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Discrete interpolating varieties in pseudoconvex open sets of $\boldsymbol{C}^n$

By Bao Qin Li

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Abstract. We give a necessary and sufficient condition for a discrete variety in a pseudoconvex open set Ω of $\boldsymbol{C}^n$ to be an interpolating variety for Hörmander's weighted algebras of holomorphic functions in Ω.

1. Introduction.

Let Ω be an open set in $\boldsymbol{C}^n$ and p a nonnegative function defined in Ω. We shall denote by $A_p(\Omega)$ the algebra of all holomorphic functions f in Ω satisfying that $|f(z)| \leq Ae^{Bp(z)}$, $z \in \Omega$, for some constants $A, B > 0$. This paper concerns the following interpolation problem for $A_p(\Omega)$ when p is a Hörmander weight function in Ω (and then Ω is necessarilly pseudoconvex. see Section 2): under what necessary and sufficient conditions is a discrete set $V = \{\zeta_k\} \subset \Omega$ an interpolating variety for $A_p(\Omega)$? That is, under what necessary and sufficient conditions is it true that for any sequence $\{a_k\}$ of complex numbers satisfying that $|a_k| < A_1 e^{B_1 p(\zeta_k)}$, $k \in \boldsymbol{N}$, for some $A_1, B_1 > 0$ there always exists a $f \in A_p(\Omega)$ such that $f(\zeta_k) = a_k$ for all $k \in \boldsymbol{N}$? We will then say that V is an interpolating variety for $A_p(\Omega)$.

It is well-known that interpolation for the algebras $A_p(\Omega)$ is intrinsically related to the idea theory for the rings $A_p(\Omega)$, systems of partial differential equations and convolution equations, etc. Many questions in harmonic analysis, like finding all distribution solutions or finding out whether there are any to a system of linear partial differential equations with constant coefficients or, more generally, convolution equations in $\boldsymbol{R}^n$, can be translated into interpolation problems for $A_p(\Omega)$. Various interesting results have been obtained by imposing conditions on the weight p, the domain Ω, and the variety V in one and several complex variables. We refer to [**BG**], [**BL**], [**BS**], [**BT**], [**E**], [**M**], [**Oh**], [**Ou**], [**S**], etc. and references therein for various results on interpolation and related problems. In [**BL**], Berenstein and the author, motivated by the previous work of Berenstein and Taylor [**BT**], obtained a necessary and sufficient interpolation condition for the case $\Omega = \boldsymbol{C}^n$ in terms of the growth o f "directional derivatives" of an entire holomorphic mapping vanishing on the variety V, which yielded an interpolation condition in terms of the Jacobian of the mapping (see [**BL**]). Since Jacobian is one of the most important and convenient quantities associated to a mapping, the condition turns out to be very useful. It, for example, plays an important role in the work of Taylor and the author ([**LT**]) on the well-known transcendental Bézout problem for entire holomorphic mappings in $\boldsymbol{C}^n$. Other applications of the result and method can be found in recent

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papers [**L**], [**M**], [**Oh**], [**Ou**], etc. The analogous result for the case that Ω is the unit ball in $\boldsymbol{C}^n$ and $p(z) = \log \frac{1}{1-|z|}$ was given in [**M**]. Despite of various interpolation conditions available in the literature, so far no both necessary and sufficient interpolation conditions without extra conditions have been known for general Hörmander algebras $A_p(\Omega)$.

The aim of this paper is to obtain a necessary and sufficient interpolation condition for general Hörmander algebras $A_p(\Omega)$, where p is an arbitrary Hörmander weight defined in an open set Ω of $\boldsymbol{C}^n$ (see Section 2 for the definition). It turns out that a discrete variety V in Ω is an interpolating variety for $A_p(\Omega)$ if and only if V is a subset of the zero set of a holomorphic mapping $f = (f_1, f_2, \ldots, f_n) : \Omega \to \boldsymbol{C}^n$, $f_j \in A_p(\Omega)$, whose Jacobian is bounded from below by $\epsilon e^{-Cp(\zeta)}$, $\zeta \in V$, for some constants $\epsilon, C > 0$. Since this characterization does not impose any extra conditions on the weight p, the domain Ω, and the discrete variety V, it is probably a most general result one may obtain for interpolation in the Hörmander algebras $A_p(\Omega)$ and would be of interests in studying related problems and applications in general domains of $\boldsymbol{C}^n$.

We will include some preliminaries and state the detailed results in Section 2, and give the proofs of the results in Section 3.

2. Preliminaries and results.

We first recall the definitions of the weight p and weighted algebra $A_p(\Omega)$, introduced by Hörmander ([**H1**]). Let Ω be an open set in $\boldsymbol{C}^n$. A plurisubharmonic function $p : \Omega \to [0, \infty)$ is called a Hörmander weight if it satisfies the following Hörmander's conditions:

(i) all polynomials belong to $A_p(\Omega)$;
(ii) there exist constants $K_1, \ldots, K_4$ such that $z \in \Omega$ and $|\zeta - z| \le e^{-K_1 p(z) - K_2}$ implies that $\zeta \in \Omega$ and

$$p(\zeta) \le K_3 p(z) + K_4. \tag{2.1}$$

Let $A(\Omega)$ be the ring of holomorphic functions in Ω. The Hörmander algebra $A_p(\Omega)$ is defined as

$$A_p(\Omega) = \{f \in A(\Omega) : \exists A,\ B > 0 \quad \text{such that} \quad |f(z)| \le Ae^{Bp(z)},\ z \in \Omega\}.$$

For the meaning of the above conditions and examples of the algebras, the reader is referred to [**H1**] and [**BT**]. In particular, we note that under the above conditions, Ω is necessarily pseudoconvex (see [**H1**]), and $A_p(\Omega)$ is closed under differentiation.

The Hörmander algebras include most classically studied algebras of entire functions in $\boldsymbol{C}^n$ such as the algebra $A_{|z|^\rho}(\boldsymbol{C}^n)$ of all entire functions of order$\le \rho$ and finite type ($p(z) = |z|^\rho$, $\rho > 0$) and the algebra $\hat{\mathscr{E}}'(\boldsymbol{R}^n)$ of Fourier transforms of distributions with compact support in $\boldsymbol{R}^n$ ($p(z) = |\Im z| + \log(1 + |z|^2)$), and algebras of holomorphic functions in the unit ball such as $A^{-\infty}$ $\big(p(z) = \log \frac{1}{1-|z|}\big)$. Many other Hörmander weights can be given. Note, however, that the conditions exclude the algebra of bounded holomorphic functions ($p(z) \equiv 0$), for which necessary and sufficient interpolation conditions and the idea theory when Ω is the unit disk in $\boldsymbol{C}$ were studied by Carleson in [**C**].

Let $V = \{\zeta_k\} \subset \Omega$ be a discrete variety in Ω. The weighted space $A_p(V)$ of sequences of complex numbers is defined as

$$A_p(V) = \big\{\{a_k\}_{k\in \boldsymbol{N}} : \exists A,\ B > 0 \quad \text{such that} \quad |a_k| \le Ae^{Bp(\zeta_k)},\ k \in \boldsymbol{N}\big\}.$$

With the above definitions, the notion of interpolation stated in the introduction may be phrased using the restriction mapping

$$\rho : \rho(f) = \{f(\zeta_k)\}_{k\in \boldsymbol{N}} \tag{2.2}$$

from $A_p(\Omega)$ to $A_p(V)$. Associated to the given discrete variety $V = \{\zeta_k\}$, there is a unique closed ideal in $A(\Omega)$,

$$I = I(V) := \big\{f \in A(\Omega) : f(\zeta_k) = 0,\ \forall k\big\}.$$

Two holomorphic functions g, h in Ω can be identified modulo I if and only $f(\zeta_k) = g(\zeta_k)$, $k \in \boldsymbol{N}$. The quotient space $A(\Omega)/I$ can be identified to the space $A(V)$, the space of all sequences $\{a_k\}_{k\in\boldsymbol{N}}$ of complex numbers. The map ρ is the natural restriction map from $A(\Omega)$ into $A(V)$. It is clear that $\rho(A_p(\Omega)) \subseteq A_p(V)$; but, in general, $A_p(V)$ is too large. The interpolation problem for $A_p(\Omega)$ is to determine when ρ is surjective from $A_p(\Omega)$ to $A_p(V)$.

If the restriction map ρ is surjective from $A_p(\Omega)$ to $A_p(V)$, V is then called an interpolating variety for $A_p(\Omega)$. This clearly means that for any $\{a_k\} \in A_p(V)$ there exists a $f \in A_p(\Omega)$ such that $f(\zeta_k) = a_k$, $k \in \boldsymbol{N}$, i.e., f has prescribe values at each $\zeta_k, k \in \boldsymbol{N}$.

In the following, we shall use $f^{-1}(0)$ to denote the zero set of a mapping f and use df to denote the Jacobian of f. We will always assume that p is a Hörmander weight defined in an open set $\Omega \subseteq \boldsymbol{C}^n$. As mentioned earlier, Ω is necessarily pseudoconvex. We have the following

THEOREM 2.1. *Let p be a Hörmander weight defined in an open set $\Omega \subseteq \boldsymbol{C}^n$. Then a discrete variety $V = \{\zeta_k\}$ in Ω is an interpolating variety for $A_p(\Omega)$ if and only if there exists a holomorphic mapping $f = (f_1, f_2, \dots, f_n) : \Omega \to \boldsymbol{C}^n$ with $f_j \in A_p(\Omega)$ such that $V \subseteq f^{-1}(0)$ and*

$$|df(\zeta_k)| \ge \epsilon e^{-Cp(\zeta_k)}, \quad k \in \boldsymbol{N} \tag{2.3}$$

for some $\epsilon, C > 0$.

The following result allows the holomorphic mapping f to be "non-equidimensional." Note that both Theorem 2.1 and the following corollary apply to arbitrary discrete varieties, which are, in particular, not required to be complete intersections.

COROLLARY 2.2. *Let p be a Hörmander weight defined in an open set $\Omega \subseteq \boldsymbol{C}^n$ and $m \ge n$ an integer. Then a discrete variety $V = \{\zeta_k\}$ in Ω is an interpolating variety for*

$A_p(\Omega)$ if and only if there exists a holomorphic mapping $f = (f_1, f_2, \dots, f_m) : \Omega \to \boldsymbol{C}^m$ with $f_j \in A_p(\Omega)$ such that $V \subseteq f^{-1}(0)$ and

$$\sum |J_{n\times n}(\zeta_k)| \geq \epsilon e^{-Cp(\zeta_k)}, \quad k \in \boldsymbol{N}$$

for some $\epsilon, C > 0$, where the sum is taken over all the $n \times n$ minors $J_{n\times n}$ of the Jacobian of f.

If $\Omega = \boldsymbol{C}^n$, the above theorem yields the results in [**BL**]. In [**BL**], we introduced the "directional derivatives" sum $\sum_{j=1}^{m} |D_u f_j|$ for every direction u in $\boldsymbol{C}^n$, which was used in both the statement and the proof of the theorem, so that certain one variable results and arguments could be applied to the restrictions of the functions to complex lines in $\boldsymbol{C}^n$. The directional derivative condition involves every direction in $\boldsymbol{C}^n$ and is fairly complicated. Unlike in [**BL**], directional derivatives will play no role in the present paper.

3. Proofs of the results.

We now give the proofs of our results. In the proofs, we shall follow the usual practice, using A, B, C, ϵ to denote positive constants, which may depend on the dimension n and the actual values of which may vary from one occurrence to the next. For completeness and for the reader's convenience, we keep careful track of the estimates required in the proof of Theorem 2.1 to lead to the conclusions of the theorem.

Proof of Theorem 2.1. We first prove the sufficiency. Suppose that $a = \{a_k\} \in A_p(V)$ is a sequence. We need to show that there is a function F in $A_p(\Omega)$ such that $F(\zeta_k) = a_k$ for each $k \in \boldsymbol{N}$. From the given condition, we know that $|f(z)| < A_1 e^{B_1 p(z)}$, $z \in \Omega$, for some constants $A_1, B_1 > 0$.

In the following, if $(a_{i,j})$ is a matrix, we let $\|(a_{i,j})\| = \max_{i,j} |a_{i,j}|$. Note that for a point $z = (z_1, z_2, \dots, z_n) \in \boldsymbol{C}^n$, we have that $\|z\| \leq |z| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\frac{1}{2}} \leq \sqrt{n}\|z\|$. Denote by Jf the Jacobian matrix of f and $\operatorname{adj}(Jf)$ the adjoint matrix of Jf. Let $\zeta_k \in V$ be an arbitrary point. Set $h(z) := e^{-K_1 p(z) - K_2}$, where K_1 and K_2 are the numbers in (ii) in the definition of the weight (see (2.1)).

When $|z - \zeta_k| \leq h(\zeta_k)$, it follows from (2.1) that $z \in \Omega$ and

$$|f(z)| \leq A_1 e^{B_1(K_3 p(\zeta_k) + K_4)} \leq A e^{Bp(\zeta_k)}, \tag{3.1}$$

for some $A, B > 0$. Let, for $1 \leq j \leq n$, $g_j(z) = f_j\big(\zeta_k + \frac{1}{2}h(\zeta_k)z\big)$, $z \in \Omega$. Then $w := \zeta_k + \frac{1}{2}h(\zeta_k)z \in \Omega$. Thus, g_j is a well-defined holomorphic function in Ω, and

$$\max_{|z|=1}\{|g_j(z)|\} \leq \max_{|z-\zeta_k|\leq h(\zeta_k))}\{|f(z)|\} \leq A e^{Bp(\zeta_k)}.$$

By the Cauchy formula in the unit ball (see e.g. [**R**, p. 39]) we have that

$$g_j(z) = \int_S \frac{g_j(w)}{(1-\langle w, z\rangle)^n} d\sigma(w),$$

where σ is the normalized rotation-invariant positive Borel measure on the unit sphere S and $\langle w, z\rangle$ is the usual inner product. Thus we have that

$$\frac{\partial g_j(z)}{\partial z_i} = n \int_S \frac{w_i g(w)}{(1-\langle z, w\rangle)^{n+1}} d\sigma(w),$$

where $w = (w_1, \ldots, w_n)$ and $1 \le i \le n$, from which we obtain that

$$\left| \frac{\partial g_j(0)}{\partial z_i} \right| \le n \int_S |g(w)| d\sigma(w) \le n e^{Ap(\zeta_k)+B}$$

in view of the fact that $\int_S d\sigma(w) = 1$. We then have that

$$\left| \frac{\partial f_j(\zeta_k)}{\partial z_i} \right| = \frac{2}{h(\zeta_k)} \left| \frac{\partial g(0)}{\partial w_i} \right| \le \frac{2n}{h(\zeta_k)} A e^{Bp(\zeta_k)}.$$

Thus,

$$\big\| Jf(\zeta_k) \big\| \le \max_{i,j} \left\{ \left| \frac{\partial f_j(\zeta_k)}{\partial z_i} \right| \right\} \le \frac{A}{h(\zeta_k)} e^{Bp(\zeta_k)} \tag{3.2}$$

and also

$$\big\| \mathrm{adj}(Jf)(\zeta_k) \big\| \le \frac{A}{\big(h(\zeta_k)\big)^{n-1}} e^{(n-1)Bp(\zeta_k)}, \tag{3.3}$$

for some constants $A, B > 0$.

For convenience, we write a point in $\boldsymbol{C}^n$ as a column vector. Let $g(z) = f(z) - Jf(\zeta_k)(z-\zeta_k)$, $z \in \Omega$. Then for $|z-\zeta_k| \le h(\zeta_k)$, we have that, in view of (3.1) and (3.2),

$$\begin{aligned} |g(z)| &\le |f(z)| + \big| Jf(\zeta_k)(z-\zeta_k) \big| \\ &\le |f(z)| + \sqrt{n} \big\| Jf(\zeta_k)(z-\zeta_k) \big\| \\ &\le |f(z)| + n\sqrt{n} \big\| Jf(\zeta_k) \big\| \times \|z-\zeta_k\| \le A e^{Bp(\zeta_k)}. \end{aligned}$$

Also, $Jg(\zeta_k) = Jf(\zeta_k) - Jf(\zeta_k) = 0$. Recall the following Schwarz Lemma (see e.g. [**G**]): If f is holomorphic in an open neighborhood of a closed ball $\bar{B}(\zeta, r)$ in $\boldsymbol{C}^n$ centered at ζ and with radius r, $|f(z)| \le M$ for $z \in B(\zeta, r)$, and $\frac{\partial^{|I|} f}{\partial z^I}(\zeta) = 0$ whenever $|I| < m$ for some $m \in \boldsymbol{N}$, where $I \in (\mathbf{Z}^+)^n$ is a multi-index, then $|f(z)| \le M r^{-m} |z-\zeta|^m$ for $z \in \bar{B}(\zeta, r)$. Applying the lemma with $m = 2$ to each component of our mapping g in $|z-\zeta_k| < h(\zeta_k)$, we have that

$$|g(z)| \le Ae^{Bp(\zeta_k)}\big(h(\zeta_k)\big)^{-2}|z-\zeta_k|^2. \tag{3.4}$$

Also,

$$\begin{aligned}\|z-\zeta_k\| &= \big\|(Jf)^{-1}(\zeta_k)Jf(\zeta_k)(z-\zeta_k)\big\| \\ &\le n\big\|(Jf)^{-1}(\zeta_k)\big\| \times \big\|Jf(\zeta_k)(z-\zeta_k)\big\|.\end{aligned}$$

Thus, by (2.3) and (3.3), we deduce that

$$\begin{aligned}\big|Jf(\zeta_k)(z-\zeta_k)\big| &\ge \big\|Jf(\zeta_k)(z-\zeta_k)\big\| \\ &\ge \frac{1}{n}\|z-\zeta_k\| \times \big\|(Jf)^{-1}(\zeta_k)\big\|^{-1} \\ &= \frac{1}{n}\|z-\zeta_k\|\frac{|\det Jf(\zeta_k)|}{\big\|\mathrm{adj}(Jf)(\zeta_k)\big\|} \ge \epsilon|z-\zeta_k|\big(h(\zeta_k)\big)^{n-1}e^{-Cp(\zeta_k)}, \end{aligned} \tag{3.5}$$

for some $\epsilon, C > 0$.

Suppose that $\zeta_j \in V$ is a zero of f in $|z-\zeta_k| < h(\zeta_k)$ with $\zeta_j \ne \zeta_k$. Then

$$g(\zeta_j) = f(\zeta_j) - Jf(\zeta_k)(\zeta_j-\zeta_k) = -Jf(\zeta_k)(\zeta_j-\zeta_k).$$

Thus, it follows from (3.4) that

$$\big|Jf(\zeta_k)(\zeta_j-\zeta_k)\big| = \big|g(\zeta_j)\big| \le Ae^{Bp(\zeta_k)}\big(h(\zeta_k)\big)^{-2}\big|\zeta_j-\zeta_k\big|^2.$$

We deduce that

$$\big|\zeta_j-\zeta_k\big|^2 \ge \big|Jf(\zeta_k)(\zeta_j-\zeta_k)\big|A^{-1}e^{-Bp(\zeta_k)}\big(h(\zeta_k)\big)^2,$$

or, by virtue of (3.5), $|\zeta_j-\zeta_k| \ge \epsilon_0(h(\zeta_k))^{n+1}e^{-C_0p(\zeta_k)}$, for some constants $0 < \epsilon_0 < 1$ and $C_0 > 0$. This inequality clearly also holds when $|\zeta_j-\zeta_k| \ge h(\zeta_k)$. Thus, it holds for each $\zeta_j \in V$ with $\zeta_j \ne \zeta_k$. Hence, we have that

$$d_k := \min\big\{h(\zeta_k), \mathrm{dist}\big\{\zeta_k, V\setminus\{\zeta_k\}\big\}\big\} \ge \epsilon_0\big(h(\zeta_k)\big)^{n+1}e^{-C_0p(\zeta_k)}. \tag{3.6}$$

Also, by the definition of g, (3.5) and (3.4), we have that for $|z-\zeta_k| < h(\zeta_k)$,

$$\begin{aligned}|f(z)| &= \big|Jf(\zeta_k)(z-\zeta_k)+g(z)\big| \\ &\ge \big|Jf(\zeta_k)(z-\zeta_k)\big| - |g(z)| \\ &\ge \epsilon|z-\zeta_k|\big(h(\zeta_k)\big)^{n-1}e^{-Cp(\zeta_k)} - Ae^{Bp(\zeta_k)}\big(h(\zeta_k)\big)^{-2}|z-\zeta_k|^2 \\ &= |z-\zeta_k|\big\{\epsilon\big(h(\zeta_k)\big)^{n-1}e^{-Cp(\zeta_k)} - Ae^{Bp(\zeta_k)}\big(h(\zeta_k)\big)^{-2}|z-\zeta_k|\big\}. \end{aligned}\tag{3.7}$$

We may increase the value of A or decrease the value of ϵ in (3.7), if necessary, so that (3.7) still holds and, meanwhile, $\frac{\epsilon}{A} < \epsilon_0$, where ϵ_0 is the number in (3.6). For the same reason, we may increase the value C in (3.7) so that $C \geq C_0$, where C_0 is the number in (3.6). Let $\eta_k = \frac{\epsilon}{2A}(h(\zeta_k))^{n+1}e^{-(B+C)p(\zeta_k)}$. Then by virtue of (3.6), we have that

$$2\eta_k < \epsilon_0\big(h(\zeta_k)\big)^{n+1}e^{-C_0p(\zeta_k)} \leq d_k \leq h(\zeta_k).$$

Thus, the ball $B(\zeta_k, \eta_k)$ does not contain any other points in the variety V and $B(\zeta_k, \eta_k) \cap B(\zeta_j, \eta_j) = \varnothing$. Furthermore, when $\frac{1}{2}\eta_k \leq |z - \zeta_k| \leq \eta_k$, it is easy to check, by virtue of (3.7) and by the definition of $h(z)$, that

$$\begin{aligned} |f(z)| &\geq \frac{1}{2}\eta_k \frac{\epsilon}{2}\big(h(\zeta_k)\big)^{n-1}e^{-Cp(\zeta_k)} \\ &= \frac{\epsilon^2}{8A}\big(h(\zeta_k)\big)^{2n}e^{-(B+2C)p(\zeta_k)} \geq \epsilon_1 e^{-C_1p(\zeta_k)} \end{aligned} \tag{3.8}$$

for some positive numbers ϵ_1 and C_1.

We next use the L^2-estimate for $\bar{\partial}$ operators ([**H1**], [**H2**]) to obtain our desired function F mentioned in the beginning. Since $\{a_k\} \in A_p(V)$, $|a_k| < A_2e^{B_2p(\zeta_k)}$ for some $A_2, B_2 > 0$. Denote $U_0 = \cup_k B(\zeta_k, \frac{\eta_k}{2})$ and $U = \cup_k B(\zeta_k, \eta_k)$. Then it is easy to verify, in view of (2.1), that there exist $\epsilon, C > 0$ such that $d(z) := \max\{d(z, U_0), d(z, U^c)\}$, where U^c is the complement of U in Ω, satisfies that $d(z) \geq \frac{\eta_k}{4} \geq \epsilon e^{-Cp(z)}$ for $z \in U \setminus U_0$. By the well-known Whitney's theorem (see e.g. [**BG1**, p. 18]), there exists a cut-off function $\chi \in C^\infty$, $0 \leq \chi \leq 1$, such that

$$\chi(z) = 1,\ z \in U_0; \quad \chi(z) = 0,\ z \in U^c$$

and

$$\big|\bar{\partial}\chi(z)\big| \leq \frac{C}{d(z)} \leq Ae^{Bp(z)}, \quad z \in U \setminus U_0,$$

for some $A, B, C > 0$. Define a function $\phi_0 \in \Omega$ satisfying that $\phi_0(z) = a_k$ on each $B(\zeta_k, \eta_k)$, $k \in \boldsymbol{N}$ (The values of ϕ_0 in $\Omega \setminus U$ can be arbitrarily defined). Let $\phi = \phi_0\bar{\partial}\chi$. Then the support of ϕ is contained in $U \setminus U_0$ and ϕ is clearly a $\bar{\partial}-$ closed form. If $z \in U \setminus U_0$, then there exists a $\zeta_k \in V$ such that $\frac{1}{2}\eta_k \leq |z - \zeta_k| \leq \eta_k$ and thus $|f(z)| \geq \epsilon_1 e^{-C_1p(\zeta_k)} \geq \epsilon e^{-Cp(z)}$ for some $\epsilon, C > 0$ by (3.8) and (2.1). Also by (2.1) again, when $|z - \zeta_k| \leq \eta_k < d_k \leq h(\zeta_k)$, we have that $|a_k| < A_2e^{B_2p(\zeta_k)} \leq Ae^{Bp(z)}$ for some $A, B > 0$, from which it follows that $|\phi(z)| = |a_k\bar{\partial}\chi(z)| \leq Ae^{Bp(z)}$ for some $A, B > 0$. Therefore, for each $m > 0$ there exist a $\alpha > 0$ such that

$$\int_\Omega \frac{|\phi(z)|^2}{|f(z)|^{2m}}e^{-\alpha p(z)}d\lambda < \infty, \tag{3.9}$$

in view of the fact that $(1 + |z|)^{2n+1} \leq Ae^{Bp(z)}$ for some $A, B > 0$ by the condition

(i) in the definition of the weight, where $d\lambda$ denotes the Lebesgue measure in $\boldsymbol{C}^n$. By Theorem 2.6 in [**KT**], (3.9) with $m \geq 2n+1$ implies that there exist $\bar\partial$ closed $(0,1)$-forms $\phi_1, \phi_2, \ldots, \phi_n$ in Ω and some $\beta > 0$ such that $\phi = \phi_1 f_1 + \cdots + \phi_n f_n$ and $\int_\Omega |\phi_j(z)|^2 e^{-\beta p(z)} d\lambda < \infty$. Thus by Hörmander's theorem [**H**, Lemma 4], there exist solutions ψ_j to the $\bar\partial$ equations $\bar\partial\psi_j = \phi_j$ in Ω satisfying the L^2-estimates:

$$\int_\Omega \left|\psi_j(z)\right|^2 e^{-\beta p(z)} (1+|z|^2)^{-2} d\lambda < \infty.$$

Define $F = \phi_0 \chi - \sum_{j=1}^n \psi_j f_j$. Then

$$\bar\partial F = \phi_o \bar\partial\chi - \sum_{j=1}^n f_j \bar\partial\psi_j = \phi - \sum_{j=1}^n f_j \phi_j = 0.$$

Thus, F is holomorphic in Ω. It is easy to verify that $\int_\Omega |F(z)|^2 e^{-Ap(z)} d\lambda < \infty$ for some $A > 0$, which implies that $F \in A_p(\Omega)$ ([**H**, Lemma 3]). Clearly, $F(\zeta_k) = \phi_0(\zeta_k)\chi(\zeta_k) = a_k$ for all $k \in \boldsymbol{N}$. This shows that V is an interpolating variety for $A_p(\Omega)$.

Next, we prove the necessity. Let $A_l(V) = \{a = \rho(f) : f \in A_l(\Omega),\ \|a\| \leq 1\}$, where $\|a\| := \sup_{k\in\boldsymbol{N}}\{|a_k| e^{-p(\zeta_k)}\}$ and $A_l(\Omega) := \{f \in A_p(\Omega) : |f(z)| \leq l e^{lp(z)},\ z \in \Omega\}$. We assert that there exist some l and $0 < \epsilon < 1$ such $A_l(V) \supset \mathscr{A}_\epsilon := \{a = \{a_k\}_{k\in\boldsymbol{N}} : \|a\| \leq \epsilon\}$. This may be proved by using the open mapping theorem for the restriction map $\rho : \rho(f) = \{f(\zeta_k)\}_{k=1}^\infty$ in (2.2) with $A_p(\Omega)$ and $A_p(V)$ endowed with inductive limit topology, or by using the Baire category theorem. For completeness, we include a proof for this assertion.

In fact, it is easy to see that the space $\mathscr{A} := \{a = \{a_k\}_{k\in\boldsymbol{N}} \subset \boldsymbol{C} : \|a\| \leq 1\}$ is complete under the metric induced by the norm $\|a\|$. Since V is an interpolating variety for $A_p(\Omega)$, for any sequence $a = \{a_k\} \in \mathscr{A}$, there exists a $f \in A_l(\Omega)$ for some l such that $\rho(f) = a$. Hence, $a \in A_l(V)$. This shows that $\mathscr{A} = \cup_{l=1}^\infty A_l(V)$. Also, for each $l, A_l(V)$ is a closed subset of $\mathscr{A}$. In fact, if $\{f_j\}$ is a sequence in $A_l(\Omega)$ such that $a_j := \rho(f_j) \to a \in \mathscr{A}$ as $j \to \infty$, then by (2.1), $\{f_j\}$ is uniformly bounded on each compact subset of Ω, and thus, by the Montel theorem and Weierstrass theorem (see e.g. [**G**]), we can assume, by passing to a subsequence, that $f_j \to f$, where f is holomorphic in Ω. Clearly, $f \in A_l(\Omega)$ and $\rho(f) = \{f(\zeta_k)\} = a$. This shows that $a \in A_l(V)$ and thus that $A_l(V)$ is closed. We can then apply the Baire-category theorem (see e.g. [**Ho**]) to obtain a l such that $A_l(V)$ has a non-empty interior, i.e., $A_l(V) \supset \mathscr{A}_\epsilon$ for some $0 < \epsilon < 1$. This shows the above assertion.

Now, for each fixed k, we consider $\mathbf{a}_k = (0, \ldots, 0, 1, 0, \ldots)$ with all the entries in $\mathbf{a}_k$ being zero except the $k-$th entry, which is 1. Then $\epsilon\mathbf{a}_k \in \mathscr{A}_\epsilon$. Therefore, there exists a sequence $\{h_k\}_{k=1}^\infty$ of functions $h_k \in A_l(\Omega)$ such that $\rho(h_k) = \epsilon\mathbf{a}_k$, i.e., $h_k(\zeta_j) = 0$ for $j \neq k$ but $h_k(\zeta_k) = \epsilon$. Let $g_k = \frac{1}{\epsilon} h_k$. Then g_k satisfies that

$$g_k(\zeta_j) = 0,\ j \neq k; \quad g_k(\zeta_k) = 1; \quad |g_k(z)| < L e^{Lp(z)},\ z \in \Omega, \tag{3.10}$$

where $L = \frac{l}{\epsilon}$, independent of k and z.

Next, we fix a positive number N such that

$$\int_{\Omega} \frac{1}{(1+|z|)^N} d\lambda \leq \int_{\boldsymbol{C}^n} \frac{1}{(1+|z|)^N} d\lambda < +\infty. \tag{3.11}$$

For each fixed integer $j (1 \leq j \leq n)$, we define for $z \in \Omega$,

$$f_j(z) = \sum_{k=1}^{\infty} (z_j - \zeta_{k,j}) g_k^2(z) \frac{1}{(1+|\zeta_k|)^{N+1}} \exp\big(-\gamma p(\zeta_k)\big), \tag{3.12}$$

where $z = (z_1, \ldots, z_n)$, $\zeta_k = (\zeta_{k,1}, \ldots, \zeta_{k,n})$, and γ is a (yet to be determined) positive constant.

Let $f = (f_1, f_2, \ldots, f_n)$. We will show that this is the desired mapping in the theorem. To this end, we first prove that f_j satisfies the right growth estimate required in the theorem, i.e., $f_j \in A_p(\Omega)$. For convenience, denote by $f_{j,k}$ the general term of the series (3.12). We then have that, in view of (3.10),

$$\begin{aligned}
|f_{j,k}(z)| &\leq \big(|z| + |\zeta_k|\big) L^2 e^{2Lp(z)} \frac{1}{\big(1+|\zeta_k|\big)^{N+1}} \exp\big(-\gamma p(\zeta_k)\big) \\
&\leq (1+|z|)\big(1+|\zeta_k|\big) L^2 e^{2Lp(z)} \frac{1}{\big(1+|\zeta_k|\big)^{N+1}} \exp\big(-\gamma p(\zeta_k)\big) \\
&= L^2 (1+|z|) e^{2Lp(z)} \frac{1}{\big(1+|\zeta_k|\big)^{N}} \exp\big(-\gamma p(\zeta_k)\big).
\end{aligned} \tag{3.13}$$

Let $q_k = \min\{\delta(\zeta_k), \inf_{l \neq k}\{|\zeta_l - \zeta_k|\}\}$, where $\delta(z) = e^{-|K_1|p(z) - |K_2|}$ and K_1, K_2 are constants in (2.1). Also, let $\mathscr{D}_k = B\big(\zeta_k, \frac{q_k}{2}\big)$ be the ball centered at ζ_k with radius $\frac{q_k}{2}$. Then $q_k \leq \delta(\zeta_k) \leq 1$ and $\mathscr{D}_k \cap \mathscr{D}_l = \varnothing$ for $k \neq l$. For $|z - \zeta_k| \leq q_k$, by (3.10) and (2.1) we have that $|g_k(z)| \leq Le^{Lp(z)} \leq Ae^{Bp(\zeta_k)}$ for some $A, B > 0$. If $q_k < \delta(\zeta_k)$, then there is a $\zeta_l \in V \cap B(\zeta_k, \delta(\zeta_k))$ such that $\zeta_l \neq \zeta_k$ and $q_k = |\zeta_l - \zeta_k|$. Applying the Schwarz lemma to $g_k(z) - g_k(\zeta_k)$ in the ball $|z - \zeta_k| < \delta(\zeta_k)$, we obtain that

$$\big|g_k(z) - g_k(\zeta_k)\big| \leq 2Ae^{Bp(\zeta_k)} \frac{1}{\delta(\zeta_k)} |z - \zeta_k|$$

and, in particular, at $z = \zeta_l$, we have that, in view of (3.10),

$$1 = g_k(\zeta_k) \leq 2Ae^{Bp(\zeta_k)} \frac{1}{\delta(\zeta_k)} |\zeta_l - \zeta_k|$$

or $q_k = |\zeta_l - \zeta_k| \geq \epsilon e^{-Cp(\zeta_k)}$ for some constants $\epsilon, C > 0$. We may assume that $\epsilon < e^{-|K_2|}$ and $C > |K_1|$. The above inequality is then also true if $q_k = \delta(\zeta_k)$. Therefore in any case, we have that $q_k \geq \epsilon e^{-Cp(\zeta_k)}$ and so that the volume of the ball $\mathscr{D}_k$ satisfies that $vol \mathscr{D}_k = \frac{\pi^n}{n!}\big(\frac{q_k}{2}\big)^{2n} \geq \epsilon_2 e^{-C_2 p(\zeta_k)}$ for some $\epsilon_2, C_2 > 0$. Also, by (2.1), for $z \in \mathscr{D}_k$ we have that $p(z) \leq K_3 p(\zeta_k) + K_4$. We deduce, by (3.13), that

$$|f_{j,k}(z)| \leq L^2(1+|z|)e^{2Lp(z)} \frac{1}{vol(\mathscr{D}_k)} \int_{\mathscr{D}_k} \frac{1}{\big(1+|\zeta_k|\big)^N} e^{(-\gamma p(\zeta_k))} d\lambda$$
$$\leq A(1+|z|)e^{2Lp(z)} \int_{\mathscr{D}_k} \frac{1}{\big(1+|\zeta_k|\big)^N} e^{(C_2 - \frac{\gamma}{K_3})p(z)} d\lambda$$

for some constant $A > 0$.

We thus take $\gamma = C_2 K_3$ so that

$$|f_{j,k}(z)| \leq A(1+|z|)e^{2Lp(z)} \int_{\mathscr{D}_k} \frac{1}{\big(1+|\zeta_k|\big)^N} d\lambda. \tag{3.14}$$

Note that if $z \in \mathscr{D}_k$, we have that $|z-\zeta_k| < \frac{q_k}{2} < \delta(\zeta_k) \leq 1$ and so that

$$1+|z| < 1+|z-\zeta_k|+|\zeta_k| < 2+|\zeta_k| < 2\big(1+|\zeta_k|\big).$$

Therefore, in view of the fact that $\mathscr{D}_k \cap \mathscr{D}_l = \varnothing$ for $k \neq l$, we have that

$$\sum_{k=1}^{\infty} \int_{\mathscr{D}_k} \frac{1}{\big(1+|\zeta_k|\big)^N} d\lambda \leq \sum_{k=1}^{\infty} \int_{\mathscr{D}_k} \frac{2^N}{(1+|z|)^N} d\lambda$$
$$\leq 2^N \int_{\Omega} \frac{1}{(1+|z|)^N} d\lambda := Q < +\infty \tag{3.15}$$

by (3.11). In view of the property (2.1) we thus have showed that the series $f_j = \sum_{k=1}^{\infty} f_{j,k}$ converges uniformly in compact sets in Ω and so that f_j is a holomorphic function in Ω. Moreover, by virtue of (3.14) and (3.15), we have that

$$|f_j(z)| \leq AQ(1+|z|)e^{2Lp(z)}. \tag{3.16}$$

But $(1+|z|)e^{2p(z)} = e^{\log(1+|z|)+2p(z)} = e^{O\{p(z)\}}$ in view of the condition (i) for the weight. We thus conclude that $f_j \in A_p(\Omega)$.

It is obvious that $V \subseteq f^{-1}(0)$ by the construction of each f_j (see (3.12) and (3.10)). Next we show that the mapping f satisfies the estimate on its Jacobian in the theorem. By (3.10) and (3.12) one can check that $f_j, 1 \leq j \leq n$, can be expanded into the following power series at each ζ_k,

$$f_j(z) = c_k(z_j - \zeta_{k,j}) + \sum_{i_1+\cdots+i_n=2}^{\infty} C_{i_1,\ldots,i_n}(z_1-\zeta_{k,1})^{i_1} \ldots (z_j-\zeta_{k,j})^{i_j} \ldots (z_n-\zeta_{k,n})^{i_n},$$

where

$$c_k = \frac{1}{\big(1+|\zeta_k|\big)^{N+1}} \exp\big(-\gamma p(\zeta_k)\big), \tag{3.17}$$

and $C_{i_1,\dots,i_n}$'s are complex numbers. It is clear that $c_k \geq \epsilon e^{-Cp(\zeta_k)}$ for some $\epsilon, C > 0$, by virtue of the condition (i) for the weight. We also see that

$$\left(\frac{\partial f_j}{\partial z_1}(\zeta_k), \dots, \frac{\partial f_j}{\partial z_j}(\zeta_k), \dots, \frac{\partial f_j}{\partial z_n}(\zeta_k)\right) = (0, \dots, 0, c_k, 0, \dots, 0)$$

with the j−th entry being c_k. Hence the Jacobian df satisfies that $|df(\zeta_k)| = (c_k)^n \geq \epsilon^n e^{-Cnp(\zeta_k)}$, which is the desired estimate in the theorem. This completes the proof. □

Proof of Corollary 2.2. For the necessity, using Theorem 2.1, we obtain a holomorphic mapping $f = (f_1, f_2, \dots, f_n)$ with $f_j \in A_p(\Omega)$ such that $V \subseteq f^{-1}(0)$ and $|df(\zeta_k)| \geq \epsilon e^{-Cp(\zeta_k)}$, $k \in \boldsymbol{N}$ for some $\epsilon, C > 0$. If $m > n$, we can easily add $m - n$ holomorphic functions $f_{n+1}, \dots, f_m \in A_p(\Omega)$ satisfying that $V \subseteq f_j^{-1}(0)$, $n+1 \leq j \leq m$. Let $F = (f_1, f_2, \dots, f_m)$. Then F clearly satisfies the conclusion of Corollary 2.2.

For the sufficiency, note that at each $\zeta_k \in V$ there exists a $n \times n$ minor $J_{n\times n}$ of the Jacobian of f such that $|J_{n\times n}(\zeta_k)| \geq \epsilon_1 e^{-C_1 p(\zeta_k)}$ for some $\epsilon_1, C_1 > 0$. This minor is the Jacobian of a mapping $G := (f_{i_1}, f_{i_2}, \dots, f_{i_n})$, where $\{f_{i_1}, f_{i_2}, \dots, f_{i_n}\} \subseteq \{f_1, f_2, \dots, f_m\}$. Thus, the proof is identical to the one of Theorem 2.1 by simply replacing f and the Jacobian in the proof of Theorem 2.1 by G and $J_{n\times n}(\zeta_k)$, respectively. □

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Bao Qin Li

Department of Mathematics
Florida International University
University Park
Miami, FL 33199
USA
E-mail: libaoqin@fiu.edu