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Foliated CR manifolds

Dedicated to Professor Philippe Tondeur on his seventieth birthday

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Abstract. We study foliations on CR manifolds and show the following. (1) For a strictly pseudoconvex CR manifold M, the relationship between a foliation $\mathscr{F}$ on M and its pullback $\pi^*\mathscr{F}$ on the total space $C(M)$ of the canonical circle bundle of M is given, with emphasis on their interrelation with the Webster metric on M and the Fefferman metric on $C(M)$, respectively. (2) With a tangentially CR foliation $\mathscr{F}$ on a nondegenerate CR manifold M, we associate the basic Kohn-Rossi cohomology of $(M, \mathscr{F})$ and prove that it gives the basis of the E_2-term of the spectral sequence naturally associated to $\mathscr{F}$. (3) For a strictly pseudoconvex domain Ω in a complex Euclidean space and a foliation $\mathscr{F}$ defined by the level sets of the defining function of Ω on a neighborhood U of $\partial\Omega$, we give a new axiomatic description of the Graham-Lee connection, a linear connection on U which induces the Tanaka-Webster connection on each leaf of $\mathscr{F}$. (4) For a foliation $\mathscr{F}$ on a nondegenerate CR manifold M, we build a pseudohermitian analogue to the theory of the second fundamental form of a foliation on a Riemannian manifold, and apply it to the flows obtained by integrating infinitesimal pseudohermitian transformations on M.

1. Introduction.

Foliations on CR manifolds appear naturally in several contexts. For instance, if a CR manifold $(M, T_{1,0}(M))$ is *Levi flat*, then the maximally complex distribution $H(M)$ of M is completely integrable so that M carries a foliation (the *Levi foliation*) by complex manifolds (cf. [**16**], [**38**]). Cf. Section 2 for notation and conventions. To see another example of this sort, let $\Omega = \{\varphi < 0\} \subset \boldsymbol{C}^{n+1}$ be a strictly pseudoconvex domain with real analytic boundary $M = \partial\Omega$. Let $\mathscr{O}(\bar{\Omega})$ be the algebra of functions on $\bar{\Omega}$ which admit a holomorphic extension to some neighborhood of $\bar{\Omega}$. Let $\Sigma \subset M$ be a real analytic submanifold which is not $\boldsymbol{C}$-tangent at any of its points. By a result in [**8**], if Σ is locally a maximum modulus set for $\mathscr{O}(\bar{\Omega})$ (cf., e.g., [**14**] for definitions), then $L = T(\Sigma) \cap H(M)$ is completely integrable and gives rise to on Σ a $\boldsymbol{C}$-tangent foliation $\mathscr{F}$ of codimension one. On the other hand, by a result in [**5**], if Σ is tangent to the characteristic direction T of a pseudohermitian structure θ on M, then Σ is a contact CR submanifold (in the sense of [**43**], and thus a CR manifold), $\mathscr{F}$ is a Riemannian foliation and the metric g_Σ induced on Σ by the Webster metric of (M, θ), where $\theta = (i/2)(\bar{\partial} - \partial)\varphi$, is bundle-like (also, Σ is Levi flat and $\mathscr{F}$ is its Levi foliation).

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Opposite to the Levi flat case, if $(M, T_{1,0}(M))$ is a nondegenerate CR manifold of hypersurface type whose pseudohermitian structure θ is a contact form on M, then the characteristic direction T of (M, θ) defines a flow on M (the *contact flow*, cf., e.g., [**17**]). Also, foliations by Riemann spheres appear (cf. [**27**]) on twistor spaces (nondegenerate 5-dimensional CR manifolds) of 3-dimensional conformal manifolds (a generalization of the example to n dimensions is due to [**39**]). A converse of this situation is known as well, namely if M is a nondegenerate CR manifold of CR dimension $n = 2m$ carrying a foliation by compact complex manifolds of complex dimension $\geq m$, then $m = 1$, the leaves are $\boldsymbol{C}P^1$'s, and M arises from a twistor construction (cf. [**28**]).

Furthermore, it should be noted that in [**10**] one considers foliations $\mathscr{F}$ on a CR manifold M such that, for any defining local submersion $f : U \to U'$ (i.e., the leaves of $\mathscr{F}|U$ are the fibres of f), the local quotient manifold U' is a CR manifold, f is a CR map, and $df : H(U) \to H(U')$ is surjective. Such $\mathscr{F}$ has a transverse CR structure (in the sense of [**6**]) and also a "tangential" CR structure (so that each leaf of $\mathscr{F}$ becomes a CR submanifold of M). While foliations with transverse CR structure have been investigated (cf. [**3**] and [**1**]), a systematic treatment of foliations with tangential CR structure is still missing in the mathematical literature.

The purpose of the present paper is to study basic properties of foliations on CR manifolds, in particular, tangentially CR foliations on nondegenerate CR manifolds, and prove the following as the first step.

If M is a strictly pseudoconvex CR manifold with a fixed contact form θ whose corresponding Levi form G_θ is positive definite, and $\mathscr{F}$ is a foliation on M which is tangent to the characteristic direction T of θ, then the pullback foliation $\pi^*\mathscr{F}$ of $\mathscr{F}$ to the total space of the canonical circle bundle $\pi : C(M) \to M$ of M is nondegenerate with respect to the Fefferman metric F_θ on $C(M)$. Furthermore, F_θ is bundle-like for $\pi^*\mathscr{F}$ if and only if the Webster metric g_θ of (M, θ) is bundle-like for $\mathscr{F}$. For a transversally oriented codimension q foliation $\mathscr{F}$ on M, we show that if (1) $\mathscr{F}$ is tangent to T, (2) the transverse volume element of $\mathscr{F}$ in (M, g_θ) is holonomy invariant, and (3) the mean curvature form κ of $\mathscr{F}$ in (M, g_θ) is d_B-exact, then the q-dimensional basic cohomology $H_B^q(\mathscr{F})$ of $\mathscr{F}$ is nonvanishing. Thus we generalize a result in [**23**] (cf. also Corollary 9.22 in [**41**], p. 125) to the case of foliations on CR manifolds.

With any tangentially CR foliation $\mathscr{F}$ on M we associate a cohomology algebra $H_B^{0,\bullet}(\mathscr{F})$, the *basic Kohn-Rossi cohomology* of $(M, \mathscr{F})$, which has the property that $H_B^{0,0}(\mathscr{F}) = CR^\infty(M)$ [the space of CR functions on M] and that $H_B^{0,1}(\mathscr{F})$ injects into the ordinary Kohn-Rossi cohomology group $H^{0,1}(M)$ of M on $(0,1)$-forms. We build a decreasing filtration $\{F^r\Omega^{0,\bullet}\}_{r\geq 0}$ of $\Omega^{0,\bullet}(M)$ by $\bar{\partial}_M$-differential ideals, and show that if $\{E_i^{r,s}\}_{i\geq 0}$ is the corresponding spectral sequence, then $E_2^{r,0} \approx H_B^{0,r}(\mathscr{F})$.

Given a smoothly bounded strictly pseudoconvex domain $\Omega = \{\varphi < 0\} \subset \boldsymbol{C}^{n+1}$ and a foliation $\mathscr{F}$ defined by the level sets of φ on a neighborhood U of $\partial\Omega$, we give a new axiomatic description of the *Graham-Lee connection*, a linear connection ∇ on U which induces the Tanaka-Webster connection on each leaf of $\mathscr{F}$, and then compute Faran's invariants $h_{\bar{\beta}}^{\alpha}$ and k^α (cf. [**15**]) in terms of the pseudohermitian torsion of the Graham-Lee connection and transverse curvature of φ, respectively. Also, for a foliation $\mathscr{F}$ on a nondegenerate CR manifold M we use the adapted connection determined by the Bott connection of $\mathscr{F}$ and the Tanaka-Webster connection (associated to a choice of contact form θ on M) to produce a pseudohermitian analogue to the theory of the second fundamental form of a foliation on a Riemannian manifold (cf. [**41**], p. 62).

The theory is applied to foliations which are tangent to the characteristic direction of θ and orthogonal to a semi-Levi foliation, and to flows obtained by integrating infinitesimal pseudohermitian transformations on a nondegenerate CR manifold.

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2. Foliations and the Fefferman metric.

Given a foliation $\mathscr{F}$ on a strictly pseudoconvex CR manifold M with a contact form θ, whose corresponding real Levi form G_θ being positive definite, our main technique in this section is to consider the pullback foliation $\pi^*\mathscr{F}$ of $\mathscr{F}$ on the total space $C(M)$ of the canonical circle bundle over M.

This pullback foliation $\pi^*\mathscr{F}$ enjoys many of the properties of the original foliation $\mathscr{F}$. For instance, $\pi^*\mathscr{F}$ is tangentially oriented $\Leftrightarrow \mathscr{F}$ is tangentially oriented, the (canonical) transverse volume element of $\pi^*\mathscr{F}$ is holonomy invariant $\Leftrightarrow$ the transverse volume element of $\mathscr{F}$ is holonomy invariant, $\pi^*\mathscr{F}$ is harmonic (with respect to the Fefferman metric F_θ on $C(M)) \Leftrightarrow \mathscr{F}$ is harmonic (with respect to the Webster metric g_θ on M), etc. Furthermore, $\pi^*\mathscr{F}$ "lives" in the presence of a Lorentz metric (the Fefferman metric F_θ). The resulting philosophy then is that one might get a better understanding of the geometry of a foliated (strictly pseudoconvex) CR manifold by establishing general theorems about foliated Lorentz manifolds.

2.1. CR and pseudohermitian geometry.

We start by recalling a few notions of CR and pseudohermitian geometry, which are needed throughout the paper. Let $(M, T_{1,0}(M))$ be a *CR manifold* of type (n,k), where M is a real $(2n+k)$-dimensional C^∞ manifold and $T_{1,0}(M)$ is its *CR structure*, that is, a complex rank n subbundle of the complexified tangent bundle $T(M) \otimes \boldsymbol{C}$ of M such that

$$T_{1,0}(M) \cap T_{0,1}(M) = \{0\},$$

i.e., $T_{1,0}(M)$ is *totally complex*, and

$$[\Gamma^\infty(T_{1,0}(M)), \Gamma^\infty(T_{1,0}(M))] \subseteq \Gamma^\infty(T_{1,0}(M)),$$

i.e., $T_{1,0}(M)$ is *involutive*, or (*formally*) *Frobenius integrable*, where $T_{0,1}(M)$ stands for the complex conjugate of $T_{1,0}(M)$. The integers n and k are called the *CR dimension* and *CR codimension* of $(M, T_{1,0}(M))$, respectively. Clearly, if $k = 0$, then $(M, T_{1,0}(M))$ is a complex manifold of complex dimension n. We shall be mainly interested in CR manifolds of type $(n,1)$, which are commonly referred to as CR manifolds of *hypersurface type*.

Let $(M, T_{1,0}(M))$ be a CR manifold of arbitrary but fixed type. Let

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$$

be the *Levi*, or *maximally complex*, *distribution* of M. It carries the complex structure $J : H(M) \to H(M)$ given by

$$J(Z + \bar{Z}) = i(Z - \bar{Z}), \quad Z \in T_{1,0}(M), \quad i = \sqrt{-1}.$$

The *Levi form* L is then defined by

$$L(Z, \overline{W}) = i\pi([Z, \overline{W}]), \quad Z, W \in T_{1,0}(M),$$

where $\pi : T(M) \otimes \boldsymbol{C} \to \{T(M) \otimes \boldsymbol{C}\}/\{H(M) \otimes \boldsymbol{C}\}$ is the natural bundle map. A CR manifold with $L = 0$ is called *Levi flat*. Note that if $k = 0$, then $L = 0$ (i.e., a complex manifold is Levi flat). A CR manifold is called *nondegenerate* if L is nondegenerate.

Assume now that $k = 1$ and M is orientable. Let θ be a *pseudohermitian structure* on M, that is, a global nowhere zero C^∞ section of $H(M)^\perp \subset T^*(M)$, the *conormal bundle* of $H(M)$ defined by $H(M)_x^\perp = \{\omega \in T_x^*(M) \mid \mathrm{Ker}(\omega) \supseteq H(M)_x\}$ for $x \in M$. Consider

$$G_\theta(X, Y) = d\theta(X, JY), \quad X, Y \in H(M),$$

(the *real* Levi form). It is also customary to consider the complex bilinear form

$$L_\theta(Z, \overline{W}) = -i\, d\theta(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

Then L_θ and the complex linear extension of G_θ to $H(M) \otimes \boldsymbol{C}$ coincide on $T_{1,0}(M) \otimes T_{0,1}(M)$. Also, L_θ and L coincide up to a bundle isomorphism $H(M)^\perp \approx T(M)/H(M)$.

A CR manifold $(M, T_{1,0}(M))$, of hypersurface type, is *strictly pseudoconvex* if G_θ is positive definite for some pseudohermitian structure θ on M.

When $(M, T_{1,0}(M))$ is nondegenerate (of hypersurface type), any pseudohermitian structure θ is a *contact form* on M so that $\theta \wedge (d\theta)^n$ is a volume form on M. If this is the case, let T be the *characteristic direction* of (M, θ), that is, a unique tangent vector field on M, transverse to $H(M)$, determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. As usual, we extend G_θ to a degenerate metric $\tilde{G}_\theta = \pi_H^* G_\theta$ on M given by $\tilde{G}_\theta(X, Y) = G_\theta(\pi_H(X), \pi_H(Y))$ for any $X, Y \in T(M)$, where $\pi_H : T(M) \to H(M)$ is the canonical projection associated to the direct sum decomposition $T(M) = H(M) \oplus \boldsymbol{R}T$ [in particular, $\tilde{G}_\theta(T, T) = 0$]. The *Webster metric* of (M, θ) is then defined by

$$g_\theta = \tilde{G}_\theta + \theta \otimes \theta.$$

If $(2r, 2s)$ is the signature of G_θ $(r + s = n)$, then g_θ is a semi-Riemannian metric on M of signature $(2r + 1, 2s)$ [and if M is strictly pseudoconvex with G_θ positive definite, then g_θ is a Riemannian metric on M]. For instance, let $\boldsymbol{H}_n = \boldsymbol{C}^n \times \boldsymbol{R}$ be the *Heisenberg group* with the multiplication law $(z, t) \cdot (w, s) = (z + w, t + s + 2\,\mathrm{Im}\langle z, w\rangle)$, where $\langle z, w\rangle = \delta_{ij} z^i \overline{w}^j$, and consider the *Lewy operators*

$$L_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha} - i z^\alpha \frac{\partial}{\partial t}, \quad 1 \le \alpha \le n.$$

Then

$$T_{1,0}(\boldsymbol{H}_n)_x = \sum_{\alpha=1}^{n} \boldsymbol{C} L_{\alpha,x}, \quad x \in \boldsymbol{H}_n,$$

where $L_\alpha = \overline{L_{\bar{\alpha}}}$, is a CR structure on $\boldsymbol{H}_n$ making $\boldsymbol{H}_n$ into a strictly pseudoconvex CR manifold of CR dimension n (and actually into a *CR Lie group*, that is, a Lie group which is a CR manifold whose CR structure is left invariant).

A C^∞ map of CR manifolds $f : M \to M'$ is a *CR map* if $df_x(T_{1,0}(M)_x) \subseteq T_{1,0}(M')_{f(x)}$ for any $x \in M$. A *CR isomorphism*, or *CR equivalence*, is a CR map which is a C^∞ diffeomorphism.

Any C^∞ real hypersurface M in $\boldsymbol{C}^{n+1}$ is a CR manifold of CR dimension n, with the CR structure $T_{1,0}(M) = \{T(M) \otimes \boldsymbol{C}\} \cap T^{1,0}(\boldsymbol{C}^{n+1})$, where $T^{1,0}(\boldsymbol{C}^{n+1})$ is the span of $\{\partial/\partial z^j \mid 1 \le j \le n+1\}$. In particular, the boundary $\partial\Omega_{n+1}$ of the *Siegel domain*

$$\Omega_{n+1} = \left\{ (z,w) \in \boldsymbol{C}^n \times \boldsymbol{C} \,\middle|\, \operatorname{Im}(w) > \sum_{\alpha=1}^{n} |z^\alpha|^2 \right\}$$

is a CR manifold which is CR isomorphic to the Heisenberg group (the CR isomorphism is given by $f(z,t) = (z, t + i|z|^2)$, $(z,t) \in \boldsymbol{H}_n$).

For any nondegenerate CR manifold M of hypersurface type, on which a contact form θ has been fixed, there is a unique linear connection ∇ (the *Tanaka-Webster connection* of (M,θ), cf., e.g., [**13**]) such that (1) $H(M)$ is ∇-parallel, (2) $\nabla g_\theta = 0$ and $\nabla J = 0$, (3) the torsion T_∇ of ∇ is *pure*, that is, $T_\nabla(Z,W) = 0$ and $T_\nabla(Z,\overline{W}) = iL_\theta(Z,\overline{W})T$ for any $Z, W \in T_{1,0}(M)$, and $\tau \circ J + J \circ \tau = 0$, where $\tau(X) = T_\nabla(T,X)$, $X \in T(M)$. The vector valued 1-form τ on M is called the *pseudohermitian torsion* of ∇ and satisfies $g_\theta(\tau(X),Y) = g_\theta(X,\tau(Y))$ for any $X, Y \in T(M)$, that is, τ is self-adjoint with respect to g_θ.

2.2. The normal bundle.

Generally, given a codimension q foliation $\mathscr{F}$ on a C^∞ manifold N, we denote by $T(\mathscr{F})$ the tangent bundle of $\mathscr{F}$ and by $\nu(\mathscr{F}) = T(N)/T(\mathscr{F})$ its normal (or transverse) bundle, and by $\Pi : T(N) \to \nu(\mathscr{F})$ the natural bundle map.

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold of CR dimension n. Let θ be a contact form on M such that G_θ is positive definite. Let $\mathscr{F}$ be a codimension q foliation of M. Note that if $2n \ge q$, then θ is not basic. Indeed, if $T(\mathscr{F}) \rfloor \theta = 0$, then $T(\mathscr{F}) \subseteq H(M)$, and if $T(\mathscr{F}) \rfloor d\theta = 0$, then for any $X \in T(\mathscr{F})$

$$0 = d\theta(X, JX) = G_\theta(X,X) \Rightarrow X = 0.$$

Hence $\mathscr{F}$ is the foliation by points, that is, $q = 2n+1$. Let us extend G_θ to the whole of $T(M)$ as a degenerate metric $\tilde{G}_\theta$, by requesting that T is orthogonal to each $V \in T(M)$, and consider

$$T(\mathscr{F})_0 = \{ Y \in T(M) \mid \tilde{G}_\theta(X,Y) = 0 \text{ for all } X \in T(\mathscr{F}) \}.$$

We collect a few elementary facts in the following

Proposition 1. *The tangent bundle* $T(\mathscr{F})$ *is nondegenerate in* $(T(M), \tilde{G}_\theta)$ *if and only if the characteristic direction* T *of* (M, θ) *is transverse to* $T(\mathscr{F})$. *In general, let* $T(\mathscr{F})_{H(M)} = \pi_H(T(\mathscr{F}))$ *be the projection of* $T(\mathscr{F})$ *to* $H(M)$. *Then we obtain*

$$T(\mathscr{F})_0 = [T(\mathscr{F})_{H(M)}]^\perp \oplus \boldsymbol{R}\, T, \tag{2.1}$$

where the orthogonal complement $[T(\mathscr{F})_{H(M)}]^\perp$ *of* $T(\mathscr{F})_{H(M)}$ *is taken in* $(H(M), G_\theta)$. *If* T *is tangent to* $\mathscr{F}$, *then the following hold*:

(1) $T(\mathscr{F})_{H(M)} = H(M) \cap T(\mathscr{F})$.

(2) *The natural bundle map* $\sigma_0 : \nu(\mathscr{F}) \to T(\mathscr{F})_0$ *is a bundle monomorphism and corestricts to a bundle isomorphism*

$$\nu(\mathscr{F}) \approx [T(\mathscr{F})_{H(M)}]^\perp.$$

(3) $H_\theta(r, s) = \tilde{G}_\theta(\sigma_0(r), \sigma_0(s))$, $r, s \in \nu(\mathscr{F})$, *is a Riemannian metric on the normal bundle* $\nu(\mathscr{F}) \to M$.

Proof. Let us prove the first statement in Proposition 1. Assume that T is transverse to $T(\mathscr{F})$. Let $X \in T(\mathscr{F})$ such that $\tilde{G}_\theta(X, Y) = 0$ for any $Y \in T(\mathscr{F})$. Then

$$0 = \tilde{G}_\theta(X, X) = G_\theta(\pi_H(X), \pi_H(X)) = \|\pi_H(X)\|^2,$$

so that $\pi_H(X) = 0$. Thus $T(\mathscr{F}) \ni X = \theta(X)T$, which yields $\theta(X) = 0$ so that $X = 0$.

Vice versa, assume that $T(\mathscr{F})$ is nondegenerate in $(T(M), \tilde{G}_\theta)$. The proof is done by contradiction. If $T_x \in T(\mathscr{F})_x$ for some $x \in M$, then $\tilde{G}_{\theta,x}(v, T_x) = 0$ for any $v \in T_x(M) \supset T(\mathscr{F})_x$, which yields $T_x = 0$ by the nondegeneracy of $T(\mathscr{F})_x$ in $(T_x(M), \tilde{G}_{\theta,x})$, a contradiction.

To prove the second statement in Proposition 1, let $T(\mathscr{F})_{H(M)}$ be the projection of $T(\mathscr{F})$ to $H(M)$, namely,

$$T(\mathscr{F})_{H(M)} = \{X - \theta(X)T \mid X \in T(\mathscr{F})\}.$$

Since

$$[T(\mathscr{F})_{H(M)}]^\perp \cap \boldsymbol{R}\, T \subseteq H(M) \cap \boldsymbol{R}\, T = \{0\},$$

the sum in (2.1) is direct. To prove (2.1), first note that $T \in T(\mathscr{F})_0$. Next, if $Z \in [T(\mathscr{F})_{H(M)}]^\perp$, then $\tilde{G}_\theta(Z, Y) = 0$ for any $Y \in T(\mathscr{F})_{H(M)}$, which is written as $Y = X - \theta(X)T$ with $X \in T(\mathscr{F})$. Thus

$$0 = \tilde{G}_\theta(Z, Y) = \tilde{G}_\theta(Z, X),$$

and hence $Z \in T(\mathscr{F})_0$. To check the opposite inclusion, let $Z \in T(\mathscr{F})_0 \subset H(M) \oplus \boldsymbol{R}\, T$. Then $Z = Y + fT$ for some $Y \in H(M)$ and $f \in C^\infty(M)$. Since $\tilde{G}_\theta(Z, X) = 0$ for any $X \in T(\mathscr{F})$, it follows that

$$G_\theta(Y, X - \theta(X)T) = \tilde{G}_\theta(Z, X) = 0,$$

which implies $Y \in [T(\mathscr{F})_{H(M)}]^\perp$.

Consider the bundle map $\sigma_0 : \nu(\mathscr{F}) \to T(\mathscr{F})_0$ defined by

$$\sigma_0(\Pi(Y)) = (Y - \theta(Y)T)^\perp, \quad Y \in T(M),$$

where $(Y - \theta(Y)T)^{\perp}$ is the $[T(\mathscr{F})_{H(M)}]^{\perp}$-component of $Y - \theta(Y)T$ in $H(M)$. To see that $\sigma_0(\Pi(Y))$ is well-defined, suppose that $\Pi(Y) = \Pi(Z)$. Then $Y - Z \in T(\mathscr{F})$, and hence $Y - Z - \theta(Y - Z)T \in T(\mathscr{F})_{H(M)}$ so that $(Y - Z - \theta(Y - Z)T)^{\perp} = 0$.

Assume now that $T \in T(\mathscr{F})$. The proof of (1) is immediate. To check that σ_0 is a bundle monomorphism, let $\sigma_0(\Pi(Y)) = 0$, that is,

$$Y - \theta(Y)T \in T(\mathscr{F})_{H(M)} = H(M) \cap T(\mathscr{F}) \subseteq T(\mathscr{F}).$$

Thus, by $T \in T(\mathscr{F})$,

$$0 = \Pi(Y - \theta(Y)T) = \Pi(Y).$$

The isomorphism claimed in (2) of Proposition 1 follows by a dimension argument. Indeed, since $\dim_{\boldsymbol{R}} \nu(\mathscr{F})_x = q$, the fact that $H(M) + T(\mathscr{F}) \supseteq H(M) + \boldsymbol{R}\, T = T(M)$ implies that

$$2n + 1 = \dim_{\boldsymbol{R}} H(M)_x + \dim_{\boldsymbol{R}} T(\mathscr{F})_x - \dim_{\boldsymbol{R}}\{H(M)_x \cap T(\mathscr{F})_x\},$$

and hence $\dim_{\boldsymbol{R}}[T(\mathscr{F})_{H(M)}]_x = 2n - q$ for any $x \in M$.

Let us now prove (3) of Proposition 1. As the image of σ_0 lies in $H(M)$, $H_\theta(r, r) = \|\sigma_0(r)\|^2 \geq 0$ and $= 0$ if and only if $Y - \theta(Y)T \in T(\mathscr{F})_{H(M)}$ for each $Y \in T(M)$ such that $\Pi(Y) = r$. Therefore, $H_\theta(r, r) = 0$ if and only if $r \in \boldsymbol{R}\, \Pi(T)$. In particular, if $T \in T(\mathscr{F})$, then H_θ is a Riemannian metric in $\nu(\mathscr{F})$. Proposition 1 is proved. □

Remark 1. As the Webster metric g_θ is a Riemannian metric on M, one may consider as well the normal bundle

$$T(\mathscr{F})^{\perp} = \{Y \in T(M) \mid g_\theta(Y, X) = 0 \text{ for all } X \in T(\mathscr{F})\}$$

with the corresponding bundle isomorphism $\sigma : \nu(\mathscr{F}) \to T(\mathscr{F})^{\perp}$ given by $\sigma(\Pi(Y)) = Y^{\perp}$, where $Y^{\perp}$ is the $T(\mathscr{F})^{\perp}$-component of $Y \in T(M) = T(\mathscr{F}) \oplus T(\mathscr{F})^{\perp}$, and the metric induced by g_θ on $\nu(\mathscr{F})$ via σ. However, *when $T \in T(\mathscr{F})$, it follows that $T(\mathscr{F})^{\perp} = [T(\mathscr{F})_{H(M)}]^{\perp}$, $\sigma = \sigma_0$, and the metric on $\nu(\mathscr{F})$ induced by g_θ is precisely H_θ.* Indeed, let $Y \in T(\mathscr{F})^{\perp}$, namely $g_\theta(Y, X) = 0$ for any $X \in T(\mathscr{F})$. Since $g_\theta(Y, T) = 0$ in particular, $Y \in H(M)$. Therefore

$$\tilde{G}_\theta(Y, X - \theta(X)T) = \tilde{G}_\theta(Y, X) = g_\theta(Y, X) - \underbrace{\theta(Y)}_{=0}\theta(X) = 0$$

for any $X \in T(\mathscr{F})$, which shows that $T(\mathscr{F})^{\perp} \subseteq [T(\mathscr{F})_{H(M)}]^{\perp}$. The opposite inclusion may be proved in a similar manner. Also, it is immediate to see

$$\begin{aligned}
\sigma(\Pi(Y)) &= \sigma(\Pi(Y - \theta(Y)T)) && [\text{as } T \in T(\mathscr{F})] \\
&= (Y - \theta(Y)T)^{\perp} && [\text{as } T(\mathscr{F})^{\perp} = [T(\mathscr{F})_{H(M)}]^{\perp}] \\
&= \sigma_0(\Pi(Y))
\end{aligned}$$

for any $Y \in T(M)$.

2.3. The Fefferman metric.

The first statement in Proposition 1 shows that, under the natural assumption that T be tangent to the leaves of $\mathscr{F}$, $T(\mathscr{F})$ is degenerate in $(T(M), \tilde{G}_\theta)$. However, the pullback of $\mathscr{F}$ to the total space of the principal S^1-bundle

$$C(M) = \{K(M)\backslash\{\text{zero section}\}\}/\boldsymbol{R}_+ \to M$$

turns out to be nondegenerate in $(C(M), F_\theta)$, where F_θ is the Fefferman metric of (M, θ). Here $K(M) = \Lambda^{n+1,0}(M)$ is the canonical line bundle over M. To be more precise, a complex p-form ω on M is said to be a $(p,0)$-*form*, or a form of *type* $(p,0)$ if $T_{0,1}(M) \rfloor \omega = 0$, and $\Lambda^{p,0}(M) \to M$ denotes the bundle of the $(p,0)$-forms on M.

We proceed by recalling a few notions regarding the Fefferman metric (cf., e.g., [**29**]). Consider the 1-form η on $C(M)$ given by

$$\eta = \frac{1}{n+2}\left\{d\gamma + \pi^*\left(i\omega_\alpha^\alpha - \frac{i}{2}h^{\alpha\bar{\beta}}\,dh_{\alpha\bar{\beta}} - \frac{R}{4(n+1)}\theta\right)\right\},$$

where γ is the (local) fibre coordinate on $C(M)$, $\pi : C(M) \to M$ is the projection, $h_{\alpha\bar{\beta}}$ are the (local) components of the Levi form with respect to a (local) frame $\{T_\alpha\}$ of $T_{1,0}(M)$, i.e., $h_{\alpha\bar{\beta}} = L_\theta(T_\alpha, \overline{T}_\beta)$, ω_α^β are the connection 1-forms of the *Tanaka-Webster connection* ∇ of (M, θ), that is,

$$\nabla T_\beta = \omega_\beta^\alpha T_\alpha,$$

and $R = h^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$ is the *pseudohermitian scalar curvature* (again cf. [**29**]). Also, $R_{\alpha\bar{\beta}}$ is the *pseudohermitian Ricci tensor*. The Fefferman metric F_θ of (M, θ) is the Lorentz metric on $C(M)$ given by

$$F_\theta = \pi^*\tilde{G}_\theta + 2(\pi^*\theta) \odot \eta,$$

where $\odot$ denotes the symmetric tensor product. Note that η is a connection 1-form on the principal S^1-bundle $\pi : C(M) \to M$ (cf. also [**18**], p. 855). Let then

$$\beta_z = \{d_z\pi : \mathrm{Ker}(\eta_z) \to T_x(M)\}^{-1}, \quad z \in C(M)_x,\ x \in M,$$

be the *horizontal lift* with respect to η. For a tangent vector field X on M we adopt the notation $X^\uparrow = \beta(X)$. Let $S = \partial/\partial\gamma$ be the tangent vector field to the S^1-action. Then $T^\uparrow - S$ is timelike, and hence $(C(M), F_\theta)$ is time-oriented by $T^\uparrow - S$, namely $(C(M), F_\theta)$ is a *space-time* (cf., e.g., [**7**], p. 17). Moreover, if M is compact, then $(C(M), F_\theta)$ is not chronological (cf. Proposition 2.6 in [**7**], p. 23).

Let $\mathscr{F}$ be a foliation of M and $\pi^*\mathscr{F}$ the pullback of $\mathscr{F}$ to $C(M)$, that is,

$$T(\pi^*\mathscr{F})_z = (d_z\pi)^{-1}T(\mathscr{F})_{\pi(z)}, \quad z \in C(M).$$

The leaves of $\pi^*\mathscr{F}$ are connected components of the inverse images (via π) of the leaves of $\mathscr{F}$. We may state the following

Proposition 2. *Let $\mathscr{F}$ be a foliation on the strictly pseudoconvex CR manifold M, carrying the contact form θ (with G_θ positive definite). Let $T(\mathscr{F})^\uparrow$ be the horizontal lift (with respect to η) of $T(\mathscr{F})$, that is, $T(\mathscr{F})_z^\uparrow = \beta_z(T(\mathscr{F})_{\pi(z)})$ for $z \in C(M)$. Then for the tangent bundle $T(\pi^*\mathscr{F})$ of the pullback foliation $\pi^*\mathscr{F}$ on $C(M)$ we obtain*

$$T(\pi^*\mathscr{F}) = T(\mathscr{F})^\uparrow \oplus \mathrm{Ker}(d\pi). \tag{2.2}$$

Let T be the characteristic direction of (M,θ). If T is tangent to $\mathscr{F}$, then the following hold:

(1) *$T(\pi^*\mathscr{F})$ is nondegenerate in $(T(C(M)),F_\theta)$ and each leaf $\tilde{L}$ of $\pi^*\mathscr{F}$ is a Lorentz manifold with the induced metric ι^*F_θ, where $\iota:\tilde{L}\hookrightarrow C(M)$ denotes the inclusion.*

(2) *The metric h_θ induced by F_θ on $\nu(\pi^*\mathscr{F})=T(C(M))/T(\pi^*\mathscr{F})$, the normal bundle of $\pi^*\mathscr{F}$, is positive definite.*

(3) *The Fefferman metric F_θ is bundle-like for $(C(M),\pi^*\mathscr{F})$ if and only if the Webster metric g_θ is bundle-like for $(M,\mathscr{F})$.*

Here, by slightly generalizing the definition in, e.g., [**32**], p. 79, given a semi-Riemannian manifold (N,g) (i.e., g is nondegenerate, of constant index) and a foliation $\mathscr{F}$ on N, we call g a *bundle-like* metric for $(N,\mathscr{F})$ if (1) $T(\mathscr{F})$ is nondegenerate in (N,g) and (2) the metric h induced by g on $\nu(\mathscr{F})$ is *holonomy invariant*, that is, $\mathscr{L}_X h=0$ for any $X\in T(\mathscr{F})$, where $\mathscr{L}_X$ stands for the Lie differentiation with respect to X.

Proof. Let us first prove (2.2) in Proposition 2. Since η is a connection 1-form, it follows that

$$T(\mathscr{F})^{\uparrow}\cap \mathrm{Ker}(d\pi)\subseteq \mathrm{Ker}(\eta)\cap\mathrm{Ker}(d\pi)=\{0\}.$$

Therefore the sum in (2.2) is direct. The inclusion "$\supseteq$" holds by the very definition of $\pi^*\mathscr{F}$. Vice versa, if

$$V\in T(\pi^*\mathscr{F})\subseteq T(C(M))=\mathrm{Ker}(\eta)\oplus\mathrm{Ker}(d\pi),$$

then $V=X^{\uparrow}+fS$ for some $X\in T(M)$ and $f\in C^\infty(C(M))$, where $X^{\uparrow}=\beta(X)$ is the horizontal lift of X with respect to η. Also, $V\in T(\pi^*\mathscr{F})$ yields that $X=d\pi(V)\in T(\mathscr{F})$. Hence $X^{\uparrow}\in T(\mathscr{F})^{\uparrow}$, that is, $V\in T(\mathscr{F})^{\uparrow}+\mathrm{Ker}(d\pi)$. The identity (2.2) is thus proved.

Assume now that $T\in T(\mathscr{F})$. To see (1), consider $V\in T(\pi^*\mathscr{F})$ such that $F_\theta(V,W)=0$ for any $W\in T(\pi^*\mathscr{F})$. Then we have

$$(\pi^*\tilde{G}_\theta)(V,W)+(\pi^*\theta)(V)\eta(W)+(\pi^*\theta)(W)\eta(V)=0,$$

which implies, by taking the decomposition $V=V_H+V_V\in\mathrm{Ker}(\eta)\oplus\mathrm{Ker}(d\pi)$ into account, that

$$\tilde{G}_\theta(d\pi(V_H),d\pi(W_H))+\theta(d\pi(V_H))\eta(W_V)+\theta(d\pi(W_H))\eta(V_V)=0 \tag{2.3}$$

for any $W\in T(\pi^*\mathscr{F})$. If $W=S\in\mathrm{Ker}(d\pi)\subset T(\pi^*\mathscr{F})$, then $W_H=0$. Since

$$\eta=\{d\gamma+\pi^*\eta_0\}/(n+2)$$

for some 1-form η_0 on M, which is determined in terms of θ, and $d\gamma(S)=1$, it follows that $\eta(W_V)=1/(n+2)$. Then from (2.3) we see that $\theta(d\pi(V_H))=0$, which implies

$$d\pi(V_H)\in H(M), \tag{2.4}$$

with the corresponding simpler form of (2.3) as

$$\tilde{G}_\theta(d\pi(V_H),d\pi(W_H))+\theta(d\pi(W_H))\eta(V_V)=0. \tag{2.5}$$

If $W = V$, then it follows from (2.4) that $\|d\pi(V_H)\|^2 = 0$, that is, $d\pi(V_H) = 0$ and hence $V_H \in \mathrm{Ker}(d\pi) \cap \mathrm{Ker}(\eta) = \{0\}$. Substituting $V_H = 0$ into (2.5), we then see

$$\theta(d\pi(W_H))\eta(V_V) = 0 \tag{2.6}$$

for any $W \in T(\pi^*\mathscr{F})$. Setting $W = T^\uparrow \in T(\mathscr{F})^\uparrow \subset T(\pi^*\mathscr{F})$ in (2.6) then yields that $\eta(V_V) = 0$, that is, $V_V = 0$. Hence we conclude that $V = 0$, that is, $T(\pi^*\mathscr{F})$ is non-degenerate in $(T(C(M)), g_\theta)$.

Note now that $F_\theta(S, S) = 0$, and hence F_θ is indefinite on $T(\pi^*\mathscr{F})$. Since F_θ is nondegenerate on $T(\pi^*\mathscr{F})$, there F_θ must have signature $(2n + 1 - q, 1)$. Yet F_θ is a Lorentz metric, therefore F_θ is positive definite on $T(\pi^*\mathscr{F})^\perp$. Consequently, the metric $h_\theta(r, s) = F_\theta(\rho(r), \rho(s))$, $r, s \in \nu(\pi^*\mathscr{F})$, induced by F_θ on the normal bundle $\nu(\pi^*\mathscr{F})$ of $\pi^*\mathscr{F}$ is positive-definite, where $\rho : \nu(\pi^*\mathscr{F}) \to T(\pi^*\mathscr{F})^\perp$ is the natural isomorphism. This proves (2).

To prove (3), note first that $\mathscr{L}_{\tilde{X}} h_\theta = 0$ if and only if

$$\tilde{X}(F_\theta(V, W)) = F_\theta([\tilde{X}, V], W) + F_\theta(V, [\tilde{X}, W]) \tag{2.7}$$

for any $\tilde{X} \in T(\pi^*\mathscr{F})$ and $V, W \in T(\pi^*\mathscr{F})^\perp$. We now need the following

Lemma 1. *$T(\pi^*\mathscr{F})^\perp \subseteq \mathrm{Ker}(\eta)$ and consequently*

$$\mathrm{Ker}(\eta) = T(\mathscr{F})^\uparrow \oplus T(\pi^*\mathscr{F})^\perp. \tag{2.8}$$

Moreover, $d\pi(T(\pi^*\mathscr{F})^\perp) \subseteq H(M)$.

Proof of Lemma 1. For any

$$V \in T(\pi^*\mathscr{F})^\perp \subset T(C(M)) = \mathrm{Ker}(\eta) \oplus \mathrm{Ker}(d\pi),$$

one has the decomposition $V = V_H + fS$ with $V_H \in \mathrm{Ker}(\eta)$. On the other hand, since

$$F_\theta(S, T^\uparrow) = (\pi^*\theta)(T^\uparrow)\eta(S) = \theta(d\pi(T^\uparrow))/(n+2) = 1/(n+2),$$

we have

$$F_\theta(V, T^\uparrow) = f/(n+2) + F_\theta(V_H, T^\uparrow).$$

As $T \in T(\mathscr{F})$, it follows that

$$T^\uparrow \in T(\mathscr{F})^\uparrow \subset T(\pi^*\mathscr{F}),$$

so that $T^\uparrow$ is orthogonal to V. Hence we obtain

$$\begin{aligned} f &= -(n+2)F_\theta(V_H, T^\uparrow) = -(n+2)(\pi^*\tilde{G}_\theta)(V_H, T^\uparrow) \\ &= -(n+2)\tilde{G}_\theta(d\pi(V_H), T) = 0, \end{aligned}$$

that is, $V \in \mathrm{Ker}(\eta)$. Then the identity (2.8) follows, by (2.2), from the facts

$$\begin{aligned} &T(C(M)) = \{T(\mathscr{F})^\uparrow \oplus \mathrm{Ker}(d\pi)\} \oplus T(\pi^*\mathscr{F})^\perp, \\ &T(\mathscr{F})^\uparrow \oplus T(\pi^*\mathscr{F})^\perp \subseteq \mathrm{Ker}(\eta). \end{aligned}$$

To prove the last statement in Lemma 1, let

$$V \in T(\pi^*\mathscr{F})^{\perp} \subset T(C(M)) = H(M)^{\uparrow} \oplus \boldsymbol{R}T^{\uparrow},$$

that is, $V = Y^{\uparrow} + fT^{\uparrow}$ for some $Y \in H(M)$. Since $S \in \mathrm{Ker}(d\pi) \subset T(\pi^*\mathscr{F})$, S and V are orthogonal. Thus we have

$$\begin{aligned} 0 = F_\theta(V, S) &= F_\theta(Y^{\uparrow}, S) + f/(n+2) \\ &= (\pi^*\theta)(Y^{\uparrow})\eta(S) + f/(n+2). \end{aligned}$$

Hence $\theta(Y) = 0$ yields $f = 0$. Lemma 1 is proved. □

By Lemma 1, (2.7) holds if and only if it is satisfied for vector fields V, W of the form $V = Y^{\uparrow}$, $W = Z^{\uparrow}$ for some $Y, Z \in H(M)$. Also, (2.7) is identically satisfied when $\tilde{X} \in \mathrm{Ker}(d\pi)$. Indeed, if this is the case, then (by a result in [**25**], Vol. I, p. 78) one has $[\tilde{X}, Y^{\uparrow}] = [\tilde{X}, Z^{\uparrow}] = 0$. Hence

$$\tilde{X}(F_\theta(Y^{\uparrow}, Z^{\uparrow})) = \tilde{X}(G_\theta(Y, Z) \circ \pi) = 0,$$

since $d\pi(X) = 0$.

Assume from now on that $\tilde{X} \in T(\mathscr{F})^{\uparrow}$, that is, $\tilde{X} = X^{\uparrow}$ for some $X \in T(\mathscr{F})$. By Proposition 1.3 in [**25**], Vol. I, p. 65, $[X, Y]^{\uparrow}$ is the $\mathrm{Ker}(\eta)$-component of $[X^{\uparrow}, Y^{\uparrow}]$. Then it follows from $\theta(Y) = \theta(Z) = 0$ that the identity (2.7) is equivalent to

$$X(\tilde{G}_\theta(Y, Z)) = \tilde{G}_\theta([X, Y], Z) + \tilde{G}_\theta(Y, [X, Z]) \tag{2.9}$$

for any $X \in T(\mathscr{F})$ and $Y, Z \in H(M)$ such that $Y^{\uparrow}, Z^{\uparrow} \in T(\pi^*\mathscr{F})^{\perp}$. Finally, note that for each $V = Y^{\uparrow} \in T(\pi^*\mathscr{F})^{\perp}$ with $Y \in H(M)$ one has

$$0 = F_\theta(X^{\uparrow}, V) = \tilde{G}_\theta(X, Y) \circ \pi$$

for any $X \in T(\mathscr{F})$, and hence

$$Y \in H(M) \cap T(\mathscr{F})^{\perp} = [T(\mathscr{F})_{H(M)}]^{\perp}.$$

Therefore, $\mathscr{L}_{\tilde{X}} h_\theta = 0$ if and only if (2.9) holds for any $X \in T(\mathscr{F})$ and $Y, Z \in [T(\mathscr{F})_{H(M)}]^{\perp}$, that is, if and only if $\mathscr{L}_X F_\theta = 0$. This completes the proof of Proposition 2. □

2.4. Foliated Lorentz manifolds.

Let N be a C^∞ manifold and $\mathscr{F}$ a codimension q foliation of N. A differential p-form ω on N is called *basic* if

$$X \rfloor \omega = 0, \quad \mathscr{L}_X \omega = 0$$

for all $X \in T(\mathscr{F})$. Note that the exterior derivative d preserves basic forms, since

$$X \rfloor d\omega = \mathscr{L}_X \omega - d(X \rfloor \omega) = 0, \quad \mathscr{L}_X \, d\omega = d\mathscr{L}_X \omega = 0.$$

Hence, denoting by $\Omega_B^p(\mathscr{F})$ the set of basic p-forms, we obtain the *basic complex* of $\mathscr{F}$ (cf. [**41**], p. 119)

$$\Omega_B^0(\mathscr{F}) \xrightarrow{d_B} \Omega_B^1(\mathscr{F}) \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Omega_B^q(\mathscr{F}) \xrightarrow{d_B} 0,$$

where $d_B = d|\Omega_B$, and the corresponding *basic cohomology* of $\mathscr{F}$

$$H_B^j(\mathscr{F}) = H^j(\Omega_B^\bullet(\mathscr{F}), d_B), \quad 0 \leq j \leq q.$$

Also, we consider the spectral sequence determined by the following multiplicative filtration of the de Rham complex $\Omega^\bullet(N)$ (a decreasing filtration by differential ideals, cf. [**41**], p. 120)

$$F^r\Omega^m = \{\omega \in \Omega^m(N) \mid X_1 \rfloor \cdots X_{m-r+1} \rfloor \omega = 0 \text{ for } X_1, \dots, X_{m-r+1} \in T(\mathscr{F})\}.$$

Let us now consider a codimension q foliation $\mathscr{F}$ on an n-dimensional connected Lorentz manifold (N, g) such that $T(\mathscr{F})$ is nondegenerate in $(T(N), g)$. The *second fundamental form* α of $\mathscr{F}$ in (N, g) is defined by

$$\alpha : T(\mathscr{F}) \otimes T(\mathscr{F}) \to \nu(\mathscr{F}), \quad \alpha(X, Y) = \Pi(\nabla_X^N Y), \quad X, Y \in T(N),$$

where ∇^N is the Levi-Civita connection of (N, g). As in the Riemannian case, the involutivity of $T(\mathscr{F})$ implies that α is symmetric, since ∇^N is torsion-free. Next, by mere linear algebra (cf., e.g., [**34**], p. 49), $T(N) = T(\mathscr{F}) \oplus T(\mathscr{F})^\perp$ and we have a bundle isomorphism

$$\sigma : Q = \nu(\mathscr{F}) \to T(\mathscr{F})^\perp, \quad \sigma(s) = \text{the } T(\mathscr{F})\text{-component of } Y_s, \quad s \in \nu(\mathscr{F}),$$

where $Y_s \in T(N)$ with $\Pi(Y_s) = s$. Let g_Q be the induced metric on Q defined by

$$g_Q(r, s) = g(\sigma(r), \sigma(s)), \quad r, s \in \nu(\mathscr{F}).$$

We set $\mathrm{ind}(\mathscr{F}) = -1$ if each leaf L of $\mathscr{F}$ is Lorentzian, and $\mathrm{ind}(\mathscr{F}) = 1$ if each leaf L of $\mathscr{F}$ is Riemannian, with respect to the induced metric $g_L = \iota^* g$ on L, where $\iota : L \hookrightarrow N$ is the inclusion, respectively. It should be remarked here that if g is a bundle-like metric for $\mathscr{F}$, then no other possibility occurs. Indeed, let ∇ be the connection in Q given by

$$\nabla_X s = \begin{cases} \Pi([X, \sigma(s)]) & \text{if } X \in \Gamma^\infty(P), \\ \Pi(\nabla_X^N \sigma(s)) & \text{if } X \in \Gamma^\infty(P^\perp), \end{cases}$$

where $s \in \Gamma^\infty(Q)$ and $P = T(\mathscr{F})$. A verbatim repetition of the proof of Theorem 5.11 in [**41**], p. 53, shows that g is bundle-like for $\mathscr{F}$ if and only if $\nabla g_Q = 0$. Let $x, y \in N$ and $\alpha(t)$ a piecewise smooth curve joining x and y. If g is bundle-like, a standard argument based on ∇-parallel translation along α then shows that $\mathrm{ind}(g_Q)_x = \mathrm{ind}(g_Q)_y$, where $\mathrm{ind}(g_Q)$ is the index of g_Q (in the sense of [**34**], p. 55). Hence (Q, g_Q) is a semi-Riemannian bundle. Now, let g_P be the leafwise metric induced by g on P. Namely, if $x \in N$ and $L \in N/\mathscr{F}$ is the leaf through x, then $g_{P,x} = (\iota^* g)_x = g_{L,x}$. Clearly, we have $\mathrm{ind}(g) = \mathrm{ind}(g_P) + \mathrm{ind}(g_Q)$ at each point of N, and hence g_P also has constant index. For any $Z \in T(\mathscr{F})^\perp$, the *Weingarten map* $W(Z) : T(\mathscr{F}) \to T(\mathscr{F})$ of $\mathscr{F}$ is given by

$$g_P(W(Z)(X), X') = g_Q(\alpha(X, X'), \sigma^{-1}(Z)), \quad X, X' \in T(\mathscr{F}),$$

where g_P is the semi-Riemannian bundle metric on $P = T(\mathscr{F})$ induced by g. Then

$W(Z)$ is self-adjoint. The *mean curvature form* of $\mathscr{F}$ in (N, g) is the 1-form $\kappa \in \Omega^1(N)$ defined by

$$\kappa(Z) = \operatorname{trace} W(Z), \quad Z \in \Gamma^\infty(P^\perp),$$

$$X \rfloor \kappa = 0, \quad X \in \Gamma^\infty(P).$$

Assume from now on that $\mathscr{F}$ is tangentially oriented, that is, $\mathscr{F}$ is equipped with a principal $GL^+(p, \boldsymbol{R})$-subbundle $B \to N$ of the principal $GL(p, \boldsymbol{R})$-bundle $L(P) \to N$, where $p = n - q$ and the fibre $L(P)_x$ is the set of $\boldsymbol{R}$-linear isomorphisms $u : \boldsymbol{R}^p \to P_x$, $x \in N$. Let $\{E_1, \dots, E_p\}$ be a local g_P-orthonormal frame of P, adapted to B, defined on an open set $U \subseteq N$, satisfying $g_P(E_i, E_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_i^2 = 1$ (thus $\operatorname{ind}(\mathscr{F}) = \varepsilon_1 \cdots \varepsilon_p$). The *characteristic form* of $\mathscr{F}$ is a p-form $\chi_{\mathscr{F}} \in \Omega^p(N)$ defined by

$$\chi_{\mathscr{F}}(Y_1, \dots, Y_p) = \det(g(Y_i, E_j)), \quad Y_1, \dots, Y_p \in \Gamma(TN).$$

Note that $P^\perp \rfloor \chi_{\mathscr{F}} = 0$. The Lorentzian analogue of *Rummler's formula* (cf., e.g., [**41**], p. 68) still holds, namely,

$$Z \rfloor d\chi_{\mathscr{F}} + \kappa(Z)\chi_{\mathscr{F}} = 0 \text{ along } P \tag{2.10}$$

for any $Z \in \Gamma^\infty(P^\perp)$. Indeed, as $\chi_{\mathscr{F}}(E_1, \dots, E_p) = \operatorname{ind}(\mathscr{F})$,

$$\begin{aligned}(\mathscr{L}_Z \chi_{\mathscr{F}})(E_1, \dots, E_p) &= -\sum_{i=1}^p \chi_{\mathscr{F}}(E_1, \dots, \pi^\perp([Z, E_i]), \dots, E_p) \\ &= -\sum_{i=1}^p \varepsilon_i \operatorname{ind}(\mathscr{F}) g([Z, E_i], E_i),\end{aligned}$$

where $\mathscr{L}_Z$ is the Lie differentiation and $\pi^\perp : T(N) \to P$ is the natural bundle morphism. On the other hand, by the definition of κ, we obtain

$$\kappa(Z) = \sum_{i=1}^p \varepsilon_i g_P(W(Z)(E_i), E_i) = \sum_{i=1}^p \varepsilon_i g([Z, E_i], E_i),$$

and thus (2.10) is proved.

Assume further that $\mathscr{F}$ is transversally oriented (i.e., $P^\perp$ is oriented), and let ν be the characteristic form of $P^\perp$ defined in a completely analogous manner with $\chi_{\mathscr{F}}$. Let $\mu = d\operatorname{vol}(g)$ be the Lorentz volume form on N. Assume also that N is oriented, and let $\{E_A \mid 1 \le A \le n\}$ be an oriented local g-orthonormal frame, satisfying $g(E_A, E_B) = \varepsilon_A \delta_{AB}$, of $T(N)$ such that $\{E_i \mid 1 \le i \le p\}$ and $\{E_\alpha \mid p+1 \le \alpha \le n\}$ are frames in P and $P^\perp$, respectively, and denote its dual coframe by $\{\omega_A \mid 1 \le A \le n\}$. Then for any $\alpha \in \Omega^r(N)$

$$(*\alpha)(E_{A_1}, \dots, E_{A_{n-r}}) \cdot \mu = \varepsilon_{A_1} \cdots \varepsilon_{A_{n-r}} \alpha \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{n-r}},$$

where $* : \Omega^r(N) \to \Omega^{n-r}(N)$ is the Hodge operator. In particular, for ν we have

$$(*\nu)(E_1, \dots, E_p) \cdot \mu = \operatorname{ind}(\mathscr{F}) \nu \wedge \omega_1 \wedge \cdots \wedge \omega_p. \tag{2.11}$$

Also, a calculation based on the identity

$$\nu(Y_{p+1}, \dots, Y_n) = \det(g(Y_\alpha, E_\beta))$$

leads to

$$\nu = \varepsilon_{p+1} \cdots \varepsilon_n \omega_{p+1} \wedge \cdots \wedge \omega_n.$$

Hence ν is proportional to the transverse volume element $\omega_{p+1} \wedge \cdots \wedge \omega_n$ of $\mathscr{F}$, and (2.11) may be written as

$$(*\nu)(E_1, \ldots, E_p) \cdot \mu = (-1)^{pq+1} \omega_1 \wedge \cdots \wedge \omega_n.$$

Since $*\nu, \chi_{\mathscr{F}} \in \Omega^p(N)$ and $P^\perp \rfloor *\nu = 0$, $P^\perp \rfloor \chi_{\mathscr{F}} = 0$ as well, there is a function $f \in C^\infty(N)$ such that $*\nu = f\chi_{\mathscr{F}}$. A calculation shows that $f = (-1)^{pq+1} \operatorname{ind}(\mathscr{F})$ so that

$$*\nu = (-1)^{pq+1} \operatorname{ind}(\mathscr{F}) \chi_{\mathscr{F}}. \tag{2.12}$$

As a corollary of (2.12), we have

$$\nu \wedge \chi_{\mathscr{F}} = (-1)^{pq} \operatorname{ind}(\mathscr{F}) \mu. \tag{2.13}$$

At this point we may prove the following

Proposition 3. *Let $\mathscr{F}$ be a transversally oriented foliation on a compact orientable Lorentz manifold (N, g). Assume that the transverse volume element ν of $\mathscr{F}$ is holonomy invariant; hence $\nu \in \Omega_B^q(\mathscr{F})$ and $d\nu = 0$. If $\mathscr{F}$ is harmonic* (*i.e.*, $\kappa = 0$), *then* $[\nu] \neq 0$ *in* $H_B^q(\mathscr{F})$.

The proof is a *verbatim* repetition of the proof of Theorem 9.21 in [**41**], p. 124 (and Proposition 3 is the Lorentzian analogue of a result by Kamber and Tondeur [**23**]). Indeed, Rummler's formula (2.10) yields (when $\kappa = 0$)

$$d\chi_{\mathscr{F}} \in F^2\Omega^{p+1},$$

and the assumption that $\nu = d_B\alpha$ for some $\alpha \in \Omega_B^{q-1}(\mathscr{F})$ leads, by (2.13), to

$$d(\alpha \wedge \chi_{\mathscr{F}}) = (-1)^{pq} \operatorname{ind}(\mathscr{F}) \mu,$$

and then, by Green's lemma, to a contradiction.

We may also establish the following

Proposition 4. *Let $\mathscr{F}$ be a foliation on a strictly pseudoconvex CR manifold M, and assume that $\mathscr{F}$ is tangent to the characteristic direction T of (M, θ) for some contact form θ on M. Then the following hold*:

(1) *$\mathscr{F}$ is transversally oriented if and only if $\pi^*\mathscr{F}$ is transversally oriented and, if this is the case, the transverse volume element ν of $\mathscr{F}$ in (M, g_θ) is holonomy invariant if and only if the transverse volume element $\tilde{\nu}$ of $\pi^*\mathscr{F}$ in $(C(M), F_\theta)$ is holonomy invariant.*

(2) *$\mathscr{F}$ is harmonic in (M, g_θ) if and only if $\pi^*\mathscr{F}$ is harmonic in $(C(M), F_\theta)$.*

Proposition 3 together with Proposition 4 then shows that *for any transversally oriented codimension q foliation $\mathscr{F}$ on a compact strictly pseudoconvex CR manifold M, if* (1) *$\mathscr{F}$ is tangent to the characteristic direction T of (M, θ)*, (2) *the transverse volume element ν of $\mathscr{F}$ in (M, g_θ) is holonomy invariant, and* (3) *$\mathscr{F}$ is harmonic in (M, g_θ), then* $[\nu] \neq 0$ *in* $H_B^q(\mathscr{F})$. Indeed, if M is compact, then so is $C(M)$ and, given a local coordinate system (U, x^A) on M, $(\pi^{-1}(U), u^A = x^A \circ \pi, u^{2n+2} = \gamma)$ yields a local coordinate on $C(M)$. Hence an orientation of M induces that of $C(M)$.

This is only illustrative of our ideas as to the use of the Fefferman metric. [The preceding statement also follows by directly applying the aforementioned result of Kamber and Tondeur (Theorem 9.21 in [**23**], p. 124) to $\mathscr{F}$ on (M, g_θ).] We may further exploit the relationship between pseudohermitian geometry and conformal Lorentzian geometry to prove the following

Corollary 1. *Let $\mathscr{F}$ be a transversally oriented codimension q foliation on a compact strictly pseudoconvex CR manifold M, which is tangent to the characteristic direction T of (M, θ) for a fixed contact form θ. Assume that the transverse volume element v of $\mathscr{F}$ in (M, g_θ) is holonomy invariant and that the mean curvature form κ of $\mathscr{F}$ in (M, g_θ) is closed (i.e., $d\kappa = 0$). If $[\kappa] = 0$ in $H^1_B(\mathscr{F})$, then $H^q_B(\mathscr{F}) \neq 0$.*

Proof. We shall need the following

Lemma 2. *Let $\mathscr{F}$ be a transversally oriented codimension q foliation on an n-dimensional Lorentz manifold (N, g), and assume that $T(\mathscr{F})$ is nondegenerate in $(T(N), g)$. Then $\mathscr{F}$ is harmonic in $(N, e^{2u}g)$, with $u \in C^\infty(N)$, if and only if*

$$du(Z) = p^{-1}\kappa(Z), \quad Z \in T(\mathscr{F})^\perp, \tag{2.14}$$

where $p = n - q$ and κ is the mean curvature form of $\mathscr{F}$ in (N, g). Furthermore, the following are equivalent:

(1) *u is a basic function, i.e., $u \in \Omega^0_B(\mathscr{F})$.*

(2) *The transverse volume element $\hat{v}$ of $\mathscr{F}$ in $(N, \hat{g} = e^{2u}g)$ is holonomy invariant if and only if the transverse volume element v of $\mathscr{F}$ in (N, g) is holonomy invariant.*

The statement (1) in Lemma 2, that is, if $du = p^{-1}\kappa$, then the leaves of $\mathscr{F}$ are minimal in $(N, e^{2u}g)$, was first discovered in [**24**] for the case of a Riemannian metric g (and our argument below follows closely the proof of Proposition 12.6 in [**41**], p. 151). The relationship between the Levi-Civita connections $\nabla^{\hat{g}}$ and ∇^g of Lorentz metrics $\hat{g} = e^{2u}g$ and g, respectively, is given by

$$\nabla^{\hat{g}} = \nabla^g + (du) \otimes I + I \otimes (du) - g \otimes \operatorname{grad}_g u,$$

which implies that the second fundamental forms $\hat{\alpha}$ and α of $\mathscr{F}$ in $(N, \hat{g})$ and (N, g), respectively, are related by

$$\hat{\alpha} = \alpha - g \otimes \Pi(\operatorname{grad}_g u).$$

Hence, for the corresponding Weingarten maps, we have

$$\hat{W}(Z) = W(Z) - du(Z)I, \quad Z \in \Gamma^\infty(P^\perp),$$

where I denotes the identity transformation. Consequently, the corresponding mean curvature forms satisfy

$$\hat{\kappa}(Z) = \kappa(Z) - pZ(u),$$

where $p = n - q$. Therefore $\hat{\kappa} = 0$ if and only if u satisfies (2.14).

The statement (2) in Lemma 2 follows from the formula

$$\mathscr{L}_X \hat{v} = e^{qu}\{q\, du(X)v + \mathscr{L}_X v\}, \quad X \in \Gamma^\infty(P).$$

This completes the proof of Lemma 2. □

Let us go back to the proof of Corollary 1. Since $[\kappa] = 0$ in $H_B^1(\mathscr{F})$ by assumption, there is a basic function $v \in \Omega_B^0(\mathscr{F})$ such that $\kappa = dv$. Set $u = (p+1)^{-1}v$, where $p = 2n + 1 - q$ and $\dim M = 2n + 1$. Then it is immediate from (2.17) in the proof of Proposition 4 that

$$d(u \circ \pi)(Z^{\uparrow}) = (p+1)^{-1}\tilde{\kappa}(Z^{\uparrow}), \quad Z \in T(\mathscr{F})^{\perp}.$$

Hence it follows from Lemma 2 that $\pi^*\mathscr{F}$ is harmonic in $(C(M), e^{2u\circ\pi}F_\theta)$, from which we see that $\pi^*\mathscr{F}$ is harmonic in $(C(M), F_{\hat{\theta}})$, since, by a result of Lee [**29**], the Fefferman metric changes conformally $F_{\hat{\theta}} = e^{2u\circ\pi}F_\theta$ under a transformation $\hat{\theta} = e^{2u}\theta$.

Now note that, by Proposition 4, the transverse volume element $\tilde{\nu}$ of $\pi^*\mathscr{F}$ in $(C(M), F_\theta)$ is holonomy invariant. Since $u \in \Omega_B^0(\mathscr{F})$, it follows that $u \circ \pi \in \Omega_B^0(\pi^*\mathscr{F})$ so that, again by Lemma 2, the transverse volume element $\hat{\nu}$ of $\pi^*\mathscr{F}$ in $(C(M), F_{\hat{\theta}})$ is holonomy invariant. In consequence, by Proposition 3, we may conclude that $0 \neq H_B^q(\pi^*\mathscr{F}) \approx H_B^q(\mathscr{F})$. $\square$

Proof of Proposition 4. First we note that

$$\nu(\mathscr{F}) \approx T(\mathscr{F})^{\perp} \xrightarrow{\beta} [T(\mathscr{F})^{\perp}]^{\uparrow} = T(\pi^*\mathscr{F})^{\perp} \approx \nu(\pi^*\mathscr{F}),$$

from which it is immediate that $\mathscr{F}$ is transversally oriented if and only if so is $\pi^*\mathscr{F}$. We only need to justify here the equality in the sequence. To this end, let $\tilde{X} \in T(\pi^*\mathscr{F})$ and write $\tilde{X} = X^{\uparrow} + V$ for some $X \in P = T(\mathscr{F})$ and $V \in \mathscr{V} = \mathrm{Ker}(d\pi)$. Then for any $Y \in P^{\perp}$ we have

$$\begin{aligned} F_\theta(\tilde{X}, Y^{\uparrow}) &= (\pi^*\tilde{G}_\theta)(\tilde{X}, Y^{\uparrow}) + (\pi^*\theta)(Y^{\uparrow})\eta(\tilde{X}) \\ &= \tilde{G}_\theta(X, Y) + \theta(Y)\eta(V) = \tilde{G}_\theta(\pi_H(X) + \theta(X)T, Y), \end{aligned}$$

since $P^{\perp} \subset H(M)$. Therefore we see

$$\begin{aligned} F_\theta(\tilde{X}, Y^{\uparrow}) &= G_\theta(\pi_H(X), Y) = g_\theta(\pi_H(X) + \theta(X)T, Y) \\ &= g_\theta(X, Y) = 0, \end{aligned}$$

which shows $[P^{\perp}]^{\uparrow} \subseteq T(\pi^*\mathscr{F})^{\perp}$ and hence the desired equality holds, for both bundles have rank q.

At this point we may relate the Weingarten maps of $\mathscr{F}$ and $\pi^*\mathscr{F}$, respectively. Let $\nabla^{C(M)}$ be the Levi-Civita connection of $(C(M), F_\theta)$. Given $Z \in P^{\perp}$, the Weingarten map $\tilde{W}(Z^{\uparrow}) : T(\pi^*\mathscr{F}) \to T(\pi^*\mathscr{F})$ of $\pi^*\mathscr{F}$ is given by

$$F_\theta(\tilde{W}(Z^{\uparrow})(\tilde{X}), \tilde{X}') = F_\theta(\nabla_{\tilde{X}}^{C(M)}\tilde{X}', Z^{\uparrow})$$

for any $\tilde{X} = X^{\uparrow} + V$ and $\tilde{X}' = X'^{\uparrow} + V'$, where $X, X' \in P$ and $V, V' \in \mathscr{V}$. As $\pi : C(M) \to M$ is a principal S^1-bundle, the projection π is a submersion. Recall, however, that for the vector field $S = \partial/\partial\gamma$ tangent to the S^1-action, $F_\theta(S, S) = 0$ and hence S is null, or isotropic, so that π is not a semi-Riemannian submersion (according to the terminology adopted in [**34**], p. 212). Nevertheless, we may relate $\nabla^{C(M)}$ to ∇^M, in the spirit of [**35**]. Another difficulty is that $\mathrm{Ker}(\eta)$ and $\mathscr{V}$ are not orthogonal (with respect to the Fefferman metric F_θ), yet $H(M)^\uparrow \perp \mathscr{V}$ does hold. Noting that $[Y^\uparrow, V] = 0$ for any $Y \in T(M)$, a calculation then leads to

$$\begin{aligned}2F_\theta(\tilde{W}(Z^\uparrow)(\tilde{X}), \tilde{X}') = &-Z(\tilde{G}_\theta(X, X')) - \tilde{G}_\theta(X, [X', Z]) - \tilde{G}_\theta(X', [X, Z]) \\ &+ \theta(X)\Omega(X'^\uparrow, Z^\uparrow) + \theta(X')\Omega(X^\uparrow, Z^\uparrow) \\ &+ d\theta(X, Z)\eta(V') + d\theta(X', Z)\eta(V),\end{aligned}$$

where $\Omega = D\eta$ is the curvature 2-form of η. Let $H(\mathscr{F})$ be the g_θ-orthogonal complement of $\boldsymbol{R}T$ in P. In particular, for any $X, X' \in H(\mathscr{F})$

$$\begin{aligned}2F_\theta(\tilde{W}(Z^\uparrow)(\tilde{X}), \tilde{X}') = &-Z(g_\theta(X, X')) - g_\theta(X, [X', Z]) - g_\theta(X', [X, Z]) \\ &+ d\theta(X, Z)\eta(V') + d\theta(X', Z)\eta(V),\end{aligned}$$

or (by exploiting the explicit expression of ∇^M, cf., e.g., [**25**], p. 160)

$$2F_\theta(\tilde{W}(Z^\uparrow)(\tilde{X}), \tilde{X}') = 2g_\theta(W(Z)(X), X') + d\theta(X, Z)\eta(V') + d\theta(X', Z)\eta(V). \tag{2.15}$$

Similarly, we also have

$$F_\theta(\tilde{W}(Z^\uparrow)(T^\uparrow), T^\uparrow) = \Omega(T^\uparrow, Z^\uparrow). \tag{2.16}$$

Next, we may calculate $\tilde{\kappa}(Z^\uparrow) = \operatorname{trace} \tilde{W}(Z^\uparrow)$. Let $\{E_1, \ldots, E_{p-1}, T\}$ be a g_θ-orthonormal frame of $T(\mathscr{F})$. Then

$$\left\{E_1^\uparrow, \ldots, E_{p-1}^\uparrow, T^\uparrow + \frac{n+2}{2}S, T^\uparrow - \frac{n+2}{2}S\right\}$$

is an F_θ-orthonormal frame of $T(\pi^*\mathscr{F})$. Note that $T^\uparrow - ((n+2)/2)S$ is timelike and $\operatorname{ind}(\pi^*\mathscr{F}) = -1$, in particular. Since $\theta(W(Z)(T)) = 0$, a straightforward calculation based on (2.15) and (2.16) now leads to

$$\tilde{\kappa}(Z^\uparrow) = \kappa(Z) \circ \pi, \quad Z \in P^\perp. \tag{2.17}$$

In particular, $\mathscr{F}$ is harmonic in (M, g_θ) if and only if $\pi^*\mathscr{F}$ is harmonic in $(C(M), F_\theta)$. Finally, if $\tilde{\nu} \in \Omega^q(C(M))$ is given by

$$\tilde{\nu}(\tilde{Y}_1, \ldots, \tilde{Y}_q) = \det(F_\theta(\tilde{Y}_\alpha, E_\alpha^\uparrow))$$

for some oriented g_θ-orthonormal frame $\{E_\alpha \mid 1 \le \alpha \le q\}$ of $P^\perp$, then $\tilde{\nu} = \pi^*\nu$ and by a simple calculation we see that

$$\mathscr{L}_{\tilde{X}}\tilde{\nu} = \pi^*(\mathscr{L}_X \nu)$$

for any $\tilde{X} = X^\uparrow + V$ with $X \in P$ and $V \in \mathscr{V}$. Proposition 4 is now proved. □

3. Tangentially CR foliations.

Let $(M, T_{1,0}(M))$ be a CR manifold and $\mathscr{F}$ a foliation on M. We say that $\mathscr{F}$ is a (*tangentially*) *CR foliation* if each leaf L of $\mathscr{F}$ is a CR submanifold of M, that is, L is a CR manifold and the inclusion $\iota : L \hookrightarrow M$ is a CR map, i.e., $d\iota_x(T_{1,0}(L)_x) \subseteq T_{1,0}(M)_x$ for each $x \in L$.

A typical example of a CR foliation is illustrated by the following

Example 1 (A CR foliation by level sets). Let $\boldsymbol{C}^{n+1}$ be the $(n+1)$-dimensional complex Euclidean space with complex coordinates $(z^1, \dots, z^n, w)$, $w = u + iv$, and $\alpha : \boldsymbol{R} \to \boldsymbol{R}$ a smooth function such that $\alpha(0) = 0$ and $\alpha'(t) < 0$ for any $t \in \boldsymbol{R}$. Define $f : \boldsymbol{C}^{n+1} \to \boldsymbol{R}$ by

$$f(z^1, \dots, z^n, w) = \alpha(|z^1|^2 + \cdots + |z^n|^2 - v)e^u.$$

Then f is a smooth submersion so that it defines a foliation $\mathscr{F}$ on $\boldsymbol{C}^{n+1}$ by the level sets of f.

Note that

$$f^{-1}(c) = \begin{cases} \{(z,w) \in \Omega_{n+1} \mid u = \log(c/\alpha(\rho))\} & \text{if } c > 0, \\ \partial\Omega_{n+1} & \text{if } c = 0, \\ \{(z,w) \in \boldsymbol{C}^{n+1} \backslash \bar{\Omega}_{n+1} \mid u = \log(c/\alpha(\rho))\} & \text{if } c < 0, \end{cases}$$

where $\rho = \sum_{\alpha=1}^{n} |z^\alpha|^2 - v$. Thus $\mathscr{F}$ is a CR foliation on $\boldsymbol{C}^{n+1}$, one of whose leaves is the Heisenberg group $\boldsymbol{H}_n \approx \partial\Omega_{n+1}$.

Now, let $\mathscr{F}$ be a CR foliation. Let $H(\mathscr{F}) \to M$ denote the subbundle of $T(\mathscr{F})$ whose portion over a leaf L of $\mathscr{F}$ coincides with the Levi distribution $H(L)$ of L. Similarly, let $T_{1,0}(\mathscr{F}) \to M$ denote the complex subbundle of $T(\mathscr{F}) \otimes \boldsymbol{C}$ whose portion over a leaf L of $\mathscr{F}$ coincides with the CR structure $T_{1,0}(L)$ of L.

Assume from now on that M is a nondegenerate CR manifold (of hypersurface type), and fix a contact form θ on M and the corresponding characteristic direction T of (M, θ). It should be remarked that if M is strictly pseudoconvex, then θ is holonomy invariant if and only if $H(\mathscr{F}) = 0$. Indeed, if $X \in H(\mathscr{F}) \subseteq H(M)$, then $\theta(X) = 0$. Hence,

$$0 = \mathscr{L}_X \theta = X \rfloor d\theta \Rightarrow G_\theta(X, X) = 0 \Rightarrow X = 0.$$

Recall that a $(0,s)$-*form* on M is a complex s-form ω on M such that $T_{1,0}(M) \rfloor \omega = 0$ and $T \rfloor \omega = 0$. Let $\Lambda^{0,s}(M) \to M$ be the bundle of $(0,s)$-forms on M and set $\Omega^{0,s}(M) = \Gamma^\infty(\Lambda^{0,s}(M))$. We recall the *tangential Cauchy-Riemann operator* $\bar{\partial}_M$, which is the first order differential operator

$$\bar{\partial}_M : \Omega^{0,s}(M) \to \Omega^{0,s+1}(M)$$

defined as follows. If ω is a $(0,s)$-form, then $\bar{\partial}_M \omega$ is the unique $(0,s+1)$-form which coincides with $d\omega$ on $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($s+1$ terms). A smooth function $f : M \to \boldsymbol{C}$ is called a CR function if it satisfies the *tangential Cauchy-Riemann equation*

$$\bar{\partial}_M f = 0.$$

The space of CR functions on M is denoted by $CR^\infty(M)$.

3.1. The basic tangentially Cauchy-Riemann complex.

We say that $\omega \in \Omega^{0,s}(M)$ is a *basic* $(0,s)$-*form* if it satisfies

$$\bar{Z} \rfloor \omega = 0, \quad \bar{Z} \rfloor \bar{\partial}_M \omega = 0$$

for any $Z \in T_{1,0}(\mathscr{F})$. Let $\Omega_B^{0,s}(\mathscr{F})$ denote the space of all basic $(0,s)$-forms on $(M, \mathscr{F})$. Since $\bar{\partial}_M^2 = 0$, we see easily that

$$\bar{\partial}_M \Omega_B^{0,s}(\mathscr{F}) \subseteq \Omega_B^{0,s+1}(\mathscr{F}).$$

Let $CR^\infty(\mathscr{F})$ be the space of smooth functions $f : M \to \boldsymbol{C}$ whose restriction $f|_L$ to each leaf L of $\mathscr{F}$ is a CR function on L, namely $f|_L \in CR^\infty(L)$. Note that $CR^\infty(M) \subseteq CR^\infty(\mathscr{F})$. Moreover, we have

$$\Omega_B^{0,0}(\mathscr{F}) = CR^\infty(\mathscr{F}).$$

Let $\bar{\partial}_B$ be the restriction of $\bar{\partial}_M$ to $\Omega_B^{0,s}(\mathscr{F})$. Then we obtain a complex

$$\Omega_B^{0,0}(\mathscr{F}) \xrightarrow{\bar{\partial}_B} \Omega_B^{0,1}(\mathscr{F}) \xrightarrow{\bar{\partial}_B} \cdots \xrightarrow{\bar{\partial}_B} \Omega_B^{0,k}(\mathscr{F}) \xrightarrow{\bar{\partial}_B} 0, \tag{3.1}$$

which is called the *basic tangentially Cauchy-Riemann complex* of $(M, \mathscr{F})$. Here we suppose that $\dim M = 2N + 1$ and $\mathscr{F}$ has codimension $q = 2k$. For the remainder of this section, we set $n = N - k$ and assume $n \geq 1$. The cohomology of the complex (3.1) given by

$$H_B^{0,s}(\mathscr{F}) = H^s(\Omega_B^{0,\bullet}(\mathscr{F}), \bar{\partial}_B) = \frac{\operatorname{Ker}\{\bar{\partial}_B \mid \Omega_B^{0,s}(\mathscr{F}) \to \Omega_B^{0,s+1}(\mathscr{F})\}}{\bar{\partial}_B \Omega_B^{0,s-1}(\mathscr{F})},$$

where $0 \leq s \leq k$, is called the *basic Kohn-Rossi cohomology* of $(M, \mathscr{F})$. In particular, we obtain

$$\begin{aligned} H_B^{0,0}(\mathscr{F}) &= \operatorname{Ker}\{\bar{\partial}_B \mid \Omega_B^{0,0}(\mathscr{F}) \to \Omega_B^{0,1}(\mathscr{F})\} = \{f \in CR^\infty(\mathscr{F}) \mid \bar{\partial}_B f = 0\} \\ &= CR^\infty(\mathscr{F}) \cap CR^\infty(M) = CR^\infty(M). \end{aligned}$$

Let

$$H^{0,s}(M) = H^s(\Omega^{0,\bullet}(M), \bar{\partial}_M)$$

be the ordinary *Kohn-Rossi cohomology* of the CR manifold M. For any CR foliation $\mathscr{F}$ on a nondegenerate CR manifold, there exists a natural injection of $H_B^{0,1}(\mathscr{F})$ into the Kohn-Rossi cohomology group $H^{0,1}(M)$, namely the map

$$H_B^{0,1}(\mathscr{F}) \hookrightarrow H^{0,1}(M), \quad [\omega] \mapsto [\omega]_{H^{0,1}(M)} \tag{3.2}$$

is a monomorphism. Here $\omega \in \Omega_B^{0,1}(\mathscr{F})$ with $\bar{\partial}_B \omega = 0$. Indeed, if $\omega, \omega' \in \operatorname{Ker}\{\bar{\partial}_B : \Omega_B^{0,1}(\mathscr{F}) \to \Omega_B^{0,2}(\mathscr{F})\}$ lie in the same Kohn-Rossi cohomology class, then $\omega' - \omega = \bar{\partial}_M f$ for some smooth function $f : M \to \boldsymbol{C}$. Then

$$0 = \bar{Z} \rfloor \omega' = \underbrace{\bar{Z} \rfloor \omega}_{=0} + \bar{Z} \rfloor \bar{\partial}_M f$$

for any $Z \in T_{1,0}(\mathscr{F})$, and hence $f \in \Omega_B^{0,0}(\mathscr{F})$. Thus $\omega' - \omega = \bar{\partial}_B f$, that is, $[\omega] = [\omega']$.

Remark 2. When M has CR codimension 0, that is, M is a complex manifold, $\Omega^{0,s}(M)$ is the space of all $(0,s)$-forms, which are locally spanned by monomials containing s anti-holomorphic differentials $d\bar{z}^\alpha$, with respect to local complex coordinates z^α on M. Note that $H^{0,s}(M)$ is then the Dolbeaut cohomology, and given a foliation $\mathscr{F}$ on M by CR submanifolds, (3.2) still holds.

EXAMPLE 1 (continued). Set $\beta(t) = \alpha'(t)/(\alpha(t) - i\alpha'(t))$. Then $T_{0,1}(\mathscr{F})$ is spanned by

$$\bar{Z}_\alpha = \frac{\partial}{\partial \bar{z}^\alpha} - 2\beta(\rho) z_\alpha \frac{\partial}{\partial \overline{w}}, \quad 1 \leq \alpha \leq n, \tag{3.3}$$

where we set $z_\alpha = z^\alpha$. Note that $\beta(0) = i$, and hence along the leaf $\partial\Omega_{n+1}$ of $\mathscr{F}$ the vector fields (3.3) correspond, under the CR isomorphism $\partial\Omega_{n+1} \cong \boldsymbol{H}_n$, to the Lewy operators.

We remark that $H_B^{0,s}(\mathscr{F}) = 0$ for $s \in \{1, 2\}$. Indeed, first it follows from (3.2) that $H_B^{0,1}(\mathscr{F}) \hookrightarrow H^{0,1}(\boldsymbol{C}^{n+1}) = 0$. On the other hand, if we define a $(0,1)$-form $\Theta \in \Omega^{0,1}(\boldsymbol{C}^{n+1})$ by

$$\Theta = d\overline{w} + 2\beta(\rho) z_\alpha \, d\bar{z}^\alpha,$$

then we obtain that

$$\Omega_B^{0,1}(\mathscr{F}) = \{\lambda\Theta \mid \lambda \in C^\infty(\boldsymbol{C}^{n+1}), \bar{\partial}_M \lambda(\bar{Z}_\alpha) = -i\beta'(\rho) z_\alpha \lambda, 1 \leq \alpha \leq n\},$$

$$\bar{\partial}_B \Omega_B^{0,0}(\mathscr{F}) = \{(\partial f/\partial \overline{w})\Theta \mid f \in CR^\infty(\mathscr{F})\},$$

and $\bar{\partial}_B \omega = 0$ for any $\omega \in \Omega_B^{0,1}(\mathscr{F})$ as seen below. Thus the meaning of the fact $H_B^{0,1}(\mathscr{F}) = 0$ is that the system

$$\frac{\partial f}{\partial \bar{z}_\alpha} = 2\beta(\rho) z_\alpha \lambda, \quad \frac{\partial f}{\partial \overline{w}} = \lambda$$

admits a solution $f \in C^\infty(\boldsymbol{C}^{n+1})$, provided that λ satisfies the compatibility relations $\bar{\partial}_M \lambda(\bar{Z}_\alpha) + i\beta'(\rho) z_\alpha \lambda = 0$.

To compute $H_B^{0,2}(\mathscr{F})$, let $\omega = \omega_{\alpha\beta} \, d\bar{z}^\alpha \wedge d\bar{z}^\beta + \omega_\alpha \, d\bar{z}^\alpha \wedge d\overline{w}$ be a basic $(0,2)$-form. Then we see that the condition

$$0 = \bar{Z}_\alpha \rfloor \omega = 2(\omega_{\alpha\beta} + \beta(\rho) z_\alpha \omega_\beta) \, d\bar{z}^\beta + \omega_\alpha \, d\overline{w}$$

yields that $\Omega_B^{0,2}(\mathscr{F}) = \{0\}$.

Similar to the above, let $\Omega^{p,0}(M)$ denote the space of $(p,0)$-forms ω such that $T \rfloor \omega = 0$, and consider the first order differential operator

$$\partial_M : \Omega^{p,0}(M) \to \Omega^{p+1,0}(M)$$

defined as follows. If $\omega \in \Omega^{p,0}(M)$, then $\partial_M \omega$ is a unique element of $\Omega^{p+1,0}(M)$ which coincides with $d\omega$ on $T_{1,0}(M) \otimes \cdots \otimes T_{1,0}(M)$ ($p+1$ terms). Then $\partial_M^2 = 0$ in all degrees and one may consider the cohomology groups

$$H^{p,0}(M) = H^p(\Omega^{\bullet,0}(M), \partial_M).$$

Moreover, if $\mathscr{F}$ is a CR foliation on M, then one may define the space of basic $(p,0)$-forms $\Omega_B^{p,0}(\mathscr{F})$, consisting of all elements $\omega \in \Omega^{p,0}(M)$ satisfying

$$T_{1,0}(\mathscr{F}) \rfloor \omega = 0, \quad T_{1,0}(\mathscr{F}) \rfloor \partial_M \omega = 0,$$

and the corresponding cohomology

$$H_B^{p,0}(\mathscr{F}) = H^p(\Omega_B^{\bullet,0}(\mathscr{F}), \partial_B),$$

where ∂_B denotes the restriction of ∂_M to $\Omega_B^{\bullet,0}(\mathscr{F})$. Then one sees that complex conjugation gives isomorphisms

$$H^{p,0}(M) \approx H^{0,p}(M), \quad H_B^{p,0}(\mathscr{F}) \approx H_B^{0,p}(\mathscr{F}).$$

Example 2 (The contact flow). Let $(M, T_{1,0})$ be a nondegenerate CR manifold of hypersurface type, and θ a contact form on M. Let T be the characteristic direction of (M, θ), and denote by $\mathscr{F}$ the flow defined by T (cf., e.g., [**41**], p. 132). Following [**17**], p. 160, let us consider the space U_h^r of all horizontal r-forms on M, where an r-form ω on M is called *horizontal* if $T \rfloor \omega = 0$ and $\mathscr{L}_T\omega = 0$. Thus U_h^r is nothing but $\Omega_B^r(\mathscr{F})$. Employing Kohn's solution (cf. [**26**]) to the Neumann problem for the $\bar\partial_M$ operator on a compact strictly pseudoconvex CR manifold, Gigante established the following

Theorem 1 (Gigante [**17**]). *Let M be a compact strictly pseudoconvex CR manifold and θ a contact form on M. Let T be the characteristic direction of (M, θ), and $\mathscr{F}$ the flow defined by T. If the Tanaka-Webster connection of (M, θ) has vanishing pseudohermitian torsion ($\tau = 0$) and strictly positive definite pseudohermitian Ricci curvature, then $H_B^1(\mathscr{F}) = 0$.*

We may give a short proof of Theorem 1, based on a result of Lee [**30**], as well as on our previous considerations. Indeed, $H_B^1(\mathscr{F}) = H^{0,1}(M) \oplus H^{1,0}(M)$. Furthermore, by a result in [**30**], if $R_{\alpha\bar\beta}\xi^\alpha\bar\xi^\beta > 0$ for any $\xi = (\xi^1, \dots, \xi^n)$, then $H^{0,1}(M) = 0$ (note that the assumption $\tau = 0$ was removed).

3.2. The filtration $\{F^r\Omega^{0,\bullet}\}_{r\geq 0}$.

We define a multiplicative filtration of the Cauchy-Riemann complex by setting

$$F^r\Omega^{0,m} = \{\omega \in \Omega^{0,m}(M) \mid \bar Z_1 \rfloor \cdots \bar Z_{m-r+1} \rfloor \omega = 0 \text{ for } Z_1, \dots, Z_{m-r+1} \in T_{1,0}(\mathscr{F})\}.$$

Note that we have

$$\Omega^{0,m}(M) = F^0\Omega^{0,m} \supseteq F^1\Omega^{0,m} \supseteq \cdots \supseteq F^m\Omega^{0,m} \supseteq F^{m+1}\Omega^{0,m} = \{0\}$$

for any $0 \leq m \leq N$. Also, the following diagram is commutative:

$$\begin{array}{ccccccc}
\Omega^{0,m}(M) & \supseteq \cdots \supseteq & F^r\Omega^{0,m} & \supseteq & F^{r+1}\Omega^{0,m} & \supseteq \cdots \\
\downarrow \bar\partial_M & & \downarrow \bar\partial_M & & \downarrow \bar\partial_M & \\
\Omega^{0,m+1}(M) & \supseteq \cdots \supseteq & F^r\Omega^{0,m+1} & \supseteq & F^{r+1}\Omega^{0,m+1} & \supseteq \cdots
\end{array}$$

Indeed, let

$$\omega \in F^r\Omega^{0,m} \subset \Omega^{0,m}(M) \xrightarrow{\bar\partial_M} \Omega^{0,m+1}(M).$$

Then, since $T_{0,1}(\mathscr{F})$ is involutive, it follows that

$$\bar Z_1 \rfloor \cdots \bar Z_{m-r+2} \rfloor (\bar\partial_M\omega) = 0$$

for any $Z_j \in T_{1,0}(\mathscr{F})$. Thus we have

$$\bar{\partial}_M F^r \Omega^{0,m} \subseteq F^r \Omega^{0,m+1}.$$

Now, setting

$$F^r \Omega^{0,\bullet} = \bigoplus_{m=0}^{N} F^r \Omega^{0,m},$$

we obtain the following

Proposition 5. *Let $\mathscr{F}$ be a CR foliation on the nondegenerate CR manifold M. Then $\{F^r \Omega^{0,\bullet}\}_{r \geq 0}$ is a decreasing filtration of $\Omega^{0,\bullet}(M)$ by differential ideals. Also, $\dim_{\boldsymbol{C}} T_{1,0}(\mathscr{F})_x = n$, $x \in M$, implies that $F^r \Omega^{0,n+r} = \Omega^{0,n+r}(M)$, and $\dim_{\boldsymbol{C}} T_{1,0}(M)_x / T_{1,0}(\mathscr{F})_x = k$, $x \in M$, yields that*

$$F^{k+1} \Omega^{0,m} = \{0\}. \tag{3.4}$$

Proof. Since we have seen that $\bar{\partial}_M F^r \Omega^{0,\bullet} \subseteq F^r \Omega^{0,\bullet}$, it remains to check that

$$\Omega^{0,\bullet}(M) \wedge F^r \Omega^{0,\bullet} \subseteq F^r \Omega^{0,\bullet}.$$

To this end, let $\omega = \omega_0 + \cdots + \omega_N$ with $\omega_m \in F^r \Omega^{0,m}$. Then we have

$$\alpha \wedge \omega_m \in F^r \Omega^{0,m+s}$$

for any $\alpha \in \Omega^{0,s}(M)$. Indeed, it is easy to see that

$$\bar{Z}_1 \rfloor \cdots \bar{Z}_{m+s-r+1} \rfloor (\alpha \wedge \omega_m) = 0$$

for any $Z_j \in T_{1,0}(\mathscr{F})$, because at most s of the $\bar{Z}_j$'s enter α, so that there are enough $\bar{Z}_j$'s left to kill ω_m. Hence we obtain

$$\Omega^{0,s}(M) \wedge F^r \Omega^{0,m} \subseteq F^r \Omega^{0,m+s},$$

from which the desired inclusion follows.

To prove (3.4), we need some local considerations. Let $\{T_1, \ldots, T_N\}$ be a local frame of $T_{1,0}(M)$ such that $\{T_1, \ldots, T_n\}$ is a local frame of $T_{1,0}(\mathscr{F})$. Let $\{\theta^1, \ldots, \theta^N\}$ be a local dual frame determined by

$$\theta^i(T_j) = \delta^i_j, \quad \theta^i(\bar{T}_j) = 0, \quad \theta^i(T) = 0.$$

Each $\omega \in F^r \Omega^{0,m}$ is then locally a sum of monomials of the form

$$\bar{\theta}^{\alpha_1} \wedge \cdots \wedge \bar{\theta}^{\alpha_p} \wedge \bar{\theta}^{j_1} \wedge \cdots \wedge \bar{\theta}^{j_q},$$

$$1 \leq \alpha_1, \ldots, \alpha_p \leq n, \quad n+1 \leq j_1, \ldots, j_q \leq N,$$

with $C^\infty(M)$-coefficients, where $0 \leq p \leq m - r$ and $q = m - p$. If $r = k+1$, then $0 \leq p \leq m - k - 1$ so that $q \geq k+1$. Hence $\bar{\theta}^{j_1} \wedge \cdots \wedge \bar{\theta}^{j_q} = 0$, and (3.4) is proved. $\square$

For a given CR foliation $\mathscr{F}$ on the nondegenerate CR manifold M with a fixed contact form θ, we set $\langle Z, W \rangle = L_\theta(Z, \overline{W})$ and define

$$T_{1,0}(\mathscr{F})^\perp = \{Z \in T_{1,0}(M) \mid \langle Z, W \rangle = 0 \text{ for any } W \in T_{1,0}(\mathscr{F})\}.$$

An argument of mere linear algebra then shows that $T_{1,0}(\mathscr{F})$ is nondegenerate in $(T_{1,0}(M), \langle\,,\rangle)$ and $T_{1,0}(\mathscr{F}) \oplus T_{1,0}(\mathscr{F})^\perp = T_{1,0}(M)$.

PROPOSITION 6. *Let $\mathscr{F}$ be a CR foliation on the nondegenerate CR manifold M. Let $\{E_i^{r,s}\}_{i\geq 0}$ be the spectral sequence associated with the filtered differential space $(\Omega^{0,\bullet}(M), \bar{\partial}_M, \{F^r\Omega^{0,\bullet}\}_{r\geq 0})$. Then we have the following isomorphisms of linear spaces*:

$$E_0^{r,s} \approx \mathrm{Hom}(\Lambda^s T_{0,1}(\mathscr{F}), \Lambda^r[T_{0,1}(\mathscr{F})^{\perp}]^*),$$

$$E_1^{r,0} \approx \Omega_B^{0,r}(\mathscr{F}), \quad E_2^{r,0} \approx H_B^{0,r}(\mathscr{F}),$$

where $T_{0,1}(\mathscr{F})^{\perp} = \overline{T_{1,0}(\mathscr{F})^{\perp}} \subset T_{0,1}(M)$.

PROOF. We set

$$Z_i^{r,m} = \{\omega \in F^r\Omega^{0,m} \mid \bar{\partial}_M \omega \in F^{r+i}\Omega^{0,m+1}\},$$

$$D_i^{r,m} = (F^r\Omega^{0,m}) \cap \bar{\partial}_M(F^{r-i}\Omega^{0,m-1}),$$

and

$$E_i^r\Omega^{0,m} = \frac{Z_i^{r,m}}{Z_{i-1}^{r+1,m} + D_{i-1}^{r,m}}.$$

(Cf., e.g., [**22**], Vol. III, p. 21.) Also, we set $E_i^{r,s} = E_i^r\Omega^{0,r+s}$. Then

$$E_0^{r,s} = \frac{F^r\Omega^{0,r+s}}{F^{r+1}\Omega^{0,r+s}} \approx \mathrm{Hom}(\Lambda^s T_{0,1}(\mathscr{F}), \Lambda^r[T_{0,1}(\mathscr{F})^{\perp}]^*). \tag{3.5}$$

With these understood, we now define

$$Z_i^r = \bigoplus_{m=0}^{N} Z_i^{r,m}, \quad E_i^r = \bigoplus_{m=0}^{N} E_i^r\Omega^{0,m}.$$

Then $\bar{\partial}_M Z_i^r \subset Z_i^{r+i}$ and $\bar{\partial}_M \mathrm{Ker}(\pi_i^r) \subset \mathrm{Ker}(\pi_i^{r+i})$, where $\pi_i^r : Z_i^r \to E_i^r$ is the natural projection, and hence $\bar{\partial}_M$ induces differentials $d_i^r : E_i^r \to E_i^{r+i}$. The resulting differential $d_0^{r,s} : E_0^{r,s} \to E_0^{r,s+1}$ corresponds, under the isomorphism (3.5), to

$$(\bar{\partial}_{\mathscr{CE}}\tilde{\omega})(\bar{Z}_1, \dots, \bar{Z}_{s+1}) = A_{s+1}[(\bar{Z}_1, \dots, \bar{Z}_{s+1}) \mapsto \bar{Z}_1 \rfloor \bar{\partial}_M(\tilde{\omega}(\bar{Z}_2, \dots, \bar{Z}_{s+1}))]$$

for any $Z_j \in T_{1,0}(\mathscr{F})$, $1 \leq j \leq s+1$. Here A_{s+1} is the alternation map (cf., e.g., [**25**], Vol. I, p. 28) and, for any $(0, r+s)$-form ω which is locally (cf. the discussion preceding Proposition 6) a sum of monomials of the form $\alpha \wedge \beta$ with $\alpha \in \Lambda^s T_{0,1}(\mathscr{F})^*$ and $\beta \in \Lambda^r[T_{0,1}(\mathscr{F})^{\perp}]^*$, we set

$$\tilde{\omega}(\bar{Z}_1, \dots \bar{Z}_s) = \bar{Z}_1 \rfloor \cdots \bar{Z}_s \rfloor \omega, \quad Z_j \in T_{1,0}(\mathscr{F}).$$

Note that, as the notation suggests, $\bar{\partial}_{\mathscr{CE}}$ is a CR analogue of the Chevalley-Eilenberg differential in [**41**], p. 122. Then we have

$$E_1^{r,s} \approx H^s(\mathrm{Hom}(\Lambda^\bullet T_{0,1}(\mathscr{F}), \Lambda^r[T_{0,1}(\mathscr{F})^{\perp}]^*), \bar{\partial}_{\mathscr{CE}}),$$

and hence

$$E_1^{r,0} \approx \Omega_B^{0,r}(\mathscr{F}). \tag{3.6}$$

Since $d_1^{r,0}$ induces, on the right hand side of (3.6), the differential $\bar{\partial}_B$, it follows that $E_2^{r,0} \approx H_B^{0,r}(\mathscr{F})$. $\square$

3.3. The Graham-Lee connection.

Let $\Omega = \{\varphi < 0\} \subset \boldsymbol{C}^{n+1}$ be a smoothly bounded strictly pseudoconvex domain and Δ_φ denote the Laplacian of the Kähler metric on Ω whose Kähler 2-form is $(i/2)\partial\bar{\partial}\log(-1/\varphi)$. Then, according to a result by Graham and Lee [**20**], if u is a local solution to $\Delta_\varphi u = 0$ which is smooth up to a portion of $\partial\Omega$, then the boundary value f of u must satisfy $\mathscr{C}_\varphi f = 0$, where $\mathscr{C}_\varphi$ is a differential operator on $\partial\Omega$ of order $2n+2$, which was first studied by Graham [**19**] on the Siegel domain and the unit ball. In order to compute $\mathscr{C}_\varphi$, one needs to understand the interrelation between the tangential pseudohermitian geometry of the leaves of the foliation $\mathscr{F}$ defined by level sets of φ, and the geometry of the ambient complex space.

One key instrument in this respect turns out to be a canonical connection on a one-sided neighborhood of $\partial\Omega$ (the *Graham-Lee connection*, cf. Theorem 2 below) which induces the Tanaka-Webster connection on each leaf of $\mathscr{F}$. We give a new axiomatic description of this connection and a hint on how one may recover Faran's results (cf. [**15**]) in this setting, namely in the presence of a fixed defining function for the foliation. Actually, we merely look at Faran's third order invariants $h^\alpha_{\bar{\beta}}$ and k^α, whereas the problem of recovering Faran's result on whether a given real hypersurface may be a leaf of a Ricci flat foliation (cf. [**15**], p. 403) is left open.

To be more precise, let $\Omega \subset \boldsymbol{C}^{n+1}$ be a strictly pseudoconvex domain. Let $V \subseteq \boldsymbol{C}^{n+1}$ be an open set, and $\varphi : V \to \boldsymbol{R}$ a smooth defining function for Ω such that $\Omega = \{x \in V \mid \varphi(x) < 0\}$ and $\partial\Omega = \{x \in V \mid \varphi(x) = 0\}$, satisfying $d\varphi(x) \neq 0$ for any $x \in \partial\Omega$. For a sufficiently small one-sided neighborhood U of the boundary $\partial\Omega$, we set $M_\varepsilon = \{x \in U \mid \varphi(x) = \varepsilon\}$ so that $M_0 = \partial\Omega$.

Consider now the foliation $\mathscr{F}$ on U whose leaves are the level sets M_ε of φ, where $\varepsilon \in \varphi(U)$. Since Ω is strictly pseudoconvex, the restriction of the real $(1,1)$-form $i\partial\bar{\partial}\varphi$ to $T_{1,0}(\mathscr{F})$ is definite, and by replacing φ by $-\varphi$ if necessary, we may assume that it is positive definite. Note that there exists a uniquely defined complex vector field ξ of type $(1,0)$ on U which is orthogonal to $T_{1,0}(\mathscr{F})$ with respect to $\partial\bar{\partial}\varphi$ and for which $\partial\varphi(\xi) = 1$ (cf. Lee and Melrose [**31**], p. 163). Let us then define a function $r : U \to \boldsymbol{R}$ by setting $r = \partial\bar{\partial}\varphi(\xi, \bar{\xi})$, so that ξ and r are characterized by

$$\xi \rfloor \partial\bar{\partial}\varphi = r\bar{\partial}\varphi, \quad \partial\varphi(\xi) = 1. \tag{3.7}$$

Let $\{W_1, \ldots, W_n\}$ be a local frame of $T_{1,0}(\mathscr{F})$. Then $\{W_\alpha, \xi\}$ is a local frame of $T_{1,0}(U)$. Let θ^α be the (local) complex 1-forms of type $(1,0)$ on U determined by $\theta^\alpha(W_\beta) = \delta^\alpha_\beta$ and $\theta^\alpha(\xi) = 0$. Then $\{\theta^\alpha, \partial\varphi\}$ is a local frame of $T^{1,0}(U)^*$ and, as a consequence of the first of the formulae (3.7),

$$\partial\bar{\partial}\varphi = h_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta + r\partial\varphi \wedge \bar{\partial}\varphi, \tag{3.8}$$

for some positive definite Hermitian matrix of functions $h_{\alpha\bar{\beta}}$. It follows from (3.8) that r is positive if and only if φ is strictly plurisubharmonic, and $r = 0$ if and only if φ satisfies the homogeneous complex Monge-Ampère equation $\det(\partial\bar{\partial}\varphi) = 0$ (cf. [**20**]). We call r the *transverse curvature* of φ.

Consider the real 1-form $\theta = (i/2)(\bar{\partial}\varphi - \partial\varphi)$ on U. Its exterior derivative is then given by

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta + r\, d\varphi \wedge \theta. \tag{3.9}$$

Thus the *Levi form* L_θ of θ, that is, the restriction of $-i\,d\theta$ to $T_{1,0}(\mathscr{F}) \otimes T_{0,1}(\mathscr{F})$ is given by

$$L_\theta(Z, \overline{V}) = h_{\alpha\bar\beta} Z^\alpha \overline{V}^\beta,$$

where $Z = Z^\alpha W_\alpha$, $V = V^\beta W_\beta \in T_{1,0}(\mathscr{F})$.

Let $j_\varepsilon : M_\varepsilon \to U$ be the inclusion. Then $\theta_\varepsilon = j_\varepsilon^* \theta$ is a pseudohermitian structure on $(M_\varepsilon, T_{1,0}(M_\varepsilon))$. If we write $\xi = (1/2)(N - iT)$, with N and T real, then we have $(d\varphi)(N) = 2$, $\theta(N) = 0$, and the restriction T_ε of T to M_ε is tangent to M_ε. Also, (3.7) shows that T_ε is the characteristic direction of $(M_\varepsilon, \theta_\varepsilon)$. Among the linear connections on U which restrict to the Tanaka-Webster connection ∇^ε on each leaf M_ε of $\mathscr{F}$, we single out a canonical one (cf. also Proposition 1.1 in [**20**], p. 701) in the following manner.

Let $\nu(\mathscr{F}) = T(U)/T(\mathscr{F})$ and $\Pi : T(U) \to \nu(\mathscr{F})$ be the projection. Given a linear connection ∇ on U, we consider the bundle map

$$\alpha : T(\mathscr{F}) \otimes T(\mathscr{F}) \to \nu(\mathscr{F}), \quad \alpha(X, Y) = \Pi(\nabla_X Y),$$

where $X, Y \in T(\mathscr{F})$. Let T_∇ be the torsion of ∇ and set $\tau(X) = T_\nabla(T, X)$ for any $X \in T(U)$. We say T_∇ is *pure* if

$$T_\nabla(Z, W) = 0, \tag{3.10}$$

$$T_\nabla(Z, \overline{W}) = iL_\theta(Z, \overline{W})T, \tag{3.11}$$

$$T_\nabla(N, Z) = rZ + i\tau(Z) \tag{3.12}$$

for any $Z, W \in T_{1,0}(\mathscr{F})$ and

$$\tau \circ J + J \circ \tau = 0. \tag{3.13}$$

Here J is the restriction of J_0 (the complex structure of U) to $H(\mathscr{F})$.

Now, we may state the following

Theorem 2. *Let $\Omega \subset \boldsymbol{C}^{n+1}$ be a strictly pseudoconvex domain and φ a smooth defining function for Ω. Let $\mathscr{F}$ be the foliation by level sets of φ on a one-sided neighborhood U of $\partial\Omega$. Then there exists a unique linear connection ∇ on U such that*

(1) *$T_{1,0}(\mathscr{F})$, N, T and L_θ are parallel with respect to ∇, and*

(2) *the torsion T_∇ of ∇ is pure.*

Consequently, one has $\alpha = 0$.

The canonical connection ∇ furnished by Theorem 2 is referred to as the *Graham-Lee connection* of (U, φ). Let $\pi_+ : T(U) \otimes \boldsymbol{C} \to T_{1,0}(\mathscr{F})$, respectively $\pi_- : T(U) \otimes \boldsymbol{C} \to T_{0,1}(\mathscr{F})$, be the canonical projections associated with the direct sum decomposition:

$$T(U) \otimes \boldsymbol{C} = T_{1,0}(\mathscr{F}) \oplus T_{0,1}(\mathscr{F}) \oplus \boldsymbol{C}T \oplus \boldsymbol{C}N. \tag{3.14}$$

Proof. Let us first prove the uniqueness statement in Theorem 2. The identity (3.11) may be written as

$$[Z, \overline{Y}] = \nabla_Z \overline{Y} - \nabla_{\overline{Y}} Z - iL_\theta(Z, \overline{Y})T.$$

Hence, by (1), we have

$$\nabla_{\bar{Y}} Z = \pi_+([\bar{Y}, Z]) \tag{3.15}$$

for any $Y, Z \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$. Next, $\nabla L_\theta = 0$ may be written as

$$X(L_\theta(Y, \bar{Z})) = L_\theta(\nabla_X Y, \bar{Z}) + L_\theta(Y, \nabla_X \bar{Z}),$$

from which we obtain

$$L_\theta(\nabla_X Y, \bar{Z}) = X(L_\theta(Y, \bar{Z})) - L_\theta(Y, \pi_-([X, \bar{Z}])) \tag{3.16}$$

for any $X, Y, Z \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$.

We now compute $\nabla_T X$. To this end, set

$$K_T X = -(1/2) J(\mathscr{L}_T J)(X)$$

for $X \in H(\mathscr{F})$. Then, since $T_{1,0}(\mathscr{F})$ is ∇-parallel and ∇ is a real differential operator, it follows that $T_{0,1}(\mathscr{F})$ is also ∇-parallel. Hence $H(\mathscr{F})$ is ∇-parallel and $\nabla J = 0$. Consequently, it follows from (3.13) that

$$\tau(X) = K_T X$$

for any $X \in H(\mathscr{F})$ (actually, it holds for any $X \in T(\mathscr{F})$ if one extends J to $T(\mathscr{F})$ by $JT = 0$). Moreover, it is verified that

$$\nabla_T X = \tau(X) + \mathscr{L}_T X \tag{3.17}$$

for any $X \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$. Finally, by (3.12) and $\nabla N = 0$, we have

$$\nabla_N Z = rZ + i\tau(Z) + [N, Z] \tag{3.18}$$

for any $Z \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$. From (3.15) through (3.18) and $\nabla T = \nabla N = 0$, it follows that ∇ is uniquely determined.

To establish the existence statement in Theorem 2, let

$$\nabla : \Gamma^\infty(T(U) \otimes \boldsymbol{C}) \times \Gamma^\infty(T(U) \otimes \boldsymbol{C}) \to \Gamma^\infty(T(U) \otimes \boldsymbol{C})$$

be given by the following identities:

$$\begin{aligned} &\nabla_{\bar{X}} Y = \pi_+([\bar{X}, Y]), \quad \nabla_X \bar{Y} = \overline{\nabla_{\bar{X}} Y}, \\ &\nabla_X Y = U_{XY}, \quad \nabla_{\bar{X}} \bar{Y} = \overline{\nabla_X Y}, \\ &\nabla T = 0, \quad \nabla N = 0, \\ &\nabla_T X = \mathscr{L}_T X + K_T X, \quad \nabla_T \bar{X} = \overline{\nabla_T X}, \\ &\nabla_N X = rX + iK_T X + \mathscr{L}_N X, \quad \nabla_N \bar{X} = \overline{\nabla_N X}, \end{aligned} \tag{3.19}$$

where $X, Y \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$ and $U_{XY} \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$ is given by

$$L_\theta(U_{XY}, \bar{Z}) = X(L_\theta(Y, \bar{Z})) - L_\theta(Y, \pi_-([X, \bar{Z}]))$$

for $Z \in \Gamma^\infty(T_{1,0}(\mathscr{F}))$. Then it is immediate that ∇ extends to define a linear connection on U.

Hence it suffices to check that ∇ obeys the axioms (1) and (2). To see this, let $\Pi^{\perp} : T(U) \to T(\mathscr{F})$ be the canonical projection associated with the direct sum decomposition $T(U) = T(\mathscr{F}) \oplus \boldsymbol{R}N$. Set $\nabla_X^{\mathscr{F}} Y = \Pi^{\perp}(\nabla_X Y)$ for any $X, Y \in \Gamma^{\infty}(T(\mathscr{F}))$. Then it is verified that $\nabla^{\mathscr{F}}$ restricts on each leaf M_{ε} of $\mathscr{F}$ to a linear connection on M_{ε}, and we have

$$\nabla_X Y = \nabla_X^{\mathscr{F}} Y + \sigma(\alpha(X, Y))$$

for any $X, Y \in \Gamma^{\infty}(T(\mathscr{F}))$, where $\sigma : \nu(\mathscr{F}) \to \boldsymbol{R}N$ denotes the natural bundle isomorphism. Then it follows that $\alpha = 0$, since by the very definition (3.19) of ∇, the N-component of $\nabla_X Y$ vanishes for all $X, Y \in \Gamma^{\infty}(T(\mathscr{F}))$. Furthermore, it follows from (3.19) again that ∇ restricts on each leaf M_{ε} of $\mathscr{F}$ to the Tanaka-Webster connection of M_{ε}. Hence $T_{1,0}(\mathscr{F})$ and L_{θ} are parallel with respect to ∇ and the identities (3.10), (3.11) and (3.13) are satisfied. Up to now, we see that ∇ obeys axiom (1). Finally, note that (3.12) follows from $\nabla_N X = rX + iK_T X + \mathscr{L}_N X$ and $\nabla N = 0$. Hence ∇ obeys (2) as well. The proof of Theorem 2 is now complete. □

Remark 3. The purity axiom (3.12) is natural in the following sense. Note that we may write $T_{\nabla}(N, Z)$ in the form

$$T_{\nabla}(N, Z) = A^{\alpha} W_{\alpha} + A^{\bar{\alpha}} \overline{W}_{\alpha} + BT + CN$$

with unknown functions $A^{\alpha}, A^{\bar{\alpha}}, B$ and C to be determined, where $\{W_{\alpha}\}$ is a local frame of $T_{1,0}(\mathscr{F})$. Then the condition that the linear connection ∇ we look for restricts to the Tanaka-Webster connection ∇^{ε} on each leaf M_{ε} of $\mathscr{F}$ together with the requirement $d^2\theta = 0$ for the exterior derivative of (3.9) and the integrability of the complex structure on U implies that

$$A^{\alpha} = rZ^{\alpha}, \quad A^{\bar{\alpha}} = i\tau^{\bar{\alpha}}(Z), \quad B = C = 0,$$

where $Z = Z^{\alpha} W_{\alpha} \in T_{1,0}(\mathscr{F})$ and $\tau = \tau^{\alpha} \otimes W_{\alpha} + \tau^{\bar{\alpha}} \otimes \overline{W}_{\alpha}$ (cf. [**20**], p. 703). Hence we require that $T_{\nabla}(N, Z) = rZ + i\tau(Z)$.

Faran [**15**] determined a complete system of local invariants under biholomorphic mappings of foliations of U by nondegenerate real hypersurfaces. His study is imitative of the work of Chern and Moser [**9**], and indeed the local invariants of a foliation by real hypersurfaces turn out to be similar to the local invariants of a single real hypersurface. There is, however, a remarkable difference between them, since one of these invariants is the intrinsic normal direction $N = 2\,\mathrm{Re}(\xi)$. *The flow along N gives a foliate map* (that is, a map sending leaves to leaves) *whose restriction to each leaf of $\mathscr{F}$ is a contact transformation*, yet in general not a CR diffeomorphism. Indeed, by (3.9) we have

$$\begin{aligned}
(\mathscr{L}_N\theta)(W_{\alpha}) &= N(\theta(W_{\alpha})) - \theta([N, W_{\alpha}]) = d\theta(N, W_{\alpha}) \\
&= ih_{\alpha\bar{\beta}}\theta^{\alpha} \wedge \bar{\theta}^{\beta}(N, W_{\alpha}) + r\, d\varphi \wedge \theta(N, W_{\alpha}) = 0, \\
(\mathscr{L}_N\theta)(T) &= N(\theta(T)) - \theta([N, T]) = d\theta(N, T) \\
&= r\, d\varphi \wedge \theta(N, T) = r\, d\varphi(N)\theta(T) = 2r, \\
(\mathscr{L}_N\theta)(N) &= d\theta(N, N) = 0.
\end{aligned}$$

Summing up, we have

$$\mathscr{L}_N \theta = 2r\theta.$$

Faran [15] has built the third order invariants $h^{\alpha}_{\bar{\beta}}$ and k^{α}, that is, invariants which may be calculated at a point by using only the 3-jet of φ at that point, and gave their geometric interpretation. Precisely, it turns out that $h^{\alpha}_{\bar{\beta}}$ measures the failure of the flow on N from being a CR map, while $k^{\alpha} = 0$ if and only if there is a defining function φ of Ω so that $\det(\partial^2\varphi/\partial z_i \partial \bar{z}_j) = 0$, namely φ has vanishing transverse curvature. As observed in [20], Graham and Lee's setting bears the same relationship to Faran's setting as does Webster's (cf. [42]) to that of Chern and Moser [9]. We need the following structure equation (cf. [20]):

$$d\theta^{\alpha} = \theta^{\beta} \wedge \varphi^{\alpha}_{\beta} - i\partial\varphi \wedge \tau^{\alpha} + ir^{\alpha}\, d\varphi \wedge \theta + \frac{1}{2} r\, d\varphi \wedge \theta^{\alpha}. \tag{3.20}$$

Here $\tau^{\alpha} = A^{\alpha}_{\bar{\beta}}\bar{\theta}^{\beta}$ and $A^{\alpha}_{\bar{\beta}}$ is given by $\tau(\overline{W}_{\beta}) = A^{\alpha}_{\bar{\beta}} W_{\alpha}$. Also, $r^{\alpha} = h^{\alpha\bar{\beta}} r_{\bar{\beta}}$ and $r_{\bar{\beta}} = dr(\overline{W}_{\beta}) = \overline{W}_{\beta} r$. Finally, φ^{α}_{β} are given by $\nabla W_{\beta} = \varphi^{\alpha}_{\beta} \otimes W_{\alpha}$, where ∇ is the Graham-Lee connection. Using (3.20), one may derive

$$(\mathscr{L}_N \theta^{\alpha})(W_{\beta}) = r\delta^{\alpha}_{\beta} - \varphi^{\alpha}_{\beta}(N), \quad (\mathscr{L}_N \theta^{\alpha})(W_{\bar{\beta}}) = -iA^{\alpha}_{\bar{\beta}},$$

$$(\mathscr{L}_N \theta^{\alpha})(T) = 2ir^{\alpha}, \quad (\mathscr{L}_N \theta^{\alpha})(N) = 0.$$

Summing up, one has

$$\mathscr{L}_N \theta^{\alpha} = (r\delta^{\alpha}_{\beta} - \varphi^{\alpha}_{\beta}(N))\theta^{\beta} - i\tau^{\alpha} + 2ir^{\alpha}\theta.$$

In particular,

$$\mathscr{L}_N \theta^{\alpha} \equiv -i\tau^{\alpha} \mod \theta, \theta^{\alpha},$$

$$\mathscr{L}_N \theta^{\alpha} \equiv 2ir^{\alpha}\theta \mod \theta^{\alpha}, \bar{\theta}^{\alpha}.$$

A comparison with (2.4) and (2.5) in [15], p. 401, shows that Faran's third order invariants $h^{\alpha}_{\bar{\beta}}$ and k^{α} are essentially $A^{\alpha}_{\bar{\beta}}$ and r^{α}, respectively. Hence, *the flow along N is a CR map* (when restricted to a leaf of $\mathscr{F}$) *if and only if* $\tau^{\alpha} = 0$ (that is, each leaf of $\mathscr{F}$ has vanishing pseudohermitian torsion).

4. Foliations and the Tanaka-Webster connection.

We adopt the following terminology. If $(M, T_{1,0}(M))$ is a CR manifold and $\mathscr{F}$ is a foliation on M, then $\mathscr{F}$ is called a *semi-Levi foliation* if $T(\mathscr{F}) \subseteq H(M)$, where $H(M)$ is the Levi distribution of M. A semi-Levi foliation is a *Levi foliation* if $JT(\mathscr{F}) = T(\mathscr{F})$ with respect to the complex structure $J : H(M) \to H(M)$. Note that if $\mathscr{F}$ is a semi-Levi foliation of codimension one, then $\mathscr{F}$ is a Levi foliation and $(M, T_{1,0}(M))$ is Levi flat, that is, $L = 0$. Let $(M, \mathscr{F})$ be a foliated CR manifold of hypersurface type, and $\theta \in \Omega^1(M)$ a pseudohermitian structure on M. If $\theta \in \Omega^1_B(\mathscr{F})$, then $\mathscr{F}$ is a semi-Levi foliation and (the Levi form of) M is degenerate.

Generally, let $(N, \mathscr{F})$ be a foliated manifold. Then $\mathscr{F}$ is called a *semi-Riemannian* *foliation* if there is a holonomy invariant semi-Riemannian bundle metric g_Q on the normal bundle $Q = \nu(\mathscr{F})$. Note that for any semi-Riemannian foliation $(\mathscr{F}, g_Q)$ on N there is a bundle-like semi-Riemannian metric h on N which induces g_Q on Q.

Our aim in this section is to study foliations on nodegenerate CR manifolds, on which a contact form θ has been fixed. Recall that with any foliation $\mathscr{F}$ of a Riemannian manifold M one has a natural connection on the normal bundle $\nu(\mathscr{F})$, induced by both the Bott (partial) connection of $\mathscr{F}$ and the Levi-Civita connection of the given Riemannian metric on M (cf., e.g., (5.3) in [**41**], p. 48). In the spirit of pseudohermitian geometry (cf. [**42**]), when M is a CR manifold, we replace the Riemannian connection by the Tanaka-Webster connection of θ and investigate the resulting theory of the "second fundamental form" of $\mathscr{F}$ in M (cf. also [**12**] and [**4**] where similar ideas lead to a study of the geometry of the second fundamental form of a CR immersion).

4.1. The second fundamental form.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold (of hypersurface type) and θ a contact form on M. Let g_θ and ∇ be the Webster metric and the Tanaka-Webster connection of (M, θ), respectively. Let $\mathscr{F}$ be a foliation on M such that the tangent bundle $P = T(\mathscr{F})$ is nondegenerate in $(T(M), g_\theta)$. We denote by $P^\perp$ the orthogonal complement of P in $T(M)$ with respect to g_θ, and by g_Q the bundle metric induced by g_θ on the normal bundle $Q = \nu(\mathscr{F})$. Let D be the connection in Q defined by

$$D_X s = \begin{cases} \mathring{\nabla}_X s & \text{if } X \in \Gamma^\infty(P), \\ \Pi(\nabla_X \sigma(s)) & \text{if } X \in \Gamma^\infty(P^\perp), \end{cases}$$

where $s \in \Gamma^\infty(Q)$, $\mathring{\nabla}$ is the Bott connection of $(M, \mathscr{F})$, $\Pi : T(M) \to Q$ is the natural bundle map and $\sigma : Q \to P^\perp$ is the natural bundle isomorphism, respectively. Then it is immediate (cf. [**41**]) to see the following

Proposition 7. *Let M be a nondegenerate CR manifold and θ a fixed contact form on M. Let $\mathscr{F}$ be a foliation on M such that $T(\mathscr{F})$ is nondegenerate in $(T(M), g_\theta)$. Then D is an adapted connection in Q and its torsion T_D satisfies*

(1) $P \rfloor T_D = 0$ *for* $P = T(\mathscr{F})$, *and*

(2) $T_D(Z, Z') = \Pi(T_\nabla(Z, Z'))$ *for any* $Z, Z' \in \Gamma^\infty(P^\perp)$.

Moreover, $\mathscr{F}$ is semi-Riemannian and g_θ is bundle-like if and only if g_Q is parallel with respect to D.

A pseudohermitian analogue $\alpha : P \otimes P \to Q$ of the second fundamental form (of a foliation on a Riemannian manifold) is given by

$$\alpha(X, X') = \Pi(\nabla_X X')$$

for any $X, X' \in \Gamma^\infty(P)$. Also, if $Z \in \Gamma^\infty(P^\perp)$, we consider the Weingarten map $W(Z) : P \to P$ given by

$$g_\theta(W(Z)(X), X') = g_Q(\alpha(X, X'), \sigma^{-1}(Z)).$$

It should be noted that in general α is not symmetric and $W(Z)$ is not self-adjoint with respect to g_θ, since the Tanaka-Webster connection ∇ has nontrivial torsion. Next, define $\alpha^\perp : P^\perp \otimes P^\perp \to P$ by setting

$$\alpha^\perp(Z, Z') = \frac{1}{2}\Pi^\perp(\nabla_Z Z' + \nabla_{Z'} Z)$$

for any $Z, Z' \in \Gamma^\infty(P^\perp)$, where $\Pi^\perp : T(M) \to P$ is the natural bundle map. Also, let $\kappa \in \Omega^1(M)$ be defined by

$$P \rfloor \kappa = 0, \quad \kappa(Z) = \operatorname{trace} W(Z)$$

for any $Z \in \Gamma^\infty(P^\perp)$. A pseudohermitian analogue $t \in \Gamma^\infty(P^\perp)$ of the mean curvature vector (of a foliation on a Riemannian manifold) is then given by

$$g_\theta(t, Z) = \kappa(Z)$$

for any $Z \in \Gamma^\infty(P^\perp)$.

Let g_P be the bundle metric induced by g_θ on P. Since $\nabla g_\theta = 0$, we have

$$(\mathscr{L}_Z g_P)(X, X') = -2g_Q(\alpha(X, X'), \sigma^{-1}(Z)) + g_\theta(T_\nabla(X, X'), Z) + g_\theta(T_\nabla(Z, X), X') + g_\theta(T_\nabla(Z, X'), X) \tag{4.1}$$

for any $X, X' \in \Gamma^\infty(P)$ and $Z \in \Gamma^\infty(P^\perp)$. Also, by a similar calculation, we have

$$(\mathscr{L}_X g_Q)(\sigma^{-1}(Z), \sigma^{-1}(Z')) = -2g_\theta(\alpha^\perp(Z, Z'), X) + g_\theta(T_\nabla(X, Z), Z') + g_\theta(T_\nabla(X, Z'), Z) \tag{4.2}$$

for any $X \in \Gamma^\infty(P)$ and $Z, Z' \in \Gamma^\infty(P^\perp)$.

Let us now extend the complex structure J to the whole $T(M)$ by setting $JT = 0$. This furnishes a bundle morphism $J : T(M) \to T(M)$ satisfying $J^2 = -I + \theta \otimes T$, I being the identity transformation of $T(M)$, and

$$g_\theta(JX, JY) = g_\theta(X, Y) - \theta(X)\theta(Y)$$

for any $X, Y \in T(M)$. Finally, we remark that the real expression of the purity axioms of the torsion T_∇ yields the identity

$$T_\nabla = \theta \wedge \tau + d\theta \otimes T, \tag{4.3}$$

where T is the characteristic direction of (M, θ) (cf. [**12**], p. 174).

We apply these notions and formulas to foliations $\mathscr{F}$ all of whose leaves are tangent to T. If this is the case, that is, if $T \in \Gamma^\infty(P)$, let $H(\mathscr{F})$ be the orthogonal complement (with respect to g_θ) of $\boldsymbol{R}T$ in P. Then $H(M) = H(\mathscr{F}) \oplus P^\perp$. Note that the flow determined by T is a subfoliation of $\mathscr{F}$, that is, each leaf of $\mathscr{F}$ is foliated by real curves which are the maximal integral curves of T. The study of the corresponding exotic characteristic classes (cf. [**11**]) is an open problem. We now obtain the following

THEOREM 3. *Let M be a nondegenerate CR manifold and θ a contact form on M with vanishing pseudohermitian torsion ($\tau = 0$). Let $\mathscr{F}$ be a foliation of M such that $T(\mathscr{F})$ is nondegenerate in $(T(M), g_\theta)$. Assume that T is tangent to the leaves of $\mathscr{F}$. Then the following hold*:

(1) *D is torsion-free.*
(2) *α is symmetric and $W(Z)$ is self-adjoint for any $Z \in \Gamma^\infty(P^\perp)$.*
(3) *The induced metric g_P along the leaves is invariant under the flows of vector fields orthogonal to the foliation if and only if $\alpha = 0$ and $H(\mathscr{F})$ is J-invariant.*
(4) *$\mathscr{F}$ is semi-Riemannian and g_θ is bundle-like if and only if $\alpha^\perp = 0$.*

Proof. The fact that $T_D = 0$ follows from $\tau = 0$ together with Proposition 7 and the purity axiom (4.3), for $\Pi(T) = 0$.

To see (2) through (4), let us drop the assumption $\tau = 0$ for the moment. First note that, since $\nabla T = 0$, one has

$$\alpha(X, T) = 0 \tag{4.4}$$

for any $X \in \Gamma^\infty(P)$. Moreover,

$$\alpha(T, X) = \Pi(\tau(X)) \tag{4.5}$$

as a consequence of (4.4) and of

$$\alpha(X, X') = \alpha(X', X) + \Pi(T_\nabla(X, X'))$$

for any $X, X' \in \Gamma^\infty(P)$. Then it is known from (4.4) that $W(Z)$ is $H(\mathscr{F})$-valued and from (4.5) together with the self-adjointness of τ with respect to g_θ that

$$W(Z)(T) = \Pi^\perp(\tau(Z)) \tag{4.6}$$

for any $Z \in \Gamma^\infty(P^\perp)$. Finally, we note that

$$\alpha(X, X') = \alpha(X', X),$$

$$g_\theta(W(Z)X, X') = g_\theta(X, W(Z)X')$$

for any $X, X' \in \Gamma^\infty(H(\mathscr{F}))$, since $\Pi(T_\nabla(X, X')) = 0$. Then (2) follows from the following more general statement that (1) *α is symmetric if and only if $\tau(P) \subseteq P$, and* (2) *$W(Z)$ is self-adjoint* (*with respect to g_θ*) *if and only if $\tau(P^\perp) \subseteq P^\perp$, provided that $T \in \Gamma^\infty(P)$.*

Using the purity axiom (4.3) and noting that τ is self-adjoint with respect to g_θ, one may compute the torsion terms in (4.1) to obtain

$$(\mathscr{L}_Z g_P)(X, T) = -g_\theta((J + 2\tau)(X), Z)$$

for any $X \in \Gamma^\infty(P)$, and

$$(\mathscr{L}_Z g_P)(X, X') = -2g_Q(\alpha(X, X'), \sigma^{-1}(Z))$$

for any $X, X' \in \Gamma^\infty(H(\mathscr{F}))$. Then (3) is a corollary of a more general statement that $\mathscr{L}_Z g_P = 0$ *along the leaves if and only if $\alpha = 0$ on $H(\mathscr{F}) \otimes H(\mathscr{F})$ and $J + \tau$ is a bundle endomorphism of $H(\mathscr{F})$.*

Finally, note that (4.2) furnishes, by a similar computation, that

$$(\mathscr{L}_T g_Q)(\sigma^{-1}(Z), \sigma^{-1}(Z')) = -2\theta(\alpha^\perp(Z, Z')) + 2g_\theta(\tau(Z), Z'),$$

$$(\mathscr{L}_X g_Q)(\sigma^{-1}(Z), \sigma^{-1}(Z')) = -2g_\theta(\alpha^\perp(Z, Z'), X)$$

for any $X \in H(\mathscr{F})$ and $Z, Z' \in \Gamma^\infty(P^\perp)$. Then $\mathscr{L}_X g_Q = 0$ for any $X \in P$ if and only if $\alpha^\perp = A \otimes T$, where $A \in \Gamma^\infty((P^\perp)^* \otimes (P^\perp)^*)$ is given by $A(Z, Z') = g_\theta(\tau(Z), Z')$. Hence (4) holds when $\tau = 0$. □

4.2. The characteristic form.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR-manifold and θ a contact form on M. If $E \to M$ is a vector bundle over M with standard fibre $\boldsymbol{R}^k$, then we denote by $L(E) \to M$ the principal $GL(k, \boldsymbol{R})$-bundle of frames (in the fibres) of E. Let $\mathscr{F}$ be a foliation on M such that the tangent bundle $P = T(\mathscr{F})$ is nondegenerate in $(T(M), g_\theta)$. Let g_P be the bundle metric induced on P by the Webster metric g_θ and $(v, p-v)$ the signature of g_P, where $p = \dim_{\boldsymbol{R}} P_x$, $x \in M$. Denote by $O(P) \to M$ the principal $O(v, p-v)$-subbundle of $L(P) \to M$ determined by g_P. From now on, we assume that $\mathscr{F}$ is tangentially oriented, namely the structure group $O(v, p-v)$ reduces to $SO(v, p-v)$. Let then $\chi_{\mathscr{F}} \in \Omega^p(M)$ be the *characteristic form* of $(M, \mathscr{F})$ defined by

$$\chi_{\mathscr{F}}(v_1, \ldots, v_p) = \det(g_\theta(v_i, u(e_j)))$$

for any $v_i \in T_x(M)$ and some frame $u : \boldsymbol{R}^p \to T_x(M)$ adapted to the "tangential" $SO(v, p-v)$-structure. Clearly, the definition of $\chi_{\mathscr{F}}(v_1, \ldots, v_p)$ is independent of the choice of adapted frames at x. Also, $P^\perp \rfloor \chi_{\mathscr{F}} = 0$, where $P^\perp$ is the orthogonal complement of P in (M, g_θ). We shall need the following

Lemma 3. *Let M be a nondegenerate CR manifold with a fixed contact form θ, and $\mathscr{F}$ a tangentially oriented foliation of M whose tangent bundle P is nondegenerate in $(T(M), g_\theta)$. Then*

$$\mathscr{L}_Z \chi_{\mathscr{F}}|_P = \{-\kappa(Z) + \theta(\operatorname{trace}(\tau_P)Z + \Pi^\perp((J-\tau)(Z)))\}\chi_{\mathscr{F}}|_P \tag{4.7}$$

for any $Z \in \Gamma^\infty(P^\perp)$.

Here $\tau_P : P \to P$ is given by $\tau_P(X) = \Pi^\perp(\tau(X))$ for any $X \in P$. The identity (4.7) is the pseudohermitian analogue of a formula in [**40**] (cf. also (6.17) in [**41**], p. 66) and will be referred to as the *pseudohermitian Rummler formula*.

Proof. Let $\{E_1, \ldots E_p\}$ be an oriented local orthonormal frame of P so that $g_P(E_i, E_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_1 = \cdots = \varepsilon_v = -1$ and $\varepsilon_{v+1} = \cdots = \varepsilon_p = 1$. Then it is immediate from

$$(\mathscr{L}_Z \chi_{\mathscr{F}})(E_1, \ldots, E_p) = -\sum_{i=1}^p \chi_{\mathscr{F}}(E_1, \ldots, \Pi^\perp([Z, E_i]), \ldots, E_p)$$

and

$$\Pi^\perp([Z, E_i]) = \sum_{j=1}^p \varepsilon_j g_\theta([Z, E_i], E_j) E_j$$

that

$$\mathscr{L}_Z \chi_{\mathscr{F}}|_P = -\sum_{i=1}^p \varepsilon_i g_\theta([Z, E_i], E_i) \chi_{\mathscr{F}}|_P. \tag{4.8}$$

On the other hand, we have

$$\kappa(Z) = \sum_{i=1}^{p} \varepsilon_i g_\theta(W(Z)E_i, E_i) = \sum_{i=1}^{p} \varepsilon_i g_Q(\Pi(\nabla_{E_i} E_i), \sigma^{-1}(Z))$$
$$= -\sum_{i=1}^{p} \varepsilon_i g_\theta(E_i, \nabla_{E_i} Z) = \sum_{i=1}^{p} \varepsilon_i g_\theta([Z, E_i] + T_\nabla(Z, E_i), E_i),$$

since $2g_\theta(E_i, \nabla_Z E_i) = Z(\varepsilon_i) = 0$. Finally, again by making use of the purity axiom (4.3), we obtain

$$\kappa(Z) = \sum_{i=1}^{p} \varepsilon_i g_\theta([Z, E_i], E_i) + \theta(\mathrm{trace}(\tau_P)Z + \Pi^\perp((J - \tau)(Z))),$$

and hence (4.8) yields (4.7). □

By the pseudohermitian Rummler formula, the p-form

$$\eta = Z \rfloor d\chi_{\mathscr{F}} + \{\kappa(Z) - \theta(\mathrm{trace}(\tau_P)Z + \Pi^\perp((J - \tau)(Z)))\}\chi_{\mathscr{F}}$$

vanishes along the leaves of $\mathscr{F}$. As an immediate application, we may look at the case of a foliation tangent to the characteristic direction of (M, θ) and orthogonal to a semi-Levi foliation. Then we obtain

Proposition 8. *Let $\mathscr{F}$ be tangent to T. If $P^\perp$ is involutive, then the following statements are equivalent*:
(1) $\kappa = 0$.
(2) $\mathscr{L}_Z \chi_{\mathscr{F}} = 0$ *for any* $Z \in \Gamma^\infty(P^\perp)$.
(3) $d\chi_{\mathscr{F}} = 0$.

The proof of this proposition mimics closely that of Theorem 6.23 in [**41**], p. 69, and is therefore omitted.

In contrast with the case of foliations on Riemannian manifolds, it should be remarked that a foliation $\mathscr{F}$ with $\kappa = 0$ is not necessarily harmonic. To see a geometric interpretation of this condition, let β denote the second fundamental form of (each leaf of) $\mathscr{F}$ in (M, g_θ) (cf., e.g., (6.1) in [**41**], p. 62). Note that the Levi-Civita connection ∇^θ of (M, g_θ) is related to the Tanaka-Webster connection ∇ as follows (cf. [**12**], p. 174):

$$\nabla^\theta = \nabla + \left(\frac{1}{2}\Omega_\theta - A\right) \otimes T + \tau \otimes \theta + \theta \odot J, \tag{4.9}$$

where Ω_θ and A are given respectively by

$$\Omega_\theta(X, Y) = g_\theta(X, JY), \quad A(X, Y) = g_\theta(\tau(X), Y)$$

for $X, Y \in T(M)$ and $\odot$ denotes the symmetric tensor product. It then follows from (4.9) that β is related to α as

$$\beta(X, X') = \alpha(X, X') + \{(1/2)\Omega_\theta(X, X') - A(X, X')\}\Pi(T) + \theta(X')\Pi(\tau(X)) + (1/2)\{\theta(X)\Pi(JX') + \theta(X')\Pi(JX)\} \tag{4.10}$$

for any $X, X' \in P$. For a given $Z \in P^{\perp}$, let $a(Z) : P \to P$ be the Weingarten map associated to β (cf., e.g., (6.3) in [**41**], p. 62). Then, by (4.10), we have

$$a(Z)(X) = W(Z)(X) - \theta(Z)\Pi^{\perp}(((1/2)J + \tau)(X)) + A(X,Z)\Pi^{\perp}(T) - (1/2)\theta(X)\Pi^{\perp}(JZ) + (1/2)g_{\theta}(JX,Z)\Pi^{\perp}(T). \tag{4.11}$$

Let $\ell \in \Omega^1(M)$ be the mean curvature of $\mathscr{F}$ in (M, g_θ) (cf. e.g. (6.13) in [**41**], p. 65). Taking traces in (4.11), we then obtain

$$\ell(Z) = \kappa(Z) - \theta(\mathrm{trace}(\tau_P)Z + \Pi^{\perp}((J - \tau)(Z))), \tag{4.12}$$

for any $Z \in P^{\perp}$.

As a immediate application, we may observe the following. *If T is tangent to $\mathscr{F}$, then $\mathscr{F}$ is harmonic if and only if $\kappa = 0$.* Or, *assume that (M, θ) has vanishing pseudohermitian torsion* (e.g., M is an odd dimensional sphere, the Heisenberg group, or the pseudoconvex locus of a pseudo-Siegel domain, cf. [**4**], p. 84–85). *Then $\mathscr{F}$ is harmonic if and only if $\kappa = \theta(\Pi^{\perp} \circ J)$.*

4.3. Flows.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold and θ a contact form on M. Let $\varepsilon \in \{\pm 1\}$ and let X be a tangent vector field on M so that $g_\theta(X, X) = \varepsilon$ everywhere on M. Let $\mathscr{F}$ be the flow determined by X, that is, the foliation whose leaves are the integral curves of X. Let $\chi_{\mathscr{F}}$ be defined by $\chi_{\mathscr{F}}(Y) = g_\theta(Y, X)$ for any $Y \in T(M)$. Then $\chi_{\mathscr{F}} \in \Omega^1(M)$ is the characteristic form of $\mathscr{F}$ on (M, g_θ).

Now, note that the pseudohermitian analogue of the mean curvature form $\kappa \in \Omega^1(M)$ is given by $\kappa(Z) = \varepsilon g_\theta(\nabla_X X, Z)$. Since $2g_\theta(\nabla_X X, X) = X(\varepsilon) = 0$, it follows that $\nabla_X X \in \Gamma^\infty(P^\perp)$ and the mean curvature vector $t \in \Gamma^\infty(P^\perp)$ is given by

$$t = \varepsilon \quad \nabla_X X. \tag{4.13}$$

We first prove the following

Theorem 4. *Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold with a fixed contact form θ, and T the characteristic direction of (M, θ). Let $\mathscr{F}$ be the flow on M defined by T. Then the following hold*:

(1) *$\mathscr{F}$ is totally geodesic in (M, g_θ).*
(2) *The orbits of T are autoparallel curves of ∇.*
(3) *$\mathscr{L}_T \chi_{\mathscr{F}} = 0$.*
(4) *g_P is invariant under flows of vector fields lying in the Levi distribution of M.*
(5) *$d\chi_{\mathscr{F}} \in F^2\Omega^2(M)$.*

Theorem 4 follows from the following

Proposition 9. *Let $\varepsilon \in \{\pm 1\}$ and $X \in T(M)$ so that $g_\theta(X, X) = \varepsilon$. Let $\mathscr{F}$ be the flow on M defined by X. Then the following statements are equivalent*:

(1) *$\kappa = 0$.*
(2) *The orbits of X are autoparallel curves of ∇.*

(3) *The characteristic form* $\chi_{\mathscr{F}}$ *satisfies*

$$\mathscr{L}_X \chi_{\mathscr{F}} = \theta(X)(JX + \tau(X))^\flat - A(X,X)\theta,$$

where $\flat$ *denotes lowering of indices by* g_θ, *that is*, $X^\flat(Y) = g_\theta(X,Y)$ *for* $X, Y \in T(M)$.

(4) *For any* $Z \in P^\perp$ *the Lie derivative* $\mathscr{L}_Z g_P$ *satisfies*

$$(\mathscr{L}_Z g_P)(X,X) = 2A(X,X)\theta(Z) - 2\theta(X)g_\theta((J+\tau)(X),Z).$$

Proof. The equivalence of (1) and (2) follows from (4.13). Moreover, again by (4.13), we have

$$\kappa(Z) = -\varepsilon g_\theta(X, \nabla_X Z),$$

and hence

$$(\mathscr{L}_X \chi_{\mathscr{F}})Z = \varepsilon\kappa(Z) + g_\theta(T_\nabla(X,Z),X) \tag{4.14}$$

for any $Z \in T(M)$. Also, by the purity axiom (4.3), we have

$$g_\theta(T_\nabla(X,Z),X) = -A(X,X)\theta(Z) + \theta(X)g_\theta((J+\tau)(X),Z) \tag{4.15}$$

for any $Z \in T(M)$. Then (4.14) together with (4.15) yields the equivalence of (1) and (3). On the other hand, it follows from (4.1) and (4.15) that

$$(\mathscr{L}_Z g_P)(X,X) = -2\varepsilon\kappa(Z) + 2A(X,X)\theta(Z) - 2\theta(X)g_\theta((J+\tau)(X),Z),$$

which implies the equivalence of (1) and (4). □

All that remains to be checked is (5) in Theorem 4. This follows from

$$F^2\Omega^2(M) = \{\omega \in \Omega^2(M) \mid X \rfloor \omega = 0\}$$

and the fact that $\kappa = 0$ if and only if

$$X \rfloor d\chi_{\mathscr{F}} = -A(X,X)\theta + \theta(X)(JX + \tau(X))^\flat.$$

This completes the proof of Theorem 4. □

Next, we restrict our attention to flows defined by infinitesimal pseudohermitian transformations. A C^∞ diffeomorphism $f : M \to M$ is called a *pseudohermitian transformation* of (M,θ) if (1) f is a CR map and (2) $f^*\theta = \theta$. Let $\mathrm{Psh}(M,\theta)$ denote the group of all pseudohermitian transformations of (M,θ), which has been studied by Webster (cf. Theorem 1.2 in [**42**]) and Musso (cf. Theorem 4.10 in [**33**]). Let $U(M,\theta) \to M$ be the principal $U(r,s)$-subbundle of $L(T(M)) \to M$ consisting of all linear frames of the form $u = (x, \{X_\alpha, J_x X_\alpha, T(x)\})$ with

$$g_{\theta,x}(X_i, X_j) = \varepsilon_i \delta_{ij}, \quad 1 \le i,\ j \le 2n,$$

where $X_{\alpha+n} = J_x X_\alpha$, $X_\alpha \in H(M)_x$, $1 \le \alpha \le n$. Then we have

Proposition 10. *A* C^∞ *diffeomorphism* f *of* M *is a pseudohermitian transformation of* (M,θ) *if and only if the induced transformation of* $L(T(M))$ *maps* $U(M,\theta)$ *into itself. Any fibre preserving transformation of* $U(M,\theta)$ *which leaves the canonical form of* $U(M,\theta)$ *invariant is induced by a pseudohermitian transformation* $f \in \mathrm{Psh}(M,\theta)$.

The proof of Proposition 10 follows from the fact that $f \in \mathrm{Psh}(M, \theta)$ if and only if f is a C^∞ diffeomorphism, $f^*\theta = \theta$, and $f^*\theta^\alpha = U^\alpha_\beta \theta^\beta$ for any local frame $\{\theta^\alpha\}$ of $T_{1,0}(M)^*$ and some (locally defined) C^∞ functions U^α_β on M (it mimics the proof of Proposition 3.1 in [**25**], p. 236).

A tangent vector field X on M is said to be an *infinitesimal pseudohermitian transformation* of (M, θ) if its local one-parameter group of local transformations consists of local pseudohermitian transformations of (M, θ). Let $\boldsymbol{i}(M, \theta)$ be the set of all infinitesimal pseudohermitian transformations of (M, θ). By analogy with Proposition 3.2 in [**25**], p. 237, we also have

Proposition 11. *For a vector field X tangent to a nondegenerate CR manifold M, the following statements are equivalent*:

(1) $X \in \boldsymbol{i}(M, \theta)$.
(2) *The natural lift of X to $L(T(M))$ is tangent to $U(M, \theta)$ at each point of $U(M, \theta)$.*
(3) *$\mathscr{L}_X \theta = 0$ and $\mathscr{L}_X \theta^\alpha = V^\alpha_\beta \theta^\beta$ for any local frame $\{\theta^\alpha\}$ of $T_{1,0}(M)^*$ and some local smooth functions V^α_β on M.*

It is verified from (3) in Proposition 11 that $\boldsymbol{i}(M, \theta)$ is a Lie algebra, as $\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y]$. Since each pseudohermitian transformation of (M, θ) preserves the Webster metric g_θ,

$$\mathrm{Psh}(M, \theta) \subseteq \mathrm{Iso}(M, g_\theta). \tag{4.16}$$

Let $\varepsilon \in \{\pm 1\}$ and $X \in \boldsymbol{i}(M, \theta)$ such that $g_\theta(X, X) = \varepsilon$. Let $\mathscr{F}$ be the flow determined by X. Then it follows from (4.16) that $[X, Z] \in P^\perp$ for any $Z \in P^\perp$. Consequently, we have

$$\mathscr{L}_X \chi_{\mathscr{F}} = 0. \tag{4.17}$$

Assume, in particular, that M is compact. Then, by virtue of Theorem 1.2 in [**42**], p. 31, $\mathrm{Psh}(M, \theta)$ is compact. Let G be the closure in $\mathrm{Psh}(M, \theta)$ of the one-parameter group of transformations obtained by integrating X. Then G is compact and abelian, and hence is a torus. Let $\Omega^k(M)^G$ denote the space of G-invariant k-forms on M. As a corollary of (4.17), there is a short exact sequence

$$0 \to \Omega^k_B(\mathscr{F}) \hookrightarrow \Omega^k(M)^G \xrightarrow{X \rfloor} \Omega^{k-1}_B(\mathscr{F}) \to 0, \tag{4.18}$$

from which one may conclude (as in [**41**], p. 139) that $H^k_B(\mathscr{F})$ are finite dimensional for $0 \le k \le 2n$, and zero for $k > 2n$.

Note that, if $\tau = 0$, then it is known by Proposition 2.2 in [**42**], p. 33, that $T \in \boldsymbol{i}(M, \theta)$. If $X = T$, then the connecting homomorphism

$$\Delta : H^{k-1}_B(\mathscr{F}) \to H^{k+1}_B(\mathscr{F}),$$

in the long exact cohomology sequence associated with (4.18), is given by $\Delta[\omega] = [(d\theta) \wedge \omega]$ for any $[\omega] \in H^{k-1}_B(\mathscr{F})$. As a corollary of Theorem 1 (and of Theorem 10.13 in [**41**], p. 139), the map $(i_T)_* : H^1(M, \boldsymbol{R}) \to H^0_B(\mathscr{F})$ is injective.

References

[1] E. Barletta, On the transverse Beltrami equation, Comm. Partial Differential Equations, **21** (1996), 1469–1485.

[2] E. Barletta and S. Dragomir, On the CR structure of the tangent sphere bundle, Matematiche (Catania), (2)L (1995), 237–249.

[3] E. Barletta and S. Dragomir, On $\mathscr{G}$-Lie foliations with transverse CR structure, Rend. Mat. Appl. (VII), **16** (1996), 169–188.

[4] E. Barletta and S. Dragomir, Pseudohermitian immersions, pseudo-Einstein structures, and the Lee class of a CR manifold, Kodai Math. J., **19** (1996), 62–86.

[5] E. Barletta and S. Dragomir, On $\boldsymbol{C}$-tangent foliations of maximum modulus sets, Atti Sem. Mat. Fis. Univ. Modena, **45** (1997), 143–153.

[6] E. Barletta and S. Dragomir, Transversally CR foliations, Rend. Mat. Appl. (VII), **17** (1997), 51–85.

[7] J. K. Beem and P. E. Ehrlich, Global Lorentzian Geometry, Marcel Dekker, New York-Basel, 1981.

[8] L. Boutet de Monvel and A. Iordan, Sur les feuilletages $\boldsymbol{C}$-tangents des sous-variétés du bord d'une variété complexe, Colloque d'Analyse Complexe et Géométrie, Marseille, 1992, Astérisque, **217** (1993), 39–52.

[9] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math., **133** (1974), 219–271.

[10] E. M. Cirka, Introduction to the geometry of CR manifolds, Russian Math. Surveys, **46** (1991), 95–197.

[11] L. A. Cordero and P. M. Gadea, Exotic characteristic classes and subfoliations, Ann. Inst. Fourier (Grenoble), **26** (1976), 225–237.

[12] S. Dragomir, Pseudohermitian immersions between strictly pseudoconvex CR manifolds, Amer. J. Math., **117** (1995), 169–202.

[13] S. Dragomir, A survey of pseudohermitian geometry, Rend. Circ. Mat. Palermo (2), **49** (1997), 101–112.

[14] T. Duchamp and E. L. Stout, Maximum modulus sets, Ann. Inst. Fourier (Grenoble), **31** (1981), 37–69.

[15] J. Faran, Local invariants of foliations by real hypersurfaces, Michigan Math. J., **35** (1988), 395–404.

[16] M. Freeman, The Levi form and local complex foliations, Trans. Amer. Math. Soc., **57** (1976), 369–370.

[17] G. Gigante, Sulla coomologia di varietà pseudohermitiane con torsione nulla, Riv. Mat. Univ. Parma (4), **2** (1986), 157–165.

[18] C. R. Graham, On Sparling's characterization of Fefferman metrics, Amer. J. Math., **109** (1987), 853–874.

[19] C. R. Graham, The Dirichlet problem for the Bergman Laplacian, I–II, Comm. Partial Differential Equations, **8** (1983), 433–476, 563–641; Compatibility operators for degenerate elliptic equations on the ball and Heisenberg group, Math. Z., **187** (1984), 289–304.

[20] C. R. Graham and J. M. Lee, Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains, Duke Math. J., **57** (1988), 697–720.

[21] P. C. Greiner, J. J. Kohn and E. M. Stein, Necessary and sufficient conditions for solvability of the Lewy equation, Proc. Natl. Acad. Sci. USA, **72** (1975), 3287–3289.

[22] W. Greub, S. Halperin and R. Vanstone, Connections, Curvature and Cohomology, Academic Press, New York-London, 1976.

[23] F. Kamber and P. Tondeur, Duality for Riemannian foliations, Proc. Sympos. Pure Math., Vol. 40, 1983, Part 1, 609–618.

[24] F. Kamber and P. Tondeur, Foliations and metrics, Proceedings of a Year in Differetial Geometry, University of Maryland, Birkhäuser, Progr. Math., **32** (1983), 103–152.

[25] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Interscience Publishers, New York, Vol. I, 1963.

[26] J. J. Kohn, Boundaries of complex manifolds, Proc. Conf. on Complex Analysis, Minneapolis, 1964, Springer-Verlag, New York, 1965, pp. 81–94.

[27] C. R. LeBrun, Twistor CR manifolds and three-dimensional conformal geometry, Trans. Amer. Math. Soc., **284** (1984), 601–616.

[28] C. R. LeBrun, Foliated CR manifolds, J. Differential Geom., **22** (1985), 81–96.

[29] J. M. Lee, The Fefferman metric and pseudohermitian invariants, Trans. Amer. Math. Soc., **296** (1986), 411–429.

[30] J. M. Lee, Pseudo-Einstein structures on CR manifolds, Amer. J. Math., **110** (1988), 157–178.

[31] J. M. Lee and R. Melrose, Boundary behaviour of the complex Monge-Ampère equation, Acta Math., **148** (1982), 159–192.

[32] P. Molino, Riemannian foliations, Progr. Math., Vol. 73, Birkhäuser, Boston-Basel, 1988.

[33] E. Musso, Homogeneous pseudo-hermitian Riemannian manifolds of Einstein type, Amer. J. Math., **113** (1990), 219–241.
[34] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York-London-Paris-San Francisco-São Paulo-Sydney-Tokyo-Toronto, 1983.
[35] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J., **13** (1966), 459–469.
[36] S. Nishikawa, On maximal spacelike hypersurfaces in a Lorentzian manifold, Nagoya Math. J., **95** (1984), 117–124.
[37] R. Penrose, Physical space-time and nonrealizable CR structures, Bull. Amer. Math. Soc. (N.S.), **8** (1983), 427–448.
[38] C. Rea, Levi-flat submanifolds and holomorphic extension of foliations, Ann. Scoula Norm. Sup. Pisa (3), **26** (1972), 665–681.
[39] H. Rossi, LeBrun's nonrealizability theorem in higher dimensions, Duke Math. J., **52** (1985), 457–474.
[40] H. Rummler, Quelques notions simples en géométrie Riemannienne et leurs applications aux feuilletages compacts, Comment. Math. Helv., **54** (1979), 224–239.
[41] P. Tondeur, Foliations on Riemannian Manifolds, Springer-Verlag, New York-Heidelberg-London-Paris-Tokyo, 1988.
[42] S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Differential Geom., **13** (1978), 25–41.
[43] K. Yano and M. Kon, CR submanifolds of Kählerian and Sasakian manifolds, Progr. Math., Vol. 30, Birkhäuser, Boston-Basel-Stuttgart, 1983.

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