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Fourier multipliers on non-Riemannian symmetric spaces

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Abstract. Restricting to K-invariant functions we prove $\boldsymbol{L}^p$ boundedness of Fourier multipliers satisfying Hörmander-Mihlin conditions on Non-Riemannian Symmetric Spaces using Flensted-Jensen duality. We also show necessity of holomorphic extension for multipliers.

1. Introduction.

Multipliers for $\boldsymbol{R}^n$ are now part of classical real harmonic analysis and we will only make a brief introduction. For more details the interested reader may consult, for instance, Stein's book [**18**].

Bounded translation invariant operators from $\boldsymbol{L}^p(\boldsymbol{R}^n)$ to itself can be viewed as convolutions with tempered distributions. It is not difficult to show that the translation invariant operators bounded on $\boldsymbol{L}^2(\boldsymbol{R}^n)$ are those that are given by convolution with a distribution whose Fourier transform is a bounded function, called a multiplier. Furthermore, it is well-known that if a translation invariant operator is bounded on $\boldsymbol{L}^p(\boldsymbol{R}^n)$, then it will also be bounded on $\boldsymbol{L}^2(\boldsymbol{R}^n)$. Thus, all translation invariant operators bounded on $\boldsymbol{L}^p(\boldsymbol{R}^n)$ are given on the Fourier transform side by multiplication with a bounded function. Except for the case $p = 2$, there is no complete characterization of the space of $\boldsymbol{L}^p$- multipliers but there are some theorems giving good sufficient conditions. One of the more familiar ones is the Hörmander-Mihlin condition which says that a function, m, is a $\boldsymbol{L}^p$-multiplier for all p, $1 < p < \infty$, if it satisfies the condition

$$|\partial_\xi^\alpha m(\xi)| \leq A_\alpha |\xi|^{-\alpha},$$

for all α such that $0 \leq \alpha \leq [(n+1)/2]$. There is also an $\boldsymbol{L}^2$ version of this.

On a non-compact Riemannian symmetric space, G/K, translation invariant operators are given by convolution with K-invariant distributions and they correspond to bounded Weyl group invariant functions, multipliers, on a maximal abelian subspace $\mathfrak{a}$ of the orthogonal complement in the Cartan decomposition. It has been shown by Clerc and Stein, see [**6**], that if the operator is bounded on $\boldsymbol{L}^p(G/K)$ then the multiplier will extend holomorphically to the tube

$$\mathscr{T} = \mathfrak{a}^* + i\,\mathrm{Conv}\left(W\left|1-\frac{2}{p}\right|\rho\right),$$

where Conv denotes convex hull, W the Weyl group and ρ is the half-sum of the positive roots. In [**1**], Anker showed that if m is holomorphic inside the tube $\mathscr{T}$ and satisfies a Hörmander-Mihlin type condition on the boundary of the tube, then m is a $\boldsymbol{L}^p$-multiplier for G/K.

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We would like to generalize the setting to non-Riemannian symmetric spaces, G/H. In accordance with the Riemannian case one could try to look at H-invariant convolutors but unfortunately this does not work. The reason is that H is non-compact so H-invariant functions cannot belong to any $\boldsymbol{L}^p$ space. So if we apply the convolutor to a function with small support then the resulting function will be almost H-invariant and hence not in $\boldsymbol{L}^p$.

Using Flensted-Jensen duality, which connects a non-Riemannian symmetric space with its non-compact Riemannian form, van den Ban, Flensted-Jensen and Schlichtkrull defined a class of operators, which they called multipliers, on the space of K-finite, $\mathscr{C}^\infty$ functions with compact support by taking convolution on the dual side. In this paper we consider $\boldsymbol{L}^p$-boundedness for this type of operators. We prove that under conditions similar to those of Anker for the Riemannian case such operators are bounded on $\boldsymbol{L}^p(K\backslash G/H)$. We also consider holomorphic extension of this kind of multipliers. The main outline of the proof that multipliers extend holomorphically is similar to that in the Riemannian case. But replacing spherical functions with Eisenstein integrals a lot of complications arise.

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2. Notation.

For more details about the general theory of non-Riemannian symmetric spaces see for example [**10**], [**4**] and [**11**] part II. Let G be a non-compact semisimple connected Lie group with finite center and an involution σ. Let H be an open subgroup of G^σ, the fixed point group of σ. So G/H is a semisimple symmetric space. Assume that we have a Cartan involution θ commuting with σ and denote by K the fixed point set for θ. Corresponding to the involutions we have the following decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}. \tag{1}$$

Since $\sigma \circ \theta = \theta \circ \sigma$, we also have

$$\mathfrak{g} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p} \oplus \mathfrak{q} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p}. \tag{2}$$

This splits into eigenspaces for $\sigma \circ \theta$

$$\mathfrak{g}_+ = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p},$$

$$\mathfrak{g}_- = \mathfrak{h} \cap \mathfrak{p} \oplus \mathfrak{q} \cap \mathfrak{k}.$$

We can now define the non-compact Riemannian form, G^d/K^d, of G/H. First we set

$$\mathfrak{g}^d = \mathfrak{h} \cap \mathfrak{k} \oplus i(\mathfrak{h} \cap \mathfrak{p}) \oplus i(\mathfrak{q} \cap \mathfrak{k}) \oplus \mathfrak{q} \cap \mathfrak{p}. \tag{3}$$

Then let G^d be the real form of $G_{\boldsymbol{C}}$(the complexification of G) with Lie algebra $\mathfrak{g}^d$. The subgroup $K^d = G^d \cap H_{\boldsymbol{C}}$ corresponding to $\mathfrak{h} \cap \mathfrak{k} \oplus i(\mathfrak{h} \cap \mathfrak{p})$ is a maximal compact subgroup of G^d. Hence the symmetric space G^d/K^d is Riemannian. We will also need $H^d = G^d \cap K_{\boldsymbol{C}}$ with Lie algebra $\mathfrak{h}^d = \mathfrak{h} \cap \mathfrak{k} \oplus i(\mathfrak{q} \cap \mathfrak{k})$. By partial holomorphic extension we obtain a map $f \mapsto f^r$ mapping

$\mathscr{C}^\infty(K;G/H)$ into $\mathscr{C}^\infty(H^d;G^d/K^d)$, i.e. it takes K-finite smooth functions on G/H to H^d-finite smooth functions on G^d/K^d. Since we are only interested in the so called "most continuous part" of the Plancherel decomposition, we take a maximal split θ-invariant Cartan subspace $\mathfrak{b}$ of $\mathfrak{q}$, i.e. such that the intersection $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{p}$ is maximal Abelian in $\mathfrak{q} \cap \mathfrak{p}$. We have two root systems $\Sigma(\mathfrak{a},\mathfrak{g})$ and $\Sigma(\mathfrak{a},\mathfrak{g}_+)$ with Weyl groups W and $W_{K\cap H}$, respectively. We will also use the quotient $\mathscr{W} = W/W_{K\cap H}$. Set $m_\alpha^+ = \dim \mathfrak{g}_\alpha \cap \mathfrak{g}_+$ and $m_\alpha^- = \dim \mathfrak{g}_\alpha \cap \mathfrak{g}_-$, where $\mathfrak{g}_\alpha$ is the root space related to $\alpha \in \Sigma(\mathfrak{a},\mathfrak{g})$. If we define

$$\mathfrak{b}^r = \mathfrak{b} \cap \mathfrak{p} + i(\mathfrak{b} \cap \mathfrak{k}) \tag{4}$$

then $\mathfrak{b}^r$ is maximal split for $\mathfrak{g}^d$. We have the generalized Cartan decompositions

$$G = K\overline{A^{++}}H, \quad G^d = H^d\overline{A^{++}}K^d.$$

Here A^{++} corresponds to the positive roots in $\Sigma(\mathfrak{a},\mathfrak{g}_+)$. We also have another generalized Cartan decomposition

$$G = \cup_{w\in\mathscr{W}} K\overline{A^+}H,$$

where A^+ is the positive Weyl chamber for $\Sigma(\mathfrak{a},\mathfrak{g})$.

3. Multipliers.

Let ψ be a function in $\boldsymbol{PW}^*(\mathfrak{b}_{\boldsymbol{C}^*})^{W(\mathfrak{b}^r)}$, i.e. a $W(\mathfrak{b}^r)$-invariant entire function of exponential type with slow growth, then by van den Ban, Flensted-Jensen and Schlichtkrull [**3**] this function has an operator $M_\psi : \mathscr{C}_c^\infty(K;G/H) \mapsto \mathscr{C}_c^\infty(K;G/H)$ associated to it. Such operators are special cases of what they call multipliers, which are operators

$$M : \mathscr{C}_c^\infty(K;G/H) \mapsto \mathscr{C}_c^\infty(K;G/H) \tag{5}$$

that are equivariant under the action of $\mathfrak{g}$, K and $\boldsymbol{D}(G/H)$ and restricts continuously to $\mathscr{C}_c^\infty(G/H)^\mu$ for all μ in $\hat{K}$. The connection to multipliers is that the Fourier transform of $M_\psi(f)$ is $\psi(\lambda)\hat{f}(\pi)$ on $v \in (\mathscr{H}_\pi^{-\infty})^H$ of type λ. Here $(\mathscr{H}_\pi^{-\infty})^H$ denotes the space of H-fixed distribution vectors for the irreducible unitary representation π of G. To construct M_ψ they use partial holomorphic extension defined by

$$(M_\psi f)^r = f^r * F, \quad f \in \mathscr{C}_c^\infty(K;G/H) \tag{6}$$

where F is the K-bi-invariant distribution with spherical transform ψ. By the Paley-Wiener theorem for Riemannian symmetric spaces, the assumptions on ψ implies that $F \in \mathscr{E}'(K\backslash G/K)$. Hence the convolution is in the image of $\mathscr{C}_c^\infty(K;G/H)$ under the isomorphism $(.)^r$, so M_ψ is well defined.

We shall be interested in $\boldsymbol{L}^p$-multipliers for K-invariant functions on G/H. Let ψ be a $W(\mathfrak{b}^r)$-invariant function on the support of the Plancherel measure. Let $\boldsymbol{L}^p(K\backslash G/H)$ be the space of K-invariant $\boldsymbol{L}^p$-functions on G/H. We can repeat the construction above, referring to the isomorphism $\boldsymbol{L}^p(K\backslash G/H) \cong \boldsymbol{L}^p(H^d\backslash G/K^d)$ instead. That is if F maps $\boldsymbol{L}^p(H^d\backslash G^d/K^d)$ to itself, the operator M_ψ will be well defined and map $\boldsymbol{L}^p(K\backslash G/H)$ to itself. We also have to show that $\widehat{M_\psi f}v = \psi(\lambda)\hat{f}v$, where v is an H-fixed distribution of type λ. This may essentially be done as in [**3**] by referring to a paper by Delorme and Flensted-Jensen [**7**], where a related result is proved. We only have to take care that all the integrals are well-defined, this follows because the representations are appearing tempered. For the convenience of the reader we will give the proof, which in our case is simpler as we only look at the trivial K-type.

PROPOSITION 1. *The following relation holds*

$$\widehat{M_\psi f}(\pi)v = \psi(\lambda)\hat{f}(\pi)v,$$

here v is an H-fixed spherical distribution vector of type λ for the representation π, and $f \in \boldsymbol{L}^p(K\backslash G/H)$.

PROOF. We want to show that

$$\langle \pi(M_\psi f)v, v'\rangle = \psi(\lambda)\langle \pi(f)v, v'\rangle,$$

for $v' \in \mathscr{H}_\pi^\infty$ with trivial left K-action and $v \in (\mathscr{H}_\pi^{-\infty})^H$ spherical of type λ. Bernstein has shown that the representations that appear in the support of the Plancherel measure are all tempered, see [**5**]. It is known, see for example Knapp [**13**] VII.11, that the K-finite matrix-coefficients of a tempered representation belong to all $\boldsymbol{L}^q$ for $q > 2$. Hence both sides of the identity above are well-defined. As $M_\psi f$ is K-invariant we can rewrite the LH-side as, using $P_{\mathbf{1}}$ to denote the projection to the trivial isotypic component,

$$\int_{K\backslash G/H} \langle P_{\mathbf{1}}\pi(x)v, M_\psi f(x)v'\rangle dx,$$

which by duality is, denoting $\phi(x) = \langle P_{\mathbf{1}}\pi(x)v, v'\rangle$

$$\int_{H^d\backslash G^d/K^d} \phi^r(x) f^r * F(x)dx.$$

As the integrand is right K^d-invariant this equals

$$\int_{H^d\backslash G^d} \phi^r(x) f^r * F(x)dx.$$

We now write the convolution as a double integral and substitute this into the last expression

$$\int_{H^d\backslash G^d} \int_{H^d\backslash G^d/K^d} f^r(y)\phi^r(x) \int_{H^d} F(y^{-1}hx)dhdydx,$$

which we rewrite using the left H^d-invariance of ϕ^r as

$$\int_{H^d\backslash G^d/K^d} f^r(y) \int_{G^d} \phi^r(x)F(y^{-1}x)dxdy.$$

After a change of variables, this becomes

$$\int_{H^d\backslash G^d/K^d} f^r(y) \int_{G^d} \phi^r(yx)F(x)dxdy.$$

We now use the right K-invariance of F to obtain

$$\int_{G^d/K^d} F(x) \int_{H^d\backslash G^d/K^d} \int_{K^d} f^r(y)\phi^r(ykx)dkdydx.$$

Let

$$\Phi(x) = \int_{H^d\backslash G^d/K^d} \int_{K^d} f^r(y)\phi^r(ykx)dkdy.$$

We then observe that Φ is bi-K^d-invariant and it is also an eigenfunction for $\boldsymbol{D}(G^d/K^d)$ with eigenvalue λ and hence it must be a constant multiple of the spherical function $\phi_{-i\lambda}$. Thus it remains to prove that this constant is equal to $\langle \pi(f)v, v'\rangle$. But this constant is just

$$\Phi(e) = \int_{H^d\backslash G^d/K^d} \int_{K^d} f^r(y)\phi^r(yk)dkdy,$$

which as ϕ^r is right-K^d-invariant is

$$\int_{H^d\backslash G^d/K^d} f^r(x)\phi^r(x)dx = \int_{K\backslash G/H} f(x)\phi(x)dx.$$

Taking into account the left-K-invariance of f and ϕ this completes the proof. □

Let $\mathscr{T}$ be the strip $(\mathfrak{b}^r)^* + i\,\mathrm{Conv}(W(\mathfrak{b}^r)\rho)$, where Conv denotes the convex hull.

THEOREM 1. Let ψ be a holomorphic function in the strip $\mathscr{T}$ and continuous up to the boundary, satisfying the following estimate

$$|\nabla^i \psi(\lambda)| \le C(1+|\lambda|)^{-i}, \quad \lambda \in \bar{\mathscr{T}}, i < \left[\frac{n}{2}\right] + 1, \tag{7}$$

where ∇ is the gradient, then M_ψ is a bounded operator on $L^p(K\backslash G/H)$ for $1 < p < \infty$.

REMARK 1. Before the proof it may be useful to recall that Anker [**1**] has proved that this assumption implies $\boldsymbol{L}^p$-boundedness in the Riemannian case. In the Riemannian case it is fairly easy to deduce from this the corresponding results also for smaller strips using a majorizing principle. In the non-Riemannian case this is not so easy and we get problems for example with the discrete series. See also example 1 in section 4.

REMARK 2. In the Riemannian case the analytic continuation is necessary since the spherical functions are in $\boldsymbol{L}^p$ for $p > 2\rho/(\mathrm{Im}\,\lambda + \rho)$ and ψ is the spherical transform of a K-invariant distribution, but in the non-Riemannian case the problem arises that although we can make a similar construction as Eisenstein integrals of trivial K-type, these may have zeros and hence may cause the function ψ to have poles. See section 4.

PROOF. By construction the corresponding operator on the dual side is convolution with the inverse spherical transform of ψ which is a K-biinvariant function F. Anker shows that F can be divided into two parts, F_0 concentrated close to the origin and $F_\infty = F - F_0$, with the following properties

- $\int_{|x|\ge 2|y|} |F_0(y^{-1}x) - F_0(x)|dx \le C$,
- $F_\infty \in \boldsymbol{L}^1(G^d/K^d)$.

He also shows that F_0 is bounded on $\boldsymbol{L}^2(G^d/K^d)$ but this we will not use. Let f be in $\boldsymbol{L}^p(K\backslash G/H)$

then by duality f^r will be in $\boldsymbol{L}^p(H^d\backslash G^d/K^d)$. Hence, if we prove that $f^r * F \in \boldsymbol{L}^p(H^d\backslash G^d/K^d)$ then the theorem follows from the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{L}^p(K\backslash G/H) & \xrightarrow{\cong} & \boldsymbol{L}^p(H^d\backslash G^d/K^d) \\
{\scriptstyle M_\psi}\Big\downarrow & & \Big\downarrow{\scriptstyle *F} \\
\boldsymbol{L}^p(K\backslash G/H) & \xrightarrow{\cong} & \boldsymbol{L}^p(H^d\backslash G^d/K^d).
\end{array}$$

Thus we want to prove that $f^r * F \in \boldsymbol{L}^p(H^d\backslash G^d/K^d)$. Considering the convolution as a K^d-invariant function on $H^d\backslash G^d$, we obtain by the second condition above:

$$\begin{aligned}
\left(\int_{H^d\backslash G^d} |f^r * F_\infty(x)|^p dx\right)^{1/p} &= \left(\int_{H^d\backslash G^d} \left|\int_{G^d/K^d} f^r(xy^{-1})F_\infty(y)dy\right|^p dx\right)^{1/p} \\
&\le \left(\int_{H^d\backslash G^d} |f^r(xy^{-1})|^p dx\right)^{1/p} \int_{G^d/K^d} |F_\infty(y)|dy \\
&\le C\left(\int_{H^d\backslash G^d} |f^r(x)|^p dx\right)^{1/p}.
\end{aligned}$$

For the local part we first use a covering lemma to reduce to functions with support in the unit ball of A, which is a space of homogeneous type and so we may use Hardy spaces. So assume that f is an atom with support in a ball of radius t. Let V_t denote the part of $H^d\backslash G^d$ where $|a| \ge 2t$ then with $\delta(a) = \prod_{\alpha\in\Sigma^+(\mathfrak{a},\mathfrak{g})} \sinh^{m_\alpha^+}(\log a)\cosh^{m_\alpha^-}(\log a)$ (see Flensted-Jensen [**9**])

$$\begin{aligned}
\int_{V_t} |f^r * F_0(x)|dx &= \int_{V_t} \left|\int_A \int_{H^d} f^r(a')F_0(a'^{-1}h^{-1}x)\delta(a')da'dh\right| dx \\
&\le \int_A |f^r(a')|\delta(a')da' \int_{|a|\ge 2t}\int_{H^d} \left|F_0(a'^{-1}h^{-1}a) - F_0(h^{-1}a)\right| \delta(a)dadh.
\end{aligned}$$

To be able to use our first condition on F_0 we need a lemma.

LEMMA 1. *If $h \in H^d$ and $a \in A$ then the following inequality is true in G^d/K^d,*

$$|ha| \ge |a|. \tag{8}$$

PROOF. Since $\mathfrak{a} \in \mathfrak{p}\cap\mathfrak{q}$ it is orthogonal to $\mathfrak{h}^d$. This implies that the orbits $A.o$ and $H^d.o$ are orthogonal at the origin in G^d/K^d, hence the result follows because G^d/K^d is a space of negative curvature. □

For the second part we use the $\boldsymbol{L}^2$-boundedness of the operator and the assumption $t \le 1$. Thus by Schwartz' inequality this part is bounded by some constant. □

REMARK 3. By the same argument we also obtain the same result for multipliers satisfying $\boldsymbol{L}^2$-conditions instead, since by Anker[**1**] the corresponding operator fulfills the same conditions.

3.1. An Example.

(for more information about the material in this section see [**14**]) Let $G = SL(2,\boldsymbol{R})$ this group has two involutions $\theta : g \mapsto (g^T)^{-1}$ and $\sigma : g \mapsto IgI$ where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The first is associated with the hyperbolic plane and to the second we have two symmetric spaces G/G^σ and G/H, where $G^\sigma = \{\text{diagonal matrices in } G\}$ is the fix point set of σ and H is the subset of all positive diagonal matrices. We will be interested in the space G/H. Clearly $\theta \circ \sigma = \sigma \circ \theta$, hence we are in the situation of our earlier study. The generalized Cartan decomposition in this case is $G = KAH$ where

$$\begin{aligned} H &= \left\{ \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}; s \in \boldsymbol{R} \right\}, \\ A &= \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}; t \in \boldsymbol{R} \right\}, \\ K &= \left\{ \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}; \phi \in \boldsymbol{R} \right\}. \end{aligned}$$

In this case we can improve our result a little because instead of restricting our attention to K-invariant function we can look at functions of a fixed K-type. The reason is that this corresponds to picking Fourier coefficients. Since M is assumed to preserve K-types we can take away the K action and then after we have applied M we may put it back, hence reducing to the K-invariant case. We want to find the Riemannian non-compact form X^r of $X = G/H$. For this we first introduce $\mathfrak{g}^d$ which is

$$\begin{aligned} \mathfrak{g}^d &= \mathfrak{k}^d(= i\mathfrak{h}) \oplus \mathfrak{h}^d(= i\mathfrak{k}) \oplus \mathfrak{a} \\ &= \boldsymbol{R}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \boldsymbol{R}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \boldsymbol{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} it & z \\ \overline{z} & -it \end{pmatrix}; t \in \boldsymbol{R}, z \in \boldsymbol{C} \right\}. \end{aligned}$$

This implies that $G^d = SU(1,1)$. We also obtain $K^d = S(U(1) \times U(1))$ and

$$H^d = \left\{ \begin{pmatrix} \cosh s & i\sinh s \\ -i\sinh s & \cosh s \end{pmatrix}; s \in \boldsymbol{R} \right\}. \tag{9}$$

So $X^r = SU(1,1)/S(U(1) \times U(1))$, i.e. the hyperbolic disc. As the maximal split Cartan subspace we take $\mathfrak{a}$ and we see that $\mathfrak{b}^r = \mathfrak{a}$ and $\mathfrak{a}_{\boldsymbol{C}} \cong \boldsymbol{C}$. To introduce the Fourier transform we need some representation spaces. Put $t^{s,\varepsilon} = |t|^s \operatorname{sgn}^\varepsilon t$. Let $D_{(s,\varepsilon)}$ be the space of $\mathscr{C}^\infty$ functions ϕ on $\boldsymbol{R}$ such that $t \mapsto \phi(-1/t)t^{s-1,\varepsilon}$ is also infinitely differentiable. When s is a positive integer the case of interest to us will be $\varepsilon \equiv s+1 \pmod 2$ and we will denote this space D_s. If we use the notation

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{10}$$

then we can define a representation of G on $D_{(s,\varepsilon)}$ as follows

$$T_{(s,\varepsilon)}(g)\phi(t) = \phi\left(\frac{\alpha t+\gamma}{\beta t+\delta}\right)(\beta t+\delta)^{s-1,\varepsilon}. \tag{11}$$

The operator

$$B_{(s,\varepsilon)}\phi(t) = \int_{-\infty}^{\infty}(t-t_1)^{-s-1,\varepsilon}\phi(t_1)dt_1$$

is an intertwining operator between $T_{(s,\varepsilon)}$ and $T_{(-s,\varepsilon)}$. For s pure imaginary the representation is irreducible but for s a positive integer and $\varepsilon \equiv s+1$ there are two invariant subspaces with intersection $E_s = \operatorname{Ker} B_{(s,\varepsilon)}$. When s is imaginary this representation is unitary for the scalar product

$$(\psi,\phi) = \int_{-\infty}^{\infty}\psi(t)\overline{\phi(t)}dt.$$

For s a positive integer we obtain a unitary representation by letting $T_{(s,\varepsilon)}$ act on the factor space D_s/E_s equipped with the scalar product

$$(\tilde{\psi},\tilde{\phi})_s = (A_s\psi,\phi),$$

where $A_s = 2^{-s}\pi^{1/2}\Gamma((1-s)/2)(d/dt)^s$. The H-invariant distribution vectors are $\theta_{(s,\varepsilon,\nu)} = t^{(s-1)/2,\nu}$. In our present situation the type of these vectors are determined by s, since $\boldsymbol{D}(G/H)$ is generated by the Laplacian

$$\Delta = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \tag{12}$$

and

$$T_{(s,\varepsilon)}(\Delta) = \frac{1}{4}(1-s^2)\cdot I. \tag{13}$$

The Fourier transform

$$F_{(s,\varepsilon,\nu)}f(t) = \int_X f(x)(\delta t-\gamma)^{((s-1)/2,\nu)}(-\beta t+\alpha)^{((s-1)/2,\nu+\varepsilon)}dx. \tag{14}$$

maps $\mathscr{C}_c^\infty(X)$ to D_χ or D_s/E_s. Since $K = \boldsymbol{S}^1$ the K-types are functions f such that $f(kaH) = e^{in\phi}f(aH)$. Even if this is not directly related to the discrete series one can show that for functions of a fixed K-type the discrete series is empty, see Appendix 1 in [**8**]. This explains why it suffices to look at the dual, which is only related to the (most) continuous part. If we integrate such functions against $e^{-in\phi}$ we obtain a K-invariant function to which we may apply Theorem 1. Then we define a new function from the resulting K-invariant function by defining it to be of the same K-type as the original function. As a result such a multiplier M will map K-finite $\boldsymbol{L}^p$-functions to K-finite functions with norm only depending on the number of K-parts of the given function. The Plancherel formula for X

$$\int_X |f(x)|^2dx = \frac{1}{16\pi^2}\sum_{s=1}^{\infty}si_s^{-1}\|F_sf\|_s^2 + \frac{1}{64\pi^2}\sum_{\varepsilon,\nu=0,1}\int_0^{\infty}\rho\tanh\left(\pi\frac{\rho+i\varepsilon}{2}\right)\|F_{i\rho,\varepsilon,\nu}f\|^2d\rho,$$

where $i_s = 2^{-s}\pi^{-1/2}\Gamma((1+s)/2)\,(\cos(\varepsilon\pi/2)+1)$, should be compared with the Plancherel formula for X^r, see [**12**] Introduction Theorem 4.2

$$\int_{X^r} |f(x)|^2 dx = \frac{1}{2\pi}\int_{K/M}\int_{\boldsymbol{R}_+} |\hat{f}(\lambda,kM)|^2 \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda dkM. \tag{15}$$

When we neglect the discrete part we observe that the formulas are of the same kind and hence it is not so surprising that the behaviour of the multipliers should be similar. (Observe, however, that there are four copies of $\boldsymbol{R}_+$ in the first case.)

Two examples of multipliers are $\psi(s) = e^{-t(1+\zeta-s^2)^{1/2}}$, where ζ is some small number, and $\psi'(s) = 4/((3/2)-s^2)$, the first one is almost the spherical transform of the Poisson Kernel for the Riemannian form and the second one is the inverse of $(1/8)I+\Delta$. They clearly satisfies the conditions of Theorem 1. The reason we are not considering the inverse of Δ itself is that then the function would have a pole at $s=\pm 1$, also to avoid poles when we differentiate we include the ε shift in the first example. Another example of a multiplier connected to the first example is the spherical transform of the Heat kernel for the Riemannian form, $\psi(s) = e^{-t(1-s^2)}$, this satisfies the conditions of theorem 1 and is entire, but it does not belong to the space $\boldsymbol{PW}^*(\mathfrak{b}^*_{\boldsymbol{C}})^{\mathscr{W}}$ as it is not of exponential type.

4. Necessity of holomorphic extension.

We shall now prove that the multipliers extend holomorphically as in the Riemannian case. Let $\{\alpha_1,\dots,\alpha_l\}$ and $\{\tilde{\alpha}_1,\dots,\tilde{\alpha}_{l'}\}$ be the sets of simple roots in $\Sigma(\mathfrak{a})^+$ and $\Sigma(\mathfrak{b}^r)^+$ respectively. We will assume that the root system $\Sigma(\mathfrak{b}^r)$ satisfies the condition

$$\text{If } \tilde{\alpha}\in\Sigma(\mathfrak{b}^r)^+ \text{ and } \tilde{\alpha}|_{\mathfrak{a}} \neq 0, \text{ then } \sigma\circ\theta(\tilde{\alpha})\in\Sigma(\mathfrak{b}^r)^+$$

and that the systems are compatible, i.e.

$$\Sigma(\mathfrak{a})^+ = \{\tilde{\alpha}|_{\mathfrak{a}};\tilde{\alpha}\in\Sigma(\mathfrak{b}^r)^+ \text{ and } \tilde{\alpha}|_{\mathfrak{a}}\neq 0\}.$$

We will denote the dual bases by $\{\omega_1,\dots,\omega_l\}$ and $\{\tilde{\omega}_1,\dots,\tilde{\omega}_{l'}\}$. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{m}$ the centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{k}$. Let

$$\begin{aligned} U_p = \Bigg\{\lambda\in\mathfrak{a}^*_c \,\Bigg|\, \Bigg(&\mathrm{Re}\Big\langle w(\lambda-\rho_m)+\left(\frac{1}{p}-\frac{1}{p'}\right)\rho,\omega_1\Big\rangle,\dots, \\ &\mathrm{Re}\Big\langle w(\lambda-\rho_m)+\left(\frac{1}{p}-\frac{1}{p'}\right)\rho,\omega_l\Big\rangle\Bigg)\in(-\infty,0)^l \text{ for all } w\in W(\mathfrak{b}^r)\Bigg\}, \end{aligned}$$

where ρ_m is the ρ-function for the root-system $\Sigma(\mathfrak{m},\mathfrak{b}\cap\mathfrak{k})$ and p' the dual index that is

$$\frac{1}{p'} = 1-\frac{1}{p},$$

and set

$$\tilde{U}_p = \{\lambda-\rho_m;\lambda\in U_p\}.$$

THEOREM 2. *If M_ψ is a $\boldsymbol{L}^p$-multiplier then $\psi(\lambda)$ has a partial holomorphic extension to the set $\tilde{U}_p$.*

PROOF. The idea of the proof is the same as in the Riemannian setting. Assume $1 < p < 2$. First we prove that if the operator is bounded on $\boldsymbol{L}^p$ then it is also bounded on $\boldsymbol{L}^{p'}$, then we show that there is a function in $\boldsymbol{L}^{p'}$, which is holomorphic in the given strip and such that the operator acts by multiplication with ψ on it. For Riemannian symmetric spaces we can use spherical functions, in the present situation we use Eisenstein integrals.

Let η_w be the w-component of $\eta \in \boldsymbol{C}^{\mathscr{W}}$. Using this we define a function on X by setting

$$\tilde{\eta}_\lambda(x) = \begin{cases} a^{i\lambda-\rho}\eta_w & \text{if } x \in HwP \\ 0 & \text{if } x \notin \bigcup_{w\in\mathscr{W}} HwP \end{cases}$$

Where P is the parabolic subgroup corresponding to the positive roots in $\Sigma(\mathfrak{a},\mathfrak{g})$. The K-invariant Eisenstein integral is now defined by

$$E(\eta,\lambda)(x) = \int_K \tilde{\eta}_\lambda(x^{-1}k)dk.$$

The normalized Eisenstein integral is obtained by a transformation of η

$$E^0(\eta,\lambda) = E(C(1,\lambda)^{-1}\eta,\lambda),$$

where $C : W \times \mathfrak{a}^* \to \mathrm{End}(\boldsymbol{C}^{\mathscr{W}})$ is a certain meromorphic transformation corresponding to Harish-Chandra's c-function. It is normalized by its asymptotic behaviour

$$E^0(\eta,\lambda)(aw^{-1}) \sim a^{\lambda-\rho}\eta_w,$$

where $a \in A^+$(corresponding to $\Sigma^+(\mathfrak{a},\mathfrak{g})$), $w \in \mathscr{W}$ and $\mathrm{Re}\,\lambda$ is strictly dominant. The normalized Eisenstein integrals are regular along the imaginary axes but might have singularities outside. To get rid of the singularities in a neighborhood of the imaginary axes one can multiply with a suitable product of linear factors. Lemma 14 in [**2**] shows that we can choose a product $p(\lambda)$ such that the function $\tilde{E}^0 := p(\lambda)E^0(\eta,\lambda)$ becomes holomorphic in a tube around the imaginary axes containing $\tilde{U}_p$ and is also a joint eigenfunction of $\boldsymbol{D}(G/H)$ with the same eigenvalue as E^0.

The last step of the proof will be to show that we can solve for ψ in a holomorphic way. In the Riemannian case this is easy because the spherical functions take the value one at the origin, in our case the space of eigenfunctions is not in general one-dimensional so it is not simple. Let us begin with the first step, to show duality.

LEMMA 2. *If a multiplier M_ψ is bounded on $\boldsymbol{L}^p$ then it is also bounded on the dual space $\boldsymbol{L}^{p'}$.*

PROOF. We want to show that the dual of $.*F$ is $((.\circ\theta)*F)\circ\theta$. The result then follows because $f\circ\theta \in \boldsymbol{L}^p \Leftrightarrow f \in \boldsymbol{L}^p$.

$$\int_{H^d\backslash G^d} f*F(x)g(x)dx = \int_{G^d}\int_A f(a)F(a^{-1}x)g(x)\delta(a)dadx$$

Here we have written the convolution as an integral over H^d and A. This in turn is equal to

$$\int_A f(a)\int_{G^d} F(x)g(ax)dx\delta(a)da = \int_A f(a)((g\circ\theta)*F)(\theta(a))\delta(a)da.$$

This equality follows easily using $\theta a = a^{-1}$. Finally as the convolution is right K-invariant this is just (as K is θ invariant)

$$\int_{H^d\backslash G^d} f(x)((g\circ\theta)*F)(\theta(x))dx. \qquad \square$$

The next step is to prove that M_ψ acts on the normalized Eisenstein integrals as multiplication by ψ.

LEMMA 3. $M_\psi(\tilde{E}^0(\eta,\lambda)) = \psi(\lambda-\rho_m)\tilde{E}^0(\eta,\lambda)$

PROOF. By definition this is the same as $(\tilde{E}^0(\eta,\lambda))^r * F = \psi(\lambda-\rho_m)(\tilde{E}^0(\eta,\lambda))^r$. We write the convolution as

$$(\tilde{E}^0)^r * F(x) = \int_{K^d}\int_{B^r}(\tilde{E}^0)^r(xkb^{-1})F(b)dbdk.$$

The K-invariant Eisenstein integrals are joint eigenfunctions of $\boldsymbol{D}(G/H)$ and hence by duality we see that $(\tilde{E}^0)^r$ is a joint eigenfunction of $\boldsymbol{D}(G^d/K^d)$ with the same eigenvalue: $\gamma_q(D;\lambda) = \gamma(D;\lambda-\rho_m)$, where γ is the Harish-Chandra isomorphism $\gamma: \boldsymbol{D}(G^d/K^d)\to S(\mathfrak{b}^r)^{W(\mathfrak{b}^r)}$ and γ_q is the corresponding algebra homomorphism from $\boldsymbol{D}(G/H)$ to $S(\mathfrak{a})^W$, see Part II, lecture 4 in [**11**]. Using this fact we obtain, by Proposition IV.2.4 in [**12**],

$$(\tilde{E}^0)^r * F(x) = \int_{B^r}\phi_{\lambda-\rho_m}(b^{-1})F(b)db(\tilde{E}^0)^r(x),$$

which is what we wanted to show. $\square$

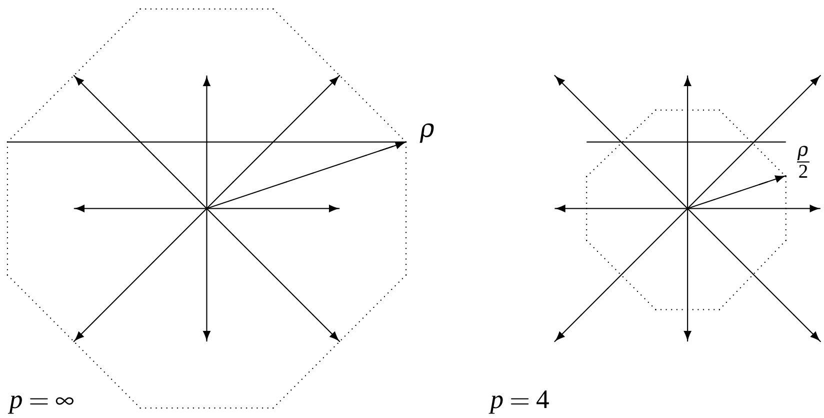
LEMMA 4. *For $\lambda\in U_p$ the regularized normalized Eisenstein integrals, $\tilde{E}^0$, are $\boldsymbol{L}^{p'}$-functions.*

PROOF. As the Eisenstein integrals are eigenfunctions of $\boldsymbol{D}(G/H)$ with eigenvalue $\gamma(D;\lambda-\rho_m)$ the same is true for the regularized version and it follows from Oshima [**15**], Corollary 4.3, that for λ in the given set they belong to $\boldsymbol{L}^{p'}$, because with λ in that set condition 2 of that Corollary is trivially satisfied. $\square$

REMARK 4. The corresponding set for spherical functions on the non-compact Riemannian form is

$$V_p = \left\{\lambda\in(\mathfrak{b}^r)^*_c \,\middle|\, \left(\mathrm{Re}\left\langle w\lambda+\left(\frac{1}{p}-\frac{1}{p'}\right)\rho,\tilde{\omega}_1\right\rangle,\ldots,\mathrm{Re}\left\langle w\lambda+\left(\frac{1}{p}-\frac{1}{p'}\right)\rho,\tilde{\omega}_l\right\rangle\right)\right.$$
$$\left.\in(-\infty,0)^l \text{ for all } w\in W(\mathfrak{b}^r)\right\}.$$

This clearly poses more conditions on the set of λ's, but in the definition of U_p we only take $\lambda\in\mathfrak{a}^*_c$. Even so, for $p=\infty$ the set $\tilde{U}_p$ is contained in V_p. This explains why Theorem 1 is valid. For other values of p the Riemannian set, V_p will not contain the set $\tilde{U}_p$ unless we have $\rho_m = 0$, which happens for example if $G/H = SL(3,\boldsymbol{C})/SL(3,\boldsymbol{R})$.



EXAMPLE 1. An example where the sets differ is the space $SO(4,1)/SO(2)\times SO(2,1)$. In the figure below the dotted polygon indicates the boundary of the set V_p and the horizontal line segment represents the set $\tilde{U}_p$. So, we can see $\tilde{U}_\infty \subset V_\infty$ but $\tilde{U}_4$ is not contained in V_4. The problem is that the shift ρ_m does not vary with p.

If we differentiate $\tilde{E}^0$ with respect to λ it is clear from the definition that this will only give some additional polynomial factor which does not affect the exponential decay. Hence, the derivative will also belong to $\boldsymbol{L}^{p'}$ and as M_ψ is continuous we find that $M_\psi(\tilde{E}^0(\eta,\lambda))$ too is holomorphic. If we use lemma 3 it follows that to finish the proof of the theorem we would like to divide by $\tilde{E}^0$, at least evaluated at some point. The problem is that we need $\tilde{E}^0(\eta,.)$ to be non-zero in a neighborhood of the point we are interested in.

EXAMPLE 2. To illustrate the problem let us consider a simple example, the real hyperboloids $SO(p,q)/SO(p-1,q)$, with $q>1$. The K-invariant Eisenstein integrals are joint eigenfunctions of $\boldsymbol{D}(G/H)$, especially of the Laplace-Beltrami operator. Hence, the restriction to A^+ is also an eigenfunction of the radial part of the Laplace-Beltrami operator. This eigenequation might be transformed into a hypergeometric equation by a change of variables and this, together with the normalization, can be used to determine the normalized Eisenstein integrals, see [**11**] example 8.1. The result is

$$E^0(\lambda)(t)=\frac{\Gamma((1/2)(\lambda+\rho))\Gamma((1/2)(\lambda-\rho+q))}{\Gamma(\lambda)\Gamma(q/2)}2^{\lambda-\rho}F\left(\frac{\rho+\lambda}{2},\frac{\rho-\lambda}{2};\frac{q}{2};-\sinh^2 t\right).$$

The hypergeometric function F takes the value 1 at the origin, just like the spherical functions in the Riemannian case. But the Γ-factors in the numerator might give poles and the Γ-factor in the denominator might give zeroes. The point is that both the poles and the zeroes might be cancelled in a neighborhood of the origin by multiplying with a suitable factor only depending on λ.

The reason that we restrict ourselves to $q>1$ above is that in those cases the relative Weyl group is trivial, $\mathscr{W}=\{e\}$. When $q=1$ we obtain $\mathscr{W}=\{\pm 1\}$ and the hypergeometric equation has a two dimensional set of regular solutions at the origin given by

$$c_1 F\left(\frac{\rho+\lambda}{2},\frac{\rho-\lambda}{2};\frac{q}{2};-\sinh^2 t\right)+c_2\sinh t\,F\left(\frac{p+\lambda}{2},\frac{p-\lambda}{2};\frac{4-q}{2};-\sinh^2 t\right).$$

If $c_1 = 0$ the function will be zero at the origin hence it is important to have the full basis.

In general rank the equation for the radial part of the Laplacian-Beltrami is no longer a classical hypergeometric equation but following the recipe in Part III, Sect 2, of [**11**] to change the root-system it can be turned into a subsystem of a generalized hypergeometric system. It shows that Eisenstein integrals become linear combinations of generalized hypergeometric functions for that system. At least for generic values of λ, using Corollary 4.1.8 in [**11**] Part I, this would give an alternative proof, but the argument following this example is simpler. The situation is particularly simple if $\mathscr{W} = \{e\}$ and the symmetric space is basic in the sense of Oshima & Sekiguchi, see [**17**] definition 6.4, i.e. that $m_\lambda^+ \geq m_\lambda^-$ for any $\lambda \in \Sigma(\mathfrak{a},\mathfrak{g})$ such that $(1/2)\lambda \notin \Sigma(\mathfrak{a},\mathfrak{g})$. In that case the Eisenstein integral will be given by a constant(depending on λ) times the generalized hypergeometric function and, as the spherical function in the Riemannian case, this function takes the value one at the origin, see [**11**] Corollary 4.4.6. Thus, the situation would be as in the case with $q > 1$ above.

Arguing as in the first part of the proof of Proposition 4.2 in [**16**], we see that for generic values of λ (more precisely for λ's such that $2\langle\lambda,\alpha\rangle$ is not an integer multiple of $\langle\alpha,\alpha\rangle$) the Eisenstein integrals, for the standard basis of $\boldsymbol{C}^{\mathscr{W}}$, are linearly independent as functions on G/H. For singular values of λ it is possible to modify the functions, to make them linearly independent as functions on G/H and holomorphic as functions of λ, by taking suitable linear combinations with meromorphic coefficients as in the proof of proposition 3.9 in the same paper, see also the example above. Thus we might assume that we are in the situation that we have a function $f(\lambda,x)$ holomorphic in λ, real-analytic in x and not identically zero as a function of x for any fixed λ, that satisfies the identity in lemma 3. (For generic λ one could use the boundary values instead, as in the example above.) Given λ_0 we want to show that there is a point x_0 such that for λ in a neighbourhood of λ_0, we have $f(\lambda,x_0) \neq 0$. Because we could then divide by $f(\lambda,x_0)$ and conclude that $\psi(\lambda)$ is holomorphic in a neighbourhood of λ_0. As the function $x \mapsto f(\lambda_0,x)$ is real-analytic and not identically zero, it cannot be identically zero in a neighbourhood of the origin e. (For generic λ's the argument in the example gives a precise characterization of the set of such points.) Let x_0 be such that $f(\lambda_0,x_0) \neq 0$. Since f is continuous as a function of λ, the set of λ's such that $f(\lambda,x_0) \neq 0$, is open. Hence, there is an open neighbourhood of λ_0 such that $f(\lambda,x_0) \neq 0$ for λ's in that neighbourhood as claimed. This concludes the proof. $\square$

REMARK 5. We have seen that for generic λ's the Eisenstein integrals corresponding to the standard basis of $\boldsymbol{C}^{\mathscr{W}}$ are linearly independent and they all satisfies the identity in lemma 3. Thus it would be natural to consider diagonal matrix multipliers that would act in the different directions independently.

REMARK 6. In lemma 4 we only used Corollary 4.3 in [**15**] in a trivial way. Oshima's result shows that the Eisenstein integrals might belong to $\boldsymbol{L}^p$ for other values as well. Of course, the multiplier has to be defined also at those points. There is the discrete series for example. By Flensted-Jensen [**8**] we can determine the discrete series in the case when the rank is one. Set $\alpha = (m_+ - 1)/2$ and $\beta = (m_- - 1)/2$. The discrete series consists of the points $\{i\eta \,|\, \eta = |\beta| - (\alpha + 1 + 2m) > 0, m \in \boldsymbol{N}\}$. Hence it is nonempty if and only if $m_- > \rho + 1$. This shows that for symmetric spaces of that kind, there exists (p,p)-multipliers for G^d/K^d which are not multipliers for G/H, if $2 \geq p \geq \rho/\beta$. For example we have the hyperbolic spaces $SO_o(r,1)/SO_o(r-1,1)$ with $r > 3$.

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