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Hyperspaces with the Hausdorff Metric and Uniform ANR's

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Abstract. For a metric space $X = (X, d)$, let $\mathrm{Cld}_H(X)$ be the space of all nonempty closed sets in X with the topology induced by the Hausdorff extended metric:

$$d_H(A,B) = \max\left\{\sup_{x\in B} d(x,A),\ \sup_{x\in A} d(x,B)\right\} \in [0,\infty].$$

On each component of $\mathrm{Cld}_H(X)$, d_H is a metric (i.e., $d_H(A,B) < \infty$). In this paper, we give a condition on X such that each component of $\mathrm{Cld}_H(X)$ is a uniform AR (in the sense of E. Michael). For a totally bounded metric space X, in order that $\mathrm{Cld}_H(X)$ is a uniform ANR, a necessary and sufficient condition is also given. Moreover, we discuss the subspace $\mathrm{Dis}_H(X)$ of $\mathrm{Cld}_H(X)$ consisting of all discrete sets in X and give a condition on X such that each component of $\mathrm{Dis}_H(X)$ is a uniform AR and $\mathrm{Dis}_H(X)$ is homotopy dense in $\mathrm{Cld}_H(X)$.

1. Introduction.

Let $X = (X, d)$ be a metric space. The set of all non-empty closed sets in X is denoted by $\mathrm{Cld}(X)$. On the subset $\mathrm{Bdd}(X) \subset \mathrm{Cld}(X)$ consisting of bounded closed sets in X, we can define the *Hausdorff metric* d_H as follows:

$$d_H(A,B) = \max\left\{\sup_{x\in B} d(x,A),\ \sup_{x\in A} d(x,B)\right\},$$

where $d(x,A) = \inf_{a\in A} d(x,a)$. We denote the metric space $(\mathrm{Bdd}(X), d_H)$ by $\mathrm{Bdd}_H(X)$. On the whole set $\mathrm{Cld}(X)$, we allow $d_H(A,B) = \infty$, but d_H induces the topology of $\mathrm{Cld}(X)$ like a metric does. The space $\mathrm{Cld}(X)$ with this topology is denoted by $\mathrm{Cld}_H(X)$. When X is bounded, $\mathrm{Cld}_H(X) = \mathrm{Bdd}_H(X)$. Even though X is unbounded, $\mathrm{Cld}_H(X)$ is metrizable. Indeed, let $\bar{d}$ be the metric on X defined by $\bar{d}(x,y) = \min\{1, d(x,y)\}$. Then, $\bar{d}_H$ is an admissible metric of $\mathrm{Cld}_H(X)$. It should be noted that each component of $\mathrm{Cld}_H(X)$ is contained in $\mathrm{Bdd}(X)$ or in the complement $\mathrm{Cld}(X) \setminus \mathrm{Bdd}(X)$. Thus, $\mathrm{Bdd}_H(X)$ is a union of components of $\mathrm{Cld}_H(X)$. On each component of $\mathrm{Cld}_H(X)$, d_H is a metric even if it is contained in $\mathrm{Cld}(X) \setminus \mathrm{Bdd}(X)$. Then, we regard every component of $\mathrm{Cld}_H(X)$ as a metric space with d_H.

When X is compact, it is well-known that $\mathrm{Cld}_H(X)$ $(= \mathrm{Bdd}_H(X))$ is an ANR (an AR)[1] if and only if X is locally connected (connected and locally connected) [**12**]. However, in case X is non-compact, this does not hold. In this paper, we construct a metric AR X such that $\mathrm{Cld}_H(X)$

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[1] An ANR (an AR) means an absolute neighborhood retract (an absolute retract) for metrizable spaces.

is not an ANR. Recently, Costantini and Kubiś [**4**] showed that $\mathrm{Bdd}_H(X)$ is an AR if X is *almost convex*, that is, for each $x,y \in X$ and for each $s,t > 0$ such that $d(x,y) < s+t$, there exists $z \in X$ with $d(x,z) < s$ and $d(y,z) < t$. Under a more mild condition on X, we show that each component of $\mathrm{Cld}_H(X)$ is a uniform AR in the sense of Michael [**9**]. For a totally bounded metric space X, in order that $\mathrm{Cld}_H(X)$ $(= \mathrm{Bdd}_H(X))$ is a uniform ANR, a necessary and sufficient condition is given. Moreover, we discuss the subspace $\mathrm{Dis}_H(X) \subset \mathrm{Cld}_H(X)$ consisting of all discrete sets in X and give a condition on X such that each component of $\mathrm{Dis}_H(X)$ is a uniform AR and $\mathrm{Dis}_H(X)$ is homotopy dense in $\mathrm{Cld}_H(X)$, where Y is *homotopy dense* in Z if there exists a homotopy $h : Z \times \boldsymbol{I} \to Z$ such that $h_0 = \mathrm{id}_Z$ and $h_t(Z) \subset Y$ for $t > 0$.

2. Main results and examples.

First of all, we shall construct a metric AR X such that $\mathrm{Cld}_H(X)$ (nor $\mathrm{Dis}_H(X)$) is not an ANR.

EXAMPLE 1. The following subspace X of Euclidean plain $\boldsymbol{R}^2$ is an AR:

$$X = [1,\infty) \times \{0\} \cup \bigcup_{n \in \boldsymbol{N}} \{n, n+2^{-n} \mid n \in \boldsymbol{N}\} \times \boldsymbol{I}.$$

Then, $\mathrm{Cld}_H(X)$ is not locally path-connected at $A = \boldsymbol{N} \times \{1\}$. Otherwise, we can find $0 < \gamma < 1$ such that if $d_H(A,B) < \gamma$ then A and B are connected by a path with diam < 1. Choose $k \in \boldsymbol{N}$ so that $2^{-k} < \gamma$, and let $B = A \cup \{(k+2^{-k}, 1)\} \in \mathrm{Cld}_H(X)$. Since $d_H(A,B) < \gamma$, there is a path $f : \boldsymbol{I} \to \mathrm{Cld}_H(X)$ such that $f(0) = A$, $f(1) = B$ and $\operatorname{diam} f(\boldsymbol{I}) < 1$. Let

$$U = \{t \in \boldsymbol{I} \mid f(t) \cap \{k+2^{-k}\} \times (0,1] \neq \varnothing\} \text{ and } V = \{t \in \boldsymbol{I} \mid f(t) \cap \{k+2^{-k}\} \times \boldsymbol{I} = \varnothing\}.$$

Then, $0 \in U$, $1 \in V$ and $U \cap V = \varnothing$. Since $f(t) \cap [0,\infty) \times \{0\} = \varnothing$ for every $t \in \boldsymbol{I}$, it is easy to see that U and V are open in $\boldsymbol{I}$ and $U \cup V = \boldsymbol{I}$. This contradicts to the connectedness of $\boldsymbol{I}$. Similarly, $\mathrm{Dis}_H(X)$ is not locally path-connected at A.

In order to state the main results, we need some notations and definitions. Let $X = (X,d)$ be a metric space. For $A \subset X$ and $\gamma > 0$, we denote

$$\mathrm{N}(A,\gamma) = \{x \in X \mid d(x,A) < \gamma\} \text{ and } \overline{\mathrm{N}}(A,\gamma) = \{x \in X \mid d(x,A) \leqslant \gamma\}.$$

When $A = \{a\}$, we write $\mathrm{N}(\{a\},\gamma) = \mathrm{B}(a,\gamma)$ and $\overline{\mathrm{N}}(\{a\},\gamma) = \overline{\mathrm{B}}(a,\gamma)$.

A metric space X is called a *uniform ANR* if for an arbitrary metric space $Z = (Z,d)$ containing X isometrically as a closed subset, there exist a uniform neighborhood U of X in Z (i.e., $U = \mathrm{N}(X,\gamma)$ for some $\gamma > 0$) and a retraction $r : U \to X$ which is uniformly continuous at X, that is, for each $\varepsilon > 0$, there is some $\delta > 0$ such that if $x \in X$, $z \in U$ and $d(x,z) < \delta$ then $d(x,r(z)) < \varepsilon$. When $U = Z$ in the above, X is called a *uniform AR*. A uniform ANR is a uniform AR if it is homotopically trivial, that is, all the homotopy groups are trivial. In [**10**], it is shown that a metric space X is a uniform ANR if and only if every metric space Z containing X isometrically as a dense subset is a uniform ANR and X is homotopy dense in Z.

A collection $\mathscr{A}$ of subsets of X is said to be *uniformly discrete* if there exists some $\delta > 0$ such that the δ-neighborhood $\mathrm{B}(x,\delta)$ of each $x \in X$ meets at most one member of $\mathscr{A}$, that is,

$$\inf\{\mathrm{dist}(A,A') \mid A \neq A' \in \mathscr{A}\} > 0,$$

where $\mathrm{dist}(A,A') = \inf\{d(x,x') \mid x \in A,\ x' \in A'\}$.

For $\eta > 0$, an η-*chain* in a metric space $X = (X,d)$ is a finite sequence $(x_i)_{i=0}^k$ of points in X such that $d(x_i,x_{i-1}) < \eta$ for each $i = 1,\dots,k$, where k is called the *length* of $(x_i)_{i=0}^k$ and $\mathrm{diam}\{x_i \mid i = 0,1,\dots,k\}$ is the *diameter* of $(x_i)_{i=0}^k$. When $x_0 = x$ and $x_k = y$, we call $(x_i)_{i=0}^k$ an η-*chain from x to y* and we say that x and y are *connected* by $(x_i)_{i=0}^k$. It is said that X is *C-connected* (or *connected in the sense of Cantor*) if each pair of points in X are connected by an η-chain in X for any $\eta > 0$.

Now, we say that X is *uniformly locally C^*-connected* if for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

ul$C^*(\varepsilon)$ For each $\eta > 0$, there is some $k \in \boldsymbol{N}$ such that each pair of δ-close points of X are connected by an η-chain with length $\leqslant k$ and diam $< \varepsilon$.

This concept is invariant under uniform homeomorphisms, that is, if a metric space is uniformly homeomorphic to a uniformly locally C^*-connected metric space then it is also uniformly locally C^*-connected. It is easy to see that every almost convex metric space is uniformly locally C^*-connected. One should note that the unit circle $\boldsymbol{S}^1 \subset \boldsymbol{R}^2$ with the Euclidean metric is uniformly locally C^*-connected but not almost convex. The following is our first main result which generalizes Costantini and Kubiś' result [**4**] mentioned in Introduction:

THEOREM A. *For every uniformly locally C^*-connected metric space X, the collection of all components of* $\mathrm{Cld}_H(X)$ *is uniformly discrete and each component of* $\mathrm{Cld}_H(X)$ *is a uniform AR, hence the spaces* $\mathrm{Cld}_H(X)$ *and* $\mathrm{Bdd}_H(X)$ *are uniform ANR's.*

Here, it should be remarked that a metric space is a uniform ANR if and only if the collection of all components is uniformly discrete and each component is a uniform ANR.

The uniformly local C^*-connectedness is stronger than the uniformly local version of C-connectedness. It is said that X is *uniformly locally C-connected* if for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

ul$C(\varepsilon)$ For each $\eta > 0$, each pair of δ-close points of X are connected by an η-chain in X with diam $< \varepsilon$.

This concept is also invariant under uniform homeomorphisms. As seen in the following example, the uniformly local C-connectedness does not imply the uniformly local C^*-connectedness.

EXAMPLE 2. For each $n \in \boldsymbol{N}$, let e_n be the unit vector in $\boldsymbol{R}^{\boldsymbol{N}}$ defined by $e_n(i) = 0$ if $i \neq n$ and $e_n(n) = 1$. We define a metric space $X = (X,d)$ as follows:

$$X = \bigcup_{n\in\boldsymbol{N}} \boldsymbol{R}e_n \subset \boldsymbol{R}^{\boldsymbol{N}},\quad d(x,y) = \sum_{n\in\boldsymbol{N}} \min\{2^{-n}, |x(n)-y(n)|\}.$$

Then, X is uniformly locally C-connected but it is not uniformly locally C^*-connected.

To see the uniformly local C-connectedness, for each $\varepsilon > 0$, let $x,y \in X$ with $d(x,y) < \varepsilon$. When $x,y \in \boldsymbol{R}e_n$ for some $n \in \boldsymbol{N}$, for each $\eta > 0$, choose $k \in \boldsymbol{N}$ so that $\eta(k-1) \leqslant |x(n)-y(n)| < \eta k$, and define $x_i = x + \eta i e_n$ for $i = 0,1,\dots,k-1$ and $x_k = y$. Then, $(x_i)_{i=0}^k$ is an η-chain from x to y in X with diam $< \varepsilon$. When $x \in \boldsymbol{R}e_n$ and $y \in \boldsymbol{R}e_m$ for $n \neq m \in \boldsymbol{N}$, since $d(x,y) = d(x,0) + d(y,0)$, for each $k \in \boldsymbol{N}$, we can obtain an η-chain from x to y with diam $< \varepsilon$ by joining two η-chains from x to 0 and from 0 to y.

To see that X is not uniformly locally C^*-connected, for each $\delta > 0$, choose $n \in \boldsymbol{N}$ so that $2^{-n} < \delta$. Then, for every $m \in \boldsymbol{N}$, $d(0, me_n) \leqslant 2^{-n} < \delta$. If $(x_i)_{i=0}^k$ is a 2^{-n}-chain from 0 to me_n then $k \geqslant m2^n$. This means that X is not uniformly locally C^*-connected.

The following is our second main result:

THEOREM B. *For a totally bounded metric space X, the space $\mathrm{Cld}_H(X)$ $(= \mathrm{Bdd}_H(X))$ is a uniform ANR if and only if X is uniformly locally C-connected, whence each component of $\mathrm{Cld}_H(X)$ is a uniform AR.*

For the space $\mathrm{Dis}_H(X)$, we have the following result:[2]

THEOREM C. *Let X be a metric space with the following two properties*:

(C_1) *Every bounded closed set in X is compact*;
(C_2) *For each $\varepsilon > 0$, there exist $k, \delta > 0$ such that any pair of δ-close points in X are connected by a k-Lipschitz path $f : \boldsymbol{I} \to X$ with* $\mathrm{diam} f(\boldsymbol{I}) < \varepsilon$.

Then, the collection of all components of $\mathrm{Dis}_H(X)$ *is uniformly discrete and each component of* $\mathrm{Dis}_H(X)$ *is a uniform AR, hence* $\mathrm{Dis}_H(X)$ *is an ANR. In this case,* $\mathrm{Dis}_H(X)$ *is homotopy dense in* $\mathrm{Cld}_H(X)$.

In the above, each component of $\mathrm{Cld}_H(X)$ is a uniform AR but this follows from Theorem A. In fact, it will be seen that the condition (C_2) above implies the uniformly local C^*-connectedness.

3. Lawson semilattices which are uniform ANR's.

A *topological semilattice* is a topological space S equipped with a continuous operation $\vee : S \times S \to S$ which is idempotent, commutative and associative (i.e., $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$). A topological semilattice S is called a *Lawson semilattice* if S admits an open basis consisting of subsemilattices [**8**]. It is known that a metrizable Lawson semilattice is k-aspherical for each $k > 0$ ([**4**, Proposition 2.3]).

In [**1**], it is shown that a metrizable Lawson semilattice is an ANR (resp. an AR) if and only if it is locally path-connected (resp. connected and locally path-connected). Here, we consider the condition that a metric Lawson semilattice is a uniform ANR. By $\boldsymbol{B}^{n+1}$ and $\boldsymbol{S}^n$, we denote the unit $(n+1)$-ball and the n-sphere, respectively. We will use the following result in [**5**]:

PROPOSITION 3.1. *For each $n \geqslant 1$, there exists a map $r : \boldsymbol{B}^{n+1} \to \mathfrak{F}_3(\boldsymbol{S}^n)$ such that $r(x) = x$ for all $x \in \boldsymbol{S}^n$, where*

$$\mathfrak{F}_3(\boldsymbol{S}^n) = \{A \subset \boldsymbol{S}^n \mid \mathrm{card}\, A \leqslant 3\} \subset \mathrm{Fin}(\boldsymbol{S}^n).$$

A metric space X is *uniformly locally k-connected* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $n \leqslant k$, every map $f : \boldsymbol{S}^n \to X$ with $\mathrm{diam} f(\boldsymbol{S}^n) < \delta$ extends to a map $\tilde{f} : \boldsymbol{B}^{n+1} \to X$ with $\mathrm{diam} \tilde{f}(\boldsymbol{B}^{n+1}) < \varepsilon$. Evidently, X is uniformly locally 0-connected if and only if

[2] This is established in [**7**].

X is uniformly locally path-connected. For each simplex σ, we denote by $\sigma^{(k)}$ the union of all k-faces of σ. The following is proved in [**10**]:

PROPOSITION 3.2. *A uniformly locally $(k-1)$-connected metric space X is a uniform ANR if X has the following property $(\tilde{e})_k$:*

$(\tilde{e})_k$ *There exists $\beta > 1$ such that every map $f : |K^{(k)}| \to X$ of the k-skeleton of an arbitrary simplicial complex K extends to a map $\overline{f} : |K| \to X$ such that* $\operatorname{diam}\overline{f}(\sigma) \leqslant \beta \operatorname{diam} f(\sigma^{(k)})$ *for each $\sigma \in K$.*

Moreover, if X is $(k-1)$-connected then X is a uniform AR.

LEMMA 3.3. *If X is uniformly locally path-connected, then the collection of all components of X is uniformly discrete in X.*

PROOF. First, note that components of X are path-connected. By the uniformly local path-connectedness of X, we have $\delta > 0$ such that a pair of δ-close points of X are connected by a path in X. Then, $\operatorname{dist}(C, C') \geqslant \delta$ for each two distinct components $C \neq C'$ of X. □

It is said that a metric space X is *uniformly locally contractible* if for each $\varepsilon > 0$, there exist $\delta > 0$ such that the δ-ball $\mathrm{B}(x, \delta)$ at each $x \in X$ is contractible in the ε-ball $\mathrm{B}(x, \varepsilon)$. Every uniform ANR is uniformly locally contractible by [**9**, Proposition 1.5 and Theorem 1.6]. And, as is easily observed, every uniformly locally contractible metric space is uniformly locally k-connected for all $k \geqslant 0$.

THEOREM 3.4. *Let $L = (L, d, \vee)$ be a metric Lawson semilattice such that*

$$d(x \vee x', y \vee y') \leqslant \max\{d(x,y), d(x',y')\} \text{ for each } x, x', y, y' \in L.$$

Then, the following are equivalent:

(a) *the collection of all components of L is uniformly discrete in L and each component of L is a uniform AR*;
(b) *L is a uniform ANR*;
(c) *L is uniformly locally contractible*;
(d) *L is uniformly locally path-connected.*

PROOF. The implication (a) ⇒ (b) is easy. The implications (b) ⇒ (c) ⇒ (d) have been observed in the above. It remains to show (d) ⇒ (a).

The first half of (a) follows from Lemma 3.3. To see the second half of (a), let C be a component of L. Then, C is a path-connected. Moreover, C is a subsemilattice of L. In fact, for each $x, y \in C$, by using a path $f : \boldsymbol{I} \to C$ from x to y, a path $g : \boldsymbol{I} \to L$ from x to $x \vee y$ can be defined by $g(t) = f(0) \vee f(t)$ for $t \in \boldsymbol{I}$, hence $x \vee y \in C$.

By Proposition 3.2, it suffices to show that C satisfies the property $(\tilde{e})_1$. Let K be a simplicial complex and $f_1 : |K^{(1)}| \to C$ be a map. Suppose we have defined maps $f_i : |K^{(i)}| \to C$, $i < n$, such that $f_i | |K^{(i-1)}| = f_{i-1}$ and $\operatorname{diam} f_i(\sigma^{(i)}) = \operatorname{diam} f_1(\sigma^{(1)})$ for all $\sigma \in K$. By Proposition 3.1, for each n-simplex $\sigma \in K$, there is a map $r_\sigma : \sigma \to \mathfrak{F}_3(\partial\sigma)$ such that $r_\sigma(x) = \{x\}$ for each $x \in \partial\sigma$. We extend f_{n-1} to a map $f_n : |K^{(n)}| \to C$ by $f_n|\sigma = f_\sigma \circ r_\sigma$ for each n-simplex $\sigma \in K$, where $f_\sigma : \mathfrak{F}_3(\partial\sigma) \to C$ is defined as follows:

$$f_\sigma(\{a_1,a_2,a_3\}) = f_{n-1}(a_1) \vee f_{n-1}(a_2) \vee f_{n-1}(a_3).$$

Then, $\operatorname{diam} f_n(\sigma^{(n)}) = \operatorname{diam} f_1(\sigma^{(1)})$ for all $\sigma \in K$. In fact, each $x,y \in \sigma^{(n)}$ are contained in n-faces σ_x and σ_y of σ, respectively. We can write

$$f_n(x) = f_{n-1}(a_1) \vee f_{n-1}(a_2) \vee f_{n-1}(a_3) \text{ and}$$
$$f_n(y) = f_{n-1}(b_1) \vee f_{n-1}(b_2) \vee f_{n-1}(b_3),$$

where $r_{\sigma_x}(x) = \{a_1,a_2,a_3\}$ and $r_{\sigma_y}(y) = \{b_1,b_2,b_3\}$. By the inductive assumption, we have

$$\begin{aligned} d(f_n(x), f_n(y)) &\leqslant \max\{d(f_{n-1}(a_i), f_{n-1}(b_j)) \mid i,j = 1,2,3\} \\ &\leqslant \sup\{d(f_n(x'), f_n(y')) \mid x',y' \in \sigma^{(n-1)}\} \\ &= \operatorname{diam} f_{n-1}(\sigma^{(n-1)}) = \operatorname{diam} f_1(\sigma^{(1)}). \end{aligned}$$

Thus, by induction, we obtain maps $f_n : |K^{(n)}| \to C$, $n \in \boldsymbol{N}$, such that $f_n||K^{(n-1)}| = f_{n-1}$ and $\operatorname{diam} f_n(\sigma^{(n)}) = \operatorname{diam} f_1(\sigma^{(1)})$ for all $\sigma \in K$. These maps induce the extension $\bar{f} : |K| \to C$ of f such that $\operatorname{diam} \bar{f}_n(\sigma^{(n)}) = \operatorname{diam} f_1(\sigma^{(1)})$ for all $\sigma \in K$. Hence, C has the property $(\tilde{e})_1$. □

4. Proof of Theorem A.

It is known that $\mathrm{Cld}_H(X) = (\mathrm{Cld}_H(X), \cup)$ is a Lawson semilattice satisfying the following condition:

$$d_H(A \cup A', B \cup B') \leqslant \max\{d_H(A,B),\ d_H(A',B')\} \text{ for each } A,A',B,B' \in \mathrm{Cld}_H(X).$$

Refer to [**4**, Proposition 2.4] (cf. the proof of [**1**, Fact 4]). By Theorem 3.4, we can reduce Theorem A to the following:

THEOREM 4.1. *For every uniformly locally C^*-connected metric space X, the space $\mathrm{Cld}_H(X)$ is uniformly locally path-connected.*

Before proving this theorem, we give a characterization of the uniformly local C^*-connectedness. For two metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, let $C(X,Y)$ be the set consisting of all continuous functions from X to Y. It is said that $\mathscr{F} \subset C(X,Y)$ is *uniformly equi-continuous* if for each $\varepsilon > 0$, there is $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for each $f \in \mathscr{F}$ and $x,x' \in X$ with $d_X(x,x') < \delta$.

THEOREM 4.2. *Let D be a countable dense subset of the unit interval $\boldsymbol{I}$ with the usual metric and $0,1 \in D$. Then, a metric space $X = (X,d)$ is uniformly locally C^*-connected if and only if for each $\varepsilon > 0$, there exist $\delta > 0$ and $\mathscr{F} \subset C(D,X)$ satisfying the following*:

(i) *$\mathscr{F}$ is uniformly equi-continuous,*
(ii) *$\operatorname{diam} f(D) < \varepsilon$ for every $f \in \mathscr{F}$,*
(iii) *for each δ-close $x,y \in X$, there is $f \in \mathscr{F}$ with $f(0) = x$ and $f(1) = y$.*

PROOF. First, we show the "if" part. For each $\varepsilon > 0$, we have $\delta > 0$ and $\mathscr{F} \subset C(D,X)$ satisfying (i), (ii) and (iii). By (i), for each $\eta > 0$, there is $k \in \boldsymbol{N}$ such that $d(f(t), f(t')) < \eta$ for each $f \in \mathscr{F}$ and $t, t' \in D$ with $|t - t'| < 2/k$. By (iii), for each δ-close $x, y \in X$, we have $f \in \mathscr{F}$ with $f(0) = x$ and $f(1) = y$. Since D is dense in $\boldsymbol{I}$, there are $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $t_i - t_{i-1} < 2/k$. Then, $(f(t_i))_{i=0}^{k}$ is an η-chain from x to y of diameter $< \varepsilon$ by (ii). Thus, X is uniformly locally C^*-connected.

Conversely, assume that X is uniformly locally C^*-connected. For each $\varepsilon > 0$ and $n \in \boldsymbol{N}$, we can choose $\delta_n > 0$ so that for each $\eta > 0$, there is $k \in \boldsymbol{N}$ such that each pair of δ_n-close points of X are connected by an η-chain of length $\leqslant k$ and diameter $< 2^{-n}\varepsilon$, where $\delta_n < 2^{-n}\varepsilon$. Then, we have $k_n \in \boldsymbol{N}$ such that each pair of δ_n-close points of X are connected by δ_{n+1}-chain of length $\leqslant k_n$ and diameter $< 2^{-n}\varepsilon$. Since D is a countable dense subset of $\boldsymbol{I}$, we can obtain $\{0,1\} = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D$ and $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that $D = \bigcup_{n \in \boldsymbol{N}} D_n$, $\lim_{n\to\infty} \varepsilon_n = 0$, each component J of $\boldsymbol{I} \setminus D_{n-1}$ contains $k_n - 1$ many points of D_n and $\operatorname{diam} J < \varepsilon_n$.

For each pair of δ_1-close points $x, y \in X$, we can easily construct $f_{xy} : D \to X$ with $f_{xy}(0) = x$, $f_{xy}(1) = y$ and the following property:

(*) if $t_0 < t_1 < \cdots < t_{k_n} \in D_n$ and (t_0, t_{k_n}) is a component of $\boldsymbol{I} \setminus D_{n-1}$ (hence $t_0, t_{k_n} \in D_{n-1}$), then $(f(t_i))_0^{k_n}$ is a δ_{n+1}-chain with the diameter $< 2^{-n}\varepsilon$.

For each $t, t' \in D$, let J and J' be components of $\boldsymbol{I} \setminus D_{n-1}$ such that $t \in \operatorname{cl} J$ and $t' \in \operatorname{cl} J$. If $|t - t'| < \varepsilon_n$ then $\operatorname{cl} J \cap \operatorname{cl} J' \neq \varnothing$, whence it is easy to see that $d(f(t), f(t')) < 2^{-n+2}\varepsilon$. Therefore, $\mathscr{F} = \{f_{xy} \mid x, y \in X,\ d(x,y) < \delta_1\}$ is uniformly equi-continuous and $\operatorname{diam} f_{xy}(D) < \varepsilon$ for every $f_{xy} \in \mathscr{F}$. □

PROOF OF THEOREM 4.1. For each $\varepsilon > 0$, we have $\delta > 0$ and $\mathscr{F} \subset C(D,X)$ which satisfies (i), (ii) and (iii) in Theorem 4.2. For each $A, B \in \mathrm{Cld}_H(X)$ with $d_H(A,B) < \delta$, let $\mathscr{F}_{A,B} = \{f \in \mathscr{F} \mid f(0) \in A,\ f(1) \in B\}$. Each $a \in A$ is δ-close to some $b \in B$, whence we have $f \in \mathscr{F}_{A,B}$ with $f(0) = a$. Similarly, for each $b \in B$, there is $f \in \mathscr{F}_{A,B}$ such that $f(1) = b$. We can define a path $\varphi : \boldsymbol{I} \to \mathrm{Cld}_H(X)$ from A to B as follows:

$$\varphi(t) = \begin{cases} \operatorname{cl}_X \bigcup \{f([0,2t] \cap D) \mid f \in \mathscr{F}_{A,B}\} & \text{if } t \in [0,1/2], \\ \operatorname{cl}_X \bigcup \{f([2t-1,1] \cap D) \mid f \in \mathscr{F}_{A,B}\} & \text{if } t \in [1/2,1]. \end{cases}$$

For each $t \in [0,1/2]$, $A \subset \varphi(t) \subset \mathrm{N}(A,\varepsilon)$, hence $d_H(A,\varphi(t)) < \varepsilon$. Similarly, $d_H(B,\varphi(t)) < \varepsilon$ for each $t \in [1/2,1]$. Therefore, $\operatorname{diam}_{d_H} \varphi(\boldsymbol{I}) < 2\varepsilon$.

We verify the continuity of φ. For each $\varepsilon' > 0$, since $\mathscr{F}_{A,B}$ is uniformly equi-continuous, there is $\delta' > 0$ such that $d(f(2t), f(2t')) < \varepsilon'$ for each $f \in \mathscr{F}_{A,B}$ and $t, t' \in D$ with $|t - t'| < \delta'$. Let $0 \leqslant t' < t \leqslant 1/2$ with $|t - t'| < \delta'$. Observe that $\varphi(t') \subset \varphi(t)$. For each $x \in \varphi(t)$ and $0 < \varepsilon'' < \varepsilon'$, we have $f \in \mathscr{F}_{A,B}$ and $s \in [0,t] \cap D$ such that $d(x, f(2s)) < \varepsilon''$. When $s \leqslant t'$, we have $d(x, \varphi(t')) < \varepsilon'' < \varepsilon'$ because $f(2s) \in \varphi(t')$. When $s > t'$, we have $d(x, \varphi(t')) < \varepsilon'' + \varepsilon'$ because $f(2t') \in \varphi(t')$ and

$$d(x, f(2t')) \leqslant d(x, f(2s)) + d(f(2s), f(2t')) < \varepsilon'' + \varepsilon'.$$

Consequently, $d(x, \varphi(t')) \leqslant \varepsilon'$. It follows that $d_H(\varphi(t), \varphi(t')) \leqslant \varepsilon'$, hence $\varphi|[0,1/2]$ is continuous. Similarly, $\varphi|[1/2,1]$ is continuous. Thus, it follows that φ is continuous. □

For a complete metric space X and a dense subset $D \subset \boldsymbol{I}$, every uniformly continuous map $f : D \to X$ extends over $\boldsymbol{I}$. Then, the following follows from Theorem 4.2:

COROLLARY 4.3. *Every uniformly locally C^*-connected complete metric space is uniformly locally path-connected.*

5. Proof of Theorem B.

Due to Theorem A, the "if" part of Theorem B follows from the following:

PROPOSITION 5.1. *Every totally bounded uniformly locally C-connected metric space X is uniformly locally C^*-connected.*

PROOF. For each $\varepsilon > 0$, we have $\delta > 0$ with $\mathrm{ul}C(\varepsilon/2)$. For each $0 < \eta < \varepsilon/4$, since X is totally bounded, we have $x_1, \dots, x_n \in X$ such that $\bigcup_{i=1}^n \mathrm{B}(x_i, \eta/3) = X$. For each $x, y \in X$ with $d(x,y) < \delta$, we show that x and y are connected by an η-chain with length $\leqslant n+1$ and diam $< \varepsilon$. We may assume that $d(x,y) \geqslant \eta$. By $\mathrm{ul}C(\varepsilon/2)$, we have an $\eta/3$-chain $(y_j)_{j=0}^k$ in X from x to y with diam $< \varepsilon/2$. For each $j = 1, \dots, k-1$, choose $x_{i(j)}$ so that $d(y_j, x_{i(j)}) < \eta/3$. Then, $(x_{i(j)})_{j=1}^{k-1}$ is an η-chain in X with diam $< \varepsilon$ and $i(1) \neq i(k-1)$ because $d(x,y) \geqslant \eta$. If $x_{i(j)} = x_{i(j')}$ for some $j < j'$ then the sequence $x_{i(1)}, \dots, x_{i(j)}, x_{i(j')+1}, \dots, x_{i(k)}$ is also an η-chain in X. Hence, we can choose $j_1 = 1 < j_2 < \cdots < j_m = k-1$ so that $i(j_\ell) = i(j_{\ell+1} - 1)$ and $i(j_\ell) \neq i(j_{\ell'})$ for $\ell < \ell'$, whence the sequence $x, x_{i(j_1)}, \dots, x_{i(j_m)}, y$ is an η-chain in X from x to y with diam $< \varepsilon$ and length $m+1 \leqslant n+1$. □

The "only if" part of Theorem B follows from the following:

PROPOSITION 5.2. *Let $\mathscr{H}$ be a subspace of $\mathrm{Cld}_H(X)$ such that $\{x\} \in \mathscr{H}$ for each $x \in X$. If $\mathscr{H}$ is uniformly locally path-connected then X is uniformly locally C-connected.*

PROOF. For each $\varepsilon > 0$, there is some $\delta > 0$ such that if $d(x,y) < \delta$ then there is a map $f : \boldsymbol{I} \to \mathscr{H}$ such that $f(0) = \{x\}$, $f(1) = \{y\}$ and $\mathrm{diam}_{d_H} f(\boldsymbol{I}) < \varepsilon/2$. By the compactness of $\boldsymbol{I}$, for each $\eta > 0$, we have $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that $d_H(f(t_i), f(t_{i-1})) < \eta$, hence we can inductively choose $x_i \in f(t_i)$ so that $d(x_i, x_{i-1}) < \eta$, whence $x_0 = x$ and $x_n = y$. Since $\mathrm{diam}_{d_H} f(\boldsymbol{I}) < \varepsilon$, it follows that $f(t) \subset \mathrm{B}(x, \varepsilon/2)$ for each $t \in \boldsymbol{I}$, hence $d(x_i, x) < \varepsilon/2$ for each $i = 1, \dots, n$. Then, $(x_i)_{i=0}^n$ is an η-chain from x to y with diam $< \varepsilon$. Thus, X is uniformly locally C-connected. □

Similarly to the above, the following can be proved:

PROPOSITION 5.3. *Let $\mathscr{H}$ be a subspace of $\mathrm{Cld}_H(X)$ such that $\{x\} \in \mathscr{H}$ for each $x \in X$. If $\mathscr{H}$ is locally path-connected then each $x \in X$ has an arbitrarily small C-connected neighborhood, namely X is locally C-connected.*

6. Proof of Theorem C.

In this section, we prove Theorem C. A subset $A \subset X$ is said to be *ε-discrete* if $d(x,y) > \varepsilon$ for $x \neq y \in A$. The following proposition shows that $\mathrm{Dis}(X)$ is dense in $\mathrm{Cld}_H(X)$.

PROPOSITION 6.1. *For $\varepsilon > 0$, each $A \in \mathrm{Cld}_H(X)$ contains an ε-discrete subset B such that $A \subset N(B,\varepsilon)$, hence $d_H(A,B) < \varepsilon$.*

PROOF. By Zorn's Lemma, A has a maximal ε-discrete subset B_0. Then $A \subset \mathrm{N}(B_0,\varepsilon)$. Otherwise, we could take a point $y \in A \setminus \mathrm{N}(B_0,\varepsilon)$, whence $B_0 \subsetneq B_0 \cup \{y\} \subset A$ and $B_0 \cup \{y\}$ is ε-discrete. This contradicts the maximality of B_0. □

Due to [**10**, Theorem 2], every uniform ANR is homotopy dense in a metric space in which it is isometrically embedded as a dense subset. Thus, by Theorem 3.4, we can reduce Theorem C to the following:

THEOREM 6.2. *Let X be a metric space with the following properties*:

- (C_1) *Every bounded closed set in X is compact*;
- (C_2) *For each $\varepsilon > 0$, there exists $k,\delta > 0$ such that any pair of δ-close points in X are connected by a k-Lipschitz path $f : \boldsymbol{I} \to X$ with $\operatorname{diam} f(\boldsymbol{I}) < \varepsilon$.*

Then, $\mathrm{Dis}_H(X)$ *is uniformly locally path-connected.*

PROOF. For each $\varepsilon > 0$, choose $k,\delta > 0$ so that any pair of δ-close points in X are connected by a k-Lipschitz path $f : \boldsymbol{I} \to X$ with $\operatorname{diam} f(\boldsymbol{I}) < \varepsilon/2$. Let $A,B \in \mathrm{Dis}(X)$ and $d_H(A,B) < \delta$. Since each $x \in B$ is δ-close to some point in A, there is a collection $\{f_x \mid x \in B\}$ of k-Lipschitz paths in X such that $f_x(0) = x$, $f_x(1) \in A$ and $\operatorname{diam} f_x(\boldsymbol{I}) < \varepsilon/2$. Then, A and $A \cup B$ are connected by the path $h_A : \boldsymbol{I} \to \mathrm{Dis}_H(X)$ defined as follows:

$$h_A(t) = A \cup \{f_x(1-t) \mid x \in B\}.$$

To verify that $h_A(t) \in \mathrm{Dis}_H(X)$ for each $t \in \boldsymbol{I}$, assume the contrary, that is, $h_A(t)$ is not discrete for some $t \in \boldsymbol{I}$. Then, $0 < t < 1$ because $h_A(0) = A$ and $h_A(1) = A \cup B$ are discrete. We have infinitely many distinct points $x_i \in B$, $i \in \boldsymbol{N}$, such that $(f_{x_i}(1-t))_{i\in\boldsymbol{N}}$ converges to some $y \in X$. Since

$$\begin{aligned} d(x_i,y) &\leqslant d(f_{x_i}(0), f_{x_i}(1-t)) + d(f_{x_i}(1-t),y) \\ &\leqslant k(1-t) + d(f_{x_i}(1-t),y), \end{aligned}$$

it follows that $d(x_i,y) < k$ for sufficiently large $i \in \boldsymbol{N}$. Thus, $\{x_i \mid i \in \boldsymbol{N}\}$ is an infinite bounded set. On the other hand, since $\{x_i \mid i \in \boldsymbol{N}\} \subset B$, it is discrete in X. This is a contradiction because every bounded closed set in X is compact.

To see the continuity of h_A and $\operatorname{diam} h_A(\boldsymbol{I}) < \varepsilon/2$, let $t,t' \in \boldsymbol{I}$. Since

$$d(f_x(1-t), f_x(1-t')) < k|t-t'| \text{ for every } x \in B,$$

we have $d_H(h_A(t),h_A(t')) < k|t-t'|$, hence h_A is continuous. On the other hand, since

$$d(f_x(1-t), f_x(1-t')) \leqslant \operatorname{diam} f(\boldsymbol{I}) < \varepsilon/2 \text{ for each } x \in B,$$

it follows that

$$h_A(t) = A \cup \{ f_x(1-t) \mid x \in B\}$$
$$\subset N(A \cup \{ f_x(1-t') \mid x \in B\}, \varepsilon/2) = N(h_A(t'), \varepsilon/2).$$

Similarly, $h_A(t') \subset N(h_A(t), \varepsilon/2)$. Hence, $d_H(h_A(t), h_A(t')) < \varepsilon/2$. Thus, we have diam $h_A(\boldsymbol{I}) < \varepsilon/2$.

Similarly there exists a path $h_B : \boldsymbol{I} \to \mathrm{Dis}_H(X)$ such that $h_B(0) = B$, $h_B(1) = A \cup B$ and diam $h_B(\boldsymbol{I}) < \varepsilon/2$. By using h_A and h_B, it is easy to obtain a path from A to B in $\mathrm{Dis}_H(X)$ with diam $< \varepsilon$. Therefore, $\mathrm{Dis}_H(X)$ is uniformly locally path-connected. □

PROPOSITION 6.3. *If a metric space X satisfies the condition* (C_2), *then X is uniformly locally C^*-connected.*

PROOF. For each $\varepsilon > 0$, take $\delta, k > 0$ as in the condition (C_2). We define

$$\mathscr{F} = \{ f \in C(D,X) \mid f \text{ is } k\text{-Lipschitz with } \operatorname{diam} f(D) < \varepsilon\}.$$

It is easily follows from the definition that $\mathscr{F}$ satisfies the conditions (i) and (ii) in Theorem 4.2. For each δ-close points $x, y \in X$, there is a k-Lipschitz path $f : \boldsymbol{I} \to X$ with $f(0) = x$, $f(1) = y$ and diam $f(\boldsymbol{I}) < \varepsilon$. Then, observe that $f|D \in \mathscr{F}$. Thus, $\mathscr{F}$ satisfies the condition (iii) in Theorem 4.2. Therefore, by Theorem 4.2, X is uniformly locally C^*-connected. □

The following example shows that the uniformly local path-connectedness of $\mathrm{Dis}_H(X)$ does not imply the condition (C_1) nor (C_2) for X.

EXAMPLE 3. The space $X = \boldsymbol{I} \setminus \{2^{-n} \mid n \in \boldsymbol{N}\}$ with the usual metric does not satisfy (C_1) nor (C_2). We show that $\mathrm{Dis}_H(X)$ is uniformly locally path-connected.

First, we prove that $\mathrm{Dis}_H((0,1))$ is path-connected. Each $A \in \mathrm{Dis}_H((0,1))$ can be written as $A = \{a_n \mid n \in \boldsymbol{Z}\}$, where $a_n \leqslant a_{n+1}$ for every $n \in \boldsymbol{Z}$. Then, we can define a path $f_A : \boldsymbol{I} \to \mathrm{Dis}_H(X)$ from A to $\{a_0\}$ as follows: $f_A(0) = A$, $f(1) = \{a_0\}$, $f_A(2^{-n}) = \{a_i \mid |i| \leqslant n\}$ for $n \in \boldsymbol{N}$ and, for $2^{-n-1} < t < 2^{-n}$,

$$f_A(t) = f_A(2^{-n}) \cup \{(2^{n+1}t - 1)a_n + (2 - 2^{n+1}t)a_{n+1}\}$$
$$\cup \{(2^{n+1}t - 1)a_{-n} + (2 - 2^{n+1}t)a_{-(n+1)}\}.$$

By connecting f_A and a path from a_0 to $1/2$, we can obtain a path from A to $\{1/2\}$ in $\mathrm{Dis}_H((0,1))$. Thus, $\mathrm{Dis}_H((0,1))$ is path-connected. Similarly, it can be seen that $\mathrm{Dis}_H([0,1))$ and $\mathrm{Dis}_H((0,1])$ are also path-connected.

Next, we prove that $\mathrm{Dis}_H((-1,0) \cup (0,1))$ is path-connected. Let

$$A_+ = \{2^{-n} \mid n \in \boldsymbol{N}\} \text{ and } A_- = \{-2^{-n} \mid n \in \boldsymbol{N}\}.$$

Then, A_+ and A_- can be connected to $A_- \cup A_+$ by paths $f_\pm : \boldsymbol{I} \to \mathrm{Dis}_H((-1,0) \cup (0,1))$ defined as follows: $f_\pm(0) = A_\pm$, $f_\pm(1) = A_- \cup A_+$, $f_+(t) = tA_- \cup A_+$ and $f_-(t) = A_- \cup tA_+$ for $0 < t < 1$, where $tA = \{tx \mid x \in A\}$. Now, let $B \in \mathrm{Dis}_H((-1,0) \cup (0,1))$. If $B \subset (-1,0)$ or $B \subset (-1,0)$ then B can be connected to A_- or A_+ by a path in $\mathrm{Dis}_H((-1,0))$ or in $\mathrm{Dis}_H((0,1))$, hence it can be connected to $A_- \cup A_+$ by a path in $\mathrm{Dis}_H((-1,0)\cup(0,1))$. When $B \not\subset (-1,0)$ and $B \not\subset (0,1)$, we have two paths $f : \boldsymbol{I} \to \mathrm{Dis}_H((-1,0))$ and $f : \boldsymbol{I} \to \mathrm{Dis}_H((0,1))$ such that $f(0) = B \cap (-1,0)$, $f(1) = A_-$, $g(0) = B \cap (0,1)$ and $g(1) = A_+$. Then, a path $h : \boldsymbol{I} \to \mathrm{Dis}_H((-1,0)\cup(0,1))$ from B to $A_- \cup A_+$ can be defined by $h(t) = f(t) \cup g(t)$. Similarly, we can see that $\mathrm{Dis}_H([-1,0)\cup(0,1])$, $\mathrm{Dis}_H([-1,0)\cup(0,1))$ and $\mathrm{Dis}_H((-1,0)\cup(0,1])$ are also path-connected.

For each $0 \leqslant a < b$, $\mathrm{Dis}_H([a,b]\cap X)$ is a closed subspace of $\mathrm{Dis}_H(X)$. By using the above facts, we can easily show that $\mathrm{Dis}_H([a,b]\cap X)$ is path-connected.

For each $\varepsilon > 0$, choose $n \in \boldsymbol{N}$ so that $2^{-n+1} < \varepsilon$, let

$$X_i = [(i-1)2^{-n}, (i+1)2^{-n}] \cap X,\ i = 0, \ldots, 2^n.$$

Note that each pair of 2^{-n}-close points of X are contained in the same X_i. For each $A, B \in \mathrm{Dis}_H(X)$ with $d_H(A,B) < 2^{-n}$, let

$$E = \{i \mid A \cap X_i \neq \varnothing,\ B \cap X_i \neq \varnothing\}.$$

Then, $A \cup B \subset \bigcup_{i \in E} X_i$. For each $i \in E$, since $\mathrm{Dis}_H(X_i)$ is path-connected, there is a path $f_i : \boldsymbol{I} \to \mathrm{Dis}_H(X_i)$ with $f_i(0) = A \cap X_i$ and $f_i(1) = B \cap X_i$. A path $f : \boldsymbol{I} \to \mathrm{Dis}_H(X)$ from A to B can be defined by $f(t) = \bigcup_{i \in E} f_i(t)$. Then, $f(t) \subset \bigcup_{i \in E} X_i$ for each $t \in \boldsymbol{I}$ and $f(t) \cap X_i \supset f_i(t) \neq \varnothing$ for each $t \in \boldsymbol{I}$ and $i \in E$. It follows that

$$d_H(f(t), f(t')) \leqslant \mathrm{diam} X_i \leqslant 2^{-n+1} < \varepsilon \ \text{ for } t, t' \in \boldsymbol{I},$$

that is, $\mathrm{diam} f(\boldsymbol{I}) < \varepsilon$. Thus, $\mathrm{Dis}_H(X)$ is uniformly locally path-connected.

7. Further problems and related results.

After the eariler version of this paper had been submitted, Banakh and Voytsitski [**2**] succeeded in proving the converse of Theorem A, that is,

THEOREM 7.1 ([**2**]). *The space* $\mathrm{Bdd}_H(X)$ (*or* $\mathrm{Cld}_H(X)$) *is a uniform ANR if and only if X is uniformly locally C^*-connected.*

In Example 1, X is not a uniform ANR but $\mathrm{Cld}_H(X)$ need not be an ANR even for a uniform AR X. In [**2**], Banakh and Voytsitski showed that $\mathrm{Cld}_H(\boldsymbol{R}^{\boldsymbol{N}})$ is not an ANR, where $\boldsymbol{R}^{\boldsymbol{N}}$ has the usual product metric. The following is unknown:

PROBLEM 1. Characterize metric spaces X such that $\mathrm{Dis}_H(X)$ are ANR's.

In case X is uniformly locally compact (i.e., there is some $\delta > 0$ such that $\overline{\mathrm{B}}(x,\delta)$ is compact for each $x \in X$), it is proved in [**2**] that $\mathrm{Dis}_H(X)$ is a uniform ANR if and only if X is uniformly locally C^*-connected. Due to Theorem A, $\mathrm{Cld}_H(\boldsymbol{Q})$ and $\mathrm{Cld}_H(\boldsymbol{R} \setminus \boldsymbol{Q})$ are ANR's with respect the usual metric. However, the following is open:

PROBLEM 2. Is $\mathrm{Dis}_H(\boldsymbol{Q})$ or $\mathrm{Dis}_H(\boldsymbol{R} \setminus \boldsymbol{Q})$ an ANR?

The following proposition is shown in [**7**], which shows the complexity of the space $\mathrm{Cld}_H(X)$.

PROPOSITION 7.2. *The space* $\mathrm{Cld}_H(\boldsymbol{R}^n)$ *has uncountably many components. Moreover, the space* $\mathrm{Comp}_H(\boldsymbol{R}^n)$ *of all compact sets is one of them and all but this component are non-separable.*

PROOF. First, we show that $\mathrm{Cld}_H(\boldsymbol{R})$ has uncountably many components. Let $A = \{n^2 \mid n \in \boldsymbol{N}\}$. We can write $A = \{k(i,j) \mid i,j \in \boldsymbol{N}\}$ such that $k(i,j) \neq k(i',j')$ if $(i,j) \neq (i',j')$. For each nonempty set $E \subset \boldsymbol{N}$, we denote $A_E = \{k(i,j) \mid i \in \boldsymbol{N}, j \in E\} \in \mathrm{Cld}_H(\boldsymbol{R})$. Observe that if $E \neq E'(\subset \boldsymbol{N})$, then $A_E \setminus A_{E'}$ or $A_{E'} \setminus A_E$ contains infinitely many points, which implies that $d_H(A_E, A_{E'}) = \infty$. Then A_E and $A_{E'}$ are contained in different components. Since $\boldsymbol{N}$ has uncountably many subsets, $\mathrm{Cld}_H(\boldsymbol{R})$ has uncountably many components.

The case $n \geqslant 2$ is simpler than the above. For each $x \in \boldsymbol{S}^{n-1} = \{x \in \boldsymbol{R}^n \mid \|x\| = 1\}$, let $A_x = \{tx \mid t \in [0,\infty)\} \in \mathrm{Cld}_H(X)$. Then, for $x \neq y \in \boldsymbol{S}^{n-1}$, evidently $d_H(A_x, A_y) = \infty$. Now, let $\mathscr{A}_x$ be the component of $\mathrm{Cld}_H(\boldsymbol{R}^n)$ containing A_x. Then $\mathscr{A}_x \cap \mathscr{A}_y = \varnothing$ for $x \neq y \in \boldsymbol{S}^{n-1}$. Hence, $\mathrm{Cld}_H(\boldsymbol{R}^n)$ has uncountably many components.

It should be note that $\mathrm{Comp}_H(\boldsymbol{R}^n)$ is connected and clopen in $\mathrm{Cld}_H(\boldsymbol{R}^n)$. Hence $\mathrm{Comp}_H(\boldsymbol{R}^n)$ is component of $\mathrm{Cld}_H(\boldsymbol{R}^n)$.

Now, let $\mathscr{H}$ be a component of $\mathrm{Cld}_H(\boldsymbol{R}^n)$ such that $\mathscr{H} \neq \mathrm{Comp}_H(\boldsymbol{R}^n)$. Then $\mathscr{H}$ contains an unbounded closed set A in $\boldsymbol{R}^n$. Choose $a_n \in A$, $n \in \boldsymbol{N}$, so that $\|a_{n+1}\| > \|a_n\| + 3$. Let

$$A_1 = A \cup \bigcup_{n \in \boldsymbol{N}} \overline{\mathrm{B}}(a_n, 1).$$

Then, we have the path $h : \boldsymbol{I} \to \mathrm{Cld}_H(\boldsymbol{R}^n)$ defined by

$$h(t) = A \cup \bigcup_{n \in \boldsymbol{N}} \overline{\mathrm{B}}(a_n, t).$$

Since $h(0) = A$ and $h(1) = A_1$, it follows that $A_1 \in \mathscr{H}$. For each $E \subset \boldsymbol{N}$, let $A_E = A_1 \setminus \bigcup_{n \in E} \mathrm{B}(a_n, 1)$. There exists the path $h_E : \boldsymbol{I} \to \mathrm{Cld}_H(\boldsymbol{R}^n)$ defined by $h_E(t) = A_1 \setminus \bigcup_{n \in E} \mathrm{B}(a_n, t)$. Since $h_E(0) = A_1$ and $h_E(1) = A_E$, it follows that $A_E \in \mathscr{H}$. It is easy to see that $d_H(A_E, A_{E'}) = 1$ if $E \neq E'$. This means that $\{A_E \mid E \subset \boldsymbol{N}\}$ is discrete subset of $\mathscr{H}$. Since $\mathrm{card}\{A_E \mid E \subset \boldsymbol{N}\} > \aleph_0$, $\mathscr{H}$ is non-separable. □

For an arbitrary Banach space X, every component of $\mathrm{Cld}_H(X)$ is a complete metric AR by Theorems A and [**3**, Theorem 3.2.4].

PROBLEM 3. For a Banach space (or a Hilbert space) X, is every component of $\mathrm{Cld}_H(X)$ homeomorphic to a Hilbert space?

Even if X is Euclidean space $\boldsymbol{R}^n$, the above is unknown, that is,

PROBLEM 4. Is each non-separable component of $\mathrm{Cld}_H(\boldsymbol{R}^n)$ homeomorphic to a Hilbert space?

In relation to above problems, some results with different topologies have been obtained in [**1**], [**6**] and [**11**]. For topologies on hyperspaces, we refer to the book [**3**].

THEOREM 7.3 ([**11**]). *For a Hausdorff space* X, *the hyperspace* $\mathrm{Cld}_F(X)$ *with the Fell* *topology is homeomorphic to* $Q \setminus \{0\}$ *if and only if* X *is locally compact, locally connected, separable metrizable and has no compact components, where* $Q = \boldsymbol{I}^{\boldsymbol{N}}$ *is the Hilbert cube.*

THEOREM 7.4 ([**1**]). *For every infinite-dimensional Banach space* X *with weight* τ*, the hyperspace* $\mathrm{Cld}_{AW}(X)$ *with the Attouch-Wets topology is homeomorphic to the Hilbert space* $\ell_2(2^\tau)$ *with weight* 2^τ*, where* $w(X)$ *is the weight of* X.

THEOREM 7.5 ([**6**]). *For every infinite-dimensional separable Banach space* X*, the hyperspace* $\mathrm{Cld}_W(X)$ *with the Wijsman topology is homeomorphic to the separable Hilbert space* ℓ_2.

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