Residues of Chern-Maslov classes

Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

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Abstract. We describe a localization theory for Maslov classes associated with two Lagrangian subbundles in a real symplectic vector bundle and give a definition of the residue of the Maslov classes. We also compute explicitly the residue of the first Maslov class in the case that the non-transversal set of the two Lagrangian subbundles have codimension 1.

1.   Introduction.

The Maslov classes is a fundamental invariant in the symplectic geometry. From the geometrical point of view, an important fact is that the Maslov classes is an invariant for a pair of Lagrangian subbundles in a symplectic vector bundle and is an obstruction to the transversality of the pair. I. Vaisman formulated the Maslov classes as the secondary characteristic classes determined from a pair of good connections and expressed it by differential forms. We call them the Chern-Maslov classes. He also described the vanishing of the Chern-Maslov classes for a pair of two transversal Lagrangian subbundles in the level of differential forms and defined the residue of the Chern-Maslov classes at the non-transversal loci of the pair by working with the theory of compactly supported differential forms. (see [**7**]).

The main aim of this paper is to formulate the general residue formula which relates the integration of the Chern-Maslov classes and the sum of its local residues. We also compute the precise residues of the Maslov classes for some important cases. For doing this, we give an analogous definition of the residue of the Chern-Maslov classes by applying the localization theory of characteristic classes developed mainly by D. Lehmann and T. Suwa. To be a little more precise, we describe the Maslov classes in the Čech-de Rham cohomology and describe the residues as the dual classes of the localized Maslov classes under the Alexander duality.

In section 2, we recall the Čech-de Rham cohomology theory and its integration theory. We give the description of the duality in the case of 1 cocycles precisely. In section 3, we recall the theory of Maslov classes in the de Rham cohomology. We define the Chern-Maslov classes of the Čech-de Rham cohomology in section 4. For doing this, we describe "the difference form" of two Chern-Maslov forms, which come from two distinct complex structures. In section 5, we describe the localization theory for the Chern-Maslov classes and give a formulation of the residue of the classes in terms of Čech-de Rham cohomology theory. In section 6, we compute precisely the residue of the Maslov class, which is the first Chern-Maslov class. We also give the residue formula for the case where the non-transversal loci of a pair of Lagrangian subbundles are possibly singular. We note that the possibility of having this kind of formula is mentioned in [**7**]. In the final section, we give an application of our formula for some basic and important

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examples.

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2. Preliminaries.

We list [**4**] as a general reference for this section. The integration theory on the Čech-de Rham cohomology is developed in [**2**], [**3**]. For the Chern-Weil theory, we refer to [**1**], [**4**], [**9**]. For computations of residues, we refer to [**5**].

2.1. Čech-de Rham cohomology.

Let M be a smooth oriented manifold of dimension m, $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of M, where I is a countable ordered set, and $A^q(U_{\alpha_0 \cdots \alpha_p})$ the space of all complex valued q-forms defined on $U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. We set $C^p(\mathscr{U}, A^q) = \prod_{(\alpha_0, \ldots, \alpha_p) \in I^{(p)}} A^q(U_{\alpha_0 \cdots \alpha_p})$ for $I^{(p)} = \{(\alpha_0, \ldots, \alpha_p) \in I^{p+1} | \alpha_0 < \cdots < \alpha_p\}$, then an element σ in $C^p(\mathscr{U}, A^q)$ can be expressed by a set $\{\sigma_{\alpha_0 \cdots \alpha_p}\}$, where $\sigma_{\alpha_0 \cdots \alpha_p} \in A^q(U_{\alpha_0 \cdots \alpha_p})$. We define the Čech-de Rham complex $(A^\bullet(\mathscr{U}), D)$ by

$$A^\bullet(\mathscr{U}) = \bigoplus_r A^r(\mathscr{U}) = \bigoplus_r (\oplus_{p+q=r} C^p(\mathscr{U}, A^q))$$

and the differential $D : A^p(\mathscr{U}) \to A^{p+1}(\mathscr{U})$ by

$$(D\sigma)_{\alpha_0 \cdots \alpha_p} = \sum_{i=0}^p (-1)^i \sigma_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p} + (-1)^p d\sigma_{\alpha_0 \cdots \alpha_p}.$$

We call the cohomology of this complex the Čech-de Rham cohomology and we denote the r-th cohomology by $H^r(A^\bullet(\mathscr{U}))$. The restriction map $A^r(M) \to C^0(\mathscr{U}, A^r) \subset A^r(\mathscr{U})$ induces an isomorphism $H^r_{DR}(M) \xrightarrow{\sim} H^r(A^\bullet(\mathscr{U}))$.

We define the cup product $\smile : A^r(\mathscr{U}) \times A^s(\mathscr{U}) \to A^{r+s}(\mathscr{U})$ by

$$(\sigma \smile \tau)_{\alpha_0 \cdots \alpha_p} = \sum_{i=0}^p (-1)^{(r-i)(p-i)} \sigma_{\alpha_0 \cdots \alpha_i} \wedge \tau_{\alpha_i \cdots \alpha_p},$$

for $\sigma \in A^r(\mathscr{U}), \tau \in A^s(\mathscr{U})$. This map induces $H^r(A^\bullet(\mathscr{U})) \times H^s(A^\bullet(\mathscr{U})) \to H^{r+s}(A^\bullet(\mathscr{U}))$.

If we assume that M is compact, by choosing a "system of honey comb cells" $\{R_\alpha\}$ adapted to $\mathscr{U}$, we can define the integration $\int : H^m(A^\bullet(\mathscr{U})) \to \boldsymbol{C}$ by

$$\int_M \sigma = \sum_{p=0}^m \left(\sum_{(\alpha_0, \cdots, \alpha_p) \in I^p} \int_{R_{\alpha_0 \cdots \alpha_p}} \sigma_{\alpha_0 \cdots \alpha_p} \right).$$

Let S be a compact subset in M admitting a regular neighborhood U. We use here one of the properties of a regular neighborhood that S is deformation retract of U. We take an open covering $\mathscr{U} = \{U_0, U_1\}$ of M as $U_0 = M \setminus S$, $U_1 =$ a regular neighborhood of S. If we define a subcomplex $A^\bullet(\mathscr{U}, U_0)$ of $A^\bullet(\mathscr{U})$ as the kernel of the projection map $A^\bullet(\mathscr{U}) \to A^\bullet(U_0)$, we have $H^r(A^\bullet(\mathscr{U}, U_0)) \simeq H^r(M, M \setminus S; \boldsymbol{C})$. The bilinear map $\int \circ \smile : A^k(\mathscr{U}, U_0) \times A^{m-k}(U_1) \to \boldsymbol{C}$ induces the Alexander duality,

$$H^k(M, M \setminus S; \boldsymbol{C}) \simeq H^k(A^\bullet(\mathscr{U}, U_0)) \xrightarrow{\sim} H^{m-k}(U_1; \boldsymbol{C})^* \simeq H_{m-k}(S; \boldsymbol{C}).$$

2.2. Duality for special case.

We compute the case that $k = 1$ and S has singularities in the above duality. This results is used in the section 6.

Let M be an m dimensional smooth manifold and S a compact subset in M. We suppose that the manifold M and S are stratified by the strata $\Sigma_0 = M \setminus S, \Sigma_1, \ldots, \Sigma_q$, which satisfy $\dim_{\boldsymbol{R}} \Sigma_i = m - i$ for $q \leq m$. i.e. $M = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_q$, $S = \Sigma_1 \cup \cdots \cup \Sigma_q$. We set $U_0 = M \setminus S, U_i =$"a sufficiently small tubular neighborhood of Σ_i" $(1 \leq i \leq q)$. We consider an open covering $\mathscr{U} = \{U_0, U_1, \cdots, U_q\}$ of M and $\mathscr{U}' = \{U_1, \cdots, U_q\}$ of $U = \cup_{i=1}^q U_i$, which may be assumed to be a regular neighborhood of S. An element μ of $A^1(\mathscr{U})$ is expressed by $\mu = (\mu_0, \mu_1, \cdots, \mu_q, \mu_{01}, \mu_{02}, \cdots, \mu_{q-1q}, 0, \cdots, 0)$.

The Alexander duality $H^1(A^\bullet(\mathscr{U}, U_0)) \xrightarrow{\sim} H_{m-1}(U)$ is induced from the bilinear map $I : A^1(\mathscr{U}, U_0) \times A^{m-1}(\mathscr{U}') \to \boldsymbol{C}$ given by

$$I(\mu, \tau) = \int_M \mu \smile \tau = \sum_{p=0}^q \left(\sum_{0 \leq \alpha_0 < \cdots < \alpha_p \leq q} \int_{R_{\alpha_0 \cdots \alpha_p}} (\mu \smile \tau)_{\alpha_0 \cdots \alpha_p} \right)$$

for $\mu = (0, \mu_1, \cdots, \mu_q, \mu_{01}, \mu_{02}, \cdots, \mu_{q-1q}, 0, \cdots, 0) \in A^1(\mathscr{U}, U_0)$ and $\tau = (\tau_1, \tau_2, \cdots, \tau_q, \tau_{12}, \tau_{13}, \cdots, \tau_{(q-1)q}, \tau_{123}, \tau_{124} \cdots, \tau_{(q-2)(q-1)q}, \cdots, \tau_{123 \cdots q}) \in A^{m-1}(\mathscr{U}')$, where $\{R_\alpha\}$ is a system of honey comb cells adapted to $\mathscr{U}$.

We assume that each submanifold $R_{\alpha_0 \cdots \alpha_l}$ for $0 \leq l \leq q$ intersect transversal with the each stratum Σ_i, i.e. $\dim_{\boldsymbol{R}}(\Sigma_i \cap R_{\alpha_0 \cdots \alpha_l}) = m - i - l$. Since the bilinear map is well defined on cohomology classes, we denote the induced bilinear map also by $I : H^1(A^\bullet(\mathscr{U}, U_0)) \times H^{m-1}(A^\bullet(\mathscr{U}')) \to \boldsymbol{C}$. For two cocycles μ and τ, we set $I_p = \sum_{0 \leq \alpha_0 < \cdots < \alpha_p \leq q} \int_{R_{\alpha_0 \cdots \alpha_p}} (\mu \smile \tau)_{\alpha_0 \cdots \alpha_p}$, then above integration is described by $I(\mu, \tau) = \sum_{p=0}^q I_p$. We compute this integration inductively. Let $\pi : U_1 \to \Sigma_1$ be a projection. Since U_i and Σ_i have the same homotopy type, we have $H^{m-1}(U_i) \simeq H^{m-1}(\Sigma_i)$. Thus we can write $\tau_1 = \pi^* \xi_1 + d\omega_1, \tau_2 = d\omega_2, \ldots, \tau_q = d\omega_q$, where $\xi_1 \in A^{m-1}(\Sigma_1)$ with $d\xi_1 = 0, \omega_i \in A^{m-2}(U_i)$. Since μ is 1-cocycle, we have $-d\mu_{\alpha_0 \alpha_1 \alpha_2} = 0 = \mu_{\alpha_1 \alpha_2} - \mu_{\alpha_0 \alpha_2} + \mu_{\alpha_0 \alpha_1}$. After doing simple computations, we have

$$I_0 + I_1 + I_2 = \int_{R_1} \mu_1 \wedge \pi^* \xi_1 + \int_{R_{01}} \mu_{01} \wedge \pi^* \xi_1 + \sum_{1 \leq \alpha_0 < \alpha_1 \leq q} \int_{R_{\alpha_0 \alpha_1}} \mu_{\alpha_0} \wedge \rho_{\alpha_0 \alpha_1}$$

$$- \sum_{0 \leq \alpha_0 < \alpha_1 < \alpha_2 \leq q} \int_{R_{\alpha_0 \alpha_1 \alpha_2}} \mu_{\alpha_0 \alpha_1} \wedge \rho_{\alpha_1 \alpha_2} + \sum_{1 \leq \alpha_0 < \alpha_1 < \alpha_2 \leq q} \int_{R_{\alpha_0 \alpha_1 \alpha_2}} \mu_{\alpha_0} \wedge \tau_{\alpha_0 \alpha_1 \alpha_2}$$

where $\rho_{\alpha_0 \alpha_1} = \omega_{\alpha_1} - \omega_{\alpha_0} - \tau_{\alpha_0 \alpha_1}$. In general, we set

$$\rho_{\alpha_0 \cdots \alpha_p} = (-1)^{p+1} \sum_{\nu=0}^p (-1)^\nu \omega_{\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_p} + (-1)^p \tau_{\alpha_0 \cdots \alpha_p}.$$

Then we have

$$d\rho_{\alpha_0 \alpha_1 \cdots \alpha_p} = \begin{cases} (-1)^p \sum_{\nu=1}^p (-1)^\nu \pi^* \xi_{1\alpha_1 \cdots \hat{\alpha_\nu} \cdots \alpha_p}, & \text{if } \alpha_0 = 1, \\ 0, & \text{if } \alpha_0 \neq 1. \end{cases}$$

Since the $(m-p)$ dimensional manifold $R_{1\alpha_1\cdots\alpha_p}$ retracts to the $(m-p-1)$ dimensional manifold $\Sigma_1 \cap R_{1\alpha_1\cdots\alpha_p}$ by the projection π, we have the following commutative diagram

$$
\begin{array}{ccc}
R_{1\alpha_1\cdots\alpha_p} & \xrightarrow{\;\tilde{\iota}\;} & U_1 \cap U_{\alpha_1\cdots\alpha_p} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\Sigma_1 \cap R_{1\alpha_1\cdots\alpha_p} & \xrightarrow{\;\iota\;} & \Sigma_1 \cap U_{\alpha_1\cdots\alpha_p},
\end{array}
$$

where $\iota, \tilde{\iota}$ are inclusions. Since $\iota^*\xi_{1\alpha_1\cdots\hat{\alpha}_v\cdots\alpha_p}$ is $(m-p)$-form on $R_{1\alpha_1\cdots\alpha_p}$, we have $d\tilde{\iota}^*\rho_{1\alpha_1\cdots\alpha_p} = 0$. Since the isomorphism $H^{m-p-1}(R_{\alpha_0\alpha_1\cdots\alpha_p}) \simeq H^{m-p-1}(\Sigma_{\alpha_0} \cap R_{\alpha_0\alpha_1\cdots\alpha_p})$ holds, we can write $\rho_{1\alpha_1\cdots\alpha_p} = \pi^*\xi_{1\alpha_1\cdots\alpha_p} + d\omega_{1\alpha_1\cdots\alpha_p}$ on $R_{1\alpha_1\cdots\alpha_p}$ and $\rho_{\alpha_0\alpha_1\cdots\alpha_p} = d\omega_{\alpha_0\alpha_1\cdots\alpha_p}$ on $R_{\alpha_0\alpha_1\cdots\alpha_p}$, where $\xi_{1\alpha_1\cdots\alpha_p} \in A^{m-p-1}(\Sigma_1 \cap R_{1\alpha_1\cdots\alpha_p})$ with $d\xi_{1\alpha_1\cdots\alpha_p} = 0$ and $\omega_{\alpha_0\alpha_1\cdots\alpha_p} \in H^{m-p-2}(R_{\alpha_0\alpha_1\cdots\alpha_p})$. Finally we obtain

$$
\begin{aligned}
\int_M \mu \smile \tau = & \int_{R_1} \mu_1 \wedge \pi^*\xi_1 + \int_{R_{01}} \mu_{01} \wedge \pi^*\xi_1 \\
& + \sum_{2\le\alpha_1\le q} \int_{R_{1\alpha_0}} \mu_1 \wedge \pi^*\xi_{1\alpha_0} - \sum_{2\le\alpha_2\le q} \int_{R_{01\alpha_2}} \mu_{01} \wedge \pi^*\xi_{1\alpha_2} \\
& + \cdots + \int_{R_{12\cdots q}} \mu_1 \wedge \pi^*\xi_{12\cdots q} + (-1)^{q+1} \int_{R_{12\cdots q}} \mu_{01} \wedge \pi^*\xi_{12\cdots q}.
\end{aligned}
$$

Now applying the projection formula (For more detail, see [**4**, pp. 57–60]), we have

$$
\begin{aligned}
\int_M \mu \smile \tau = & \int_{R_1\cap S} ((\pi_1)_*\mu_1 - (\partial\pi_1)_*\mu_{01})\,\xi_1 \\
& + \sum_{2\le\alpha_1\le q} \int_{R_{1\alpha_1}\cap S} ((\pi_{1\alpha_1})_*\mu_1 - (\partial\pi_{1\alpha_1})_*\mu_{01})\,\xi_{1\alpha_1} \\
& + \cdots + \int_{R_{12\cdots q}\cap S} ((\pi_{1\cdots q})_*\mu_1 - (\partial\pi_{1\cdots q})_*\mu_{01})\xi_{12\cdots q},
\end{aligned}
$$

where $\pi_I : R_I \to R_I \cap S$, $\partial\pi_I = \pi_{I0} : R_{I0} \to R_{I0} \cap S$ and $(\pi_I)_*$, $(\partial\pi_I)_*$ are integrations along the fiber. Here we note that functions $(\pi_1)_*\mu_1 - (\partial\pi_1)_*\mu_{01}, (\pi_{1\alpha_1})_*\mu_1 - (\partial\pi_{1\alpha_1})_*\mu_{01}, \cdots, (\pi_{1\cdots q})_*\mu_1 - (\partial\pi_{1\cdots q})_*\mu_{01}$ are locally constant.

In the Alexander duality, the class $[\mu] \in H^1(M, M\setminus S; \boldsymbol{C})$ corresponds to $[X] \in H_{m-1}(S; \boldsymbol{C})$, such that

$$
\begin{aligned}
\int_M \mu \smile \tau &= \sum_{p=0}^q \left(\sum_{0\le\alpha_0<\cdots<\alpha_p\le q} \int_{R_{\alpha_0\cdots\alpha_p}\cap X} \tau_{\alpha_0\cdots\alpha_p} \right) \\
&= \int_{R_1\cap X} \xi_1 - \sum_{2\le\alpha_1\le q} \int_{R_{1\alpha_1}\cap X} \xi_{1\alpha_1} + \cdots + (-1)^{q-1} \int_{R_{12\cdots q}\cap X} \xi_{12\cdots q},
\end{aligned}
$$

for any $\tau \in A^{m-1}(\mathscr{U}')$ with $D\tau = 0$. We assume that the third stratum is an empty set, i.e. $\Sigma_2 = \emptyset$. If we let $\{V_i\}$ be connected components of Σ_1, we have $H_{m-1}(S; \boldsymbol{C}) = \bigoplus_{i=1}^r H_{m-1}(V_i; \boldsymbol{C})$. We can write the cycle $[X] = \sum_i k_i[V_i]$ for $k_i \in \boldsymbol{C}$. Summarizing the above argument, we have $(\pi_1)_*\mu_1 - (\partial\pi_1)_*\mu_{01} = k_i$ on $V_i \cap R_1$. Similarly we have $(\pi_{1\alpha_1})_*\mu_1 - (\partial\pi_{1\alpha_1})_*\mu_{01} = -k_i$ on $V_i \cap R_{1\alpha_1}$, $\cdots$, and $(\pi_{1\cdots q})_*\mu_1 - (\partial\pi_{1\cdots q})_*\mu_{01} = (-1)^{q-1}k_i$ on $V_i \cap R_{1\cdots q}$. These results imply that for determining the coefficient k_i, we only have to compute the corresponding integration $(\pi_1)_*\mu_1 - (\partial\pi_1)_*\mu_{01} = k_i$ locally at the generic point $p \in V_i \cap R_1$.

3. Chern-Maslov classes in de Rham cohomology.

Let $(E \to M, \omega)$ be a symplectic vector bundle of real rank $2n$ with symplectic form ω and L', L'' two Lagrangian subbundles. By taking a positive and ω-compatible complex structure J on E, E becomes a complex vector bundle of complex rank n. We denote it by a pair (E, J). In the above, positive means $\omega(x, Jx) \geq 0$ and ω-compatible means $\omega(Jx, Jy) = \omega(x, y)$ for any $x, y \in E$. We note that such a complex structure always exists, so we always take a complex structure as positive and ω-compatible. Using the complex structure J, we define a Riemannian metric g on E by $g(x, y) = \omega(x, Jy)$, where $x, y \in E$. Now we let $E' = \{v - \sqrt{-1}Jv | v \in E\}$ be the $\sqrt{-1}$-eigenspace of J in $E \otimes \boldsymbol{C}$. Then we can easily prove that E' is isomorphic to (E, J) as a complex vector bundle. So in the following, we do all computations in E'.

DEFINITION 1. Let L be a Lagrangian subbundle of E. We say that a connection ∇ on E' is an L-orthogonal unitary connection, if ∇ is a natural extension of a metric connection on L.

An L-orthogonal unitary connection is locally expressed as follows. Let $\{e_1, \cdots, e_n\}$ be a g-orthonormal frame of L, then $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ is a g-orthonormal frame of E. If we set $\varepsilon_i = (e_i - \sqrt{-1}Je_i)/\sqrt{2}$ for $i = 1, \cdots, n$, the set $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is an unitary frame of E'. If we take a metric connection $\bar{\nabla}$ on L, the connection matrix θ of $\bar{\nabla}$ associated to the frame $\{e_1, \cdots, e_n\}$ is skew-symmetric. Let ∇ be a natural extension of $\bar{\nabla}$ to E' which is defined by $\nabla \varepsilon_i = (\bar{\nabla} e_i - \sqrt{-1}J \bar{\nabla} e_i)/\sqrt{2}$. The connection matrix associated to the unitary frame $\{\varepsilon_1, \cdots, \varepsilon_n\}$ becomes the same matrix θ.

Since the connection matrix is skew-symmetric, we see that the Chern forms of odd degree vanish, i.e. $c^{2h-1}(\nabla) = 0$. We let ∇' and ∇'' be L' and L''-orthogonal unitary connections respectively. Since we have $dc^{2h-1}(\nabla', \nabla'') = c^{2h-1}(\nabla'') - c^{2h-1}(\nabla') = 0$, the $(4h-3)$-form $c^{2h-1}(\nabla', \nabla'')$ define a cohomology class. We cite [**1**] or [**4**, pp. 69–70] for the definition of c^{2h-1} for two or three connections.

DEFINITION 2. The $(4h-3)$-form $c^{2h-1}(\nabla', \nabla'')$ is called the h-th Chern-Maslov form and the corresponding class

$$\mu^h(E, L', L'') := [c^{2h-1}(\nabla', \nabla'')] \in H_{DR}^{4h-3}(M, \boldsymbol{C})$$

is called the h-th Chern-Maslov class.

PROPOSITION 3([**7**]). *If the two Lagrangian subbundles are transversal, i.e. $E = L' \oplus L''$, then by taking the complex structure J so that $L'' = JL'$, the Chern-Maslov form vanish.*

PROOF. Let $\{e_1, \cdots, e_n\}$ be a g-orthonormal frame of L' and $\{f_1, \cdots, f_n\}$ a g-orthonormal frame of L''. If we choose a complex structure J as $e_i \mapsto f_i$ and $f_i \mapsto -e_i$, the associated unitary frames of E' are given by $\varepsilon_i'' = (f_i - \sqrt{-1}Jf_i)/\sqrt{2}$ and $\varepsilon_i' = (e_i - \sqrt{-1}Je_i)/\sqrt{2}$. We denote by θ' the connection matrix of L'-orthogonal unitary connection associated to the frames $\{\varepsilon_1', \cdots, \varepsilon_n'\}$ and θ'' the connection matrix of L''-orthogonal unitary connection associated to the frames $\{\varepsilon_1'', \cdots, \varepsilon_n''\}$, i.e. $\nabla' \varepsilon_i' = \sum_{j=1}^n \theta_{ji}' \varepsilon_j'$, $\nabla'' \varepsilon_i'' = \sum_{j=1}^n \theta_{ji}'' \varepsilon_j''$. Since the transition matrix from $\varepsilon' = (\varepsilon_1', \cdots, \varepsilon_n')$ to $\varepsilon'' = (\varepsilon_1'', \cdots, \varepsilon_n'')$ is $\sqrt{-1}I_n$, where I_n is an identity matrix, the second equation becomes $\nabla'' \varepsilon_i' = \sum_{j=1}^n \theta_{ji}'' \varepsilon_j'$. Let $\tilde{\nabla}$ be a linear combination $t\nabla' + (1-t)\nabla''$ of ∇' and ∇''. Since both θ' and θ'' are skew-symmetric for the same frame ε', the connection matrix $\tilde{\theta}$ of $\tilde{\nabla}$ for the frame ε' is also skew-symmetric. So we have $c^{2h-1}(\nabla', \nabla'') = \pi_* c^{2h-1}(\tilde{\nabla}) = 0$. $\square$

Let ∇' and $\bar{\nabla}'$ be two L'-orthogonal unitary connections for E', and ∇'' and $\bar{\nabla}''$ two L''-orthogonal unitary connections for E'. After simple computations, we have $c^{2h-1}(\nabla', \bar{\nabla}') = 0$ and $c^{2h-1}(\nabla'', \bar{\nabla}'') = 0$. So now we have the following transgression formula.

PROPOSITION 4. *If all connections are defined above, then we have*

$$c^{2h-1}(\bar{\nabla}', \bar{\nabla}'') - c^{2h-1}(\nabla', \nabla'') = d\left\{c^{2h-1}(\bar{\nabla}', \nabla'', \bar{\nabla}'') - c^{2h-1}(\nabla', \bar{\nabla}', \nabla'')\right\}.$$

This proposition implies that the Chern-Maslov classes does not depend on the choice of a pair of L', L''-orthogonal unitary connections of E'.

4. Chern-Maslov classes in the Čech-de Rham cohomology.

As discussed in section 3, we compute the Chern-Maslov classes in terms of a fixed complex structure. But the Chern-Maslov classes depends only on the pair of Lagrangian subbundles. So if we take another complex structure and compute the Chern-Maslov classes, then those determine the same cohomology classes, which means the difference of two forms are exact. In this section, we establish a transgression formula for two Chern-Maslov forms in terms of two distinct complex structures. For doing this we have to compare two pairs of L', L''-orthogonal unitary connections defined on two complex vector bundles. So we consider a deformation of two complex structures and by using the integration along the deformation parameter we give a direct expression of the "difference form" of two Chern-Maslov forms.

Here we give some remarks on basic facts about property of the complex structure.

PROPOSITION 5([**7**]).   *Let γ be a Riemannian metric on E, then there is a positive and ω-compatible complex structure J on E canonically associated with γ.*

PROOF.   We construct a complex structure as follows.

Since $\tilde{\omega} := \omega(x, \bullet) \in E^*$ for $x \in E$ is linear functionals, we have an unique bundle map $a : E \to E$ which satisfies $\omega(x, y) = \gamma(ax, y)$ by using the Riesz representation theorem. Since the bundle map a satisfies the condition $\gamma(a^2 x, y) = \gamma(x, a^2 y)$, a^2 is diagonalizable and all eigenvalues are real. Since a^2 also satisfies that $\gamma(a^2 x, x) = -\gamma(ax, ax) \leq 0$, all eigenvalues are negative. We denote the eigenvalues of a^2 by $-\lambda_1^2, \cdots, -\lambda_{2n}^2$. We define a bundle map $p : E \to E$ by the diagonal matrix $\mathrm{diag}(\lambda_1, \cdots, \lambda_{2n})$ for same eigenspace of a^2. Since the bundle map a^2 and p have the same eigenspace, the bundle map p is positive and self-adjoint. Now we define a bundle map $J : E \to E$ by $J = ap^{-1}$. Since it satisfies $ap^{-1} = p^{-1}a$, $a^2 = -p^2$, the map J satisfies $J^2 = -\mathrm{id}$, $\omega(Jx, Jy) = \omega(x, y)$ and $\omega(x, Jx) \geq 0$. $\square$

REMARK 6.   Let J_0 and J_1 be two complex structures. We define two metrics g_0 and g_1 by $g_0(x, y) = \omega(x, J_0 y)$ and $g_1(x, y) = \omega(x, J_1 y)$. We also define a linear combination of metrics by $\tilde{g} = (1-t)g_0 + tg_1$ for $t \in [0, 1]$. Then the complex structure $\tilde{J}$ which obtains from the above algorithm satisfies $\tilde{J}|_{t=0} = J_0, \tilde{J}|_{t=1} = J_1$.

REMARK 7.   Let J_0 and J_1 be two complex structures. We define two metrics g_0 and g_1 by the same way in the above remark. We choose a bump function $\chi : M \to \boldsymbol{R}$ which satisfies $\chi|_{M \setminus \bar{U}} \equiv 0$ and $\chi|_V \equiv 1$ for some open set $U \subset M$ and a relatively compact subset V in U. If we define a new metric $\tilde{g}$ by $\tilde{g} = (1-\chi)g_0 + \chi g_1$, the complex structure $\tilde{J}$ which obtains from the above algorithm satisfies $\tilde{J}|_{M \setminus \bar{U}} = J_0, \tilde{J}|_V = J_1$.

Now we compute a transgression formula for pairs of connections. Let J_0 and J_1 be two complex structures of E, ∇'_0 (∇''_0) a L' (L'')-orthogonal unitary connection on (E, J_0), and ∇'_1 (∇''_1) a L' (L'')-orthogonal unitary connection on (E, J_1) respectively. We make a homotopy deformation of complex structures J_0 and J_1 as follows.

First, we put Riemannian metrics as $g_0(x, y) = \omega(x, J_0 y), g_1(x, y) = \omega(x, J_1 y)$ and consider the linear combination of metrics $\tilde{\gamma} = (1-t)g_0 + tg_1$. Then it follows from the remark 6 that we have the corresponding complex structure $\tilde{J}$ of $E \times I$ satisfying $J|_{t=0} = J_0, J|_{t=1} = J_1$. The complex vector bundle $(E \times I \to M \times I, \omega, \tilde{J})$ is the trivial extension of $E \to M$ to $M \times I$, we also have $L' \times I$ and $L'' \times I$ as Lagrangian subbundles of $E \times I$. We take an $L' \times I, L'' \times I$-orthogonal unitary connections $D'(\tau), D''(\tau)$. $D'(0)$ is an L'-orthogonal unitary connection on (E, J_0) and $D'(1)$ is an L'-orthogonal unitary connection on (E, J_1), but there is no guarantee that $D'(0)$, $D'(1)$ coincides the original connection ∇'_0, ∇'_1 respectively. Thus we take linear combinations $\bar{\nabla}'_0 = (1-t)\nabla'_0 + tD'(0)$ and $\bar{\nabla}'_1 = (1-s)\nabla'_1 + sD'(1)$. We make all of notation for L'' in the same manner. By applying the projection formula to $c^{2h-1}(D'(\tau), D''(\tau))$, $c^{2h-1}(\bar{\nabla}'_0, \bar{\nabla}''_0)$ and $c^{2h-1}(\bar{\nabla}'_1, \bar{\nabla}''_1)$ (For more detail, see [**4**, pp. 57–60]), we have the following three equations;

$$c^{2h-1}(D'(1), D''(1)) - c^{2h-1}(D'(0), D''(0)) = \pi_* dc^{2h-1}(D'(\tau), D''(\tau)) + d\pi_* c^{2h-1}(D'(\tau), D''(\tau))$$
$$c^{2h-1}(D'(0), D''(0)) - c^{2h-1}(\nabla'_0, \nabla''_0) = \pi_* dc^{2h-1}(\bar{\nabla}'_0, \bar{\nabla}''_0) + d\pi_* c^{2h-1}(\bar{\nabla}'_0, \bar{\nabla}''_0)$$
$$c^{2h-1}(D'(1), D''(1)) - c^{2h-1}(\nabla'_1, \nabla''_1) = \pi_* dc^{2h-1}(\bar{\nabla}'_1, \bar{\nabla}''_1) + d\pi_* c^{2h-1}(\bar{\nabla}'_1, \bar{\nabla}''_1).$$

Summarizing the above arguments, we obtain the transgression formula.

PROPOSITION 8. *The transgression formula is given by,*

$$c^{2h-1}(\nabla'_1, \nabla''_1) - c^{2h-1}(\nabla'_0, \nabla''_0) = dm_{01},$$

in the above the transgression form m_{01} is described by

$$m_{01} = \pi_* c^{2h-1}(D'(\tau), D''(\tau)) + \pi_* c^{2h-1}(\bar{\nabla}'_0, \bar{\nabla}''_0) - \pi_* c^{2h-1}(\bar{\nabla}'_1, \bar{\nabla}''_1),$$

where $\pi : M \times I \to M$ and the first term of m_{01} is a fiber integration of the parameter τ, the second is that of t and the third is that of s.

We describe a $(4h-3)$-class in Čech-de Rham cohomology corresponding to the Maslov class $\mu^h(E, L', L'') = [c^{2h-1}(\nabla', \nabla'')] \in H_{DR}^{4h-3}(M; \mathbf{C})$ in de Rham cohomology.

Let $\mathscr{U} = \{U_0, U_1\}$ be an open covering of M and J_0, J_1 a pair of complex structure defined on $E|_{U_0}, E|_{U_1}$ respectively. We denote by ν the restriction of the global Chern-Maslov form to U_0, U_1, more precisely $\nu = (c^{2h-1}(\nabla', \nabla''), c^{2h-1}(\nabla', \nabla''), 0) \in C^0(\mathscr{U}, A^{4h-3}) \subset A^{4h-3}(\mathscr{U})$.

In the following we will prove that the transgression form in the proposition can be regarded as the difference form of Chern-Maslov class in Čech-de Rham cohomology.

It is easily seen that the local defined complex structure J_1 is obtained by the restriction of a global complex structure. For any pair of L', L''-orthogonal unitary connections ∇'_0, ∇''_0 on $E|_{U_0}$ and L', L''-orthogonal unitary connections ∇'_1, ∇''_1 on $E|_{U_1}$, the Chern-Maslov form can be expressed by

$$\sigma := (c^{2h-1}(\nabla'_0, \nabla''_0), c^{2h-1}(\nabla'_1, \nabla''_1), m_{01}) \in A^{4h-3}(\mathscr{U}),$$

which should satisfy $dm_{01} = c^{2h-1}(\nabla'_1, \nabla''_1) - c^{2h-1}(\nabla'_0, \nabla''_0)$ so that they define the same cohomology class $[\sigma] = [\nu] \in H^{4h-3}(\mathscr{U})$. The second condition is the coboundary condition, that is, there is a $(4h-4)$-form $\tau = (\tau_0, \tau_1, \tau_{01}) \in A^{4h-4}(\mathscr{U})$ which satisfy $D\tau = \nu - \sigma$, i.e.

$$(d\tau_0, d\tau_1, -d\tau_{01} + \tau_1 - \tau_0)$$
$$= (c^{2h-1}(\nabla', \nabla'') - c^{2h-1}(\nabla'_0, \nabla''_0), c^{2h-1}(\nabla', \nabla'') - c^{2h-1}(\nabla'_1, \nabla''_1), -m_{01}).$$

For the second condition, we let $\bar{\nabla}', \bar{\nabla}''$ be a pair of L', L''-orthogonal unitary connection on E. We set a pair of global connections ∇', ∇'' as $\nabla' = (1-\chi)\bar{\nabla}' + \chi\nabla'_1$, $\nabla'' = (1-\chi)\bar{\nabla}'' + \chi\nabla''_1$, where $\chi: M \to \mathbf{R}$ is a bump function which satisfies $\chi \equiv 1$ on U_1. These are again L', L''-orthogonal unitary connections on E.

On U_0, from the transgression formula, we have

$$\tau_0 = \pi_* c^{2h-1}(D'_0(\tau), D''_0(\tau)) + \pi_* c^{2h-1}(\tilde{\nabla}'_0, \tilde{\nabla}''_0) - \pi_* c^{2h-1}(\tilde{\nabla}', \tilde{\nabla}''),$$

where $D'_0(\tau), D''_0(\tau)$ is a pair of $L' \times I, L'' \times I$-orthogonal unitary connection, and other connections are defined as follows;

$$\tilde{\nabla}'_0 = (1-t)\nabla'_0 + tD'(0), \qquad\qquad \tilde{\nabla}''_0 = (1-t)\nabla''_0 + tD''(0),$$
$$\tilde{\nabla}' = (1-s)\nabla' + sD'(1), \qquad\qquad \tilde{\nabla}'' = (1-s)\nabla'' + sD''(1).$$

On U_1, since $\nabla'|_{U_1} = \nabla'_1, \nabla''|_{U_1} = \nabla''_1$, we have

$$c^{2h-1}(\nabla', \nabla'') - c^{2h-1}(\nabla'_1, \nabla''_1) = c^{2h-1}(\nabla'_1, \nabla''_1) - c^{2h-1}(\nabla'_1, \nabla''_1) = 0.$$

On U_{01}, we set $\tau_{01} = 0$, then $m_{01} = \tau_0 - \tau_1$. From the above we can set $\tau_1 = 0$. Since the connections and the complex structure on U_{01} are the same one on U_1, we can choose the $L' \times I, L'' \times I$-orthogonal unitary connection on $(E \times I, \tilde{J})|_{U_0 \times I}$ as

$$D'_0(\tau) = (1 - \chi \circ pr)\bar{D}'(\tau) + \chi \circ pr D'(\tau),$$
$$D''_0(\tau) = (1 - \chi \circ pr)\bar{D}''(\tau) + \chi \circ pr D''(\tau),$$

where $pr: M \times I \to M$ is a natural projection, $\circ$ is a composite operator of mappings and $\bar{D}'(\tau), \bar{D}''(\tau)$ a pair of $L' \times I, L'' \times I$-orthogonal unitary connections on $(E \times I, \tilde{J})|_{U_0 \times I}$. Thus we restrict the connections to $U_{01} \times I$, we have $D'_0(\tau)|_{U_{01} \times I} = D'(\tau), D''_0(\tau)|_{U_{01} \times I} = D''(\tau), \tilde{\nabla}'_0 = \bar{\nabla}'_0, \tilde{\nabla}''_0 = \bar{\nabla}''_0, \tilde{\nabla}'|_{U_{01} \times I} = \bar{\nabla}'_1$ and $\tilde{\nabla}''|_{U_{01} \times I} = \bar{\nabla}''_1$.

From the above argument, we conclude the following.

THEOREM 9. *The Chern-Maslov class $[c^{2h-1}(\nabla', \nabla'')]$ in $H^{4h-3}_{DR}(M; \mathbf{C})$ corresponds to the following Čech-de Rham cohomology class,*

$$[(c^{2h-1}(\nabla'_0, \nabla''_0), c^{2h-1}(\nabla'_1, \nabla''_1), m_{01})] \in H^{4h-3}(A^\bullet(\mathscr{U})).$$

The difference form m_{01} can be written by

$$m_{01} = \pi_* c^{2h-1}(D'(\tau), D''(\tau)) + \pi_* c^{2h-1}(\bar{\nabla}'_0, \bar{\nabla}''_0) - \pi_* c^{2h-1}(\bar{\nabla}'_1, \bar{\nabla}''_1).$$

Let $\tilde{g}$ be a Riemannian metric on $E \times I$ which satisfies $\tilde{g}(x, y) = \omega(x, \tilde{J}y)$ for $x, y \in E \times I$.

We also let $\{e'_1, \ldots, e'_n\}$ and $\{e''_1, \ldots, e''_n\}$ a $\tilde{g}$-orthonormal frame of $L' \times I$ and $L'' \times I$ respectively. We set $\varepsilon' = \{\varepsilon'_1, \ldots, \varepsilon'_n\}$ and $\varepsilon'' = \{\varepsilon''_1, \ldots, \varepsilon''_n\}$, where $\varepsilon'_i = (e'_i - \sqrt{-1}\tilde{J}e'_i)/\sqrt{2}$ and $\varepsilon''_i = (e''_i - \sqrt{-1}\tilde{J}e''_i)/\sqrt{2}$ for $1 \le i \le n$. Let $A(\tau)$ be a change of frame which satisfies $\varepsilon'' = \varepsilon' A(\tau)$. Then the first Chern-Maslov class in the Čech-de Rham cohomology is expressed as follows:

COROLLARY 10. *The Chern-Maslov class $[c^1(\nabla', \nabla'')]$ in $H^1_{DR}(M; \boldsymbol{C})$ corresponds to the following Čech-de Rham cohomology class,*

$$[(c^1(\nabla'_0, \nabla''_0), c^1(\nabla'_1, \nabla''_1), m_{01})] \in H^1(A^\bullet(\mathscr{U})).$$

The difference form m_{01} can be written by

$$m_{01} = \frac{1}{2\pi\sqrt{-1}} \pi_* \frac{d(\det A(\tau))}{\det A(\tau)}.$$

PROOF. We set $h = 1$ in theorem 9. Then m_{01} is expressed by

$$m_{01} = \pi_* c^1(D'(\tau), D''(\tau)) + \pi_* c^1(\bar{V}'_0, \bar{V}''_0) - \pi_* c^1(\bar{V}'_1, \bar{V}''_1).$$

The first term is a fiber integration of the parameter τ, the second is that of t and the third is that of s. Since $c^1(\bar{V}'_0, \bar{V}''_0)$ does not have terms involving dt, it will vanish. By a similar reason, we also have $\pi_* c^1(\bar{V}'_1, \bar{V}''_1) = 0$.

We set $\tilde{D} = (1-t)D'(\tau) + tD''(\tau)$, where $0 \le t \le 1$. The matrix representation $\tilde{\theta}$ of $\tilde{D}$ with respect to the frame ε'' is

$$\tilde{\theta} = (1-t)\theta'_{\varepsilon''} + t\theta''_{\varepsilon''},$$

where $\theta'_{\varepsilon''}, \theta''_{\varepsilon''}$ is the connection matrix of $D'(\tau), D''(\tau)$ with respect to the frame ε'' respectively. We also denote the connection matrix of $D'(\tau)$ with respect to the frame ε' by $\theta'_{\varepsilon'}$. Note that $\theta'_{\varepsilon'}$ and $\theta''_{\varepsilon''}$ are skew-symmetric. Then the difference form is given by;

$$
\begin{aligned}
c^1(D'(\tau), D''(\tau)) &= \pi_* c^1(\tilde{D}) \\
&= -\frac{1}{2\pi\sqrt{-1}} \pi_* \mathrm{Trace}\left(d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta}\right) \\
&= -\frac{1}{2\pi\sqrt{-1}} \pi_* \mathrm{Trace}\left(dt \wedge (-\theta'_{\varepsilon''} + \theta''_{\varepsilon''})\right) \\
&= -\frac{1}{2\pi\sqrt{-1}} \mathrm{Trace}\left(-\theta'_{\varepsilon''} + \theta''_{\varepsilon''}\right) \\
&= \frac{1}{2\pi\sqrt{-1}} \mathrm{Trace}\left(A(\tau)^{-1} dA(\tau) + A(\tau)^{-1}\theta'_{\varepsilon'}A(\tau)\right) \\
&= \frac{1}{2\pi\sqrt{-1}} \frac{d\det A(\tau)}{\det A(\tau)}.
\end{aligned}
$$
$\square$

5. Localization of Chern-Maslov classes and Residue formula.

Let M be a smooth oriented manifold of dimension m, and (E, ω) a symplectic vector bundle of real rank $2n$ with a symplectic form ω. Let S be a compact set which has a regular neighborhood in M, and $U_0 = M \setminus S$, $U_1 = $"regular neighborhood of S" in M. Then the set $\mathscr{U} = \{U_0, U_1\}$ is an open covering of M.

We assume that the two Lagrangian subbundles are transversal on U_0. Then we can localize the Chern-Maslov classes by taking the complex structure J_0 on U_0 as $L'' = J_0 L'$, that is;

$$[(0, c^{2h-1}(\nabla_1', \nabla_1''), m_{01})] \in H^{4h-3}(A^\bullet(\mathscr{U}, U_0)).$$

We decompose the set S to connected components $\{S_\alpha\}$. By Alexander duality, the Chern-Maslov class corresponds to the dual homology class in $\bigoplus_\alpha H_{m-(4h-3)}(S_\alpha; \boldsymbol{C})$. So we give the following definition.

DEFINITION 11.   We call this homology class in $H_{m-(4h-3)}(S_\alpha; \boldsymbol{C})$ the residue of Chern-Maslov class on S_α, and denote it by $\mathrm{Res}_{\mu^h}(S_\alpha)$.

If we further assume that M is compact, we have the residue formula.

THEOREM 12.   *If M is compact, then we have*

$$\sum_\alpha (\iota_\alpha)_* \mathrm{Res}_{\mu^h}(S_\alpha) = \mu^h \frown [M] \qquad in \quad H_{m-(4h-3)}(M; \boldsymbol{C}),$$

where $\iota_\alpha : S_\alpha \to M$ is the inclusion.

PROOF.   This follows from the commutativity of the diagram;

$$[(0, c^{2h-1}(\nabla_1', \nabla_1''), m_{01})] \in H^{4h-3}(A^\bullet(\mathscr{U}, U_0)) \xrightarrow{\ \simeq\ } H_{DR}^{4h-3}(M; \boldsymbol{C}) \ni \mu^h$$

$$\left\downarrow {\simeq} \qquad\qquad\qquad\qquad \right\downarrow {\simeq}$$

$$\sum_\alpha \mathrm{Res}_{\mu^h}(S_\alpha) \in \bigoplus_\alpha H_{m-(4h-3)}(S_\alpha; \boldsymbol{C}) \xrightarrow{\ \simeq\ } H_{m-(4h-3)}(M; \boldsymbol{C}) \ni \sum_\alpha (\iota_\alpha)_* \mathrm{Res}_{\mu^h}(S_\alpha).$$

$$\square$$

6.   The residue formula for the Maslov classes.

6.1.   Smooth case.

We let $E \to M$ be a symplectic vector bundle of real rank $2n$ over a smooth manifold of real dimension m with a symplectic form ω, and L' and L'' two Lagrangian subbundles.

We assume that the non-transversal set S of the two Lagrangian subbundles is smooth and of codimension 1 in M. We further assume that the dimension of the sum of two Lagrangian subbundles for each fiber on the non-transversal set is $2n - 1$, i.e.,

$$\dim_{\boldsymbol{R}}(L'|_x + L''|_x) = 2n - 1 \qquad x \in S.$$

We consider a neighborhood U of $x \in S$. From assumption we can choose a symplectic frame $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for E on $U \setminus S$, where $\{e_1, \ldots, e_n\}$ is a frame for L' and $\{f_1, \ldots, f_n\}$ for L''. Since it holds $e_n \neq 0$, there is an element $v \in E$ which satisfies $\omega(e_n, v) = 1$. The set $\{e_1, \ldots, e_n, f_1, \ldots, f_{n-1}, v\}$ becomes a symplectic frame of E on U. Then the two symplectic frames satisfy the relation $f_n = v + 1/\varphi e_n$ for some function φ, which satisfies $\varphi = 0$ on S. We set $U_0 = U \setminus S$ and $U_1 = $ a neighborhood of S. On U_0, since the two Lagrangian subbundles are transversal, we can set the complex structure J_0 by $e_i \mapsto f_i$, $f_i \mapsto -e_i$ for $1 \leq i \leq n$. On U_1, we set the complex structure J_1 by $e_i \mapsto f_i$, $f_i \mapsto -e_i$ for $1 \leq i \leq n-1$ and $e_n \mapsto v$, $v \mapsto -e_n$.

We let γ_0 be a Riemannian metric on U_0 and γ_1 on U_1. We recall that the Euclidean scalar

product γ is defined by $\gamma(x,y) = \omega(x, Jy)$ for $x, y \in E$, where J is a complex structure. Thus the metric γ_0 which associates to the frame $\{e_1, \ldots, e_n, f_1, \ldots, f_{n-1}, v\}$ is expressed by the matrix

$$
\gamma_0 = \left(
\begin{array}{cc|cc}
I_{n-1} & & 0 & \\
& 1 & & -1/\varphi \\
\hline
0 & & I_{n-1} & \\
& -1/\varphi & & 1/\varphi^2 + 1
\end{array}
\right),
$$

where I_{n-1} denotes the identity matrix of rank $n-1$. The description of above matrix means that there are nonzero elements on $(n,n), (n,2n), (2n,n), (2n,2n)$, and diagonals. The metric γ_1 which associates to the frame $\{e_1, \ldots, e_n, f_1, \ldots, f_{n-1}, v\}$ is expressed by $\gamma_1 = I_{2n}$, where I_{2n} denotes an identity matrix of rank $2n$. We define a linear combination of two metrics by $\tilde{\gamma} = (1-t)\gamma_0 + t\gamma_1$. Since the endomorphism $a : E \to E$ was defined by $\tilde{\gamma}(av, w) = \omega(v, w)$ for $v, w \in E$, we have the diagonal matrix

$$
a^2 = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{n-1 \text{ times}}, -1/\beta, \underbrace{-1, \ldots, -1}_{n-1 \text{ times}}, -1/\beta)
$$

where $\beta = 1 + t(1-t)/\varphi^2$. If we set $p = \mathrm{diag}(1, \ldots, 1, 1/\sqrt{\beta}, 1, \ldots, 1, 1/\sqrt{\beta})$, this matrix is a positive definite. We also define an endomorphism $\tilde{J} : E \to E$ by $p^{-1}a$. Then we have

$$
\tilde{J} = \left(
\begin{array}{cc|cc}
0 & & \text{-}I_{n-1} & \\
& (1-t)/(\sqrt{\beta}\varphi) & & -\left(1 + (1-t)/\varphi^2\right)/\sqrt{\beta} \\
\hline
I_{n-1} & & 0 & \\
& 1/\sqrt{\beta} & & -(1-t)/(\sqrt{\beta}\varphi)
\end{array}
\right),
$$

this is a positive and ω-compatible complex structure. We let $\tilde{g}$ a Riemannian metric associate to the complex structure $\tilde{J}$, we have

$$
\tilde{g} = {}^t p \tilde{\gamma} = \left(
\begin{array}{cc|cc}
I_{n-1} & & 0 & \\
& 1/\sqrt{\beta} & & -(1-t)/(\sqrt{\beta}\varphi) \\
\hline
0 & & I_{n-1} & \\
& -(1-t)/(\sqrt{\beta}\varphi) & & \left(1 + (1-t)/\varphi^2\right)/\sqrt{\beta}
\end{array}
\right).
$$

Using this metric, the frame $\{e_1, \ldots, e_n\}$ for $L' \times I$ normalizes to $\{e_1, \ldots, e_{n-1}, \sqrt{D}e_n\}$ and the frame $\{f_1, \ldots, f_n\}$ for $L'' \times I$ to $\{f_1, \ldots, f_{n-1}, \sqrt{D/(1 + t/\varphi^2)}f_n\}$, where $D = \sqrt{\beta}$. We define unitary frames $\varepsilon' = \{\varepsilon_1', \ldots, \varepsilon_n'\}$ and $\varepsilon'' = \{\varepsilon_1'', \ldots, \varepsilon_n''\}$ for $E \times I$ by

$$
\varepsilon_i' = \left(e_i - \sqrt{-1}\tilde{J}e_i \right)/\sqrt{2}, \quad \varepsilon_i'' = \left(f_i - \sqrt{-1}\tilde{J}f_i \right)/\sqrt{2}
$$

for $1 \le i \le n-1$ and

$$
\varepsilon_n' = \sqrt{\frac{D}{2}} \left(e_n - \sqrt{-1}\tilde{J}e_n \right), \quad \varepsilon_n'' = \sqrt{\frac{D}{2(1 + t/\varphi^2)}} \left(f_n - \sqrt{-1}\tilde{J}f_n \right).
$$

If we define an unitary matrix A by $\varepsilon'' = \varepsilon' A$, we have

$$A = \text{diag}\left(\underbrace{\sqrt{-1},\cdots,\sqrt{-1}}_{n-1 \text{ times}}, \sqrt{\frac{t/\varphi + \sqrt{-1}D}{t/\varphi - \sqrt{-1}D}}\right).$$

If we let A_{nn} be a (n,n)-element of the matrix A, from the corollary 10, the difference form m_{01} of the first Chern-Maslov class $\mu^1 = (0, m_1, m_{01})$ is given by,

$$
\begin{aligned}
m_{01} &= \frac{1}{2\pi\sqrt{-1}}\pi_*\frac{d(\det A)}{\det A} \\
&= \frac{1}{2\pi\sqrt{-1}}\int_0^1 \frac{\partial A_{nn}}{\partial t}A_{nn}^{-1}dt \\
&= -\frac{\sigma}{2\pi}\left(\text{Tan}^{-1}\sqrt{\frac{k+1/2}{k-1/2}} - \text{Tan}^{-1}\left(\frac{k-1/2}{k+1/2}\right)\sqrt{\frac{k-1/2}{k+1/2}}\right)
\end{aligned}
$$

where $k = \sqrt{\varphi^2 + 1/4}$, $\sigma = \varphi/|\varphi|$.

Next for computing the form m_1 of the class $\mu^1 = (0, m_1, m_{01})$, we normalize the frames $\{e_1, \ldots, e_n\}$ for $L' \times I$ and $\{f_1, \ldots, f_n\}$ for $L'' \times I$ by the metric γ_1. Since it satisfy $\gamma_1(f_n, f_n) = 1 + 1/\varphi^2$, we have the frames $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_{n-1}, \sqrt{1+1/\varphi^2}f_n\}$. We also have the pair of unitary frames $\varepsilon' = \{\varepsilon_1', \ldots, \varepsilon_n'\}$ and $\varepsilon'' = \{\varepsilon_1'', \ldots, \varepsilon_n''\}$ for (E, J_1), where $\varepsilon_i' = \left(e_i - \sqrt{-1}J_1e_i\right)/\sqrt{2}$, $\varepsilon_i'' = \left(f_i - \sqrt{-1}J_1f_i\right)/\sqrt{2}$ for $1 \le i \le n-1$ and $\varepsilon_n' = \left(e_n - \sqrt{-1}J_1e_n\right)/\sqrt{2}$, $\varepsilon_n'' = \left(f_n - \sqrt{-1}J_1f_n\right)/\sqrt{2(1+1/\varphi^2)}$. If we define an unitary matrix B by $\varepsilon'' = \varepsilon'B$, we have

$$B = \text{diag}\left(\underbrace{\sqrt{-1},\cdots,\sqrt{-1}}_{n-1 \text{ times}}, \sqrt{\frac{1/\varphi + \sqrt{-1}}{1/\varphi - \sqrt{-1}}}\right).$$

Then we have the form

$$m_1 = \frac{1}{2\pi\sqrt{-1}}\frac{d(\det B)}{\det B} = \frac{1}{2\pi}\frac{d\varphi}{\varphi^2 + 1}.$$

Now we denote by $\{S_\alpha\}$ the connected components of the non-transversal locus S. The residue of the Chern-Maslov class $[\mu^1]$ is corresponding to the cycle $\Sigma_\alpha k_\alpha[S_\alpha]$ under the Alexander duality $H^1(M, M \setminus S) \simeq \bigoplus_\alpha H_{m-1}(S_\alpha)$. Since k_α's are complex numbers, it is enough to compute at a point $p \in S_\alpha$. Since the dimension of the normal direction at the point p is 1, this is the same situation with the later case (Figure 1), and we have the same form $(0, m_1, m_{01})$, only swapping f for $1/\varphi$ in the case of S^1. We consider a small neighborhood U_p of p in M. Then the open set $U_p \setminus (U_p \cap S_\alpha)$ have two connected components. Using the result of the case $y = x^n$ in the later section, we have $k_\alpha = 0$ if the sign of φ is the same on the both connected components and $k_\alpha = \pm 1/2$ if it is not, where the sign depends on the orientation of S_α.

6.2. Singular case.

All notations are the same as in the smooth case. Here we suppose that the manifold M and the non-transversal set S are stratified by the strata $\Sigma_0 = M \setminus S, \Sigma_1, \ldots, \Sigma_q$, which satisfy $\dim_{\mathbf{R}}\Sigma_i = m - i$ for $q \le m$. i.e. $M = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_q$, $S = \Sigma_1 \cup \cdots \cup \Sigma_q$. We assume that the third stratum is an empty set, i.e. $\Sigma_2 = \emptyset$. If we let $\{V_i\}$ be a connected component of Σ_1. we have $H^1(M, M \setminus S; \mathbf{C}) \simeq H_{m-1}(S; \mathbf{C}) = \bigoplus_i H_{m-1}(V_i; \mathbf{C})$. We can write the cycle $[X] = \sum_i k_i[V_i]$ for
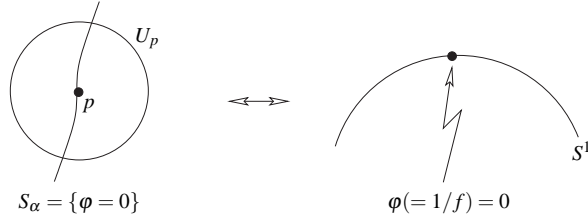
Figure 1. Comparing with the case of S^1.

$k_i \in \boldsymbol{C}$. It follows from the preliminaries 2.2 that for determining the constant coefficients $\{k_i\}$, we only have to compute it at any generic point $p \in V_i$, which means that we have the following theorem.

THEOREM 13. *On the isomorphism*

$$H^1(M, M \setminus S; \boldsymbol{C}) \simeq \bigoplus_{i=1}^{r} H_{m-1}(V_i; \boldsymbol{C}),$$

the first Chern-Maslov class $[\mu^1]$ in $H^1(M, M \setminus S; \boldsymbol{C})$ is corresponding to the cycle $\sum_i k_i [V_i]$ in $\bigoplus_{i=1}^{r} H_{m-1}(V_i; \boldsymbol{C})$, where k_i's are equal to 0 or $\pm 1/2$, the sign depends on the orientation of V_i.

7. Application.

7.1. Case of $T\boldsymbol{R}^2|_{S^1} \to S^1$.

Let $(T\boldsymbol{R}^2|_{S^1}, \omega)$ be a symplectic vector bundle over $S^1 \subset \boldsymbol{R}^2$ with symplectic form $\omega = dx \wedge dy$, where (x, y) is a global coordinate of $\boldsymbol{R}^2$. We set $L' = \partial/\partial x|_{S^1}$ and $L'' = TS^1$. Then both of them are two Lagrangian subbundles on S^1. It is obvious that those Lagrangian subbundles are not transversal only at north and south poles of S^1. We denote the north pole by p, south pole by q, and let $S = \{p, q\}$. Using the angular parameter θ of S^1, these points are expressed by $\theta = \pi/2$ and $3\pi/2$. If we set $U_0 = S^1 \setminus S$ and $U_1 =$ a neighborhood of S, $\mathscr{U} = \{U_0, U_1\}$ is an open covering of S^1. We choose a frame $e_0 = \partial/\partial x, e_1 = \partial/\partial y$ for $T\boldsymbol{R}^2|_{S^1}$. Since the rank of symplectic vector bundle is 2, we only compute the first Chern-Maslov class $[(0, m_1, m_{01})]$ and its residue.

On U_0, we choose a frame e_0 for L', and $1/f e_0 + e_1$ for L'', where $f = -\cot \theta$. We define a complex structure J_0 by $e_0 \mapsto 1/f e_0 + e_1$ and $1/f e_0 + e_1 \mapsto -r_0$. The Riemannian metric g_0 with respect to the frame $\langle e_0, e_1 \rangle$ becomes

$$\begin{pmatrix} 1 & -1/f \\ -1/f & 1 + 1/f^2 \end{pmatrix}.$$

On U_1, we choose a frame e_0 for L', and $(e_0 + f e_1)/\sqrt{1 + f^2}$ for L''. We define a complex structure J_1 by $e_0 \mapsto e_1$ and $e_1 \mapsto -e_0$. The Riemannian metric g_1 with respect to the frame $\langle e_0, e_1 \rangle$ is the identity matrix. The linear combination $\tilde{\gamma} = (1-t)g_0 + t g_1$ is expressed by

$$\begin{pmatrix} 1 & -(1-t)/f \\ -(1-t)/f & 1 + (1-t)/f^2 \end{pmatrix}.$$

Since the symplectic form ω for the frame $\langle e_0, e_1 \rangle$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the matrix representation of the complex structure $\tilde{J}$ of $T\boldsymbol{R}^2|_{S^1} \times I$ is given by

$$\frac{1}{\sqrt{\xi}} \begin{pmatrix} f\xi & -1-\xi \\ 1 & -f\xi \end{pmatrix},$$

where $\xi = 1 + t(1-t)/f^2$, and the Riemannian metric $\tilde{g}$ of $T\boldsymbol{R}^2|_{S^1} \times I$ is also given by

$$\frac{1}{\sqrt{\xi}} \begin{pmatrix} 1 & -f\xi \\ -f\xi & 1+\xi \end{pmatrix}.$$

By normalizing e_0 and $e_0 + fe_1$ by $\tilde{g}$, we have the orthonormal frame $\bar{e}_0 = \sqrt{D}e_0$ and $\bar{e}_1 = \sqrt{D/(f^2+t)}(e_0 + fe_1)$ for $L' \times I$ and $L'' \times I$ respectively, where $D = \sqrt{\xi}$. We set unitary frames by $\varepsilon' = (\bar{e}_0 - \sqrt{-1}\tilde{J}\bar{e}_0)/\sqrt{2}$ and $\varepsilon'' = (\bar{e}_1 - \sqrt{-1}\tilde{J}\bar{e}_1)/\sqrt{2}$. A transition function A which satisfies $\varepsilon'' = \varepsilon'A$ is given by

$$A = \frac{t + \sqrt{-1}\sigma\sqrt{f^2+t(1-t)}}{\sqrt{f^2+t}},$$

where $\sigma = f/|f|$ is the sign of f. This transition function satisfies the unitary condition $A\bar{A} = 1$, thus $A \in U(1)$. From the corollary 10, we have

$$m_{01} = \frac{1}{2\pi\sqrt{-1}}\pi_* \frac{dA}{A} = -\frac{\sigma}{4\pi} \int_0^1 \frac{2f^2+t}{(f^2+t)\sqrt{f^2+t(1-t)}} dt$$

$$= -\frac{\sigma}{2\pi}\left(\mathrm{Tan}^{-1}\sqrt{\frac{k+1/2}{k-1/2}} - \mathrm{Tan}^{-1}\left(\frac{k-1/2}{k+1/2}\sqrt{\frac{k-1/2}{k+1/2}}\right)\right),$$

where $k = \sqrt{f^2+1/4}$. This in fact has the form,

$$m_{01} = \begin{cases} \theta/2\pi, & \text{if} \quad 0 < \theta < \pi/2, \\ -1/2 + \theta/2\pi, & \text{if} \quad \pi/2 < \theta < 3\pi/2, \\ -1 + \theta/2\pi, & \text{if} \quad 3\pi/2 < \theta < 2\pi. \end{cases}$$

Next we compute m_1. Since e_0 and $\tilde{e}_1 := (e_0 + fe_1)/\sqrt{1+f^2}$ is normalized frame for L' and L'' respectively, the unitary frames of symplectic vector bundle is given by $\varepsilon_1' = (e_0 - \sqrt{-1}J_1e_0)/\sqrt{2}$ and $\varepsilon_1'' = (\tilde{e}_1 - \sqrt{-1}J_1\tilde{e}_1)/\sqrt{2}$. A transition function B, which is satisfying $\varepsilon'' = \varepsilon'B$, is given by $(1 + \sqrt{-1}f)/\sqrt{1+f^2}$. The 1-form m_1 is expressed as follows,

$$m_1 = \frac{1}{2\pi}\frac{df}{f^2+1} = \frac{1}{2\pi}d\theta.$$

The residue of the first Chern-Maslov class has the following form,

$$\mathrm{Res}_{\mu^1}(p) + \mathrm{Res}_{\mu^1}(q) = k_p[p] + k_q[q],$$

where k_p, k_q are constant. If we take a system of honey comb cells $\{R_0, R_1\}$ adapted to $\mathscr{U}$, then R_1 consists of two connected components. One of them is a closed neighborhood of p and the other is an also closed neighborhood of q. We denote by R_p and R_q those connected components of R_1. We compute the residue $\mathrm{Res}_{\mu^1}(p)$ by integrating the Chern-Maslov class on R_p and we have

$$\mathrm{Res}_{\mu^1}(p) = \int_{R_p} m_1 - \int_{\partial R_p} m_{01} = \frac{1}{2}.$$

We also have $\mathrm{Res}_{\mu^1}(q) = 1/2$. Finally we see that the residue of the first Chern-Maslov class is given by:

$$\mathrm{Res}_{\mu^1}(p) + \mathrm{Res}_{\mu^1}(q) = \frac{1}{2}[p] + \frac{1}{2}[q].$$

The Maslov index is defined by integrating the Maslov class on a closed curve [**7**, p. 140]. In the same way we can define the Chern-Maslov index by integrating the Chern-Maslov class on a closed curve. In this case the Chern-Maslov index can be obtained by adding the local index $1/2$, thus we have the index 1. Since the Maslov class m and the first Chern-Maslov class μ^1 have the relation $m = 2\mu^1$ ([**7**, p. 140]), it is proved the Chern-Maslov index 1 is meaningful.

7.2. Case of $y = x^n$.

Let us recall that the residue of the localized Chern-Maslov classes consists only of the local data of connections around the non-transversal set. So even if the ambient space is not compact, we can define the "local" index for the compact non-transversal set. In fact in the previous case, we computed the residue in the neighborhood of p and q. Following examples are the simplest case where M is not compact.

Let $M = \{(x,y) \in \mathbf{R}^2 \mid y = x^n\}$ be a curve in $\mathbf{R}^2$ and $(\mathrm{T}\mathbf{R}^2|_M, \omega)$ a symplectic vector bundle over M with symplectic form $\omega = dx \wedge dy$. We also let $L' = \partial/\partial x|_M$ and $L'' = \mathrm{T}M$. These line bundles are Lagrangian subbundles of $\mathrm{T}\mathbf{R}^2|_M$. If we parametrize the curve M by $(x,y) = (t,t^n)$, the line bundle L'' is spanned fiberwise by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} + nt^{n-1}\frac{\partial}{\partial y}.$$

The case $n = $ even. If we set $\theta = \mathrm{Tan}^{-1}(-1/(nt^{n-1}))$, it satisfy $nt^{n-1} = -1/\tan\theta$. Thus we have the same result as the case of S^1, i.e.

$$\mathrm{Res}_{\mu^1}(0) = 1/2[0].$$

The case $n = $ odd. We set $f(t) = nt^{n-1}$. We compute the first Chern-Maslov form μ^1 similarly as the case of S^1, then we have

$$\mu^1 = (0, m_1, m_{01}) = \left(0, \frac{1}{2\pi}d\mathrm{Tan}^{-1}f, -\frac{1}{2\pi}\left(\mathrm{Tan}^{-1}\xi^{-1} - \mathrm{Tan}^{-1}\xi^3\right)\right),$$

where $\xi = \sqrt{(k-1/2)/(k+1/2)}$ and $k = \sqrt{f^2 + 1/4}$. If we set $\varphi = \mathrm{Tan}^{-1}\xi^{-1} - \mathrm{Tan}^{-1}\xi^3$, we have $\tan\varphi = 1/|f| = 1/f$. Since $|\varphi| \leq \pi/2$, we have $\varphi = \mathrm{Tan}^{-1}(1/f) = \pi/2 - \mathrm{Tan}^{-1}f$.

We use the convention $\mathrm{Tan}^{-1}(\infty) = \pi/2, \mathrm{Tan}^{-1}(-\infty) = -\pi/2$ in above computations. If we set $U_0 = M \setminus \{0\}$ and $U_1 =$ a neighborhood of 0, $\mathscr{U} = \{U_0, U_1\}$ is an open covering of M. We let $\{R_0, R_1\}$ be a system of honey comb cells adapted to $\mathscr{U}$. We have the residue of the first Chern-Maslov class μ^1 by

$$\int_M \mu^1 = \int_{R_1} m_1 - \int_{\partial R_1} m_{01} = \frac{1}{2\pi} \int_{\partial R_1} \mathrm{Tan}^{-1} f + \frac{1}{2\pi} \int_{\partial R_1} \left(\frac{\pi}{2} - \mathrm{Tan}^{-1} f \right) = 0.$$

We conclude the residue of the first Chern-Maslov class is given by

$$\mathrm{Res}_{\mu^1}(0) = \begin{cases} 1/2[0], & \text{if } n \text{ is even,} \\ 0[0], & \text{if } n \text{ is odd.} \end{cases}$$

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