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On a mean value formula for the approximate functional equation of $\zeta(s)$ in the critical strip

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Abstract. In a recent paper, Isao Kiuchi and Naoki Yanagisawa studied the even power moments of the error term in the approximate functional equation for $\zeta(s)$. They got a mean value formula with an error term $O(T^{1/2-k\sigma})$, and then they conjecture that this term could be replaced by $E_{k,\sigma}T^{1/2-k\sigma}(1+o(1))$ with constant $E_{k,\sigma}$ depending on k and σ. In this paper, we disprove this conjecture by showing that the error term should be $f(T)T^{1/2-k\sigma}+o(T^{1/2-k\sigma})$ with $f(T)$ oscillating.

1. Introduction.

Let $s=\sigma+it$ $(0\le\sigma\le 1, t\ge 1)$ be a complex variable, $\zeta(s)$ the Riemann zeta function. The error term $R(s)$ in the approximate functional equation for $\zeta(s)$ is defined by

$$\zeta(s)=\sum_{n\le\sqrt{t/(2\pi)}}n^{-s}+\chi(s)\sum_{n\le\sqrt{t/(2\pi)}}n^{s-1}+R(s), \tag{1.1}$$

where

$$\chi(s)=2^s\pi^{s-1}\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s). \tag{1.2}$$

In [**1**], I. Kiuchi and N. Yanagisawa derived the even power moments of the function $|R(s)|$ for $0\le\sigma\le 1$. They proved

THEOREM A. *Let k be a positive integer,*

$$h(y)=\frac{\cos 2\pi(y^2-y-1/16)}{\cos(2\pi y)},\quad C_k=\int_0^1 h(y)^{2k}dy, \tag{1.3}$$

then

$$\int_1^T|R(s)|^{2k}dt=\begin{cases}\dfrac{(2\pi)^{k\sigma}C_k}{1-k\sigma}T^{1-k\sigma}+O(T^{1/2-k\sigma}) & \text{if } 0\le\sigma\le\dfrac{1}{2k},\\ \dfrac{(2\pi)^{k\sigma}C_k}{1-k\sigma}T^{1-k\sigma}+D_{k,\sigma}+O(T^{1/2-k\sigma}) & \text{if } \dfrac{1}{2k}<\sigma\le 1 \text{ and } \sigma\ne\dfrac{1}{k},\\ 2\pi C_k\log T+D_{k,1/k}+O(T^{-1/2}) & \text{if } \sigma=\dfrac{1}{k},\end{cases} \tag{1.4}$$

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where the constant $D_{k,\sigma}$ depends on k and σ.

REMARK 1. In fact, for $\sigma = 1/(2k)$, according to their proof (see the following Theorem 1 also), the error term in Theorem A should be $O(\log T)$. Because error terms of $O(1)$ for $\int_{T_1}^{T_2} |R(s)|^{2k} dt$ with $T_1 \le T_2 \le 2T_1$ only imply an error term of $O(\log T)$ for $\int_1^T |R(s)|^{2k} dt$ rather than an error term of $O(1)$.

For (1.4), they conjecture that

CONJECTURE B. *For $0 \le \sigma \le 1$ and positive integer k, the error term $O(T^{1/2-k\sigma})$ could be replaced by $E_{k,\sigma} T^{1/2-k\sigma}(1+o(1))$ as $T \to +\infty$, where the constant $E_{k,\sigma}$ depends on k and σ.*

In this paper we will disprove Conjecture B by showing the following

THEOREM 1. *Let $k, h(y)$ and C_k be as in Theorem A,*

$$g_k(y) = h(y)^{2k} - C_k, \quad G_k(y) = \int_{\sqrt{1/(2\pi)}}^{y} g_k(x - [x]) dx, \tag{1.5}$$

where $[x]$ is the integer part of x,

$$C'_k = \int_{\sqrt{1/(2\pi)}}^{\sqrt{1/(2\pi)}+1} G_k(y) dy, \tag{1.6}$$

$$C''_k = \int_0^1 h^{2k-1}(x) h'''(x) dx. \tag{1.7}$$

We have

$$\int_1^T |R(s)|^{2k} dt$$
$$= \begin{cases} C_k T + \left(2\sqrt{2\pi} G_k\left(\sqrt{\frac{T}{2\pi}}\right) - 2\sqrt{2\pi} C'_k - \frac{k}{6(2\pi)^{3/2}} C''_k\right) T^{1/2} + O(\log(T)) & \text{if } \sigma = 0, \\ \frac{(2\pi)^{k\sigma} C_k}{1-k\sigma} T^{1-k\sigma} + \left(2(2\pi)^{k\sigma+1/2} G_k\left(\sqrt{\frac{T}{2\pi}}\right) - 2(2\pi)^{k\sigma+1/2} C'_k \right. & \\ \qquad \left. + \frac{k(2\pi)^{k\sigma-3/2}}{12k\sigma - 6} C''_k\right) T^{1/2-k\sigma} + D_{k,\sigma} + O(T^{-k\sigma}) & \\ \qquad\qquad \text{if } 0 < \sigma \le 1 \text{ and } \sigma \ne \frac{1}{2k}, \frac{1}{k}, & \\ 2\sqrt{2\pi} C_k T^{1/2} - \frac{k}{24\pi} C''_k \log T + D_{k,1/(2k)} + 4\pi G_k\left(\sqrt{\frac{T}{2\pi}}\right) + O(T^{-1/2}) & \text{if } \sigma = \frac{1}{2k}, \\ 2\pi C_k \log T + D_{k,1/k} + \left(\frac{k}{6\sqrt{2\pi}} C''_k - 2(2\pi)^{3/2} C'_k \right. & \\ \qquad \left. + 2(2\pi)^{3/2} G_k\left(\sqrt{\frac{T}{2\pi}}\right)\right) T^{-1/2} + O(T^{-1}) & \text{if } \sigma = \frac{1}{k}, \end{cases} \tag{1.8}$$

where $D_{k,\sigma}$ are constants depending on K and σ.

Since $G_k(\sqrt{T/(2\pi)})$ is a non-constant periodic function of $\sqrt{T}$ with period $\sqrt{2\pi}$, which doesn't convergent, Theorem 1 is a disproof of Conjecture B.

In [**1**], they studied cases $0 \le \sigma \le 1/(2k)$ and $1/(2k) < \sigma \le 1$ separately and estimate $\int_{T_1}^{T_2} |R(s)|^{2k} dt$ for $T_1 \le T_2 \le 2T_1$ firstly. Here we will deal with these two cases in one framework and study $\int_1^T |R(s)|^{2k} dt$ directly.

2. Approximate functional equation.

We begin with an approximate functional equation due to Siegel.

LEMMA 1 (see [**2**] or [**3**, Theorem 4.16]). *If $0 \le \sigma \le 1$, $N < At$, where A is a sufficiently small constant,*

$$\begin{aligned}\zeta(s) = &\sum_{n \le \sqrt{t/(2\pi)}} n^{-s} + \chi(s) \sum_{n \le \sqrt{t/(2\pi)}} n^{s-1} \\ &+ (-1)^{[\sqrt{t/(2\pi)}]-1} e^{-(1/2)i\pi(s-1)} (2\pi t)^{(1/2)s-(1/2)} e^{-(1/2)it-(i\pi)/8} \Gamma(1-s) \\ &\times \left\{ S_N + O\left(\left(\frac{AN}{t} \right)^{N/6} \right) + O(e^{-At}) \right\}, \end{aligned} \tag{2.1}$$

where

$$S_N = \sum_{n=0}^{N-1} \sum_{\nu \le (n/2)} \frac{n! i^{\nu-n}}{\nu!(n-2\nu)! 2^n} \left(\frac{1}{2\pi} \right)^{(n/2)-\nu} a_n h^{(n-2\nu)} \left(\sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}} \right] \right), \tag{2.2}$$

a_n is the coefficients of the Taylor expansion of

$$\phi(z) = \exp\left\{ (s-1)\log\left(1 + \frac{z}{\sqrt{t}}\right) - iz\sqrt{t} + \frac{1}{2} it^2 \right\}$$

with

$$a_0 = 1, \quad a_1 = \frac{\sigma - 1}{\sqrt{t}}, \quad a_3 = \frac{i}{3} t^{-1/2} + O(t^{-1}), \tag{2.3}$$

$$a_k = O(t^{-1}), \quad \text{if } k \ge 2,\ k \ne 3. \tag{2.4}$$

We need the following somewhat weak form

LEMMA 2. *For $0 \le \sigma \le 1$,*

$$\chi(s)^{-1/2} R(s) = (-1)^{[\sqrt{t/(2\pi)}]-1} (F_1(t) t^{-1/4} + F_2(\sigma, t) t^{-3/4} + R_{\sigma,t}), \tag{2.5}$$

where

$$F_1(t) = (2\pi)^{1/4} h\left(\sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}} \right] \right), \tag{2.6}$$

$$F_2(\sigma,t) = (2\pi)^{-1/4} i\left(\frac{1}{4}-\frac{\sigma}{2}\right) h'\left(\sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}}\right]\right) - \frac{1}{24}(2\pi)^{-5/4} h'''\left(\sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}}\right]\right), \tag{2.7}$$

$$R_{\sigma,t} = O(t)^{-5/4} \tag{2.8}$$

as $t \to +\infty$.

PROOF. Apply Lemma 1 with $N = 6$ and use (2.3), (2.4), we have for $t \to +\infty$

$$R(s) = (-1)^{[\sqrt{t/(2\pi)}]-1} e^{-(1/2)i\pi(s-1)} (2\pi t)^{(1/2)s-(1/2)} e^{-(1/2)it-(i\pi)/8} \times \Gamma(1-s)(2\pi)^{-1/4}(F_1(t) + F_2(\sigma,t)t^{-1/2} + O(t^{-1})). \tag{2.9}$$

For $t > 0$ (see [**3**, P. 78, P. 80]),

$$\chi(s) = (2\pi)^{s-1} e^{(1/2-s/2)i\pi}\Gamma(1-s)(1+O(e^{-\pi t})), \tag{2.10}$$

$$\Gamma(1-s) = (2\pi)^{1/2} t^{(1/2)-s} e^{it+(1/2)i\pi s-(1/4)i\pi}(1+O(t^{-1})). \tag{2.11}$$

It follows from (2.9), (2.10) and (2.11) that

$$\begin{aligned}\chi(s)^{-1/2}R(s) &= (-1)^{[\sqrt{t/(2\pi)}]-1} e^{-(1/2)i\pi(s-1)} (2\pi t)^{(1/2)s-(1/2)} e^{-(1/2)it-(i\pi)/8} (2\pi)^{(1/2)-(s/2)} \\ &\quad \times e^{((s/4)-(1/4))i\pi} (2\pi)^{1/4} t^{(1/4)-(s/2)} e^{(1/2)it+(1/4)i\pi s-(1/8)i\pi} (2\pi)^{-1/4} \\ &\quad \times (F_1(t) + F_2(\sigma,t)t^{-1/2} + O(t^{-1}))(1+O(e^{-\pi t}))^{-1/2}(1+O(t^{-1}))^{1/2} \\ &= (-1)^{[\sqrt{t/(2\pi)}]-1}(F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + O(t)^{-5/4}).\end{aligned} \tag{2.12}$$

□

3. A basic lemma.

The following lemma is the key to the proof of Theorem 1.

LEMMA 3. *Let $f(x)$ be a integrable periodic function with period 1, $C = \int_0^1 f(x)dx$, $a > 0$, $g(x) = \int_a^x (f(y) - C)dy$, $C' = \int_a^{a+1} g(x)dx$, then as $M \to +\infty$,*

$$\int_a^M f(x)x^\alpha dx = \begin{cases} \frac{C}{2}M^2 + (g(M) - C')M + O(1) & \text{if } \alpha = 1, \\ \frac{C}{\alpha+1}M^{\alpha+1} + C(\alpha) + (g(M) - C')M^\alpha + O(M^{\alpha-1}) & \text{if } \alpha < 1, \alpha \neq 0, -1, \\ CM + C(0) + g(M) & \text{if } \alpha = 0, \\ C\log M + C(-1) + (g(M) - C')M^{-1} + O(M^{-2}) & \text{if } \alpha = -1, \end{cases} \tag{3.1}$$

where $C(\alpha)$ is a constant depends on α.

PROOF. For $\alpha \neq -1$,

$$\int_a^M f(x)x^\alpha dx = C\int_a^M x^\alpha dx + \int_a^M (f(x)-C)x^\alpha dx$$
$$= \frac{C}{\alpha+1}x^{\alpha+1}\Big|_a^M + g(x)x^\alpha\Big|_a^M - \alpha\int_a^M g(x)x^{\alpha-1}dx. \tag{3.2}$$

$$\int_a^M f(x)x^{-1}dx = C\int_a^M x^{-1}dx + \int_a^M (f(x)-C)x^{-1}dx$$
$$= C\log x\Big|_a^M + g(x)x^{-1}\Big|_a^M + \int_a^M g(x)x^{-2}dx. \tag{3.3}$$

Since $f(x)$ is a periodic function with period 1,

$$g(x+1) = \int_a^{x+1}(f(y)-C)dy = \int_a^x (f(y)-C)dy + \int_x^{x+1} f(y)dy - C$$
$$= \int_a^x (f(y)-C)dy = g(x). \tag{3.4}$$

That is, $g(x)$ is also a periodic function with period 1. Similarly, we have $\int_a^x (g(y)-C')dy$ is a (continuous) periodic function and therefore bounded. So, for $\alpha \leq 1$, $\alpha \neq 0$,

$$\int_a^M g(x)x^{\alpha-1}dx = C'\int_a^M x^{\alpha-1}dx + \int_a^M (g(x)-C')x^{\alpha-1}dx$$
$$= \frac{C'}{\alpha}x^\alpha\Big|_a^M + \int_a^M (g(x)-C')dxM^{\alpha-1} - (\alpha-1)\int_a^M\int_a^x (g(y)-C')dyx^{\alpha-2}dx$$
$$= \frac{C'}{\alpha}x^\alpha\Big|_a^M + O(M^{\alpha-1}) - (\alpha-1)\int_a^{+\infty}\int_a^x (g(y)-C')dyx^{\alpha-2}dx$$
$$+ (\alpha-1)\int_M^{+\infty}\int_a^x (g(y)-C')dyx^{\alpha-2}dx$$
$$= \frac{C'}{\alpha}M^\alpha + C_1(\alpha) + O(M^{\alpha-1}). \tag{3.5}$$

Combine (3.2), (3.3) and (3.5), we get (3.1). □

4. Proof of Theorem 1.

For $t \geq t_0 > 0$,

$$|\chi(x)|^k = \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} + G_{k,\sigma}(t) \tag{4.1}$$

with

$$G_{k,\sigma}(t) = O(t^{k((1/2)-\sigma)-1}). \tag{4.2}$$

It follows from Lemma 2 and (4.1) that

$$\begin{aligned}
\int_1^T |R(s)|^{2k} dt &= \int_1^T |\chi(x)|^k |F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + R_{\sigma,t}|^{2k} dt \\
&= \int_1^T \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} |F_1^{2k}(t)t^{-k/2} + 2kF_1^{2k-1}(t)F_2(\sigma,t)t^{-(k/2)-(1/2)} \\
&\quad + \sum_{l=2}^{2k} \binom{2k}{l} F_1^{2k-l}(t)F_2^l(\sigma,t)t^{-(k/2)-(l/2)} \\
&\quad + \sum_{l=1}^{2k} \binom{2k}{l} (F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4})^{2k-l} R_{\sigma,t}^l | dt \\
&\quad + \int_1^T G_{k,\sigma}(t) |F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + R_{\sigma,t}|^{2k} dt \\
&= \int_1^T \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} |F_1^{2k}(t)t^{-k/2} + 2kF_1^{2k-1}(t)F_2(\sigma,t)t^{-(k/2)-(1/2)}| dt \\
&\quad + \int_1^T \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} H_{k,\sigma}(t) dt + \int_1^T \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} L_{k,\sigma}(t) dt \\
&\quad + \int_1^T G_{k,\sigma}(t) |F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + R_{\sigma,t}|^{2k} dt \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.3}$$

say, where

$$H_{k,\sigma}(t) = O\left(\sum_{l=2}^{2k} \binom{2k}{l} |F_1^{2k-l}(t)F_2^l(\sigma,t)| t^{-(k/2)-(l/2)} \right), \tag{4.4}$$

$$L_{k,\sigma}(t) = O\left(\sum_{l=1}^{2k} \binom{2k}{l} |(F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4})^{2k-l} R_{\sigma,t}^l| \right). \tag{4.5}$$

To estimate I_1, we need the following Lemma.

LEMMA 4. *Let $u_1(t)$ be a real function, $u_1(t) \geq A > 0$, $u_2(t)$ be a bounded complex function, then as $t \to +\infty$,*

$$|u_1(t) + u_2(t)t^{-1/2}| = u_1(t) + \mathrm{Re}(u_2(t))t^{-1/2} + O(t^{-1}), \tag{4.6}$$

where $\mathrm{Re}(z)$ *is the real part of z.*

PROOF. Since $u_1(t) \geq A > 0$ and $u_2(t)$ are bounded, we have for t large enough,

$$u_1(t)+\mathrm{Re}(u_2(t))t^{-1/2}>0$$

and therefore

$$\begin{aligned}&|u_1(t)+u_2(t)t^{-1/2}|-u_1(t)-\mathrm{Re}(u_2(t))t^{-1/2}\\&=\frac{(\mathrm{Im}(u_2(t)))^2t^{-1}}{|u_1(t)+u_2(t)t^{-1/2}|+u_1(t)+\mathrm{Re}(u_2(t))t^{-1/2}}=O(t^{-1}),\end{aligned}\tag{4.7}$$

where $\mathrm{Im}(z)$ is the imaginary part of z. □

For $0\le x<1$, $h(x)\ne 0$, then $F_1^{2k}(t)>0$. Notice that both $F_1(t)$ and $F_2(\sigma,t)$ are piecewise continuous periodic function of $\sqrt{t}$, we have $F_1^{2k}(t)\ge A>0$ and $2kF_1^{2k-1}(t)F_2(\sigma,t)$ is bounded. Thus by Lemma 4,

$$\begin{aligned}I_1&=\int_1^T\left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)}t^{-k/2}|F_1^{2k}(t)+2kF_1^{2k-1}(t)F_2(\sigma,t)t^{-1/2}|dt\\&=\int_1^T(2\pi)^{k(\sigma-(1/2))}t^{-k\sigma}F_1^{2k}(t)dt+\int_1^T(2\pi)^{k(\sigma-(1/2))}t^{-k\sigma-(1/2)}\mathrm{Re}(2kF_1^{2k-1}(t)F_2(\sigma,t))dt\\&\quad+\int_1^T(2\pi)^{k(\sigma-(1/2))}t^{-k\sigma}P_k(\sigma,t)dt\\&=I_{11}+I_{12}+I_{13},\end{aligned}\tag{4.8}$$

say, where

$$P_k(\sigma,t)=O(t^{-1}).\tag{4.9}$$

Now we estimate $I_{11},\ I_{12},\ I_{13},\ I_2,\ I_3,\ I_4$ respectively. By Lemma 3,

$$\begin{aligned}I_{11}&=\int_1^T\left(\frac{t}{2\pi}\right)^{-k\sigma}h^{2k}\left(\sqrt{\frac{t}{2\pi}}-\left[\sqrt{\frac{t}{2\pi}}\right]\right)dt=4\pi\int_{\sqrt{1/(2\pi)}}^{\sqrt{T/(2\pi)}}x^{1-2k\sigma}h^{2k}(x-[x])dx\\&=\begin{cases}C_kT+2\sqrt{2\pi}\left(G_k\left(\sqrt{\frac{T}{2\pi}}\right)-C_k'\right)T^{1/2}+O(1) & \text{if } \sigma=0,\\ \frac{(2\pi)^{k\sigma}C_k}{1-k\sigma}T^{1-k\sigma}+D_{k,\sigma}'+2(2\pi)^{k\sigma+1/2}\left(G_k\left(\sqrt{\frac{T}{2\pi}}\right)-C_k'\right)T^{1/2-k\sigma} & \\ \quad+O(T^{-k\sigma}) & \text{if } 0<\sigma\le 1 \text{ and } \sigma\ne\frac{1}{2k},\ \frac{1}{k},\\ 2\sqrt{2\pi}C_kT^{1/2}+D_{k,1/(2k)}'+4\pi G_k\left(\sqrt{\frac{T}{2\pi}}\right) & \text{if } \sigma=\frac{1}{2k},\\ 2\pi C_k\log T+D_{k,1/k}'+2(2\pi)^{3/2}\left(G_k\left(\sqrt{\frac{T}{2\pi}}\right)-C_k'\right)T^{-1/2}+O(T^{-1}) & \text{if } \sigma=\frac{1}{k},\end{cases}\end{aligned}\tag{4.10}$$

here and after $D'_{k,\sigma}, D''_{k,\sigma}, D'''_{k,\sigma}, D''''_{k,\sigma}, D'''''_{k,\sigma}, D''''''_{k,\sigma}$ are constants depending on k and σ.

Similarly,

$$\begin{aligned} I_{12} &= \frac{-k}{48\pi^2}\int_1^T \left(\frac{t}{2\pi}\right)^{-k\sigma-(1/2)} h^{2k-1}\left(\sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}}\right]\right) h'''\left(\sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}}\right]\right) dt \\ &= \frac{-k}{12\pi}\int_{\sqrt{1/(2\pi)}}^{\sqrt{T/(2\pi)}} x^{-2k\sigma} h^{2k-1}(x-[x]) h'''(x-[x])\, dx \\ &= \begin{cases} \dfrac{-kC''_k}{6(2\pi)^{3/2}} T^{1/2} + O(1) & \text{if } \sigma = 0, \\ \dfrac{k(2\pi)^{k\sigma-(3/2)} C''_k}{12k\sigma - 6} T^{(1/2)-k\sigma} + D''_{k,\sigma} + O(T^{-k\sigma}) & \text{if } 0 < \sigma \le 1 \text{ and } \sigma \ne \dfrac{1}{2k}, \\ \dfrac{-kC''_k}{24\pi} \log T + D''_{k,1/(2k)} + O(T^{-1/2}) & \text{if } \sigma = \dfrac{1}{2k}. \end{cases} \end{aligned} \tag{4.11}$$

By (4.8) and (4.9), we see that

$$I_{13} = \begin{cases} O\left(\displaystyle\int_1^T t^{-1} dt\right) = O(\log T) & \text{if } \sigma = 0, \\ \displaystyle\int_1^{+\infty} (2\pi)^{k(\sigma-(1/2))} t^{-k\sigma} P_k(\sigma,t)\, dt - \int_T^{+\infty} (2\pi)^{k(\sigma-(1/2))} t^{-k\sigma} P_k(\sigma,t)\, dt & \\ \quad = D'''_{k,\sigma} + O(T^{-k\sigma}) & \text{if } 0 < \sigma \le 1. \end{cases} \tag{4.12}$$

Since $F_1(t)$ and $F_2(\sigma,t)$ is bounded function of t, we have

$$H_{k,\sigma}(t) = O(t^{-(k/2)-1}), \tag{4.13}$$

$$L_{k,\sigma}(t) = O(t^{-(k/2)-1}), \tag{4.14}$$

and

$$G_{k,\sigma}(t)|F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + R_{\sigma,t}|^{2k} = O(t^{-k\sigma-1}). \tag{4.15}$$

Hence

$$I_2 = \begin{cases} O\left(\displaystyle\int_1^T t^{-1} dt\right) = O(\log T) & \text{if } \sigma = 0, \\ \displaystyle\int_1^{+\infty} \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} H_{k,\sigma}(t)\, dt - \int_T^{+\infty} \left(\frac{t}{2\pi}\right)^{k((1/2)-\sigma)} H_{k,\sigma}(t)\, dt & \\ \quad = D''''_{k,\sigma} + O(T^{-k\sigma}) & \text{if } 0 < \sigma \le 1, \end{cases} \tag{4.16}$$

$$I_3 = \begin{cases} O\left(\int_1^T t^{-1}dt\right) = O(\log T) & \text{if } \sigma = 0, \\ \int_1^{+\infty}\left(\frac{t}{2\pi}\right)^{k(1/2-\sigma)} L_{k,\sigma}(t)dt - \int_T^{+\infty}\left(\frac{t}{2\pi}\right)^{k(1/2-\sigma)} L_{k,\sigma}(t)dt & \\ \quad = D_{k,\sigma}''''' + O(T^{-k\sigma}) & \text{if } 0 < \sigma \le 1, \end{cases} \tag{4.17}$$

$$I_4 = \begin{cases} O\left(\int_1^T t^{-1}dt\right) = O(\log T) & \text{if } \sigma = 0, \\ \int_1^{+\infty} G_{k,\sigma}(t)|F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + R_{\sigma,t}|^{2k}dt - \int_T^{+\infty} G_{k,\sigma}(t) & \\ \quad \cdot|F_1(t)t^{-1/4} + F_2(\sigma,t)t^{-3/4} + R_{\sigma,t}|^{2k}dt = D_{k,\sigma}'''''' + O(T^{-k\sigma}) & \text{if } 0 < \sigma \le 1. \end{cases} \tag{4.18}$$

Combine (4.10), (4.11), (4.12), (4.16), (4.17) and (4.18), we obtain (1.8). □

References

[1] I. Kiuchi and N. Yanagisawa, On mean value formulas for the approximate functional equation of the Riemann Zeta function, Arch. Math. (Basel), **78** (2002), 378–385.

[2] C. L. Siegel, Über Riemanns Nachlass zur analytischen Zahlentheorie, Quellen Stud. Geschichte der Math. Aster. Physik, Abt. B. Studien, **2** (1932), 45–80.

[3] E. C. Titchmarsh, The Theory of the Riemann Zeta Function, 2nd edition, revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.

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