_{2,v}(\mathfrak{z})^{j_{2,v}} \right) \tag{4.2}$$

with $\boldsymbol{C}$-valued holomorphic functions $c_{(f;j_1,j_2)}$ on $\mathfrak{D}$. Put

$$P_f = \left\{ j \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \,\middle|\, \begin{array}{l} \text{There exist } l_1, l_2 \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ \text{such that } j \le l_1 + l_2 \text{ and } c_{(f;l_1,l_2)} \not\equiv 0. \end{array} \right\}.$$

Then P_f is a finite set contained in $\prod_{v \in \boldsymbol{a}} \{0, 1, \ldots, p_v\}$. Take $j_0 = (j_{0,v})_{v \in \boldsymbol{a}} \in P_f$ so that $\sum_{v \in \boldsymbol{a}} j_{0,v}$ is maximum. (Of course, it is not uniquely determined in general.) Take the set

$$Q_{j_0} = \left\{ \prod_{v \in \boldsymbol{a}} (\phi_{1,v}^{q_v} \phi_{2,v}^{j_{0,v} - q_v}) \mid q = (q_v)_{v \in \boldsymbol{a}} \in \boldsymbol{Z}^{\boldsymbol{a}},\ 0 \le q \le j_0 \right\} \subset \mathcal{N}^{j_0}_{6|\boldsymbol{a}|(\sum_{v \in \boldsymbol{a}} j_{0,v})\cdot\boldsymbol{1} + j_0}.$$

Then Q_{j_0} contains $\prod_{v\in\boldsymbol{a}}(j_{0,v}+1)$ elements. Consider the rational representation (ω, V_ω) of $GL(2,\boldsymbol{C})^{\boldsymbol{a}}\times(\boldsymbol{C}^\times)^{\boldsymbol{a}}$ so that $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})Q_{j_0}\subset\mathscr{M}_\omega$. The space V_ω is $2^{\sum_{v\in\boldsymbol{a}} j_{0,v}}$-dimensional, but, as stated in section 2, the elements contained in $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})Q_{j_0}$ are $\bigotimes_{v\in\boldsymbol{a}}\mathrm{Sym}_{j_{0,v}}(\boldsymbol{C}_2^1)$ (which is $\prod_{v\in\boldsymbol{a}}(j_{0,v}+1)$-dimensional)-valued functions on $\mathfrak{D}$. Hence we can define the "determinant" of $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})Q_{j_0}$ if we fix a basis of $\bigotimes_{v\in\boldsymbol{a}}\mathrm{Sym}_{j_{0,v}}(\boldsymbol{C}_2^1)$. Put $d(j_0)=\prod_{v\in\boldsymbol{a}}(j_{0,v}+1)$. Then, by a computation, the determinant is (up to non-zero constant times) $\prod_{v\in\boldsymbol{a}}\det\begin{pmatrix}E_v\phi_{1,v}\\ E_v\phi_{2,v}\end{pmatrix}^{2^{-1}d(j_0)j_{0,v}}$, which is a non-zero holomorphic modular form of weight $\sum_{v\in\boldsymbol{a}}2^{-1}d(j_0)j_{0,v}(12|\boldsymbol{a}|\cdot\mathbf{1}-v)$ on $\mathfrak{D}^{(1,1)}(s,\boldsymbol{\Psi})$. Put this function φ_0. Consider the holomorphic function $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})f$, which is also $\bigotimes_{v\in\boldsymbol{a}}\mathrm{Sym}_{j_{0,v}}(\boldsymbol{C}_2^1)$-valued. Then we can write

$$\left\{\left(\prod_{v\in\boldsymbol{a}}E_v^{j_{0,v}}\right)f\right\}(\mathfrak{z})=\varphi_0(\mathfrak{z})^{-1}\sum_{\substack{q\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le q\le j_0}}h_q(\mathfrak{z})\left\{\left(\prod_{v\in\boldsymbol{a}}E_v^{j_{0,v}}\right)\left(\prod_{v\in\boldsymbol{a}}\phi_{1,v}^{q_v}\phi_{2,v}^{j_{0,v}-q_v}\right)\right\}(\mathfrak{z})$$

with holomorphic $\boldsymbol{C}$-valued functions h_q on $\mathfrak{D}^{(1,1)}(s,\boldsymbol{\Psi})$. Clearly each h_q is uniquely determined. Since $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})f$, φ_0 and $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})(\prod_{v\in\boldsymbol{a}}\phi_{1,v}^{q_v}\phi_{2,v}^{j_{0,v}-q_v})$ are all modular forms with respect to a certain sufficiently small congruence subgroup of $G^{(1,1)}(s,\boldsymbol{\Psi})(\boldsymbol{Q})$, the $\boldsymbol{C}$-valued functions h_q are also. By a formal calculation, we obtain that h_q is a holomorphic modular form of weight $6|\boldsymbol{a}|(d(j_0)-1)(\sum_{v\in\boldsymbol{a}} j_{0,v})\cdot\mathbf{1}+2^{-1}(d(j_0)-2)j_0+k$ on $\mathfrak{D}^{(1,1)}(s,\boldsymbol{\Psi})$. Put

$$\hat{f}(\mathfrak{z})=\varphi_0(\mathfrak{z})f(\mathfrak{z})-\sum_{\substack{q\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le q\le j_0}}h_q(\mathfrak{z})\left(\prod_{v\in\boldsymbol{a}}\phi_{1,v}^{q_v}\phi_{2,v}^{j_{0,v}-q_v}\right)(\mathfrak{z}),$$

and consider

$$P_{\hat{f}}=\left\{j\in(\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}\ \middle|\ \begin{array}{l}\text{There exist } l_1,l_2\in(\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}\\ \text{so that } j\le l_1+l_2 \text{ and } c_{(\hat{f},l_1,l_2)}\not\equiv 0.\end{array}\right\}.$$

Considering the usual expansions of $\prod_{v\in\boldsymbol{a}}\phi_{1,v}^{q_v}\phi_{2,v}^{j_{0,v}-q_v}$ as (2.3), we can verify $P_{\hat{f}}\subset P_f$. Since $(\prod_{v\in\boldsymbol{a}} E_v^{j_{0,v}})\hat{f}=0$, we can get $P_{\hat{f}}\subset P_f-\{j_0\}$ from Lemma 2.1. Hence we can write f as (4.1) by using an induction. $\square$

Now we can construct the Galois action of Theorem 4.1. For $\bigcup_{v\in\boldsymbol{a}}\{\phi_{1,v},\phi_{2,v}\}$ constructed in section 3, write f as (4.1) and set

$$f^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})=\varphi^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})^{-1}\sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le j\le p}}\sum_{\substack{q\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le q\le j}}\left[\prod_{v\in\boldsymbol{a}}\{\phi_{1,v}^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})^{q_v}\phi_{2,v}^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}})^{j_v-q_v}\}\right]h_{q,j}^{(\sigma,\Psi,a)}(\tilde{\mathfrak{z}}).\quad(4.3)$$

We can easily verify $f^{(\sigma,\Psi,a)}$ is independent of the choice of $\bigcup_{v\in\boldsymbol{a}}\{\phi_{1,v},\phi_{2,v}\}$. It is because, if f can also be written

$$f(\mathfrak{z})=\hat{\varphi}(\mathfrak{z})^{-1}\sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le j\le p}}\sum_{\substack{q\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le q\le j}}\left[\prod_{v\in\boldsymbol{a}}\{\hat{\phi}_{1,v}(\mathfrak{z})^{q_v}\hat{\phi}_{2,v}(\mathfrak{z})^{j_v-q_v}\}\right]\hat{h}_{q,j}(\mathfrak{z}),$$

with another $\bigcup_{v\in\boldsymbol{a}}\{\hat{\phi}_{1,v},\hat{\phi}_{2,v}\}$, $\hat{\varphi},\hat{h}_{q,j}$, then we have

$$\begin{aligned}\varphi(\mathfrak{z})&\sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le j\le p}}\sum_{\substack{q\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le q\le j}}\left[\prod_{v\in\boldsymbol{a}}\{\hat{\phi}_{1,v}(\mathfrak{z})^{q_v}\hat{\phi}_{2,v}(\mathfrak{z})^{j_v-q_v}\}\right]\hat{h}_{q,j}(\mathfrak{z})\\&=\hat{\varphi}(\mathfrak{z})\sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le j\le p}}\sum_{\substack{q\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le q\le j}}\left[\prod_{v\in\boldsymbol{a}}\{\phi_{1,v}(\mathfrak{z})^{q_v}\phi_{2,v}(\mathfrak{z})^{j_v-q_v}\}\right]h_{q,j}(\mathfrak{z}).\end{aligned}$$

Considering the Fourier-Jacobi expansions of both hands sides of this equation, we can easily see that $f^{(\sigma,\Psi,a)}$ is well defined.

From the definition, the function $f^{(\sigma,\Psi,a)}$ is expressed as a polynomial of $\bigcup_{v\in\boldsymbol{a}}\{r_{1,v}^{(1,1)}(\imath(\sigma,a)s,\Psi\sigma),r_{2,v}^{(1,1)}(\imath(\sigma,a)s,\Psi\sigma)\}$ with coefficients in meromorphic functions on $\mathfrak{D}^{(1,1)}(\imath(\sigma,a)s,\Psi\sigma)$. But at each $\mathfrak{z}_0\in\mathfrak{D}^{(1,1)}(\imath(\sigma,a)s,\Psi\sigma)$, take $\bigcup_{v\in\boldsymbol{a}}\{\tilde{\phi}_{1,v},\tilde{\phi}_{2,v}\}$ (functions on $\mathfrak{D}^{(1,1)}(\imath(\sigma,a)s,\Psi\sigma)$) so that $\det\begin{pmatrix}E_v\tilde{\phi}_{1,v}\\E_v\tilde{\phi}_{2,v}\end{pmatrix}\neq 0$ at $\mathfrak{z}_0$, and take $\bigcup_{v\in\boldsymbol{a}}\{\tilde{\phi}_{1,v\sigma}^{(\sigma^{-1},\Psi\sigma,a^{-1})},\tilde{\phi}_{2,v\sigma}^{(\sigma^{-1},\Psi\sigma,a^{-1})}\}$ instead of $\bigcup_{v\in\boldsymbol{a}}\{\phi_{1,v},\phi_{2,v}\}$ in Lemma 4.2. Then by Lemma 3.6, each "coefficient" is holomorphic at $\mathfrak{z}_0$. This means $f^{(\sigma,\Psi,a)}\in\mathcal{N}_{k^\sigma}^{p^\sigma,(1,1)}(\imath(\sigma,a)s,\Psi\sigma)$. The Fourier-Jacobi expansion of $f^{(\sigma,\Psi,a)}$ can easily be computed.

5. Holomorphic projections.

In section 15 of [**12**], the projection map of nearly holomorphic modular forms to holomorphic ones was constructed. In this section, we will prove that the projection is compatible with the Galois action constructed in the previous section.

For any $k\in\boldsymbol{Z}^{\boldsymbol{a}}$, $p\in(\boldsymbol{N}\cup\{0\})^{\boldsymbol{a}}$ and $v\in\boldsymbol{a}$, define an operator $L_{k,v}$ of $\mathcal{N}_k^p$ to itself by

$$L_{k,v}=-\Theta_k\circ D_{\omega_k\otimes\pi_v}\circ E_v. \tag{5.1}$$

Put the Fourier-Jacobi expansion of $f\in\mathcal{N}_k^{p,(1,1)}(s,\Psi)$ as

$$f(\mathfrak{z})=\sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le j\le p}}\sum_{0\le m\in F}\left\{\prod_{v'\in\boldsymbol{a}}((\pi\sqrt{-1})^{-1}r_{1,v'}(\mathfrak{z}))^{j_{v'}}\right\}g_{(f,j,m)}(w)\boldsymbol{e}_{\boldsymbol{a}}(mz).$$

Then by a computation, we obtain

$$\begin{aligned}(L_{k,v}f)(\mathfrak{z})=-&\sum_{\substack{j\in\boldsymbol{Z}^{\boldsymbol{a}}\\0\le j\le p}}\sum_{0\le m\in F}\left\{\prod_{v'\in\boldsymbol{a}}((\pi\sqrt{-1})^{-1}r_{1,v'}(\mathfrak{z}))^{j_{v'}}\right\}\boldsymbol{e}_{\boldsymbol{a}}(mz)\\&\times[j_v(j_v-k_v+2)g_{(f,j,m)}(w)+2(j_v+1)m_vg_{(f,j+v,m)}(w)\\&\quad-(\pi\sqrt{-1})^{-1}(s^{\Psi_v})^{-1}(\kappa_v\circ(\partial/\partial\overline{w_v})\circ g_{(f,j+v,m)})(w)].\end{aligned}\tag{5.2}$$

This explicit expansion implies the following lemma.

LEMMA 5.1. *For $f \in \mathcal{N}_k^p$, we have*

$$L_{k,v} f + p_v(p_v - k_v + 2) f \in \mathcal{N}_k^{p-v}.$$

This operator $L_{k,v}$ is compatible with the Galois action constructed in the previous section.

THEOREM 5.2. *For any $f \in \mathcal{N}_k^{p,(1,1)}(s, \Psi)$ and any $(\sigma, \Psi, a) \in C_\Psi(\boldsymbol{C})$, we have*

$$(L_{k,v} f)^{(\sigma, \Psi, a)} = L_{k^\sigma, v\sigma}(f^{(\sigma, \Psi, a)}).$$

(*Note that the $L_{k^\sigma, v\sigma}$ in the right hand side is an operator on $\mathcal{N}_{k^\sigma}^{p^\sigma,(1,1)}(\iota(\sigma, a)s, \Psi\sigma)$.*)

To prove this theorem we need the following lemma.

LEMMA 5.3. *Take any $\theta \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})$ with any $\boldsymbol{Z}$-lattice L in K and any CM-type Ψ of K. For any $j = (j_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ and $v_0 \in \boldsymbol{a}$, we have*

$$(\pi\sqrt{-1})^{-1}\left(\kappa_{v_0} \circ \frac{\partial}{\partial \overline{u_{v_0}}}\right) \circ \left\{\left(\prod_{v \in \boldsymbol{a}} \kappa_v^{j_v}\right) \circ \theta\right\} = 2 j_{v_0} t^{\Psi_{v_0}} \left(\prod_{v \in \boldsymbol{a}} \kappa_v^{j_v}\right) \circ \theta.$$

This lemma can easily be verified by a computation, since

$$\left(\prod_{v \in \boldsymbol{a}} \kappa_v^{j_v}\right) \circ \theta = \sum_{\substack{l \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \\ 0 \le l \le j}} \left(\prod_{v \in \boldsymbol{a}} {}_{j_v}C_{l_v}\right)\left(\prod_{v \in \boldsymbol{a}} (2\pi\sqrt{-1} t^{\Psi_v} \overline{u_v})^{l_v}\right)\left(\prod_{v \in \boldsymbol{a}} \left(\frac{\partial}{\partial u_v}\right)^{j_v - l_v}\right) \circ \theta,$$

where ${}_m C_n = m!/((m-n)!n!)$.

Hence we can obtain the following lemma.

LEMMA 5.4. *For $g \in \mathfrak{T}^p(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})$ and any $v_0 \in \boldsymbol{a}$, we have*

$$(*) \qquad \left\{(\pi\sqrt{-1})^{-1}(t^{\Psi_{v_0}})^{-1}\left(\kappa_{v_0} \circ \frac{\partial}{\partial \overline{u_{v_0}}} \circ g\right)\right\}^{(\sigma, \Psi, a)}$$
$$= (\pi\sqrt{-1})^{-1}(\iota(\sigma, a)_{v_0\sigma} t^{\Psi_{v_0\sigma}})^{-1}\left(\kappa_{v_0\sigma} \circ \frac{\partial}{\partial \overline{u_{v_0\sigma}}} \circ g^{(\sigma, \Psi, a)}\right),$$

where $(\sigma, \Psi, a) \in C_\Psi(\boldsymbol{C})$.

PROOF. In case $g = (\prod_{v \in \boldsymbol{a}} \kappa_v^{j_v}) \circ \theta$ with $\theta \in \mathfrak{T}(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})$ and $j = (j_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$, the left hand side of $(*)$ is equal to $2 j_{v_0} g^{(\sigma, \Psi, a)}$ from Lemma 5.3. Using Lemma 3.4, we have

$$g^{(\sigma, \Psi, a)} = c\left(\prod_{v \in \boldsymbol{a}} \kappa_{v\sigma}^{j_v}\right) \circ \theta^{(\sigma, \Psi, a)},$$

with some non-zero constant $c \in \boldsymbol{C}^\times$. By Lemma 5.3, the right hand side of $(*)$ is equal to $2 j_{v_0} g^{(\sigma, \Psi, a)}$. Lemma 3.1 shows that any $g \in \mathfrak{T}^p(\boldsymbol{C}^{\boldsymbol{a}}, L^\Psi, H_{t,\Psi})$ can be expressed as a finite sum of such functions, hence we can get this lemma. □

Looking at (5.2) carefully and using this lemma, we can easily verify Theorem 5.2.

Now we can define a projection of nearly holomorphic modular forms to holomorphic ones using this $L_{k,v}$.

THEOREM 5.5. *Take* $p = (p_v)_{v \in \boldsymbol{a}}$, $k = (k_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ *so that* $p_v \le k_v - 3$ *for each* $v \in \boldsymbol{a}$. *Put*

$$\mathfrak{A} = \prod_{v \in \boldsymbol{a}^*} \left\{ \prod_{i=0}^{p_v - 1} (1 + (p_v - i)^{-1}(p_v - i - k_v + 2)^{-1} L_{k,v}) \right\},$$

where $\boldsymbol{a}^* = \{v \in \boldsymbol{a} \mid p_v > 0\}$. *Take any* $f \in \mathscr{N}_k^{p,(1,1)}(s, \Psi)$. *Then* $\mathfrak{A}f \in \mathscr{M}_k^{(1,1)}(s, \Psi)$. *Moreover, for such* f *and any* $(\sigma, \Psi, a) \in C_\Psi(\boldsymbol{C})$, *we have*

$$(\star) \qquad (\mathfrak{A}f)^{(\sigma, \Psi, a)} = \mathfrak{A}(f^{(\sigma, \Psi, a)}),$$

where the $\mathfrak{A}$ *in the right hand side is an operator of* $\mathscr{N}_{k^\sigma}^{p^\sigma,(1,1)}(\iota(\sigma, a)s, \Psi\sigma)$ *to* $\mathscr{M}_{k^\sigma}^{(1,1)}(\iota(\sigma, a)s, \Psi\sigma)$.

REMARK. This $\mathfrak{A}$ is essentially same as that introduced in section 15 of [**12**], and so the inner product of $(\mathfrak{A}f - f)$ and any cusp form with respect to $G^{(1,1)}(s, \Psi)$ is always zero.

PROOF. Using Lemma 5.1, we can easily obtain $\mathfrak{A}f \in \mathscr{M}_k$. The property $(\star)$ is obvious from Theorem 5.2. $\square$

The above theorem needs the assumption $p_v \le k_v - 3$, but it is not so difficult. The next lemma says that the degree of near holomorphy is bounded by the weight of a modular form.

LEMMA 5.6. *Take* $k = (k_v)_{v \in \boldsymbol{a}} \in \boldsymbol{Z}^{\boldsymbol{a}}$ *so that* $k_v \ge 5$ *for each* $v \in \boldsymbol{a}$. *Then* $\mathscr{N}_k^p \subset \mathscr{N}_k^{k - 3 \cdot \boldsymbol{1}}$ *for any* $p = (p_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$.

PROOF. First let us prove $\mathscr{N}_k^{p,(1)} \subset \mathscr{N}_k^{k - 3 \cdot \boldsymbol{1},(1)}$, the same assertion for Hilbert modular forms. For any $\boldsymbol{C}$-valued nearly holomorphic Hilbert modular form h of weight k, take $p = (p_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ which satisfies $h \in \mathscr{N}_k^{p,(1)}$ and $\sum_{v \in \boldsymbol{a}} p_v$ is minimum. Then $E_v^{p_v} h \not\equiv 0$ for each $v \in \boldsymbol{a}$. Put

$$Q_h = \left\{ l = (l_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}} \,\middle|\, \left(\prod_{v \in \boldsymbol{a}} E_v^{l_v} \right) h \not\equiv 0 \right\}.$$

Take any $v_0 \in \boldsymbol{a}$ and fix it. Since $E_{v_0}^{p_{v_0}} h \not\equiv 0$, we can choose $q \in Q_h$ so that $q_{v_0} \ge p_{v_0}$ and there exists no other $q' \in Q_h$ which satisfies $q' \ge q$. Then $(\prod_{v \in \boldsymbol{a}} E_v^{q_v})h$ is contained in $\bigcap_{v \in \boldsymbol{a}} \operatorname{Ker} E_v$. From the definition of E_v, it means $(\prod_{v \in \boldsymbol{a}} E_v^{q_v})h$ is a (non-zero) holomorphic Hilbert modular form. Hence we have $(\prod_{v \in \boldsymbol{a}} E_v^{q_v})h \in \mathscr{M}_{k-2q}^{(1)}$. As is well known, a holomorphic Hilbert modular form of weight $l = (l_v)_{v \in \boldsymbol{a}}$ is non-zero only if $l_v \ge 0$ for each $v \in \boldsymbol{a}$. Hence we have $k_{v_0} \ge 2q_{v_0} \ge 2p_{v_0}$. We obtain $p_{v_0} \le k_{v_0} - 3$ from $k_{v_0} \ge 5$.

For any nearly holomorphic modular form f of weight $k = (k_v)_{v \in \boldsymbol{a}}$ with respect to $G^{(1,1)}(s, \Psi)$, take $p = (p_v)_{v \in \boldsymbol{a}} \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ so that $f \in \mathcal{N}_k^{p,(1,1)}(s, \Psi)$ and $\sum_{v \in \boldsymbol{a}} p_v$ is minimum. Express f as

$$f(\mathfrak{z}) = \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{0 \le m \in F} \left(\prod_{v \in \boldsymbol{a}} ((\pi\sqrt{-1})^{-1} r_{1,v}(\mathfrak{z}))^{j_v} \right) g_{(f,j,m)}(w) \boldsymbol{e}_{\boldsymbol{a}}(mz).$$

Consider a nearly holomorphic Hilbert modular form $(f|_k \mathsf{h}_{y,0}) \circ \varepsilon_0$ for $\mathsf{h}_{y,0} = \begin{pmatrix} 1 & sy^\rho & (1/2)syy^\rho \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ with $y \in K$. Clearly it is of weight k, and since $r_{1,v}(\varepsilon_0(z)) = (\sqrt{-1}/2)\,\mathrm{Im}(z_v)^{-1}$, we have $(f|_k \mathsf{h}_{y,0}) \circ \varepsilon_0 \in \mathcal{N}_k^{p,(1)}$. Its Fourier expansion is

$$(f|_k \mathsf{h}_{y,0}) \circ \varepsilon_0(z) = \sum_{\substack{j \in \boldsymbol{Z}^{\boldsymbol{a}} \\ 0 \le j \le p}} \sum_{0 \le m \in F} \left\{ \prod_{v \in \boldsymbol{a}} (\pi\,\mathrm{Im}(z_v))^{-j_v} \right\} 2^{-\sum_{v \in \boldsymbol{a}} j_v} (g_{(f,j,m)})_*(y^{\Psi}) \boldsymbol{e}_{\boldsymbol{a}}(mz).$$

Now for each $v \in \boldsymbol{a}$, there exists $j_0 \in (\boldsymbol{N} \cup \{0\})^{\boldsymbol{a}}$ which satisfies $j_{0,v} = p_v$ and $g_{(f,j_0,m)} \not\equiv 0$ with some $m \in F$. (If not, we have $f \in \mathcal{N}_k^{p-v}$.) For such j_0 and m, we can choose a suitable $y \in K$ so that $(g_{(f,j_0,m)})_*(y^{\Psi}) \neq 0$ for some $m \in F$. Note that $j_{0,v} = p_v$. For such y, $(f|_k \mathsf{h}_{y,0}) \circ \varepsilon_0 \in \mathcal{N}_k^{q,(1)}$ only if $q_v \ge p_v$. This implies $p_v \le k_v - 3$ since $k_v \ge 5$. $\square$

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