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Concealed generalized canonical algebras and standard stable tubes

Dedicated to Daniel Simson on the occasion of his 65th birthday

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Abstract. We introduce the class of concealed generalized canonical algebras and prove that their Auslander-Reiten quiver admits an infinite family of faithful standard stable tubes. Moreover, a new wide class of selfinjective algebras whose Auslander-Reiten quiver admits an infinite family of standard stable tubes is exhibited.

1. Introduction.

Throughout the paper K will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and connected. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules and by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$. We shall denote by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. Further, we denote by Γ_A the Auslander-Reiten quiver of A, and by τ_A and τ_A^{-} the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We will not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. A component in Γ_A of the form $\boldsymbol{Z}\boldsymbol{A}_{\infty}/(\tau^{r})$ $r \geq 1$, is called a stable tube of rank r. Therefore, a stable tube of rank r in Γ_A is an infinite component consisting of τ_A-periodic indecomposable A-modules having period r. Moreover, a stable tube of Γ_A is a regular component (contains neither a projective nor an injective module). For a component $\mathscr{C}$ of Γ_A, we denote by $\operatorname{ann}_A \mathscr{C}$ the intersection of the annihilators of all modules from $\mathscr{C}$. A component $\mathscr{C}$ in Γ_A with $\operatorname{ann}_A \mathscr{C} = 0$ is said to be faithful. We note that an arbitrary component $\mathscr{C}$ of Γ_A is a faithful component of $\Gamma_{A/\operatorname{ann}_A \mathscr{C}}$. Moreover, a component $\mathscr{C}$ of Γ_A is said to be sincere if any simple A-module occurs as a composition factor of a module in $\mathscr{C}$. It is known that every faithful component of Γ_A is sincere.

The Auslander-Reiten quiver is an important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A. In the paper, we are concerned with the problem of describing the structure of standard components of the Auslander-Reiten quiver Γ_A of an algebra A, raised more then 20 years ago by Ringel [**18**]. Recall that a component $\mathscr{C}$ of Γ_A is called standard if the full subcategory of $\operatorname{mod} A$ formed by modules

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from $\mathscr{C}$ is equivalent to the mesh category $K(\mathscr{C})$ of $\mathscr{C}$ [**3**]. Further, following [**22**], a component $\mathscr{C}$ of Γ_A is called generalized standard if $\operatorname{rad}^\infty(X,Y)=0$ for all modules X and Y in $\mathscr{C}$. It is known [**13**] that every standard component of Γ_A is generalized standard. In general, the converse implication is not true. However, the generalized standard stable tubes are standard (see [**26**, Lemma 1.3]). We also note that a component $\mathscr{C}$ of Γ_A is generalized standard if and only if $\mathscr{C}$ is a generalized standard component of $\Gamma_{A/\operatorname{ann}_A\mathscr{C}}$. It has been proved in [**22**, Theorem 2.3] that every generalized standard component $\mathscr{C}$ of Γ_A is quasi-periodic, that is, all but finitely many τ_A-orbits in $\mathscr{C}$ are periodic. In particular, this implies that every regular (generalized) standard component of Γ_A is either a stable tube or is of the form $\boldsymbol{Z}\Delta$ for a finite connected quiver Δ without oriented cycles (solution of Problem 3 from [**18**]). Moreover, the faithful (generalized) standard regular components without oriented cycles of the Auslander-Reiten quivers are exactly the connecting components of the tilted algebras of hereditary algebras by regular tilting modules (see [**22**, Corollary 3.3]). We refer also to [**21**] for a complete description of arbitrary faithful (generalized) standard components without oriented cycles.

It is expected that the infinite (generalized) standard components with oriented cycles can be obtained from faithful standard stable tubes by a sequence of admissible operations (see [**14**], [**15**]). Therefore, the description of algebras whose Auslander-Reiten quiver admits a faithful standard stable tube is an important open problem (see [**23**, Problem 3]). Recently this problem has attracted much attention (see [**8**], [**9**], [**10**], [**11**], [**16**], [**22**], [**24**], [**25**], [**26**], [**27**]). In [**17**] Ringel introduced the class of canonical algebras whose Auslander-Reiten quiver admits a separating family of standard stable tubes, and this was extended in [**9**] to the class of concealed canonical algebras. It has been proved in [**10**] (see also [**24**]) that the class of concealed canonical algebras coincides with the class of all algebras with a separating family of standard stable tubes. This was deepened in [**16**], [**26**], where a characterization of concealed canonical algebras in terms of external short paths (cycles) has been established. We note that all concealed canonical algebras are quasitilted [**6**], that is, algebras of global dimension at most two and with every indecomposable module either of projective dimension at most one or of injective dimension at most one. In [**26**] the second named author introduced the class of generalized canonical algebras whose Auslander-Reiten quiver contains an infinite family of faithful standard stable tubes. This is a wide class of algebras, containing the class of canonical algebras. We only note that there are generalized canonical algebras of arbitrary high global dimension and every basic algebra is a factor algebra of a generalized canonical algebra. We also mention that a (very special) class of generalized canonical algebras, called supercanonical algebras, has been studied in [**11**].

In the paper we introduce the class of concealed generalized canonical algebras, which are the endomorphism algebras of special tilting modules over generalized canonical algebras, containing the class of concealed canonical algebras.

The following theorem is the first main result of the paper.

THEOREM 1.1. *Let B be a concealed generalized canonical algebra. Then the Auslander-Reiten quiver Γ_B of B admits an infinite family of pairwise orthogonal faithful stable tubes.*

It would be interesting to know if the class of concealed generalized canonical algebras coincides with the class of all algebras whose Auslander-Reiten quiver admits a faithful generalized standard stable tube, and hence provides the solution of Problem 3 from [**23**].

There is a related problem (see [**23**, Problem 7]) concerning the structure of selfinjective algebras for which the Auslander-Reiten quiver admits a generalized standard stable tube. We mention that the selfinjective algebras for which the Auslander-Reiten quiver admits a nonperiodic generalized standard component have been described completely in [**28**, Section 5] and [**29**].

An algebra A is said to be a selfinjective algebra of generalized canonical type if A admits a positive Galois covering $\widehat{B} \to \widehat{B}/G = A$ (in the sense of [**30**]) by the repetitive algebra $\widehat{B}$ of a concealed generalized canonical algebra B. Examples of such algebras are provided by the trivial extensions $T(B) = B \ltimes D(B)$ of concealed generalized canonical algebras B by the minimal injective cogenerators $D(B)$, which are symmetric algebras.

The following theorem is the second main result of the paper.

THEOREM 1.2. *Let A be a selfinjective algebra of generalized canonical type. Then Γ_A admits an infinite family of pairwise orthogonal standard stable tubes.*

As a consequence we also obtain the following fact.

COROLLARY 1.3. *Let B be a concealed generalized canonical algebra. Then $\Gamma_{T(B)}$ admits an infinite family of pairwise orthogonal sincere standard stable tubes.*

For basic background on the representation theory of algebras applied here we refer to [**1**], [**17**], [**19**], [**20**], [**31**].

2. Generalized canonical algebras.

The aim of this section is to recall the main result of [**26**] on the generalized canonical algebras.

Let $m \geq 1$ be a fixed positive integer, $\mathscr{B} = \{B_0, \dots, B_m\}$ a family of basic connected nonsimple algebras, and $\mathscr{P} = \{P_0, \dots, P_m\}$ a family of modules such that, for each $i \in \{0, \dots, m\}$, P_i is a faithful indecomposable projective-injective B_i-module with injective top and projective socle. Moreover, let $\Lambda = \{\lambda_0, \dots, \lambda_m\}$ be a set of $m+1$ pairwise different elements of $\boldsymbol{P}_1(K) = K \cup \{\infty\}$, normalized such that $\lambda_0 = \infty$, $\lambda_1 = 0$, $\lambda_2 = 1$. Write each B_i as a bound quiver algebra

$$B_i = K\Delta^{(i)}/I^{(i)},$$

where $K\Delta^{(i)}$ is the path algebra of a connected quiver $\Delta^{(i)}$ and $I^{(i)}$ is an admissible ideal in $K\Delta^{(i)}$. Then

$$P_i = P_{B_i}((\omega, i)) = I_{B_i}((0, i))$$

for some vertices (ω, i) and $(0, i)$ of $\Delta^{(i)}$. Further, denote by u_i a fixed path in $\Delta^{(i)}$ with source (ω, i) and target $(0, i)$. Finally, denote by $\Delta = \Delta(\mathscr{B}, \mathscr{P})$ the quiver obtained from the disjoint union of the quivers $\Delta^{(0)}, \dots, \Delta^{(m)}$ by identifying the vertices

$(\omega, 0), \ldots, (\omega, m)$ with a vertex ω, and the vertices $(0,0), \ldots, (0,m)$ with a vertex 0. Then in Δ we have paths $u_0, \ldots, u_m$ with source ω and target 0.

For $m = 1$, let $C(\mathscr{B}, \mathscr{P}, \Lambda)$ be the bound quiver algebra $K\Delta(\mathscr{B}, \mathscr{P})/I(\mathscr{B}, \mathscr{P})$, where $I(\mathscr{B}, \mathscr{P})$ is the ideal in $K\Delta(\mathscr{B}, \mathscr{P})$ generated by $I^{(0)}$ and $I^{(1)}$.

For $m \geq 2$, we assume additionally that each $\Delta^{(i)}$ is different from the quiver $(\omega, i) \to (0, i)$. Consider the ideal $I(\mathscr{B}, \mathscr{P}, \Lambda)$ in $K\Delta(\mathscr{B}, \mathscr{P})$ generated by $I^{(0)}, \ldots, I^{(m)}$, and the elements

$$u_i + u_0 + \lambda_i u_1, \quad i = 2, \ldots, m,$$

and put $C(\mathscr{B}, \mathscr{P}, \Lambda) = K\Delta(\mathscr{B}, \mathscr{P})/I(\mathscr{B}, \mathscr{P}, \Lambda)$.

Following [**26**], the algebra $C(\mathscr{B}, \mathscr{P}, \Lambda)$ is said to be a *generalized canonical algebra* of type $(\mathscr{B}, \mathscr{P}, \Lambda)$. Denote by $\Phi = \Phi(B)$ the set of all $i \in \{0, \ldots, m\}$ for which the algebra B_i is the path algebra $K\Delta^{(i)}$ of an equioriented linear quiver

$$\Delta^{(i)} : (\omega, i) \xrightarrow{\alpha_{1p_i}} (p_{i-1}, i) \to \cdots \to (1, i) \xrightarrow{\alpha_{1i}} (0, i)$$

and P_i is the unique indecomposable projective-injective B_i-module. Moreover, denote by $\Omega = \Omega(B)$ the set of all $\lambda_i \in \Lambda$ with $i \in \Phi$, and set $\Sigma = \Lambda \setminus \Omega$. Therefore, if $\Lambda = \Omega$, then $C(\mathscr{B}, \mathscr{P}, \Lambda)$ is the canonical algebra $C(\boldsymbol{p}, \boldsymbol{\lambda})$ of type $(\boldsymbol{p}, \boldsymbol{\lambda})$, where $\boldsymbol{p} = (p_0, \ldots, p_m)$ and $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m)$, as defined in [**17**, (3.7)].

Theorem 2.1. *Let $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ be a generalized canonical algebra. Then Γ_C admits a family $\mathscr{T}^C = (\mathscr{T}^C_\mu)_{\mu \in \boldsymbol{P}_1(K) \setminus \Sigma}$ of pairwise orthogonal faithful standard stable tubes such that*

(i) *The tubes $\mathscr{T}^C_\xi, \xi \in \boldsymbol{P}_1(K) \setminus \Lambda$, are of rank* 1.
(ii) *For each $\lambda_i \in \Omega$, $\mathscr{T}^C_{\lambda_i}$ is a stable tube of rank p_i.*

Proof. See [**26**, Theorem 2.1]. □

The family $\mathscr{T}^C$ is said to be the *canonical family of stable tubes* of Γ_C. We refer to [**26**, Section 2] for the precise definition of the family $\mathscr{T}^C$, as well as further facts on the generalized canonical algebras.

We refer also to [**12**] for a complete description of all tame generalized canonical algebras.

3. Concealed generalized canonical algebras.

The aim of this section is to introduce the class of concealed generalized canonical algebras and recall some related results.

Let A be an algebra and X, Y be modules in $\operatorname{mod} A$. Then X is said to be cogenerated (respectively, generated) by Y if there is a monomorphism $X \to Y^d$ (respectively, epimorphism $Y^d \to X$) for some $d \geq 1$, where Y^d is the direct sum of d copies of Y. Moreover, a module M in $\operatorname{mod} A$ is said to be faithful if $\operatorname{ann}_A M = \{a \in A | Ma = 0\} = 0$. The following two lemmas are well known.

Lemma 3.1. *Let A be an algebra and M be a module in* $\operatorname{mod} A$. *The following conditions are equivalent*:

(i) *M is a faithful A-module.*
(ii) *A_A is cogenerated by M.*
(iii) *$D(A)_A$ is generated by M.*

Proof. See [**1**, (VI.2.2)]. □

Lemma 3.2. *Let A be an algebra and Γ a stable tube in Γ_A. Then Γ is faithful if and only if all but finitely many indecomposable modules in Γ are faithful.*

Proof. See [**19**, (XII.3.8)]. □

We need also a characterization of standard stable tubes. Recall that an indecomposable A-module X is called a brick provided $\operatorname{End}_A(X) \cong K$. Further, by the mouth of a stable tube $\mathscr{T}$ of Γ_A we mean the unique τ_A-orbit in $\mathscr{T}$ formed by the modules having exactly one predecessor (and exactly one successor) in $\mathscr{T}$.

Lemma 3.3. *Let A be an algebra and $\mathscr{T}$ a stable tube of Γ_A. The following conditions are equivalent*:

(i) *$\mathscr{T}$ is standard.*
(ii) *The mouth of $\mathscr{T}$ consists of pairwise orthogonal bricks.*
(iii) $\operatorname{rad}^{\infty}(M, M) = 0$ *for all modules M in $\mathscr{T}$.*
(iv) *$\mathscr{T}$ is generalized standard.*

Proof. See [**26**, Lemma 1.3]. □

The following homological fact will be also useful.

Lemma 3.4. *Let A be an algebra and $\mathscr{T}$ be a faithful generalized standard stable tube of Γ_A. Then, for any module X in $\mathscr{T}$, we have* $\operatorname{pd}_A X \leq 1$ *and* $\operatorname{id}_A X \leq 1$.

Proof. See [**22**, Lemma 5.9]. □

A module T in $\operatorname{mod} A$ is said to be a tilting module if $\operatorname{pd}_A T \leq 1$, $\operatorname{Ext}^1_A(T, T) = 0$, and T is a direct sum of n pairwise nonisomorphic indecomposable A-modules, where n is the rank of the Grothendieck group $K_0(A)$ of A. We refer to [**1**, Chapter VI] for a basic tilting theory.

Let $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ be a generalized canonical algebra and

$$\mathscr{T}^C = \left(\mathscr{T}^C_\mu\right)_{\mu \in \boldsymbol{P}_1(K) \setminus \Sigma}$$

be the canonical family of pairwise orthogonal faithful (generalized) standard stable tubes. We say that a module M in $\operatorname{mod} A$ is cogenerated by the family $\mathscr{T}^C$, if, for any $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$, M is cogenerated by a direct sum N_μ of indecomposable modules from $\mathscr{T}^C_\mu$. We note also that such a module M has no direct summand from $\mathscr{T}^C$, because the different tubes in $\mathscr{T}^C$ are orthogonal. In fact, then M is cogenerated by all but finitely many indecomposable modules of any fixed tube $\mathscr{T}_\mu$.

Definition 3.5. An algebra B is said to be a *concealed generalized canonical algebra* of type $(\mathscr{B}, \mathscr{P}, \Lambda)$ if B is the endomorphism algebra $\operatorname{End}_C(T)$ for a generalized canonical algebra $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ and a tilting C-module T cogenerated by the tubular family $\mathscr{T}^C$.

We show now that the class of concealed generalized canonical algebras contains the class of concealed canonical algebras, defined in [**9**]. Indeed, let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra of type $(\boldsymbol{p}, \boldsymbol{\lambda})$. Then the canonical family $\mathscr{T}^C$ of stable tubes in Γ_C is a separating family $\mathscr{T}^C = (\mathscr{T}_\mu^C)_{\mu \in \boldsymbol{P}_1(K)}$ and hence the category $\operatorname{ind} C$ of indecomposable modules in $\operatorname{mod} C$ has a decomposition

$$\operatorname{ind} C = \mathscr{P}^C \vee \mathscr{T}^C \vee \mathscr{Q}^C$$

where $\operatorname{Hom}_C(\mathscr{T}^C, \mathscr{P}^C) = 0$, $\operatorname{Hom}_C(\mathscr{Q}^C, \mathscr{T}^C) = 0$, $\operatorname{Hom}_C(\mathscr{Q}^C, \mathscr{P}^C) = 0$, and any nonzero map from $\mathscr{P}^C$ to $\mathscr{Q}^C$ factors through any of the stable tubes of $\mathscr{T}^C$. This implies that all modules in the additive category $\operatorname{add} \mathscr{P}^C$ of $\mathscr{P}^C$ are cogenerated by the family $\mathscr{T}^C$. Therefore, the class of concealed generalized canonical algebras of type $(\boldsymbol{p}, \boldsymbol{\lambda})$ coincide with the class of tilted algebras of the form $\operatorname{End}_C(T)$ for tilting modules T from $\operatorname{add} \mathscr{P}^C$, which are exactly the concealed canonical algebras defined in [**9**].

Finally, we note that a generalized canonical algebra $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ is itself a concealed generalized canonical algebra. Indeed, $C = \operatorname{End}_C(T, T)$ for the projective tilting C-module $T = C_C$, which is cogenerated by any of the faithful stable tubes of the canonical family $\mathscr{T}^C$ (see Lemma 3.1).

A module T in $\operatorname{mod} A$ is said to be a *cotilting module* if $\operatorname{id}_A T \leq 1$, $\operatorname{Ext}_A^1(T, T) = 0$, and T is a direct sum of n pairwise nonisomorphic indecomposable A-modules, where n is the rank of $K_0(A)$.

Lemma 3.6. *Let $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ be a generalized canonical algebra, T be a cotilting C-module generated by the canonical family $\mathscr{T}^C$ of stable tubes of Γ_C, and $B = \operatorname{End}_C(T)$. Then B is a concealed generalized canonical algebra.*

Proof. Consider the opposite algebra C^{op} of C and the standard duality $D : \operatorname{mod} C \to \operatorname{mod} C^{\mathrm{op}}$. Observe that $C^* = C^{\mathrm{op}}$ is a generalized canonical algebra. Indeed, if $\mathscr{B} = \{B_0, \ldots, B_m\}$ and $\mathscr{P} = \{P_0, \ldots, P_m\}$, then $C^* = C(\mathscr{B}^*, \mathscr{P}^*, \Lambda^*)$, where $\mathscr{B}^* = \{B_0^{\mathrm{op}}, \ldots, B_m^{\mathrm{op}}\}$, $\mathscr{P}^* = \{D(P_0), \ldots, D(P_m)\}$, $\Lambda^* = \Lambda$. Moreover, $T^* = D(T)$ is a tilting C^*-module cogenerated by the canonical family $\mathscr{T}^{C^*} = (\mathscr{T}_\mu^{C^*})_{\mu \in \boldsymbol{P}_1(K) \setminus \Sigma}$, with $\mathscr{T}_\mu^{C^*} = D(\mathscr{T}_\mu^C)$, of stable tubes of Γ_{C^*}. Finally, we have $B = \operatorname{End}_C(T) \cong \operatorname{End}_{C^*}(T^*)$, and hence B is a concealed generalized canonical algebra. □

4. Proof of Theorem 1.1.

Let $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ be a generalized canonical algebra, T a tilting module in $\operatorname{mod} C$ cogenerated by $\mathscr{T}^C$, and $B = \operatorname{End}_C(T)$. We will prove that the family $\mathscr{T}^B = (\mathscr{T}_\mu^B)_{\mu \in \boldsymbol{P}_1(K) \setminus \Sigma}$ of indecomposable B-modules formed by the images $\mathscr{T}_\mu^B = \operatorname{Hom}_C(T, \mathscr{T}_\mu^C)$ of the stable tubes $\mathscr{T}_\mu^C$ is a family of pairwise orthogonal faithful standard stable tubes of Γ_B.

The tilting C-module T induces a torsion pair $(\mathscr{T}(T), \mathscr{F}(T))$ in $\operatorname{mod} C$, where

$$\mathscr{F}(T) = \{X \in \operatorname{mod} C \mid \operatorname{Hom}_C(T, X) = 0\}$$

is the torsion-free class, and

$$\mathscr{T}(T) = \{X \in \operatorname{mod} C \mid \operatorname{Ext}^1_C(T, X) = 0\}$$

is the torsion class. Further, T induces also a torsion pair $(\mathscr{X}(T), \mathscr{Y}(T))$ in $\operatorname{mod} B$, where

$$\mathscr{Y}(T) = \{Y \in \operatorname{mod} B \mid \operatorname{Tor}^B_1(Y, T) = 0\}$$

is the torsion-free class, and

$$\mathscr{X}(T) = \{Y \in \operatorname{mod} B \mid X \otimes_B T = 0\}$$

is the torsion class. Then by the Brenner-Butler theorem (see [**1**, (VI.3.8)]) the functor $\operatorname{Hom}_C(T, -)$ induces an equivalence of categories $\mathscr{T}(T) \xrightarrow{\sim} \mathscr{Y}(T)$ and the functor $\operatorname{Ext}^1_C(T, -)$ induces on equivalence of categories $\mathscr{F}(T) \xrightarrow{\sim} \mathscr{X}(T)$. We note that in our situation the torsion pair $(\mathscr{X}(T), \mathscr{Y}(T))$ in $\operatorname{mod} B$ is usually not splitting, that is there are indecomposable B-modules which are neither in $\mathscr{X}(T)$ nor in $\mathscr{Y}(T)$. Therefore, it is not clear how we may recover the Auslander-Reiten quiver Γ_B of B from the Auslander-Reiten quiver Γ_C of C. However, we will show that $\mathscr{T}^B_\mu = \operatorname{Hom}_C(T, \mathscr{T}^C_\mu)$, $\mu \in \boldsymbol{P}_1(K)\setminus\Sigma$, form components of Γ_B. We divide the prove into several steps.

(1) $\operatorname{Hom}_C(\mathscr{T}^C, T) = 0$. Suppose that X is a module in the family $\mathscr{T}^C$ with $\operatorname{Hom}_C(X, T) \neq 0$, and let $\mathscr{T}^C_\mu$ be the tube of $\mathscr{T}^C$ containing X. Since T is cogenerated by $\mathscr{T}^C$, there are a module N_μ in $\operatorname{add} \mathscr{T}^C_\mu$ and a monomorphism $f : T \to N_\mu$. Then, for a nonzero morphism $g : X \to T$, we have the composed nonzero morphism $fg : X \to N_\mu$. Moreover, T has no direct summands for the tube $\mathscr{T}^C_\mu$, and hence $fg \in \operatorname{rad}^\infty(X, N_\mu)$ (see [**1**, (IV.5.1)]). Therefore, the stable tube $\mathscr{T}^C_\mu$ is not generalized standard and hence not standard, a contradiction with Theorem 2.1.

(2) $\mathscr{T}^C \subseteq \mathscr{T}(T)$. Let X be a module in $\mathscr{T}^C$. Then, applying the Auslander-Reiten formula [**1**, (IV.2.13)], and (1), we obtain

$$\operatorname{Ext}^1_C(T, X) \cong D\underline{\operatorname{Hom}}_C\big(\tau^-_C X, T\big) = 0,$$

because $\tau^-_C X \in \mathscr{T}^C$, and hence $X \in \mathscr{T}(T)$. In particular, the images $\operatorname{Hom}_C(T, X)$ of all modules X from $\mathscr{T}^C$ are indecomposable modules in the torsion-free part $\mathscr{Y}(T)$ of $\operatorname{mod} B$.

(3) $\operatorname{pd}_C X \le 1$ and $\operatorname{id}_C X \le 1$ for any module X in $\mathscr{T}^C$. This follows from Lemma 3.4 and the fact that, by Theorem 2.1, all stable tubes in $\mathscr{T}^C$ are faithful and generalized standard.

(4) $\operatorname{Ext}^1_C(X,T) \cong D\operatorname{Hom}_C(T,\tau_C X)$. This is a direct consequence of the Auslander-Reiten formula and the fact that $\operatorname{pd}_C X \le 1$ is equivalent to $\operatorname{Hom}_C(D(C),\tau_C X) = 0$ (see [**1**, (IV.2.7), (IV.2.14)]).

(5) $\operatorname{pd}_B \operatorname{Hom}_C(T,X) \le 1$ for any module X in $\mathscr{T}^C$. This follows from (3) and the fact that $\operatorname{pd}_B \operatorname{Hom}_B(T,M) \le \operatorname{pd}_C M$ for any module M in $\mathscr{T}(T)$ (see [**1**, (VI.4.1)]).

(6) $\tau_B \operatorname{Hom}_C(T,X) \cong \operatorname{Hom}_C(T,\tau_C X)$ for any module X in $\mathscr{T}^C$. This follows from [**17**, p. 171, 6(c)], (2) and (3). For convenience of the reader we present an alternative proof. Observe that the projective B-modules are the modules of the form $\operatorname{Hom}_C(T,T')$ for all modules T' from the additive category $\operatorname{add} T$ of T. Let X be a module in $\mathscr{T}^C$. Then it follows from (5) that there is an exact sequence

$$(*) \qquad 0 \to T_1 \to T_0 \to X \to 0$$

in $\operatorname{mod} C$ with T_0, T_1 from $\operatorname{add} T$ such that induced exact sequence

$$0 \to \operatorname{Hom}_C(T,T_1) \to \operatorname{Hom}_C(T,T_0) \to \operatorname{Hom}_C(T,X) \to 0$$

is a minimal projective resolution of $M = \operatorname{Hom}_C(T,X)$ in $\operatorname{mod} B$. Observe that the above sequence is exact because $\operatorname{Ext}^1_C(T,T_1) = 0$. Set $P_0 = \operatorname{Hom}_C(T,T_0)$ and $P_1 = \operatorname{Hom}_C(T,T_1)$. Consider the Nakayama functor

$$\nu = D\operatorname{Hom}_B(-,B) : \operatorname{mod} B \to \operatorname{mod} B.$$

Then, by [**1**, (IV.2.4)], there exists an exact sequence

$$0 \to \tau_B M \to \nu P_1 \to \nu P_0 \to \nu M \to 0.$$

On the other hand, applying the functor $\operatorname{Hom}_C(-,T)$ to the exact sequence $(*)$ we obtain the exact sequence

$$0 \to \operatorname{Hom}_C(X,T) \to \operatorname{Hom}_C(T_0,T) \to \operatorname{Hom}_C(T_1,T) \to \operatorname{Ext}^1_C(X,T) \to 0.$$

Hence, by (1) and (4), we get an exact sequence of left B-modules

$$0 \to \operatorname{Hom}_C(T_0,T) \to \operatorname{Hom}_C(T_1,T) \to D\operatorname{Hom}_C(T,\tau_C X) \to 0,$$

and consequently an exact sequence

$$0 \to \operatorname{Hom}_C(T,\tau_C X) \to D\operatorname{Hom}_C(T_1,T) \to D\operatorname{Hom}_C(T_0,T) \to 0$$

in $\operatorname{mod} B$. Finally, there are natural isomorphisms

$$\nu P_0 = D\operatorname{Hom}_B(\operatorname{Hom}_C(T,T_0),\operatorname{Hom}_C(T,T)) \cong D\operatorname{Hom}_C(T_0,T),$$
$$\nu P_1 = D\operatorname{Hom}_B(\operatorname{Hom}_C(T,T_1),\operatorname{Hom}_C(T,T)) \cong D\operatorname{Hom}_C(T_1,T),$$

in $\operatorname{mod} B$, because the functor $\operatorname{Hom}_C(T,-)$ induces an equivalence $\mathscr{T}(T) \xrightarrow{\sim} \mathscr{Y}(T)$ and clearly $T_0, T_1 \in \mathscr{T}(T)$. Therefore, comparing the above exact sequences, we obtain the required isomorphism of B-modules $\tau_B \operatorname{Hom}_C(T,X) \cong \operatorname{Hom}_C(T,\tau_C X)$.

(7) For each $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$, $\mathscr{T}_\mu^B = \operatorname{Hom}_C(T, \mathscr{T}_\mu^C)$ is a stable tube of Γ_B. Indeed, let X be a module in a tube $\mathscr{T}_\mu^C$ and

$$0 \to \tau_C X \to E \to X \to 0$$

be an almost split sequence in $\operatorname{mod} C$. Since $X \in \mathscr{T}(T)$, we have then the induced exact sequence

$$0 \to \operatorname{Hom}_C(T,\tau_C X) \to \operatorname{Hom}_C(T,E) \to \operatorname{Hom}_C(T,X) \to 0$$

with $\operatorname{Hom}_C(T,\tau_C X) \cong \tau_B \operatorname{Hom}_C(T,X)$, by (6). Then it is an almost split sequence in $\operatorname{mod} B$, because the torsion-free part $\mathscr{Y}(T)$ of $\operatorname{mod} B$ is closed under extensions. Therefore, $\mathscr{T}_\mu^B$ is a stable tube of Γ_B and has the same rank as $\mathscr{T}_\mu^C$.

(8) For each $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$, $\mathscr{T}_\mu^B$ is a standard stable tube. Observe that the mouth of the tube $\mathscr{T}_\mu^B$ consists of the images of the C-modules forming the mouth of $\mathscr{T}_\mu^C$ via the functor $\operatorname{Hom}_C(T,-) : \mathscr{T}(T) \xrightarrow{\sim} \mathscr{Y}(T)$. Since the tube $\mathscr{T}_\mu^C$ is standard its mouth consists of pairwise orthogonal bricks. Therefore the mouth of $\mathscr{T}_\mu^B$ also consists of pairwise orthogonal bricks. Applying now Lemma 3.3, we conclude that the tube $\mathscr{T}_\mu^B$ is standard.

(9) For each $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$, $\mathscr{T}_\mu^B$ is a faithful stable tube. Take $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$. Since, by assumption, the tilting C-module T is cogenerated by the family $\mathscr{T}^C$ of stable tubes, there exists a monomorphism $f : T \to N_\mu$ for some module N_μ from $\operatorname{add} \mathscr{T}_\mu^C$. Then we get a monomorphism $\operatorname{Hom}_C(T,f) : \operatorname{Hom}_C(T,T) \to \operatorname{Hom}_C(T,N_\mu)$ with $M_\mu = \operatorname{Hom}_C(T,N_\mu)$ from $\operatorname{add} \mathscr{T}_\mu^B$. Hence, the algebra $B = \operatorname{End}_C(T)$ is cogenerated by the module M_μ, and applying Lemma 3.1, we conclude that M_μ is a faithful B-module. Therefore, $\mathscr{T}_\mu^B$ is a faithful stable tube of Γ_B.

(10) $\mathscr{T}^B = (\mathscr{T}_\mu^B)_{\mu \in \boldsymbol{P}_1(K)\setminus\Sigma}$ is a family of pairwise orthogonal faithful standard stable tubes of Γ_B. This follows from (7), (8), (9) and the fact that the tubes $\mathscr{T}_\mu^C$, $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$, are pairwise orthogonal (see Theorem 2.1).

5. Selfinjective orbit algebras.

In this section we recall needed background on selfinjective orbit algebras and introduce the class of selfinjective algebras of generalized canonical type.

An algebra A is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is the projective A-modules are injective. Moreover, A is called *symmetric* if A and $D(A)$ are isomorphic as $A-A$-bimodules. An important class of selfinjective algebras is formed by the orbit algebras of the form $\widehat{B}/G$ where $\widehat{B}$ is the *repetitive algebra* (locally finite dimensional, without identity)

$$\widehat{B} = \bigoplus_{m \in \boldsymbol{Z}} (B_m \oplus Q_m)$$

of an algebra B, where $B_m = B$ and $Q_m = D(B)$ for all $m \in \boldsymbol{Z}$, the multiplication in $\widehat{B}$ is defined by

$$(a_m, f_m)_m \cdot (b_m, g_m)_m = (a_m b_m, a_m g_m + f_m b_{m-1})_m$$

for all $a_m, b_m \in B_m$, $f_m, g_m \in Q_m$, and G is an admissible group of automorphisms of $\widehat{B}$. Observe that $\widehat{B}$ is the matrix algebra

$$\widehat{B} = \begin{bmatrix} \ddots & \ddots & & & & 0 \\ & Q_{m-1} & B_{m-1} & & & \\ & & Q_m & B_m & & \\ & & & Q_{m+1} & B_{m+1} & \\ & 0 & & & \ddots & \ddots \end{bmatrix}$$

as defined in [**7**].

Let B be an algebra and $\mathscr{E} = \{e_i \mid 1 \leq i \leq n\}$ be a fixed set of orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$. Then we have the canonical set $\widehat{\mathscr{E}} = \{e_{j,k} \mid 1 \leq j \leq n, k \in \boldsymbol{Z}\}$ of orthogonal primitive idempotents of $\widehat{B}$ such that $e_{j,k}\widehat{B} = (e_j B)_k \oplus (e_j D(B))_k$ for $1 \leq j \leq n$ and $k \in \boldsymbol{Z}$. By an *automorphism* of $\widehat{B}$ we mean a K-algebra automorphism of $\widehat{B}$ which fixes the chosen set $\widehat{\mathscr{E}}$ of orthogonal primitive idempotents of $\widehat{B}$. A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if the induced action of G on $\widehat{\mathscr{E}}$ is free and has finitely many orbits. Then the *orbit algebra* $\widehat{B}/G$ is a selfinjective algebra and the G-orbits in $\widehat{\mathscr{E}}$ form a canonical set of orthogonal primitive idempotents of $\widehat{B}/G$ whose sum is the identity of $\widehat{B}/G$ (see [**5**]). Moreover, there are a *Galois covering* $F : \widehat{B} \to \widehat{B}/G$ and the associated *push-down functor* $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}/G$ (see [**3**], [**5**]). We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ such that $\nu_{\widehat{B}}(e_{j,k}) = e_{j,k+1}$ for all $1 \leq j \leq n$, $k \in \boldsymbol{Z}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the *trivial extension* $T(B) = B \ltimes D(B)$ of B by the injective cogenerator $D(B)$. Further, an automorphism φ of $\widehat{B}$ is said to be *positive* when, for each $j \in \{1, \ldots, n\}$, $k \in \boldsymbol{Z}$, we have $\varphi(e_{j,k}) = e_{m,r}$ for some $m \in \{1, \ldots, n\}$ and $r \geq k$. We note that, if φ is a positive automorphism of $\widehat{B}$, then the infinite cyclic group $(\varphi\nu_{\widehat{B}})$ is an admissible group of automorphisms of $\widehat{B}$. Following [**30**], a Galois covering of the form $\widehat{B} \to \widehat{B}/(\varphi\nu_{\widehat{B}})$, where φ is a positive automorphism of $\widehat{B}$ is said to be *positive*. We refer to [**28**] and [**30**] for results on selfinjective algebras having positive Galois coverings by repetitive algebras of algebras. We also note that the class of selfinjective algebras for which the Auslander-Reiten quiver admits a nonperiodic generalized standard component coincides with the class of selfinjective algebras having positive Galois coverings by the repetitive algebras of representation-infinite tilted algebras (see [**28**], [**29**]).

We will use the following consequence of [**5**, Theorem 3.6].

PROPOSITION 5.1. *Let $F : \widehat{B} \to \widehat{B}/(\varphi\nu_{\widehat{B}}) = A$ be a positive Galois covering. Then the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ preserves the indecomposable modules and the almost split sequences. Moreover, the orbit quiver $\Gamma_{\widehat{B}}/G$ is a full translation subquiver* *of* Γ_A.

We note that in general the push-down functor F_λ is not dense (see [**2**], [**4**]).

DEFINITION 5.2. Let $C = C(\mathscr{B}, \mathscr{P}, \Lambda)$ be a generalized canonical algebra, T a tilting C-module cogenerated by the canonical family $\mathscr{T}^C$ of stable tubes in Γ_C, and $B = \operatorname{End}_C(T)$ the associated concealed generalized canonical algebra. A selfinjective algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where φ is a positive automorphism of $\widehat{B}$, is said to be a *selfinjective algebra of generalized canonical type* $(\mathscr{B}, \mathscr{P}, \Lambda)$.

6. Proof of Theorem 1.2 and Corollary 1.3.

We will use the following general lemma on almost split sequences over matrix algebras.

LEMMA 6.1. *Let R and S be algebras, N an $S-R$-bimodule and $\Lambda = \left[\begin{smallmatrix} S & N \\ 0 & R \end{smallmatrix}\right]$ the associated matrix algebra. Let*

$$\eta : 0 \to X \to Y \to Z \to 0$$

be an almost split sequence in $\operatorname{mod} R$ *such that* $\operatorname{Hom}_R(N, X) = 0$. *Then η is an almost split sequence in* $\operatorname{mod} \Lambda$.

PROOF. See [**17**, (2.5)]. □

Let B be a concealed generalized canonical algebra, φ a positive automorphism of $\widehat{B}$, and $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$ the associated selfinjective algebra of generalized canonical type. We identify $\operatorname{mod} B$ with the full subcategory $\operatorname{mod} B_0$ of $\operatorname{mod} \widehat{B}$.

It follows from Theorem 1.1 (and its proof) that Γ_B admits a family $\mathscr{T}^B = (\mathscr{T}^B_\mu)_{\mu \in \boldsymbol{P}_1(K)\setminus\Sigma}$, for a finite subset Σ of $\boldsymbol{P}_1(K)$, of pairwise orthogonal faithful standard stable tubes. We shall prove that, under the canonical embedding $\operatorname{mod} B = \operatorname{mod} B_0$ into $\operatorname{mod} \widehat{B}$, $\mathscr{T}^B$ remains a family of standard stable tubes. Since $\operatorname{mod} B_0$ is closed under extensions in $\operatorname{mod} \widehat{B}$, it is enough to prove that $\tau_{\widehat{B}} Z \cong \tau_B Z$ for any indecomposable B-module Z from $\mathscr{T}^B$.

For each positive integer r, denote by $\Lambda(r)$ the matrix algebra

$$\Lambda(r) = \begin{bmatrix} B_r & Q_r & & & & & & & \\ & B_{r-1} & Q_{r-1} & & & & & \mathbf{0} & \\ & & \ddots & \ddots & & & & & \\ & & & B_1 & Q_1 & & & & \\ & & & & B_0 & Q_0 & & & \\ & & & & & B_{-1} & Q_{-1} & & \\ & & & & & & \ddots & \ddots & \\ & \mathbf{0} & & & & & & B_{-r+1} & Q_{-r+1} \\ & & & & & & & & B_{-r} \end{bmatrix}$$

that is, the factor algebra of $\widehat{B}$ by the ideal

$$I(r) = \bigoplus_{s \geq r+1} (B_s \oplus Q_s) \oplus Q_{-r} \oplus \bigoplus_{s \leq -r-1} (B_s \oplus Q_s).$$

Observe that an arbitrary almost split sequence in $\operatorname{mod} \widehat{B}$ is an almost split sequence in $\operatorname{mod} \Lambda(r)$, for some $r \geq 1$. Moreover, denote by $\Omega(r)$ the matrix algebra

$$\Omega(r) = \begin{bmatrix} B_r & Q_r & & & 0 \\ & B_{r-1} & Q_{r-1} & & \\ & & \ddots & \ddots & \\ & 0 & & B_1 & Q_1 \\ & & & & B_0 \end{bmatrix}$$

and observe that $\Omega(r)$ is a factor algebra of $\Lambda(r)$.

We first prove that $\mathscr{T}^B$ is a family of stable tubes of the Auslander-Reiten quiver $\Gamma_{\Omega(r)}$ of any algebra $\Omega(r)$, $r \geq 1$. Consider the matrix algebra

$$\Omega(1) = \begin{bmatrix} B_1 & Q_1 \\ 0 & B_0 \end{bmatrix} = \begin{bmatrix} B & D(B) \\ 0 & B \end{bmatrix}.$$

In order to prove that $\mathscr{T}^B$ is a family of stable tubes of $\Gamma_{\Omega(1)}$, it is enough to show, by Lemma 6.1, that $\operatorname{Hom}_B(D(B), Z) = 0$ for any module Z in $\mathscr{T}^B$. Suppose that there is a nonzero morphism $f : D(B) \to Z$ for an indecomposable module Z in a tube $\mathscr{T}^B_\mu$ of $\mathscr{T}^B$. Since $\mathscr{T}^B_\mu$ is a faithful stable tube of Γ_B, by Lemma 3.2, all but finitely many modules in $\mathscr{T}^B_\mu$ are faithful. Therefore, applying Lemma 3.1, we obtain that $D(B)$ is generated by an indecomposable module M of $\mathscr{T}^B_\mu$, and hence there is an epimorphism $g : M^t \to D(B)$, for some $t \geq 1$. Then the composed morphism fg is a nonzero morphism in $\operatorname{rad}^\infty(M^t, Z)$, because the injective cogenerator $D(B)$ has no direct summands in the stable tube $\mathscr{T}^B_\mu$. This implies that $\operatorname{rad}^\infty(M, Z) \neq 0$, a contradiction because the standard stable tube $\mathscr{T}^B_\mu$ is generalized standard. Therefore, indeed $\mathscr{T}^B$ is a family of standard stable tubes of $\Gamma_{\Omega(1)}$.

Let $r \geq 2$. Then $\Omega(r)$ is the matrix algebra

$$\Omega(r) = \begin{bmatrix} B_r & Q_r \\ 0 & \Omega(r-1) \end{bmatrix}.$$

Moreover, Q_r is an injective $\Omega(r-1)$-module having no simple composition factors from $\operatorname{mod} B_0$, and hence without common simple composition factors with the modules in $\mathscr{T}^B$. Therefore, we have $\operatorname{Hom}_{\Omega(r-1)}(Q(r), Z) = 0$ for any module Z in $\mathscr{T}^B$ and $r \geq 2$. Since, $\Omega(r)$ can be obtained from $\Omega(1)$ by iterated matrix extensions, we conclude inductively, applying Lemma 6.1, that $\mathscr{T}^B$ is a family of standard stable tubes of $\Gamma_{\Omega(r)}$.

Consider now the matrix algebra

$$D(1) = \begin{bmatrix} \Omega(r) & Q_{-1} \\ 0 & B_{-1} \end{bmatrix} = \begin{bmatrix} \Omega(r) & D(B) \\ 0 & B \end{bmatrix},$$

which is the coextension algebra of $\Omega(r)$ by the projective B-module B_B. Since $\mathscr{T}^B$ is a family of faithful stable tubes of Γ_B, we know that B is cogenerated by each of the stable tubes $\mathscr{T}^B_\mu$ of $\mathscr{T}^B$. Moreover, the tubes $\mathscr{T}^B_\mu$ are standard, and hence generalized standard in $\operatorname{mod} B$, so applying dual arguments we conclude that $\operatorname{Hom}_B(Z, B) = 0$ for any indecomposable module Z in $\mathscr{T}^B$. Therefore, by the lemma dual to Lemma 6.1, we conclude that $\mathscr{T}^B$ is a family of stable tubes of $\Gamma_{D(1)}$.

For $r \geq 2$ and $s = \{2, \ldots, r\}$, consider the coextension algebras

$$D(s) = \begin{bmatrix} D(s-1) & Q_{-s+1} \\ 0 & B_{-s} \end{bmatrix}.$$

Observe that we have $\operatorname{Hom}_{D(s-1)}(Z, B_{-s+1}) = 0$ for any module Z in $\mathscr{T}^B$, because the simple B_0-modules are not composition factors of B_{-s+1}. Therefore, applying the dual of Lemma 6.1, we conclude inductively that $\mathscr{T}^B$ is a family of stable tubes of the Auslander-Reiten quivers $\Gamma_{D(s)}$, $1 \leq s \leq r$. Since $D(r) = \Lambda(r)$, we finally infer that $\mathscr{T}^B$ is a family of stable tubes of $\Gamma_{\Lambda(r)}$, as required.

Therefore, we proved that $\tau_{\widehat{B}} Z \cong \tau_B Z$ for any indecomposable module in $\mathscr{T}^B$, and hence $\mathscr{T}^B$ is a family of stable tubes of $\widehat{B}$. Moreover, since $\operatorname{mod} B$ is a full subcategory of $\operatorname{mod} \widehat{B}$, $\mathscr{T}^B$ is a family of pairwise orthogonal standard stable tubes of $\Gamma_{\widehat{B}}$. Finally, applying Proposition 5.1, we conclude that $\mathscr{T}^A = (\mathscr{T}^A_\mu)_{\mu \in \boldsymbol{P}_1(K) \setminus \Sigma}$ for $\mathscr{T}^A_\mu = F_\lambda(\mathscr{T}^B_\mu)$, for any $\mu \in \boldsymbol{P}_1(K) \setminus \Sigma$, is a family of stable tubes in Γ_A.

We will show now that $\mathscr{T}^A$ is a generalized standard family of stable tubes of Γ_A. This will imply, by Lemma 3.3 (applied to each tube of the family $\mathscr{T}^A$), that $\mathscr{T}^A$ is a family of pairwise orthogonal standard stable tubes of Γ_A. Consider the Galois covering $F : \widehat{B} \to \widehat{B}/(\varphi\nu_{\widehat{B}}) = A$ and the associated push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$. Let $g = \varphi\nu_{\widehat{B}}$ and $G = (g)$. Then G acts on the category $\operatorname{mod} \widehat{B}$ such that, for any shift ${}^hM$ of a module M from $\operatorname{mod} \widehat{B}$ by an element $h \in G$, we have $F_\lambda M \cong F_\lambda {}^hM$ (see [**5**, Lemma 3.2]). Moreover, for any modules M and N in $\operatorname{mod} \widehat{B}$, we have a canonical K-linear isomorphism

$$\operatorname{Hom}_A(F_\lambda M, F_\lambda N) \xrightarrow{\sim} \bigoplus_{h \in G} \operatorname{Hom}_{\widehat{B}}(M, {}^hN)$$

(see [**5**, Lemma 3.2, Theorem 3.6]). Take now two indecomposable modules X and Y from the family $\mathscr{T}^B$ of stable tubes of $\Gamma_{\widehat{B}}$. Recall that $\mathscr{T}^B$ is a family of pairwise orthogonal standard stable tubes in the full subcategory $\operatorname{mod} B_0 = \operatorname{mod} B$ of $\operatorname{mod} \widehat{B}$. Because the generator g of the group G is of the form $g = \varphi\nu_{\widehat{B}}$, for a positive automorphism φ of $\widehat{B}$, we conclude that the simple composition factors of the module X and any shift ${}^hY$ of Y by an element $h \in G \setminus \{1\}$ are disjoint, and consequently we obtain that

$$\bigoplus_{h\in G} \operatorname{Hom}_{\widehat{B}}(X, {}^hY) \xrightarrow{\sim} \operatorname{Hom}_{\widehat{B}}(X,Y) = \operatorname{Hom}_B(X,Y).$$

Hence, we have $\operatorname{Hom}_A(F_\lambda X, F_\lambda Y) \xrightarrow{\sim} \operatorname{Hom}_B(X,Y)$, and consequently $\operatorname{rad}_A^\infty(F_\lambda X, F_\lambda Y) \xrightarrow{\sim} \operatorname{rad}_B^\infty(X,Y) = 0$. This shows that indeed $\mathscr{T}^A$ is a generalized standard family of stable tubes, as required. This finishes the proof of Theorem 1.2.

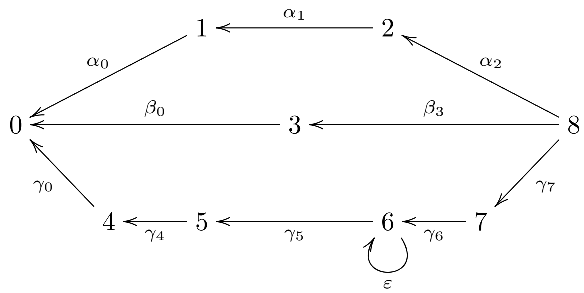
For the proof of Corollary 1.3, take φ the identity automorphism of $\widehat{B}$ and $A = \widehat{B}/(\nu_{\widehat{B}}) = T(B)$. Observe that the simple $T(B)$-modules coincide with the simple B-modules. Since every faithful B-module is sincere, we conclude that the constructed family $\mathscr{T}^{T(B)} = F_\lambda(\mathscr{T}^B)$, for the Galois covering $F : \widehat{B} \to \widehat{B}/(\nu_{\widehat{B}}) = T(B)$, is a family of pairwise orthogonal sincere stable tubes of $\Gamma_{T(B)}$.

7. Examples.

The aim of this section is to present some examples illustrating the above considerations.

We exhibit first examples of concealed generalized canonical algebras which are not generalized canonical algebras.

EXAMPLES 7.1. Let C be the bound quiver algebra KQ/I where Q is the quiver



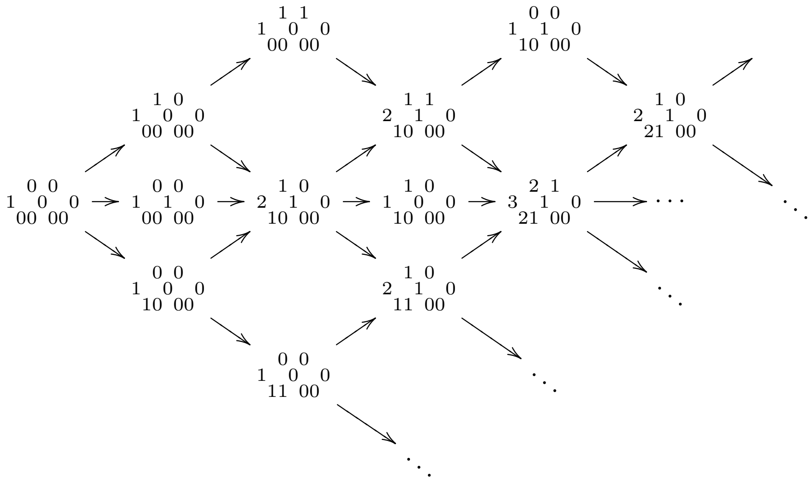
and I is the ideal in the path algebra KQ generated by the elements

$$\alpha_2\alpha_1\alpha_0 + \beta_3\beta_0 + \gamma_7\gamma_6\gamma_5\gamma_4\gamma_0,\ \varepsilon^2,\ \gamma_6\gamma_5 - \gamma_6\varepsilon\gamma_5.$$

Observe first that C is a generalized canonical algebra $C(\mathscr{B}, \mathscr{P}, \Lambda)$, where $m = 2$, $\Lambda = \{\infty, 0, 1\}$, $\mathscr{B} = \{B_0, B_1, B_2\}$, with $B_0 = K\Delta^{(0)}$ the path algebra of the quiver $\Delta^{(0)}$ given by the arrows $\alpha_0, \alpha_1, \alpha_2$, $B_1 = K\Delta^{(1)}$ the path algebra of the quiver $\Delta^{(1)}$ given by the arrows β_0, β_3, $B_2 = K\Delta^{(2)}/I^{(2)}$ the bound quiver algebra of the quiver $\Delta^{(2)}$ given by the arrows $\gamma_0, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \varepsilon$ and $I^{(2)}$ the ideal of $K\Delta^{(2)}$ generated by ε^2, $\gamma_6\gamma_5 - \gamma_6\varepsilon\gamma_5$, and $\mathscr{P} = \{P_0, P_1, P_2\}$ for the unique faithful indecomposable projective-injective B_i-modules P_i, $i \in \{0,1,2\}$. Moreover, in the above notation, we have $\Omega = \{\infty, 0\}$ and $\Sigma = \Lambda \setminus \Omega = \{1\}$. Therefore, by Theorem 2.1, the Auslander-Reiten quiver Γ_C admits the canonical family $\mathscr{T}^C = (\mathscr{T}_\mu^C)_{\mu\in \boldsymbol{P}(K)\setminus\{1\}}$ of pairwise orthogonal faithful standard stable tubes, with $\mathscr{T}_\infty^C$ the stable tube of rank 3, $\mathscr{T}_0^C$ the stable tube of rank 2, and the remaining stable tubes $\mathscr{T}_\mu^C$, $\mu \in \boldsymbol{P}_1(K)\setminus\{0,1,\infty\}$, of rank 1. We note that $\operatorname{gl\,dim} C = \infty$

because the simple C-module $S(6)$ at the vertex 6 has infinite projective dimension.

Denote by $\mathscr{C}$ the component of Γ_A containing the unique simple projective C-module $S(0)$. Then the standard calculation shows that the left hand part of the component $\mathscr{C}$ looks as follows



where the indecomposable modules are represented by their dimension vectors and

$$P(0) = S(0) = \begin{smallmatrix} & 0\ 0 & \\ 1 & 0 & 0 \\ & 00\ 00 & \end{smallmatrix},\ P(1) = \begin{smallmatrix} & 1\ 0 & \\ 1 & 0 & 0 \\ & 00\ 00 & \end{smallmatrix},\ P(2) = \begin{smallmatrix} & 1\ 1 & \\ 1 & 0 & 0 \\ & 00\ 00 & \end{smallmatrix},$$
$$P(3) = \begin{smallmatrix} & 0\ 0 & \\ 1 & 1 & 0 \\ & 00\ 00 & \end{smallmatrix},\ P(4) = \begin{smallmatrix} & 0\ 0 & \\ 1 & 0 & 0 \\ & 10\ 00 & \end{smallmatrix},\ P(5) = \begin{smallmatrix} & 0\ 0 & \\ 1 & 0 & 0 \\ & 11\ 00 & \end{smallmatrix}$$

are the indecomposable projective C-modules at the vertices $0, 1, 2, 3, 4, 5$, respectively.

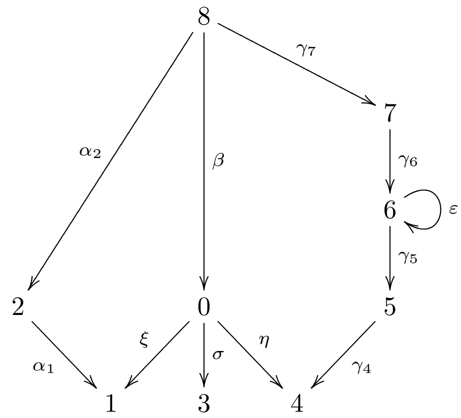
Consider the C-module

$$T = \tau_C^{-1} P(0) \oplus \left(\bigoplus_{i=1}^{8} P(i) \right)$$

Then T is the APR-tilting module at the simple projective module $S(0) = P(0)$ (see [**1**, (VI.2.8)]). Moreover, $\tau_C^{-1} P(0)$ is a submodule of the projective module

$$P(8) = \begin{smallmatrix} & 1\ 1 & \\ 2 & 1 & 1 \\ & 11\ 11 & \end{smallmatrix}.$$

Since $\mathscr{T}^C$ consists of faithful stable tubes, C_C is cogenerated by $\mathscr{T}^C$, and consequently the tilting C-module T is also cogenerated by $\mathscr{T}^C$. Hence the tilted algebra $B = \mathrm{End}_C(T)$ is a concealed generalized canonical algebra. Further, by Theorem 1.1, Γ_B admits a family $\mathscr{T}^B = (\mathscr{T}^B_\mu)_{\mu \in \boldsymbol{P}_1(K)\setminus\{1\}}$, with $\mathscr{T}^B_\mu = \mathrm{Hom}_C(T, \mathscr{T}^C_\mu)$, of pairwise orthogonal faithful standard stable tubes. Observe also that B is the bound quiver algebra KQ_B/J where Q_B is the quiver



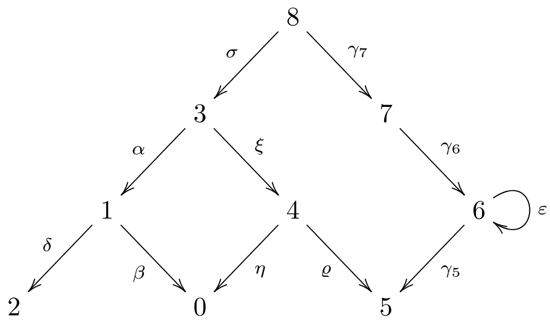
and J is the ideal in KQ_B generated by the elements

$$\alpha_2\alpha_1 - \beta\xi,\ \beta\eta - \gamma_7\gamma_6\gamma_5\gamma_4,\ \varepsilon^2,\ \gamma_6\gamma_5 - \gamma_6\varepsilon\gamma_5.$$

Consider also the C-module

$$T' = P(3) \oplus \tau_C^{-1}P(1) \oplus \tau_C^{-1}P(4) \oplus \tau_C^{-2}P(3) \oplus P(2) \oplus P(5) \oplus P(6) \oplus P(7) \oplus P(8).$$

Then it follows from the described above left hand part of $\mathscr{C}$ that T' is a tilting C-module, and $\tau_C^{-1}P(1)$, $\tau_C^{-1}P(4)$, $\tau_C^{-2}P(3)$ are submodules of the projective module $P(8)$. Hence T' is cogenerated by the family $\mathscr{T}^C$. The associated concealed generalized canonical algebra $B' = \mathrm{End}_C(T')$ is the bound quiver algebra $KQ_{B'}/J'$ where $Q_{B'}$ is the quiver



and J' is the ideal of $KQ_{B'}$ generated by the elements

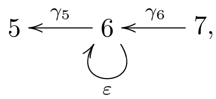
$$\alpha\beta - \xi\eta,\ \sigma\xi\varrho - \gamma_7\gamma_6\gamma_5,\ \varepsilon^2,\ \gamma_6\gamma_5 - \gamma_6\varepsilon\gamma_5.$$

Again, by Theorem 1.1, $\Gamma_{B'}$ admits a family $\mathscr{T}^{B'} = (\mathscr{T}_\mu^{B'})_{\mu\in\boldsymbol{P}_1(K)\setminus\{1\}}$, with $\mathscr{T}_\mu^{B'} = \mathrm{Hom}_C(T', \mathscr{T}_\mu^C)$, of pairwise orthogonal faithful standard stable tubes.

Observe also that $\mathrm{gl\,dim}\, B = \infty$ and $\mathrm{gl\,dim}\, B' = \infty$, because $\mathrm{pd}_B S(6) = \infty$ and $\mathrm{pd}_{B'} S(6) = \infty$. Moreover, the algebras B and B' are not generalized canonical algebras, because they have three simple projective modules. We also note that although B and B' are of infinite global dimension, all modules in the tubular families $\mathscr{T}^B$ and $\mathscr{T}^{B'}$ have the projective dimension one and the injective dimension one, because $\mathscr{T}^B$ and $\mathscr{T}^{B'}$ consist of generalized standard faithful stable tubes (see Lemma 3.4).

We may generalize the above examples as follows. Let $F = K\Delta/R$ be an arbitrary basic connected nonsimple algebra having a faithful indecomposable projective-injective module P with injective top and projective socle. We note that for an arbitrary basic algebra D and an arbitrary faithful D-module M we may take such an algebra F with D a factor algebra of F and $M = \mathrm{rad}P/\mathrm{soc}P$ (see [**26**, Corollary 2.5]). Moreover, we may take such an algebra F with $\mathrm{gl\,dim}\, F = n$ for any fixed $n \geq 1$ (see [**26**, Corollary 2.4]). Denote by 5 the unique sink of Δ, by 7 the unique source of Δ, and by u a path in Δ from 7 to 5.

Denote by E the bound quiver algebra $K\Theta/L$ given by the quiver



and the ideal L of $K\Theta$ generated by ε^2 and $\gamma_6\gamma_5 - \gamma_6\varepsilon\gamma_5$. Further, denote by Q^*, Q^*_B, $Q^*_{B'}$ the quiver obtained from the quiver Q, Q_B, $Q_{B'}$, respectively, by replacing the subquiver Θ by the quiver Δ. Moreover, let I^* be the ideal of KQ^* generated by $\alpha_2\alpha_1\alpha_0 + \beta_3\beta_0 + \gamma_7 u\gamma_4\gamma_0$ and R, J^* the ideal of KQ^*_B generated by $\alpha_2\alpha_1 - \beta\xi$, $\beta\eta - \gamma_7 u\gamma_4$ and R, and by $(J')^*$ the ideal of $KQ^*_{B'}$ generated by $\alpha\beta - \xi\eta$, $\sigma\xi\varrho - \gamma_7 u$ and R. Consider the bound quiver algebras $C^* = KQ^*/I^*$, $B^* = KQ^*_B/J^*$, and $(B')^* = KQ^*_{B'}/(J')^*$. Then C^* is a generalized canonical algebra and B^*, $(B')^*$ are concealed generalized canonical algebras of the forms $B^* = \mathrm{End}_{C^*}(T^*)$, $(B')^* = \mathrm{End}_{C^*}((T')^*)$, where T^* is the tilting C^*-module of the form

$$T^* = \tau^{-1}_{C^*} P(0) \oplus \left(\bigoplus_{i \in Q^*_0 \setminus \{0\}} P^*(i) \right),$$

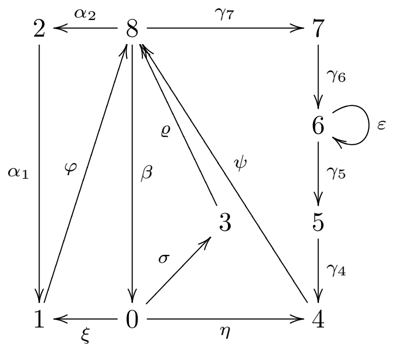
$(T')^*$ is the tilting C^*-module of the form

$$(T')^* = P^*(3) \oplus \tau^{-1}_{C^*} P^*(1) \oplus \tau^{-1}_{C^*} P^*(4) \oplus \tau^{-2}_{C^*} P^*(3) \oplus P^*(2) \oplus P^*(8) \oplus \overline{P},$$

and, for each vertex i of Q^*, $P^*(i)$ is the indecomposable projective C^*-module at i, $\overline{P} = \bigoplus_{i \in \Delta_0} P^*(i)$. Observe also that the algebra F is a factor algebra of any of the algebras C^*, B^*, $(B')^*$.

Finally we exhibit also an example of the trivial extension of a concealed generalized canonical algebra.

Example 7.2. Let B be the concealed generalized canonical algebra defined in Examples 7.1. Then the trivial extension $T(B)$ is the bound quiver algebra $K\Gamma/S$, where Γ is the quiver



and S is the ideal of $K\Gamma$ generated by the elements $\alpha_2\alpha_1 - \beta\xi$, $\beta\eta - \gamma_7\gamma_6\gamma_5\gamma_4$, ε^2, $\gamma_6\gamma_5 - \gamma_6\varepsilon\gamma_5$, $\varrho\alpha_2$, $\varrho\gamma_7$, $\varphi\gamma_7$, $\psi\alpha_2$, $\varphi\beta\sigma$, $\psi\beta\sigma$, $\varrho\beta\sigma\varrho$, $\alpha_1\varphi\alpha_2\alpha_1$, $\gamma_4\psi\gamma_7\gamma_6\gamma_5\gamma_4$, $\gamma_6\gamma_5\gamma_4\psi\gamma_7\gamma_6$, $\gamma_5\gamma_4\psi\gamma_7\gamma_6\varepsilon - \varepsilon\gamma_5\gamma_4\psi\gamma_7\gamma_6$.

References

[1] I. Assem, D. Simson and A. Skowroński, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Mathematical Society Student Texts, **65**, Cambridge Univ. Press, 2006.

[2] I. Assem and A. Skowroński, On tame repetitive algebras, Fund. Math., **142** (1993), 59–84.

[3] K. Bongartz and P. Gabriel, Covering spaces in representation theory, Invent. Math., **65** (1982), 331–378.

[4] P. Dowbor and A. Skowroński, Galois coverings of representation-infinite algebras, Comment. Math. Helv., **62** (1987), 311–337.

[5] P. Gabriel, The universal cover of a representation-finite algebra, Lecture Notes in Math., Springer Verlag, **903** (1981), 68–105.

[6] D. Happel, I. Reiten and S. O. Smalø, Tilting in abelian categories and quasitilted algebras, Memoirs Amer. Math. Soc., **575** (1996).

[7] D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras, J. London Math. Soc., **46** (1983), 347–364.

[8] O. Kerner and A. Skowroński, Quasitilted one-point extensions of wild hereditary algebras, J. Algebra, **244** (2001), 785–827.

[9] H. Lenzing and H. Meltzer, Tilting sheaves and concealed-canonical algebras, In: Representations of Algebras, CMS Conference Proceedings, **18**, Amer. Math. Soc., 1996, pp. 455–473.

[10] H. Lenzing and J. A. de la Peña, Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc., **78** (1999), 513–540.

[11] H. Lenzing and J. A. de la Peña, Supercanonical algebras, J. Algebra, **282** (2004), 298–348.

[12] Z. Leszczyński and A. Skowroński, Tame generalized canonical algebras, J. Algebra, **273** (2004), 412–433.

[13] S. Liu, Infinite radicals in standard Auslander-Reiten components, J. Algebra, **166** (1994), 245–254.

[14] P. Malicki and A. Skowroński, Almost cyclic coherent components of an Auslander-Reiten quiver, J. Algebra, **229** (2000), 695–749.

[15] P. Malicki and A. Skowroński, Algebras with separating almost cyclic coherent Auslander-Reiten components, J. Algebra, **291** (2005), 208–237.

[16] I. Reiten and A. Skowroński, Sincere stable tubes, J. Algebra, **232** (2000), 64–75.

[17] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math., Springer Verlag, **1099** (1984).

[18] C. M. Ringel, Representation theory of finite-dimensional algebras, Representations of Algebras, London Math. Soc. Lecture Note Series, **116**, Cambridge Univ. Press, 1986, pp. 7–79.

[19] D. Simson and A. Skowroński, Elements of the Representation Theory of Associative Algebras 2:

Tubes and Concealed Algebras of Euclidean Type, London Mathematical Society Student Texts, 71, Cambridge Univ. Press, 2007, in press.

[20] D. Simson and A. Skowroński, Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras, London Mathematical Society Student Texts, 72, Cambridge Univ. Press, 2007, in press.

[21] A. Skowroński, Generalized standard Auslander-Reiten components without oriented cycles, Osaka J. Math., **30** (1993), 515–527.

[22] A. Skowroński, Generalized standard Auslander-Reiten components, J. Math. Soc. Japan, **46** (1994), 517–543.

[23] A. Skowroński, Cycles in module categories, Finite Dimensional Algebras and Related Topics, NATO ASI Series, Series C: Mathematical and Physical Sciences, **424**, Kluwer Acad. Pub., 1994, pp. 309–345.

[24] A. Skowroński, On omnipresent tubular families of modules, In: Representations of Algebras, CMS Conference Proceedings, **18**, Amer. Math. Soc., 1996, pp. 641–657.

[25] A. Skowroński, On the structure of periodic modules over tame algebras, Proc. Amer. Math. Soc., **127** (1999), 1941–1949.

[26] A. Skowroński, Generalized canonical algebras and standard stable tubes, Colloq. Math., **90** (2001), 77–93.

[27] A. Skowroński, A construction of complex syzygy periodic modules over symmetric algebras, Colloq. Math., **90** (2005), 61–69.

[28] A. Skowroński and K. Yamagata, Galois coverings of selfinjective algebras by repetitive algebras, Trans. Amer. Math. Soc., **351** (1999), 715–734.

[29] A. Skowroński and K. Yamagata, On selfinjective artin algebras having nonperiodic generalized standard Auslander-Reiten components, Colloq. Math., **96** (2003), 235–244.

[30] A. Skowroński and K. Yamagata, Positive Galois coverings of selfinjective algebras, Advances Math., **194** (2005), 398–436.

[31] K. Yamagata, Frobenius algebras, Handbook of Algebra, **1**, Elsevier Science B. V., 1996, pp. 841–887.

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