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Zero sets of solutions of second order parabolic equations with two spatial variables

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Abstract. We consider local properties of zero sets dependent of time variable of analytic solutions of linear second order parabolic equations with two spatial variables. Our aim is to give several necessary conditions for the zero set at the present to be homeomorphic to those at the recent past and at the recent future.

1. Introduction.

Let u be a real valued analytic solution of the following equation in a neighbourhood $(-T,T)\times U$ of the origin in $\boldsymbol{R}^3$:

$$\frac{\partial u}{\partial t} = A(t,x,y;\partial/\partial x,\partial/\partial y)u. \tag{1.1}$$

Here $A(t,x,y;\partial/\partial x,\partial/\partial y)$ is a linear second order partial differential operator with analytic coefficients whose principal part at the origin is equal to the Laplacian $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$. We consider local properties of its zero sets defined by

$$Z(u(t)) = \{(x,y)\in U\,;\,u(t,x,y)=0\}$$

and its singular part defined by

$$S(u(t)) = \{(x,y)\in Z(u(t))\,;\,u_x(t,x,y)=u_y(t,x,y)=0\}.$$

Watanabe [**3**] set about a study concerning change of $Z(u(t))$ in t and this study in the case of space dimension ≥ 2 is yet to be investigated. When the space dimension is equal to one, such problem for not necessarily analytic solutions is studied, for example, by Angenent [**1**] and Watanabe [**4**]. Their concern was the non-increasing property of number of zero points. By other point of view the decreasing property states that the zero set $Z(u(t))$ certainly is different from $Z(u(-t))$ for sufficiently small $t>0$ when $u(0)$ has a singular point. But in our case it does not necessarily occur. As a simple example let $u_\lambda(t,x,y)$ be solutions of (1.1) such that $u_\lambda(t,x,y) = y^2+\lambda x^2+(2+2\lambda)t+O(t^2+|y|^3+|x|^3)$ as $|t|+|y|+|x|\to 0$. When $-1\neq\lambda<0$, for sufficiently small $t>0$ and for suitable small neighbourhood U_o of the origin both $Z(u(t))\cap U_o$ and $Z(u(-t))\cap U_o$ are one-dimensional smooth manifolds and have two connected components. When $\lambda=0$ and $u_0(t,x,y)=y^2+x^3+(2+6x)t+O(t^2+|y|^3+y^2|x|+|y|x^2+x^4)$

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as $|t|+|y|+|x| \to 0$, they are connected one-dimensional smooth manifolds.

We are concerned with the problem of deciding whether $Z(u(\pm t)) \cap U_o$ are homeomorphic to $Z(u(0)) \cap U_o$. In the section 3 we shall prove the following results. When $Z(u(-t)) \cap U_o$ is homeomorphic to $Z(u(0)) \cap U_o$ for sufficiently small $t > 0$, $Z(u(t)) \cap U_o$ is also homeomorphic to $Z(u(0)) \cap U_o$ (Theorem 3.1). Moreover, under some assumption the first non vanishing homogeneous term of Taylor expansion of $u(0,\cdot,\cdot)$ in (x,y) at the origin, called the initial form and denoted by p, is either harmonic or $(x^2+y^2)q(x,y)$ for some harmonic polynominal q (Theorem 3.4). If $\{\theta \in [0,\pi);\, p(\cos\theta, \sin\theta) = 0\}$ is a set of m-points where m is the degree of p, the converse statement of Theorem 3.1 holds and p is harmonic (a corollary of Theorem 3.2).

We discuss this problem with the aid of properties of approximate polynominal solution $W(t,x,y)$ of u which is given by the following.

$$\begin{cases} \dfrac{\partial W}{\partial t} = \Delta W & \text{in} \quad \boldsymbol{R}^3, \\ W\big|_{t=0} = p(x,y) & \text{in} \quad \boldsymbol{R}^2. \end{cases} \tag{1.2}$$

Here p is the polynominal stated above. Then we define the Hermite polynominal H and its conjugate Hermite polynominal H^* by the following.

$$H(x,y) = W(-1,x,y), \quad H^*(x,y) = W(1,x,y). \tag{1.3}$$

We shall use their properties of zero sets due to Watanabe in [**5**] in order to examine $Z(u(t)) \cap \{(x,y);\, x^2+y^2 \le M|t|\}$ for large constant M.

2. General properties of zero sets.

In this section we recall the general properties of zero sets that are proved in [**3**] and we give the notation and some propositions in order to use in the next section.

Without loss of generality we always assume that

$$\left.\frac{\partial^m u}{\partial y^m}\right|_{(0,0,0)} \neq 0, \quad m = \nu(u(0),(0,0)) \ge 2. \tag{2.1}$$

Here and in the following we denote by $\nu(f,P)$ the vanishing order of f at $P \in \boldsymbol{R}^2$.

For such solution u we define the numbers $m(0)$, $m(\pm 0)$ as follows.

DEFINITION 2.1. If the origin is an isolated point of $Z(u(0))$, then we put $m(0) = 0$. If this is not the case, then by virtue of Puiseux expansion the germ of set at the origin defined by $Z(u(0))$ is the union of finite number, say $m(0)$, of curves $\gamma_j, 1 \le j \le m(0)$, of the form $(\varepsilon_j x^{\sigma_j}, f_j(x))$ where $\gamma_j \neq \gamma_k\, (j \neq k)$, $\varepsilon_j = \pm 1$, $\sigma_j \ge 1$ are integers and where f_j are analytic near the origin such that $f_j(0) = 0$. Here we take f_j so that it is not even in x if σ_j is even. We denote by $\nu(\gamma_j)$ the vanishing order of $u(0)$ on the germ at the origin given by $\gamma_j \setminus \{(0,0)\}$. We define $m(\pm 0)$ by the following.

$$m(+0) = \text{the number of } \{j;\, \nu(\gamma_j) \text{ is odd}\}, \quad m(-0) = \sum_{j=1}^{m(0)} \nu(\gamma_j).$$

Put $U(\delta) = \{(x,y);\, |x| < \delta^2, |y| < \delta\}$ and take $\delta > 0$ sufficiently small and fix. Then the following is proved in [**3**].

PROPOSITION 2.2. *Let $|t|>0$ be sufficiently small. Then the following holds.*

(1) *$S(u(t))\cap U(\delta)$ is at most finite and for each point P in this set the initial form of $u(t)$ at P is harmonic after a linear change of coordinates around P.*

(2) *When $t>0$, $Z(u(t))\cap U(\delta)$ is the union of $m(+0)$ regular curves in $U(\delta)$ defined over the interval $(0,1)$ whose end points are on the boundary of $U(\delta)$.*

(3) *When $t<0$, $Z(u(t))\cap U(\delta)$ is not empty and it is the union of $\Gamma(t)$ and $\Gamma_c(t)$ where $\Gamma(t)$ consists of $m(-0)$ regular curves in $U(\delta)$ defined over $(0,1)$ whose end points are on the boundary of $U(\delta)$ and where $\Gamma_c(t)$ is either empty or the union of finite number of closed regular curves in $U(\delta)$ defined over the circle.*

(4) *When $t_1t_2>0$ and $|t_k|$, $k=1,2$, are small enough, $Z(u(t_1))\cap U(\delta)$ is homeomorphic to $Z(u(t_2))\cap U(\delta)$.*

In the following we always denote by p the initial form of $u(0)$ at the origin which can be written in the following form.

$$p(x,y)=\prod_{j=1}^{d}(y-\lambda_j x)^{d_j}, \quad \lambda_j\neq\lambda_k \text{ for } j\neq k. \tag{2.2}$$

We also define the numbers $m(p,0)$, $m(p,\pm 0)$ for p in the same way as Definition 2.1. Namely, $m(p,0)$ is the number of j such that λ_j is real. If $m(p,0)>0$, by rearrangement we assume that λ_j, $j=1,\ldots,m(p,0)$, are real. We also denote by $m(p,+0)$ the number of $j\le m(p,0)$ such that d_j is odd and by $m(p,-0)$ the sum of d_j over $j\le m(p,0)$. Note that

$$m(p,+0)\le m(+0)\le m(0)\le m(-0)\le m(p,-0). \tag{2.3}$$

Set $\Gamma(\theta,\varepsilon)=\{(x,y)\in \boldsymbol{R}^2\setminus\{(0,0)\}\,;\,|\arg(x+iy)-\theta|<\varepsilon\}$ and $\lambda_j=\tan\theta_j$. Then we have the following.

PROPOSITION 2.3. *For sufficiently small $\varepsilon>0$ there is a constant $M>0$ such that for sufficiently small $|t|>0$ the following holds.*

$$\begin{aligned} Z(u(t))\cap U(\delta) &\subset \left\{(x,y)\,;\,|x|^2\le M|t|\right\} \\ &\quad\bigcup \left\{\bigcup_{j\le m(p,0)} \left\{(x,y)\in\Gamma(\theta_j,\varepsilon)\cup\Gamma(\pi+\theta_j,\varepsilon)\,;\,|x|^2>M|t|\right\}\right\}. \end{aligned}$$

Moreover, if $t<0$, then for each $j\le m(p,0)$ both

$$\left\{(x,y)\in Z(u(t))\cap\Gamma(\theta_j,\varepsilon)\,;\,|x|^2=M|t|\right\}$$

and

$$\left\{(x,y)\in Z(u(t))\cap\Gamma(\pi+\theta_j,\varepsilon)\,;\,|x|^2=M|t|\right\}$$

consist of d_j points. If $t>0$, then they consist of one point in the case where d_j is odd and they are empty in the case where d_j is even.

Using the polynominals W,H,H^* given by (1.2), (1.3) , we prove the following lemma in order to use in a proof of Proposition 2.3.

LEMMA 2.4. *Let $(\rho(s),\gamma(s))$ be an analytic curve in $\boldsymbol{R}^3$ defined near $s=0$ such that $\gamma(0)=(0,0)$, $\rho(0)=0$, $\rho\not\equiv 0$ and $\lim_{s\to 0}\nu(u(\rho(s)),\gamma(s))=n\geq 1$. Then the following holds.*

(1) *Suppose that $\lim_{s\to 0}|\gamma(s)|/|\rho(s)|^{1/2}<\infty$. If we put $\omega=\lim_{s\to+0}\gamma(s)/|\rho(s)|^{1/2}$, then $\nu(H,\omega)\geq n$ (resp. $\nu(H^*,\omega)\geq n$) in the case where $\rho(s)<0$ (resp. $\rho(s)>0$) for sufficiently small $s>0$.*

(2) *Suppose that $\lim_{s\to 0}|\gamma(s)|/|\rho(s)|^{1/2}=\infty$. If we put $\omega'=\lim_{s\to+0}\gamma(s)/|\gamma(s)|$, then $\nu(p,\omega')\geq n$ and $2n<m+2$.*

PROOF. (1) Suppose that $\rho(s)>0$ for sufficiently small $s>0$. Since we have by definition that as $|t|+|x|+|y|\to 0$

$$u(t,x,y)=W(t,x,y)+O(\{|t|^{1/2}+|x|+|y|\}^{m+1}), \tag{2.4}$$

we obtain that as $s\to+0$

$$|\rho(s)|^{-m/2}u(\rho(s),\gamma(s))=H^*(\gamma(s)/|\rho(s)|^{1/2})+O(|\rho(s)|^{1/2})$$

and analogous relations for partial derivatives in (x,y) of u. So these imply the conclusion. When $\rho(s)<0$ for sufficiently small $s>0$, by using H instead of H^* we have the desire result.

(2) At first we note that the solution W of (1.2) can be written in the form:

$$W(t,x,y)=\sum_{k\geq 0}\frac{t^k}{k!}\Delta^k p(x,y). \tag{2.5}$$

By calculating the derivatives in s of $u(\rho(s),\gamma(s))$, we obtain that $\nu(u_t(\rho(s)),\gamma(s))\geq n-1$ and that as $s\to 0$

$$\begin{aligned}|\gamma(s)|^{-m+2}u_t(\rho(s),\gamma(s))&=|\gamma(s)|^{-m+2}\Delta W(\rho(s),\gamma(s))+O(|\gamma(s)|)\\&=\Delta p(\gamma(s)/|\gamma(s)|)+O(|\rho(s)|^{1/2}/|\gamma(s)|)+O(|\gamma(s)|)\end{aligned}$$

and analogous relations for partial derivatives in (t,x,y) of u. By the same arguments as that for the assertion (1) we have that $\nu(\Delta^k p,\omega')\geq n-k$ for each $k=0,\dots,n-1$.

Suppose that $2n-2\geq m$. If $2n-2=m$, by the fact $\nu(\Delta^{n-1}p,\omega')\geq 1$ the constant function $\Delta^{n-1}p$ is zero and hence $\Delta^{n-1}p$ always is identically zero. Take the minimun integer $l\geq 1$ such that $\Delta^l p\equiv 0$ and put $q=\Delta^{l-1}p$. Since $l\leq n-1$, this harmonic homogeneous polynominal q has a singular point $\omega'\neq(0,0)$ and hence $q\equiv 0$. This is a contradiction against the choice of l.

□

REMARK 2.5. If $m(p,0)=m(p,-0)\geq 1$, then $m(p,0)=m(p,+0)=m(-0)=m(0)=m(+0)$ and for large constant $M>0$

$$Z(u(t))\cap Z(u_y(t))\cap\{(x,y)\in U(\delta);x^2>M|t|\}=\emptyset. \tag{2.6}$$

Because, the assumption implies that $d_j=1$ for each $j=1,\dots,m(p,0)$ and so by definition $m(p,0)=m(p,+0)$ and hence we obtain by (2.3) that $m(p,0)=m(0)=m(+0)=m(-0)$. Since $Z(p)\cap Z(p_y)=\{(0,0)\}$, the latter assertion can be proved by the same arguments as that in the proof of Lemma 2.4.

Using Lemma 2.4 and the notation for a polynominal f in $\boldsymbol{R}^2$,

$$Z(f) = \{P \in \boldsymbol{R}^2\,;\, f(P) = 0\},\ \ S(f) = \{P \in Z(f)\,;\, f_x(P) = f_y(P) = 0\},$$

we show Proposition 2.3.

PROOF. It follows from the behavior at the infinity of Hermite and conjugated Hermite polynominals proved in [**5**] that for each $\varepsilon > 0$ there is a constant $M > 0$ such that the following holds.

$$\{Z(H) \cup Z(H^*)\} \cap \{(\xi,\eta)\,;\, |\xi| > M\} \subset \bigcup_{j \le m(p,0)} \left\{\Gamma\left(\theta_j, \frac{\varepsilon}{2}\right) \cup \Gamma\left(\pi + \theta_j, \frac{\varepsilon}{2}\right)\right\}. \tag{2.7}$$

Consider the germ V of semi analytic set at $(0,0,0)$ defined by

$$\begin{aligned} V = \bigcap_{j \le m(p,0)} &\{(t,x,y)\,;\,(x,y) \in Z(u(t)),\\ &|x|^2 > 2M^2|t| > 0\,,\,(x,y) \notin \Gamma(\theta_j, 2\varepsilon) \cup \Gamma(\pi+\theta_j, 2\varepsilon)\}. \end{aligned}$$

Then we claim that this germ is empty. If this is not the case, by Curve Selection Lemma, (see, for example, [**2**]), there is an analytic curve $\sigma(s)$ such that $\sigma(s) = (\rho(s), Q(s)) \in V$ for sufficiently small $s > 0$ and $\sigma(0) = (0,0,0)$.

Suppose that $\lim_{s\to 0} |Q(s)|/|\rho(s)|^{1/2} < \infty$. Putting $(\xi,\eta) = \lim_{s\to+0} Q(s)/|\rho(s)|^{1/2}$, we obtain that $|\xi| > M$ and from Lemma 2.4 that $H^*(\xi,\eta)H(\xi,\eta) = 0$, which is a contradiction against (2.7). Nextly we suppose that $\lim_{s\to 0} |Q(s)|/|\rho(s)|^{1/2} = \infty$. Put $(\xi',\eta') = \lim_{s\to+0} Q(s)/|Q(s)|$, then it follows from Lemma 2.4 that $p(\xi',\eta') = 0$ and so $\arg(\xi' + i\eta') \equiv \theta_j (\mathrm{mod}\,\pi)$ for some $j \le m(p,0)$, which is a contradiction we seek. The assertion in the latter half is a consequence of the behavior at the infnity of Hermite and conjugate Hermite polynominals that is proved in [**5**]. $\square$

It is worth while to say that the germ of set at the origin in $\boldsymbol{R}^3$ given by $u(t,x,y) = u_x(t,x,y) = u_y(t,x,y) = 0$ is at most one-dimensional. On the other hand the germ at the origin in $\boldsymbol{R}^3$ given by $u_x(t,x,y) = u_y(t,x,y) = 0$ may be two-dimensional. It is proved in [**5**] that a necessary and sufficient condition for $\{(x,y) \in \boldsymbol{R}^2\,;\, H_x(x,y) = H_y(x,y) = 0\}$ to be one-dimensional is that p is a multiple of either $(ax+by)^m$ for $m \ge 2$ or $(x^2+y^2)^{m/2}$ for even $m \ge 4$. Moreover, in this case both $S(H)$ and $S(H^*)$ are empty. Using this result, we prove the following.

PROPOSITION 2.6. *Suppose that $S(H) \ne \emptyset$. Then the germ $\mathscr{V}$ of semi-analytic set at the origin in $\boldsymbol{R}^3$ defined by*

$$\{(t,x,y)\,;\,|t| > 0,\, u_x(t,x,y) = u_y(t,x,y) = 0\}$$

is either empty or the union of finite number of germs defined by analytic curves of the form: $(\varepsilon t^\sigma, \gamma(t))$, where $\varepsilon = \pm 1$, $\gamma(0) = (0,0)$ and σ is an integer ≥ 1.

PROOF. Without loss of generality we assume that $\sum_{j=1}^d d_j\lambda_j \ne 0$. Then by Weierstrauss preparation theorem u_x and u_y can be expressed in a small neigbourhood of the origin in the following form.

$$u_x(t,x,y) = u_o(t,x,y)\, f(t,x;y), \ \ f(t,x;y) = y^{m-1} + \sum_{k=0}^{m-2} u_k(t,x) y^k,$$

$$u_y(t,x,y) = v_o(t,x,y)\, g(t,x;y), \ \ g(t,x;y) = y^{m-1} + \sum_{k=0}^{m-2} v_k(t,x) y^k.$$

Here $u_o(0,0,0)\, v_o(0,0,0) \neq 0$ and u_k, v_k, $k = 1,\ldots,m-1$, are real analytic near the origin. Since the result proved in [**5**] ensures that the resultant as function in x of polynominals in y of $H_x(x,y)$ and $H_y(x,y)$ does not identitically vanish, also is the resultant as function in (t,x), denoted by $\delta(t,x)$, of polynominals in y of $f(t,x;y)$ and $g(t,x;y)$. Hence the germ $\mathscr{V}$ is at most one-dimensional. Moreover, by virtue of Puiseux expansion the equation $\delta(t,x) = 0$, $|t| > 0$, can be expressed by $x = h(t^{1/n})$ for several holomorphic functions h and for integers $n > 0$ and hence by considering the equations $f(t^n, h(t); y) = 0$ we arrive at the conclusion. □

3. Necessary conditions for $Z(u(t))$ to be homeomorphic to $Z(u(0))$.

In this section we give several necessary conditions for $Z(u(t)) \cap U(\delta)$ to be homeomorphic to $Z(u(0))$. For the sake of convenience we write $A \approx B$ when A is homeomorphic to B.

Before stating our results, we notice the following facts. (1) By virtue of maximum principle there is no relatively compact (in $U(\delta)$) connected component of $U(\delta) \setminus Z(u(t))$ for sufficiently small $t > 0$. (2) If $m(0) = 0$, from Proposition 2.2 we have that $Z(u(t)) \cap U(\delta)$ is empty (resp. the union of closed regular curves) for sufficiently small $t > 0$ (resp. $-t > 0$). Namely, in the case of $m(0) = 0$ $Z(u(t)) \cap U(\delta)$ is not homeomorphic to $Z(u(0)) \cap U(\delta)$ for sufficiently small $|t| > 0$.

We use several times an analytic curve $\lambda(t)$ defined near $t = 0$ such that $\lambda(0) = (0,0)$, $\nu(u(t^\sigma), \lambda(t)) = m(0) \geq 2$ for sufficiently small $|t| > 0$ where $\sigma > 0$ is an odd integer. Such curves play an important role in the proof mentioned below and we denote them by λ in this section.

THEOREM 3.1. *When $Z(u(-t)) \cap U(\delta) \approx Z(u(0)) \cap U(\delta)$ for sufficiently small $t > 0$, then $Z(u(t)) \cap U(\delta) \approx Z(u(0)) \cap U(\delta)$.*

PROOF. By the assumption and Definition 2.1 we have that $m(0) = m(-0) \geq 1$ and $\nu(\gamma_j) = 1$ for each $j = 1,\ldots,m(0)$, which implies $m(0) = m(+0)$.

Suppose that $m(0) = 1$. If $S(u(t)) \cap U(\delta) \neq \emptyset$ for sufficiently small $t > 0$, by Proposition 2.2 there must be a relatively compact component of $U(\delta) \setminus Z(u(t))$, which is a contradiction against the claim (1) stated in the begining of this section. Hence $Z(u(t)) \cap U(\delta)$ is a one-dimensional connected manifold.

Suppose that $m(0) > 1$. By the assumption and Proposotion 2.2 there is a curve mentioned above $\lambda(t)$ such that $S(u(t^\sigma)) \cap U(\delta) = \{\lambda(t)\}$ for sufficiently small $-t > 0$. Hence we obtain the following for sufficiently small $t > 0$.

$$\begin{aligned} 2m(+0) &\geq N(U(\delta) \setminus Z(u(t))) \\ &= 1 + \sum\{\nu(u(t),P) - 1;\, P \in S(u(t)) \cap U(\delta)\} + m(+0) \geq 2m(+0). \end{aligned}$$

Here we have denoted by $N(A)$ the number of connected components of a subset A of $\boldsymbol{R}^2$. As a result we have that $S(u(t^\sigma)) \cap U(\delta) = \{\lambda(t)\}$ for sufficiently small $t > 0$. This fact and Proposition 2.2 thus complete the proof. □

Concerning the converse statement of Theorem 3.1, we have the following.

THEOREM 3.2. *Suppose that $Z(u(t))\cap U(\delta)\approx Z(u(0))\cap U(\delta)$ for sufficiently small $t>0$. If $m(0)=m$, then $Z(u(-t))\cap U(\delta)\approx Z(u(0))\cap U(\delta)$ and p is harmonic.*

PROOF. By assumption we have that $m(0)=m(+0)=m$ and that there is a curve $\lambda(t)$ such that $S(u(t))\cap U(\delta)=\{\lambda(t)\}$ for sufficiently small $t>0$. It follows from the assertion (2) in Lemma 2.4 that $\lim_{t\to 0}|\lambda(t)|/|t|^{\sigma/2}<\infty$ and that $\nu(H^*,\omega^*)=m$ where $\omega^*=\lim_{t\to+0}\lambda(t)/|t|^{\sigma/2}$. Using the following fact

$$
\begin{aligned}
2m(p,+0) &\geq N(\boldsymbol{R}^2\setminus Z(H^*))\\
&= 1+\sum\{\nu(H^*,P)-1; P\in S(H^*)\}+m(p,+0),
\end{aligned}
\tag{3.1}
$$

we have that $m(p,+0)=m$ and $S(H^*)=\{\omega^*\}$. Since $H^*(x,y)$ is even (resp. odd) in (x,y) in the case where m is even (resp. odd), ω^* is the origin. By virtue of (2.5) we conclude that $H^*=p$, namely p is harmonic. Using Weierstrauss preparation theorem, we have from $\nu(u(t^\sigma),\lambda(t))=m$ that in a small neigbourhood of the origin u can be written in the form.

$$
u(t^\sigma,\lambda(t)+(\xi,\eta))=v_o(t,\xi,\eta)\left\{\eta^m+\sum_{k=1}^m v_k(t,\xi)\xi^k\eta^{m-k}\right\}.
$$

Here $v_o(0,0,0)=1$ and v_j, $j=1,\dots,m$, are real analytic near the origin. Since $p(\xi,\eta)=\eta^m+\sum_{k=1}^m v_k(0,0)\xi^k\eta^{m-k}$ is harmonic, there exist m analytic functions $f_j(t,\xi)$ near the origin such that $f_j(0,0)$, $j=1,\dots,m$, are mutually distinct and

$$
u(t^\sigma,\lambda(t)+(\xi,\eta))=v_o(t,\xi,\eta)\prod_{k=1}^m\{\eta-f_j(t,\xi)\xi\},
$$

which completes the proof. □

Nextly we consider a necessary condition on the conjugate Hermite polynominals H^* and on the initial form p for $Z(u(t))\cap U(\delta)$ to be homeomorphic to $Z(u(0))\cap U(\delta)$.

THEOREM 3.3. *Suppose that $Z(u(t))\cap U(\delta)\approx Z(u(0))\cap U(\delta)$ for sufficiently small $t>0$. Then $Z(H^*)$ is either homeomorphic to $Z(u(0))\cap U(\delta)$ or a one-dimensional, connected, analytic manifold or empty.*

PROOF. Suppose that $m(0)=1$. By Proposition 2.3 we have $m(p,+0)=1$ and hence $Z(H^*)$ is a one-dimensional, connected, analytic manifold.

Nextly we suppose that $m(0)\geq 2$. Then there is a curve $\lambda(t)$ such that $S(u(t))\cap U(\delta)=\{\lambda(t)\}$ for sufficiently small $t>0$. When $\lim_{t\to 0}|\lambda(t)|/|t|^{\sigma/2}<\infty$, it follows from Lemma 2.4 that $\nu(H^*,\omega)\geq m(0)$ where $\omega=\lim_{t\to+0}\lambda(t)/|t|^{\sigma/2}$. Using (2.3) and (3.1), we obtain

$$
m(+0)=m(0)\leq\nu(H^*,\omega)\leq m(p,+0)\leq m(+0),
$$

and so $\nu(H^*,\omega)=m(p,+0)=m(0)$, $S(H^*)=\{\omega\}=\{(0,0)\}$. As a result Proposition 2.2 completes the proof in this case.

Finally we consider the case where $\lim_{t\to 0}|\lambda(t)|/|t|^{\sigma/2} = \infty$. It follows from Proposition 2.3 that for some $j \le m(p,0)$ and for sufficiently small $t > 0$

$$\lambda(t) \in \{\Gamma(\theta_j, \varepsilon) \cup \Gamma(\pi+\theta_j, \varepsilon)\} \cap \{(x,y); |x|^2 \ge M|t|^{\sigma}\}.$$

When d_j is even, $Z(u(t)) \cap U(\delta)$ is contained in either $\Gamma(\theta_j, \varepsilon) \cap \{(x,y); |x|^2 \ge M|t|\}$ or $\Gamma(\pi + \theta_j, \varepsilon) \cap \{(x,y); |x|^2 \ge M|t|\}$, namely $m(p,+0) = 0$. This means that $Z(H^*)$ is empty. When d_j is odd, we obtain $m(p,+0) = 1$ and hence we have the same conclusion as that of the case $m(0) = 1$. In this case $Z(H^*)$ is not homeomorphic to $Z(u(0)) \cap U(\delta)$. □

THEOREM 3.4. *Suppose that $Z(u(-t)) \cap U(\delta) \approx Z(u(0)) \cap U(\delta)$ for sufficiently small $t > 0$. If $m(0) \ge m-2 \ge 4$, then $p(x,y)$ is either harmonic or $(x^2+y^2)\,q(x,y)$ for some harmonic polynominal q.*

In order to prove this Theorem we study the singular points of biharmonic homogeneous polynominals. Consider the functions in the following form.

$$F(\theta) = C\sin(m\theta+\alpha) + \sin\{(m-2)\theta+\beta\}. \tag{3.2}$$

Here $C \ne 0$, α and β are real constants. Put

$$\mathscr{M} = \{\theta \in \boldsymbol{R}; \sin(m\theta+\alpha) = \sin\{(m-2)\theta+\beta\} = 0\}. \tag{3.3}$$

LEMMA 3.5. *Let $m \ge 3$. (1) F satisfies one of the following conditions.*
$(1-1)$ *F has exactly m simple zeros in $[0, \pi)$.*
$(1-2)$ *F has exactly both $(m-2)$ simple zeros and one double zero in $[0, \pi)$.*
$(1-3)$ *F has exactly $(m-3)$ simple zeros and one triple zero in $[0, \pi)$.*
$(1-4)$ *F has exactly $(m-2)$ simple zeros in $[0, \pi)$.*
(2) *When $\mathscr{M}$ is not empty, it consists of one point (resp. two points θ_k, $k = 1,2$, such that $|\theta_1-\theta_2| = \pi/2$) in the case where m is odd (resp. even).*
(3) *For each $\theta^* \in \mathscr{M}$ we have $F(\theta^*-\theta) = -F(\theta^*+\theta)$.*

PROOF. (1) Consider $F(t,\theta) = e^{-tm^2}\sin(m\theta+\alpha) \pm e^{-t(m-2)^2}\sin\{(m-2)\theta+\beta\}$ which satisfies $\partial F/\partial t = \partial^2 F/\partial\theta^2$. Then it follows from the non increasing property of number of zero points that there is a constant t_o such that for each $t < t_o$ (resp. $t > t_o$) $F(t)$ satisfies $(1-1)$ (resp. $(1-4)$) and $F(t_o)$ has the unique singular point in $[0, \pi)$ at which it vanishes at most of order 3. In the last case either $(1-2)$ or $(1-3)$ holds.

(2) Putting $\mathscr{N} = \{\lambda > 0; m\lambda, (m-2)\lambda \in \pi\boldsymbol{Z}\}$, $\lambda^* = \min\mathscr{N}$ and $m\lambda^* = a\pi$, $(m-2)\lambda^* = b\pi$, we have $2 \ge 2\lambda^*/\pi = (a-b)$ and so $a-b = 2$ (resp. 1) implies $\lambda^* = \pi$ (resp. $\pi/2$). It is easy to see that $\mathscr{N} = \lambda^*\boldsymbol{N}$ and that $\mathscr{M} = \theta^* \pm \{\mathscr{N} \cup \{0\}\}$ for each $\theta^* \in \mathscr{M}$.

(3) Put $g(\theta) = F(\theta^*+\theta) + F(\theta^*-\theta)$. Then we have from $\theta^* \in \mathscr{M}$ that $g^{(2n)}(0) = 0$ for each $n \ge 0$ and hence from the fact that $g^{(2n-1)}(0) = 0$ we obtain $g(\theta) \equiv 0$. □

New we prove Theorem 3.4.

PROOF. We divide the proof into several steps.

Step (1). By the assumption and by Theorem 3.1 there is a curve $\lambda(t)$ such that $S(u(t^{\sigma})) \cap U(\delta) = \{\lambda(t)\}$ for sufficiently small $|t|$. Since $2m(0)-2 \ge 2(m-2)-2 \ge m$, we obtain from Lemma 2.4 that $\lim_{t\to 0}|\lambda(t)|/|t|^{\sigma/2} < \infty$. As we have mentioned in the proof of

Theorem 3.3, we have that $Z(H^*) \approx Z(u(0)) \cap U(\delta)$, $S(H^*) = \{(0,0)\}$, $\lim_{t\to 0} |\lambda(t)|/|t|^{\sigma/2} = 0$ and $\nu(H^*, (0,0)) \geq m(0) \geq m-2$. It follows from the last fact and (2.5) that $m(0)$ is either m or $m-2$. When $m(0) = m$, we have from Theorems 3.1 and 3.2 that p is harmonic.

Suppose that $m(0) = m-2$ and that p is neither harmonic nor $(x^2+y^2)q(x,y)$ for every harmonic polynominal q. Then p is biharmonic, namely $\Delta^2 p = 0$, and so without loss of generality we assume that $F(\theta) := r^{-m} p(r\cos\theta, r\sin\theta)$ is given by (3.2). If $F(\theta)$ satisfies (1–1) in Lemma 3.5, we obtain by Remark 2.5 that $m(0) = m$.

Step (2). We claim that $S(H) = \{(0,0)\}$. Put $\phi_n = (n\pi - \beta)/(m-2)$ for $n \in \boldsymbol{Z}$. We notice from the assertion (3) in Lemma 3.5 that H is odd with respect to the line $\arg(x+iy) = \theta^*$, $\pi + \theta^*$ for each $\theta^* \in \mathscr{M}$ and that for $r > 0, \theta \neq \phi_n$ the solutions of $H(r\cos\theta, r\sin\theta) = 0$ (resp. $H^*(r\cos\theta, r\sin\theta) = 0$) are given by the following.

$$\frac{4(m-1)}{r^2} = \frac{F(\theta)}{\sin\{(m-2)\theta + \beta\}}, \left(\text{resp. } \frac{4(m-1)}{r^2} = \frac{-F(\theta)}{\sin\{(m-2)\theta + \beta\}}\right) \tag{3.4}$$

Suppose that $S(H) \neq \{(0,0)\}$. Then from (3.4) there is $\phi_n \in \mathscr{M}$ such that $F'(\phi_n) \neq 0$, namely $r(\phi_n)(\cos\phi_n, \sin\phi_n) \in S(H)$ for some $r(\phi_n) > 0$. Put

$$\phi^* = \begin{cases} \pi/(m-2) & \text{if } F(\theta) \neq 0 \text{ in } (\phi_n, \phi_{n+1}), \\ \min\{\phi > 0; F(\phi_n + \phi) = 0\} & \text{if otherwise.} \end{cases} \tag{3.5}$$

When $\phi^* = \pi/(m-2)$, from Lemma 3.5 we have $\phi_{n+1} \notin \mathscr{M}$ and so $F(\phi_{n+1}) \neq 0$. Hence $\boldsymbol{R}^2 \setminus Z(H)$ has a compact component contained in $\Gamma(\phi_n, \phi^*)$ and hence $U(\delta) \setminus Z(u(t))$ has also a compact component. When $\phi^* < \pi/(m-2)$, from the fact that $Z(H) \cap \Gamma(\phi_n, \phi^*)$ is the union of two unbounded regular curves intersecting at the point in $S(H)$ we arrive at one of the following two cases. One is that $Z(u(t)) \cap U(\delta)$ has at least two components. The other is that $u(t)$ has a singular point different from $\lambda(t)$. This is a contradiction we seek.

Step (3). We prove that $F(\theta)$ does not satisfy (1–3) in Lemma 3.5. Suppose that there is its triple zero ψ. Then it is easy to see that $\psi = \phi_n \in \mathscr{M}$ for some n. Using the notation (3.5), we have by Propostion 2.3, (3.4) and the claim in Step (2) that $Z(H^*) \cap \Gamma(\phi_n, \phi^*) = \{(x,y); \arg(x+iy) = \phi_n\}$ and $Z(H) \cap \Gamma(\phi_n, \phi^*)$ has three unbounded components. For large $M > 0$, small ε, $\rho > 0$ and for sufficiently small $-t > 0$ we have thus that $\{(x,y) \in Z(u(t)) \cap U(\delta) \cap \Gamma(\phi_n, \phi^* + \varepsilon); \rho|t| < x^2 < M|t|\}$ has three components σ_k, $k = 1,2,3$, which satisfy the following. One end point of each σ_k is on $\{(x,y) \in \Gamma(\phi_n, \varepsilon); x^2 = M|t|\}$. When $\phi^* = \pi/(m-2)$, the other end point of each σ_k is on $\{(x,y); x^2 = \rho|t|\}$. When $\phi^* < \pi/(m-2)$, the other end point of σ_2 (resp. σ_3) is on $\{(x,y) \in \Gamma(\phi_n + \phi^*, \varepsilon); x^2 = M|t|\}$ (resp. $\{(x,y) \in \Gamma(\phi_n - \phi^*, \varepsilon); x^2 = M|t|\}$). It follows from Proposition 2.3 that for each $j = 1, \ldots, m-2$, $\{(x,y) \in Z(u(t)) \cap U(\delta) \cap \Gamma(\theta_j, \varepsilon); x^2 > M|t|\}$ is a regular curve over (0, 1) and thus we have the following. When $\phi^* = \pi/(m-2)$, $Z(u(t)) \cap U(\delta)$ has either a compact component or a singular point. When $\phi^* < \pi/(m-2)$, $Z(u(t)) \cap U(\delta)$ has either has two component or a singular point. This is a contradiction.

Step (4). We claim that $\mathscr{M} = \emptyset$. If $\phi_n \in \mathscr{M}$, then $F(\phi_n) = F''(\phi_n) = 0$ and it follows from (3.4) and $S(H) = S(H^*) = \{(0,0)\}$ that $F'(\phi_n) = 0$. This is a contradiction against the claim of Step (3).

Step (5). For each n we claim that F has one and only one simple zero in (ϕ_n, ϕ_{n+1}). It follows from Lemma 3.5 and the claim (4) that zeros of F are at most of order two and that $F(\phi_n) \neq 0$ for every n. If $F(\theta) \neq 0$ in (ϕ_n, ϕ_{n+1}), then $\{Z(H^*) \cup Z(H)\} \cap \{(x,y); \phi_n < \arg(x+$

$iy) < \phi_{n+1}\}$ is bounded and so $\boldsymbol{R}^2 \setminus Z(H)$ has a compact component contained in $\{(x,y)\,;\, \phi_n < \arg(x+iy) < \phi_{n+1}\}$ and hence $U(\delta) \setminus Z(u(t))$ has also a compact component.

Suppose that there exist two simple zeros $\psi_1,\, \psi_2$ of F in $(\phi_n,\, \phi_{n+1})$ such that it has no simple zero in $(\psi_1,\, \psi_2)$. When $F(\theta) \neq 0$ in $(\psi_1,\, \psi_2)$, $\{Z(H) \cup Z(H^*)\} \cap \{(x,y)\,;\, \psi_1 < \arg(x+iy) < \psi_2\}$ is an unbounded regular curve over (0,1) and it does not contain the origin. We have thus that either $Z(u(t)) \cap U(\delta)$ or $Z(u(-t)) \cap U(\delta)$ has a components that does not contain $\lambda(t)$. When F has a double zero ψ_0 in $(\psi_1,\, \psi_2)$, then it follows from (3.4) that $Z(H) \cap \{(x,y)\,;\, \psi_1 < \arg(x+iy) < \psi_2\}$ has two unbounded components which do not contain the origin and from Lemma 3.5 we can find other $m-4$ simple zeros ψ_k, $k = 3, \ldots, m-2$, in $[0,\pi)$. By Proposition 2.3 we obtain for each $k = 1, \ldots, m-2$, that $Z(u(t)) \cap \{(x,y) \in U(\delta) \cap \Gamma(\psi_k, \varepsilon)\,;\, x^2 > M|t|\}$ is a regular curve over $(0,1)$ for small $\varepsilon > 0$ and for large $M > 0$. This means that

$$Z(u(t)) \cap \{U(\delta) \setminus U(\delta/2)\} \cap \Gamma(\psi_o, \varepsilon) = \emptyset. \tag{3.6}$$

Hence $Z(u(t)) \cap U(\delta)$ contains a regular cuves over $(0,1)$ such that it does not contain $\lambda(t)$ and one of its end points is in $\Gamma(\psi_1, \varepsilon)$ and the other in $\Gamma(\psi_2, \varepsilon)$. This is a contradiction.

Finally we consider the case where the double zero ψ_0 is the unique zero of F in $(\phi_n,\, \phi_{n+1})$. Then $Z(H) \cap \{(x,y)\,;\, \phi_n < \arg(x+iy) < \phi_{n+1}\}$ has two unbounded components σ_k, $k = 1,2$, such that their closures contain the origin and $\{\sigma_1 \cup \sigma_2\} \cap \{(x,y)\,;\, x^2 > M\} \subset \Gamma(\psi_0, \varepsilon)$. Hence it follows from (3.6) that $U(\delta) \setminus Z(u(t))$ has a compact component.

Final step. From the claim in Step (5) we have that for each n

$$C^2 \sin(m\phi_n + \alpha) \sin(m\phi_{n+1} + \alpha) = F(\phi_n) F(\phi_{n+1}) < 0$$

and hence we can find an integer k such that the equation $\sin(m\theta + \alpha) = 0$ has at least 3 zeros in $(\phi_k,\, \phi_{k+1})$, which implies that $1/(m-2) > 2/m$. This completes the proof of Theorem 3.4. $\square$

Finally we give an example $u(t,x,y)$ of polynominal solution of the heat equation (1.2) with initial value $u(0,x,y) = r^6 \cos 4\theta - r^{12} \sin 12\theta$ in terms of polar coordinates, that is, it is given by the following.

$$u(t,x,y) = \{r^6 + 20\, t\, r^4\} \cos 4\theta - r^{12} \sin 12\theta$$

By parameterizing its zeros by θ it can be shown that this solution satisfies the assumption in Theorem 3.4 and that $Z(u(t)) \approx Z(u(0))$ for each t.

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