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Convergence of stochastic integrals with respect to Hilbert-valued semimartingales

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Abstract. For sequences of stochastic integrals $\int_0^{\cdot} K_{s-}^n dX_s^n$, functional limit theorems are presented. And stability of strong solutions of stochastic differential equations of type

$$X^n = H^n + \int_0^{\cdot} f(X_{s-}^n) dY_s^n, \quad \forall n \geq 1$$

is discussed under jointly weak convergence of driving processes $\{(H^n, Y^n)\}_{n\geq 1}$. Where Y^n is an $\boldsymbol{H}$-valued semimartingale, H^n is a $\boldsymbol{G}$-valued càdlàg adapted process, K^n is an $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$-valued càdlàg adapted process and $f : \boldsymbol{G} \mapsto \mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$ satisfies a Lipschitz condition.

1. Introduction.

For all $n \geq 1$, let $(\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, P^n)$ be a filtered probability space satisfying the "usual conditions" and let $\boldsymbol{H}$ and $\boldsymbol{G}$ be real separable Hilbert spaces and X^n and K^n be $(\mathscr{F}_t^n)_{t\geq 0}$ adapted $\boldsymbol{H}$-valued semimartingale and $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$-valued càdlàg adapted process, respectively. Here $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$ is the space of bounded linear operators from $\boldsymbol{H}$ to $\boldsymbol{G}$. The tightness criteria of càdlàg Hilbert-valued adapted processes has been discussed by Joffe and Métivier ([**6**]) and Métivier and Nakao ([**8**]) and Xie ([**14**]). And Xie also discussed limit theorems of sequences of Hilbert-valued semimartingales. By these results, the one of purposes in this paper is to discuss functional limit theorems of sequences of stochastic integrals $\int_0^{\cdot} K_{s-}^n dX_s^n$ under jointly weak convergence of driving processes $\{(K^n, X^n)\}_{n\geq 1}$. This problem has been studied earlier by Duffie and Protter ([**1**]), Jakubowski and Mémin and Pages ([**5**]) under simple natural conditions in real processes.

The other purpose of this paper is to discuss the stability of the strong solutions of stochastic differential equations (SDE) of the type

$$X^n = H^n + \int_0^{\cdot} f(X_{s-}^n) dY_s^n, \quad n \geq 1, \tag{1.1}$$

$$X = H + \int_0^{\cdot} f(X_{s-}) dY_s, \tag{1.2}$$

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under jointly weak convergence of driving processes $\{(H^n, Y^n)\}_{n\geq 1}$, where $f: \boldsymbol{G} \mapsto \mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$ satisfies a Lipschitz condition, i.e. there exists a constant $L(>0)$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \forall x, y \in \boldsymbol{G}$$

and H^n, H and Y^n, Y are $\boldsymbol{G}$-valued càdlàg processes and $\boldsymbol{H}$-valued semimartingales, respectively. It is well known by Theorem 34.7 of Métivier ([**7**]) that there exists a unique strong solution of (1.1) and (1.2) for all $n \geq 1$. We will give some sufficient conditions under which $\{X^n\}_{n\geq 1}$ converges to X in law. This problem has been discussed earlier by several authors: Emery ([**2**], [**3**]) and Protter ([**10**]) have proved the stability of solutions of (SDE) (1.1) and (1.2) using a very strong topology in the space of semimartingales, and Slominski ([**11**], [**12**]) has proved the stability of solutions of (SDE) (1.1) under jointly weak convergence of driving processes $\{(H^n, Y^n)\}_{n\geq 1}$ in $\boldsymbol{R}^k$-valued processes. We cannot expect the above problems under only the assumption of the convergence of driving processes since they are not satisfied even in the deterministic case. Therefore, some additional conditions on sequences of semimartingales are indispensable.

In this paper we assume that sequences of $\boldsymbol{H}$-valued semimartingales satisfy the condition called (UT) introduced by Stricker ([**13**]) in $\boldsymbol{R}^k$-valued processes (where the Meyer and Zheng's convergence of semimartingales is considered in [**9**]). In Section 2, we will prove some preparative results under the condition (UT). In Section 3, the convergence of stochastic integrals with respect to Hilbert-valued semimartingales will be discussed, which is an extension of Jakubowski and Mémin and Pages's appropriate results in [**5**] to Hilbert-valued processes. Finally, the sufficient conditions will be provided for the stability of (SDE) (1.1), which is an extension of Slominski's appropriate results in [**11**].

2. Preparative results under the condition (UT).

Let $\boldsymbol{H}$ and $\boldsymbol{G}$ be real separable Hilbert spaces with scalar product $x \cdot y$ and norm $\|\cdot\|$ and let $\{e_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be orthogonal bases of $\boldsymbol{H}$ and $\boldsymbol{G}$, respectively. For $x \in \boldsymbol{H}$, put $x = \sum_{k=1}^{\infty} x_k e_k$. If Π_m maps $\boldsymbol{H}$ onto $\boldsymbol{R}^m$ of vectors $(x_1, \cdots, x_m)$, $\Pi_m x = (x_1, \cdots, x_m)$, then there exists a continuous mapping V_m of $\boldsymbol{R}^m$ into $\boldsymbol{H}$, where $\mathrm{V}_m(x_1, \cdots, x_m) = \sum_{k=1}^{m} x_k e_k$ and clearly $\|x - \mathrm{V}_m \Pi_m x\| \to 0$ when $m \to \infty$ for all $x \in \boldsymbol{H}$. There is the same result for $x \in \boldsymbol{G}$. We will use the same mappings Π_m and V_m on $\boldsymbol{H}$ and $\boldsymbol{G}$ if there is no confusion.

The setting is as follows: for every $n \geq 1$ we consider a stochastic basis $\mathscr{B}^n = (\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, P^n)$, E^n denotes the expectation with respect to P^n. All sets, variables, processes, etc, with the superscript n are defined on $\mathscr{B}^n$, and if there is no superscript, they are defined on stochastic basis $\mathscr{B} = (\Omega, \mathscr{F}, \mathscr{F}_t, P)$, usually without mentioning.

Definition 2.1. Let X^n be an $\boldsymbol{H}$-valued semimartingale. Denote by $\mathscr{H}_t^n(\boldsymbol{H}, \boldsymbol{G})$ the set of elementary predictable processes of the form

$$H_s^n = Y_0^n + \sum_{i=1}^{k-1} Y_{t_i}^n I_{]\!]t_i, t_{i+1}]\!]}(s)$$

such that $0 = t_0 < t_1 < \cdots < t_k = t$ and $Y^n_{t_i}$ is $\mathscr{F}^n_{t_i}$ measurable, $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$-valued random variable such that $\|Y^n_{t_i}\| \leq 1$. A sequence of $\{X^n\}_{n\geq 1}$ is called satisfying the condition (UT) if, for any fixed $t > 0$ and real separable Hilbert space $\boldsymbol{G}$ and for every $\epsilon > 0$, there exists an $N > 0$ such that

$$\sup_n \sup_{H^n \in \mathscr{H}^n_t(\boldsymbol{H},\boldsymbol{G})} P^n\left(\left\|\int_0^t H^n_s dX^n_s\right\| \geq N\right) < \epsilon. \tag{2.1}$$

Remark. When $\boldsymbol{H} = \boldsymbol{R}^k$ for some $k \geq 1$, this is similar to Stricker's definition.

Definition 2.2. Let X be an $\boldsymbol{H}$-valued semimartingale. An increasing positive regular right continuous adapted process A will be called a control process of X, if for every $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$-valued elementary predictable process H and every stopping time T,

$$E\left\{\sup_{t<T}\left\|\int_0^t H_s dX_s\right\|^2\right\} \leq E\left(A_{T-}\int_{]0,T[}\|H_s\|^2 dA_s\right) \tag{2.2}$$

for every real separable Hilbert space $\boldsymbol{G}$.

Theorem 2.1. *Let X be an $\boldsymbol{H}$-valued càdlàg adapted process.*

(i) *If X is a semimartingale, then X satisfies the condition (UT);*

(ii) *If $\left\{\int_0^t H_s \cdot dX_s, H \in \mathscr{H}_t(\boldsymbol{H}, \boldsymbol{R})\right\}$ is tight for all $t > 0$, then X is a semimartingale.*

Proof. (i) Suppose that X is a semimartingale. By Theorem 23.14 of Métivier ([**7**]), there exists a control process A such that, for any stopping time T, (2.2) is true. For every $n \geq 1$, put $S_n = \inf\{s > 0 : A_s \geq n\}$, then S_n is a stopping time and $S_n \uparrow \infty$ as $n \uparrow \infty$. This means that, for all $t > 0$ and $\epsilon > 0$, there exists an $n_0 \in \boldsymbol{N}$ such that $P(S_{n_0} \leq t) < \epsilon$. Hence, for all $H \in \mathscr{H}_t(\boldsymbol{H}, \boldsymbol{G})$,

$$\begin{aligned} P\left(\left\|\int_0^t H_s dX_s\right\| \geq N\right) &\leq \frac{1}{N^2} E\left[\sup_{t<S_{n_0}}\left\|\int_0^t H_s dX_s\right\|^2\right] + P(S_{n_0} \leq t) \\ &\leq \frac{1}{N^2} E(A^2_{S_{n_0}-}) + \epsilon \leq \frac{n_0^2}{N^2} + \epsilon. \end{aligned}$$

This implies that X satisfies the condition (UT).

(ii) Since X is a càdlàg adapted process, we deduce that $\sup_{s\leq t}\|X_s\| < \infty, a.s., \forall t > 0$. Hence we may suppose $E\left(\sup_{s\leq t}\|X_s\|\right) < \infty$ (if it is necessary, we may replace P by a equivalent probability measure with P). Put $K = \left\{\int_0^t H_s \cdot dX_s, H \in \mathscr{H}_t(\boldsymbol{H}, \boldsymbol{R})\right\}$. K is a convex subset of $L^1(\Omega, \mathscr{F}, P)$. The hypothesis yields $\xi_n/n \xrightarrow{P} 0$ for every sequence $\{\xi_n\}_{n\geq 1}$ in K. Theorem 12.23 of Yan ([**17**]) implies that there exists a bounded random variable $Z > 0$ such that $E(Z) = 1$ and $\sup_{\xi\in K} E(Z\xi) < \infty$. Put $\widehat{P} = Z \cdot P$. We have that $\widehat{P}$ and P are equivalent probability measures each other and $\widehat{E}\left(\sup_{s\leq t}\|X_s\|\right) = E\left(Z \cdot \sup_{s\leq t}\|X_s\|\right) < \infty$.

For every $t > 0$, we will prove that the stopping process $X_s^t = X_s I_{[\![0,t[\![}(s) + X_t I_{[\![t,\infty[\![}(s)$ is a quasimartingale under $\widehat{P}$. The definition of quasimartingale can see Definition 8.12 of He, Wang and Yan ([**4**]). Let $\tau : 0 = t_0 < t_1 < \cdots < t_k$ be a division of $[\![0,t]\!]$ and put

$$H_s^\tau = \sum_{i=0}^{k-1} \frac{\widehat{E}(X_{t_{i+1}}|\mathscr{F}_{t_i}) - X_{t_i}}{\|\widehat{E}(X_{t_{i+1}}|\mathscr{F}_{t_i}) - X_{t_i}\|} I_{]\!]t_i,t_{i+1}]\!]}(s),$$

we have that $H^\tau \in \mathscr{H}_t(\boldsymbol{H},\boldsymbol{R})$ and $\widehat{E}\big[\sum_{i=0}^{k-1} \|\widehat{E}(X_{t_{i+1}}|\mathscr{F}_{t_i}) - X_{t_i}\|\big] = \widehat{E}\int_0^t H_s^\tau \cdot dX_s$. This implies that

$$\begin{aligned}
\mathrm{Var}(X^t)_t &\,\widehat{=}\, \sup_\tau \sum_{i=0}^{k-1} \widehat{E}\|\widehat{E}(X_{t_{i+1}}|\mathscr{F}_{t_i}) - X_{t_i}\| + \widehat{E}(\|X_t\|) \\
&= \sup_\tau \widehat{E}\left(\int_0^t H_s^\tau \cdot dX_s\right) + \widehat{E}(\|X_t\|) \\
&= \sup_\tau E\left(Z\int_0^t H_s^\tau \cdot dX_s\right) + E(Z\|X_t\|) < \infty.
\end{aligned}$$

This means that X^t is a quasimartingale under $\widehat{P}$. Hence X^t is a semimartingale under P. Since $t > 0$ is arbitrary, we have X is a semimartingale under P. □

DEFINITION 2.3. Let X be an $\boldsymbol{H}$-valued càdlàg adapted process. X is called to satisfy the condition (C), if for any $\epsilon > 0$ and $N > 0$, there exists $m_0 \geq 1$ such that, for all $m \geq m_0$,

$$\sup_{H\in\mathscr{H}_t(\boldsymbol{H},\boldsymbol{G})} P\left(\left|\int_0^t H_s \cdot d(X_s - \mathrm{V}_m\Pi_m X_s)\right| \geq N\right) < \epsilon. \tag{2.3}$$

PROPOSITION 2.1. *Let X be an $\boldsymbol{H}$-valued càdlàg adapted process and satisfy the condition (C). X is a semimartingale if and only if the every component of X is a real semimartingale.*

PROOF. We only prove the sufficiency. Suppose that every component of X is a semimartingale. For every $H \in \mathscr{H}_t(\boldsymbol{H},\boldsymbol{R})$ and $N > 0$, we have

$$\begin{aligned}
&P\left(\left|\int_0^t H_s \cdot dX_s\right| \geq 2N\right) \\
&\quad\leq P\left(\left|\int_0^t H_s \cdot d(X_s - \mathrm{V}_m\Pi_m X_s)\right| \geq N\right) + P\left(\left|\int_0^t \Pi_m H_s \cdot d(\Pi_m X_s)\right| \geq N\right).
\end{aligned}$$

Since $\Pi_m X$ is a $\boldsymbol{R}^m$-valued semimartingale, we have that $\left\{\int_0^t H_s \cdot dX_s, H \in \mathscr{H}(\boldsymbol{H},\boldsymbol{R})\right\}$ is tight by the condition (C) and Theorem 12.24 of Yan ([**17**]). Theorem 2.1 implies that

X is a semimartingale. □

PROPOSITION 2.2. *Let $\{X^n\}_{n\geq 1}$ be a sequence of $\boldsymbol{H}$-valued semimartingales. $\{X^n\}_{n\geq 1}$ satisfies the condition (UT) if and only if $\{\sup_{s\leq t}\|\int_0^s H_u^n dX_u^n\|,$ $H^n\in\mathscr{H}_t^n(\boldsymbol{H},\boldsymbol{G}), n\geq 1\}$ is tight for every real separable Hilbert space $\boldsymbol{G}$.*

PROOF. We only prove the necessity. Suppose that $\{X^n\}_{n\geq 1}$ satisfies the condition (UT). For any $\epsilon>0$, there exists an $N>0$ such that, for every real separable Hilbert space $\boldsymbol{G}$, (2.1) is true. For any fixed $t>0$, $n\in\boldsymbol{N}$, real separable Hilbert space $\boldsymbol{G}$ and $H^n\in\mathscr{H}_t^n(\boldsymbol{H},\boldsymbol{G})$, put $T^n=\inf\{s>0:\|\int_0^s H_u^n dX_u^n\|\geq N\}\wedge t$, then T^n is a stopping time and

$$P^n\left(\sup_{s\leq t}\left\|\int_0^s H_u^n dX_u^n\right\|\geq N\right)\leq P^n\left(\left\|\int_0^{t\wedge T^n} H_s^n dX_s^n\right\|\geq N\right). \tag{2.4}$$

Choose a sequence of stopping times $\{T^{n,p}\}_{p\geq 1}$ such that $T^{n,p}$ takes its values in finite subset of $\{T^n(\omega),\omega\in\Omega^n\}$ and $T^{n,p}\uparrow T^n$ as $p\to\infty$. Hence we have $\int_0^{t\wedge T^{n,p}} H_s^n dX_s^n \xrightarrow{P^n} \int_0^{t\wedge T^n} H_s^n dX_s^n$. From $H^n I_{]\!]0,T^{n,p}]\!]}\in\mathscr{H}_t^n(\boldsymbol{H},\boldsymbol{G})$ and $P^n(\|\int_0^t H_s^n I_{]\!]0,T^{n,p}]\!]}(s)dX_s^n\|\geq N)$ $<\epsilon$, we have, by limit theorem,

$$P^n\left(\left\|\int_0^t H_s^n dX_s^n\right\|\geq N\right)\leq\epsilon. \tag{2.5}$$

This yields that $\{\int_0^t H_s\cdot dX_s, H\in\mathscr{H}_t(\boldsymbol{H},\boldsymbol{G})\}$ is tight by (2.4) and (2.5). □

Let X^n be an $\boldsymbol{H}$-valued semimartingale. So is $X^n-\mathrm{V}_m\Pi_m X^n$ for every $m\in\boldsymbol{N}$. Theorem 23.14 of Métivier ([**7**]) implies that there exists a control process $A^{n,m}$ of $X^n-\mathrm{V}_m\Pi_m X^n$ for all $m\in\boldsymbol{N}$.

THEOREM 2.2. *For every $t>0$, suppose that there exists $m_0\in\boldsymbol{N}$ such that*

$$\sup_n E^n(A_t^{n,m})^2<\infty,\quad \forall m\geq m_0. \tag{2.6}$$

We then have that $\{X^n\}_{n\geq 1}$ satisfies the condition (UT) if and only if $\{X^{n,k}\}_{n\geq 1}$ satisfies the condition (UT) for all $k\geq 1$, where $X^{n,k}$ is the kth component of X^n.

PROOF. The necessity is trivial. We only prove the sufficiency. For any $t>0$ and $H^n\in\mathscr{H}_t^n(\boldsymbol{H},\boldsymbol{G})$,

$$\begin{aligned}
&P^n\left(\left\|\int_0^t H_s^n dX_s^n\right\|\geq 2N\right)\\
&\leq P^n\left(\left\|\int_0^t H_s^n d(\mathrm{V}_m\Pi_m X_s^n)\right\|\geq N\right)+\frac{1}{N^2}E^n\left(A_t^{n,m}\int_0^t\|H_s^n\|^2 dA_s^{n,m}\right)\\
&\leq P^n\left(\left\|\int_0^t H_s^n d(\mathrm{V}_m\Pi_m X_s^n)\right\|\geq N\right)+\frac{1}{N^2}E^n(A_t^{n,m})^2
\end{aligned}$$

and the hypothesis imply that (2.1) is true for sufficient large N. Hence $\{X^n\}_{n\geq 1}$ satisfies the condition (UT). □

THEOREM 2.3. *Let* $\mathbf{K}$ *be a Polish space, and* K^n *and* X^n *be a* $\mathbf{K}$*-valued càdlàg adapted process and an* $\boldsymbol{H}$*-valued semimartingale, respectively. If* $(K^n, X^n) \xrightarrow{\mathscr{L}} (K, X)$ *and* $\{X^n\}_{n\geq 1}$ *satisfies the condition* (*UT*)*, then* X *is a semimartingale with respect to the natural filtration of* (K, X) *under* X *satisfying the condition* (*C*).

PROOF. By Theorem 2.1 of Jakubowski and Mémin and Pages ([**5**]), we have that every component of X is a real semimartingale with respect to the natural filtration of (K, X). Since X satisfies the condition (C), Proposition 2.1 implies that X is a semimartingale. □

3. Convergence of stochastic integrals under UT.

THEOREM 3.1. *Let* X^n *and* K^n *be an* $\boldsymbol{H}$*-valued semimartingale and a càdlàg adapted process, respectively; and let* $\{X^n\}_{n\geq 1}$ *satisfy the condition* (*UT*)*. If* $(K^n, X^n) \xrightarrow{\mathscr{L}} (K, X)$ *and* X *satisfies the condition* (*C*)*, we then have* $\int_0^\cdot K^n_{s-} \cdot dX^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot K_{s-} \cdot dX_s$ *and* $(K^n, X^n, \int_0^\cdot K^n_{s-} \cdot dX^n_s) \xrightarrow{\mathscr{L}} (K, X, \int_0^\cdot K_{s-} \cdot dX_s)$.

PROOF. Theorem 2.3 yields that X is a semimartingale with respect to the natural filtration of (K, X). Hence the stochastic integral $\int_0^\cdot K_{s-} \cdot dX_s$ may be defined. This theorem can be proved with the same method as the proof of Theorem 2.6 in [**5**]. We omit it. □

COROLLARY 3.1. *Let* X^n *be an* $\boldsymbol{H}$*-valued semimartingale and let* $\{X^n\}_{n\geq 1}$ *satisfy the condition* (*UT*)*. If* $X^n \xrightarrow{\mathscr{L}} X$ *and* X *satisfies the condition* (*C*)*, we have* $[X^n] \xrightarrow{\mathscr{L}} [X]$.

PROOF. Theorem 26.11 of Métivier ([**7**]) implies that

$$[X^n]_t = \|X^n_t\|^2 - \|X^n_0\|^2 - 2\int_{]0,t]} X^n_{s-} \cdot dX^n_s, \quad \forall t > 0.$$

By continuous theorem and Theorem 3.1, we deduce $[X^n] \xrightarrow{\mathscr{L}} [X]$. □

THEOREM 3.2. *Let* X^n *and* K^n *be the same processes as in Theorem* 3.1*. We write the semimartingale decomposition of* X^n *as* $X^n = M^n + A^n$*, where* M^n (*resp.* A^n) *is the martingale* (*resp. finite variation*) *part of* X^n*. Suppose that*

(i) $\{\mathrm{Var}(A^n)\}_{n\geq 1}$ *is tight, where* $\mathrm{Var}(A^n)$ *denotes the total variation process of* A^n;

(ii) *for every* $t > 0$, $\sup_n E^n \left(\sup_{s\leq t} \|\Delta M^n_s\|\right) < \infty$.

If $(K^n, X^n) \xrightarrow{\mathscr{L}} (K, X)$ *and* X *satisfies the condition* (*C*)*, then* X *is a semimartingale,* $\{X^n\}_{n\geq 1}$ *satisfies the condition* (*UT*) *and* $\int_0^\cdot K^n_{s-} \cdot dX^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot K_{s-} \cdot dX_s$.

PROOF. It is the same as the proof of Theorem 3 of Duffie and Protter in [**1**]. We omit it. □

LEMMA 3.1. *Let α_n and α be $\mathscr{L}(\boldsymbol{H},\boldsymbol{G})$-valued càdlàg functions on $\boldsymbol{R}_+$. If $\alpha_n \to \alpha$ in the Skorohod topology, then $(\alpha_n,\alpha_n^*) \to (\alpha,\alpha^*)$ in the Skorohod topology, where $\alpha^*(t)$ is the conjugate operator of $\alpha(t)$.*

PROOF. It is easy to prove by $\|\alpha_n(t)-\alpha(t)\| = \|\alpha_n^*(t)-\alpha^*(t)\|$. □

THEOREM 3.3. *Let an $\boldsymbol{H}$-valued semimartingale X^n have a decomposition $X^n = M^n + A^n$ and let K^n be an $\mathscr{L}(\boldsymbol{H},\boldsymbol{G})$-valued càdlàg adapted process. Suppose that*

(i) *$\{\langle M^n\rangle\}_{n\geq 1}$ and $\{\mathrm{Var}(A^n)\}_{n\geq 1}$ are C-tight;*

(ii) *for every $\epsilon>0$, $\delta>0$ and $N>0$, there exists an $n_0\in\boldsymbol{N}$ such that, for $n,\ m\geq n_0$,*

$$P^n\left(\sup_{t\leq N}\left\|\int_0^t K^n_{s-}dM^n_s - \mathrm{V}_m\Pi_m\int_0^t K^n_{s-}dM^n_s\right\|\geq\delta\right)<\epsilon, \tag{3.1}$$

$$P^n\left(\sup_{t\leq N}\left\|\int_0^t K^n_{s-}dA^n_s - \mathrm{V}_m\Pi_m\int_0^t K^n_{s-}dA^n_s\right\|\geq\delta\right)<\epsilon. \tag{3.2}$$

If $(K^n,X^n)\xrightarrow{\mathscr{L}}(K,X)$ and X satisfies the condition (C), then $\int_0^t K^n_{s-}dX^n_s \xrightarrow{\mathscr{L}} \int_0^t K_{s-}dX_s$ and $\left(K^n,X^n,\int_0^t K^n_{s-}dX^n_s\right)\xrightarrow{\mathscr{L}}\left(K,X,\int_0^t K_{s-}dX_s\right)$.

PROOF. Let $\boldsymbol{H}_1$ be any fixed real separable Hilbert space. For any $K^n\in\mathscr{H}^n_t(\boldsymbol{H},\boldsymbol{H}_1)$, $L>0$ and $\eta>0$, Lenglart's inequality implies

$$\begin{aligned}
&P^n\left(\left\|\int_0^t K^n_{s-}dX^n_s\right\|\geq 2L\right)\\
&\quad\leq P^n\left(\left\|\int_0^t K^n_{s-}dM^n_s\right\|\geq L\right)+P^n\left(\left\|\int_0^t K^n_{s-}dA^n_s\right\|\geq L\right)\\
&\quad\leq \frac{\eta}{L^2}+P^n\left(\langle M^n\rangle_t\geq\eta\right)+P^n\left(\mathrm{Var}(A^n)_t\geq L\right).
\end{aligned}\tag{3.3}$$

First let η and then L be sufficient large in (3.3), we can deduce that $\{X^n\}_{n\geq1}$ satisfies the condition (UT). Since X satisfies the condition (C), we get that X is a semimartingale by Theorem 2.3.

$K^n\xrightarrow{\mathscr{L}}K$ implies $(K^n,(K^n)^*)\xrightarrow{\mathscr{L}}(K,K^*)$ by Lemma 3.1. Moreover, for every $g\in\boldsymbol{G}$, $(K^n,(K^n)^*,(K^n)^*(g),X^n)\xrightarrow{\mathscr{L}}(K,K^*,K^*(g),X)$ and $\left(\int_0^\cdot K^n_{s-}dX^n_s\right)\cdot g = \int_0^\cdot (K^n_{s-})^*(g)\cdot dX^n_s$ imply $\int_0^\cdot (K^n_{s-})^*(g)\cdot dX^n_s\xrightarrow{\mathscr{L}}\int_0^\cdot K^*_{s-}(g)\cdot dX_s$ by Theorem 3.1. Hence, we have $\left(\int_0^\cdot K^n_{s-}dX^n_s\right)\cdot g\xrightarrow{\mathscr{L}}\left(\int_0^\cdot K_{s-}dX_s\right)\cdot g$. This means that $\int_0^\cdot K_{s-}dX_s$ is the only possible limit point of the sequence $\left\{\int_0^\cdot K^n_{s-}dX^n_s\right\}_{n\geq1}$ in the Skorohod topology. Therefore it remains to prove that the sequence $\left\{\int_0^\cdot K^n_{s-}dX^n_s\right\}_{n\geq1}$ is tight.

(a) For any $\epsilon>0$ and $N>0$, $K^n\xrightarrow{\mathscr{L}}K$ yields that there exist $L>0$ and $n_0\in\boldsymbol{N}$ such that $P^n\left(\sup_{s\leq N}\|K^n_s\|\geq L\right)<\epsilon$ for $n\geq n_0$. Put $T^{n,L}=\inf\{t>0:\|K^n_t\|\geq L\}\wedge N$. Then $T^{n,L}$ is a stopping time and $P^n(T^{n,L}\leq N)<\epsilon$. For $C>0$ and $\eta>0$, Lenglart's inequality implies

$$P^n\left(\sup_{t\le N}\left\|\int_0^t K^n_{s-}dM^n_s\right\|\ge C\right)$$
$$\le \frac{\eta}{C^2}+P^n\biggl(\int_0^N \|K^n_{s-}\|^2 d\langle M^n\rangle_s\ge \eta, T^{n,L}\ge N\biggr)+P^n(T^{n,L}\le N).$$

We have

$$P^n\left(\sup_{t\le N}\left\|\int_0^t K^n_{s-}dM^n_s\right\|\ge C\right)\le \frac{\eta}{C^2}+P^n\left(\langle M^n\rangle_N\ge\frac{\eta}{L^2}\right)+\epsilon,\quad \forall n\ge n_0. \tag{3.4}$$

First let η and then C be sufficient large in (3.4), we know that $\{\int_0^\cdot K^n_{s-}dX^n_s\}_{n\ge1}$ satisfies Theorem 3.3 (i) of Xie ([**14**]).

(b) Let S_n and T_n be stopping times with $S_n\le T_n\le N$. Put $N^n_t=\int_{t\wedge S_n}^t K^n_{s-}dM^n_s$. Then N^n is a $\boldsymbol{G}$-valued locally square integrable martingale and $\left(\int_{t\wedge S_n}^t\|K^n_{s-}\|^2 d\langle M^n\rangle_s\right)_{t\ge0}$ strongly majorizes $\|N^n\|^2$. We then have, for any $\epsilon_1>0$ and $\delta>0$,

$$P^n\left(\left\|\int_{]S_n,T_n]}K^n_{s-}dM^n_s\right\|\ge\epsilon_1\right)\le\frac{\delta}{\epsilon_1^2}+P^n\left(\langle M^n\rangle_{T_n}-\langle M^n\rangle_{S_n}\ge\frac{\delta}{L^2}\right)+\epsilon,\quad\forall n\ge n_0. \tag{3.5}$$

Since $\{\langle M^n\rangle\}_{n\ge1}$ is C-tight, there exist $n_1\ge n_0$ and $\eta_0>0$ such that

$$\sup_{n\ge n_1}P^n\left(w_N(\langle M^n\rangle,\eta_0)\ge\frac{\delta}{L^2}\right)<\epsilon, \tag{3.6}$$

where $w_N(\langle M^n\rangle,\eta_0)=\sup\{\sup_{t\le s_1,s_2\le t+\eta_0}[\langle M^n\rangle_{s_2}-\langle M^n\rangle_{s_1}]:0\le t\le t+\eta_0\le N\}$. If the above S_n and T_n also satisfy $T_n\le S_n+\eta_0$, we have $\langle M^n\rangle_{T_n}-\langle M^n\rangle_{S_n}\le w_N(\langle M^n\rangle,\eta_0)$. Hence, (3.5) and (3.6) yield

$$\sup_{n\ge n_1}\sup_{T_n\le N,S_n\le T_n\le S_n+\eta_0}P^n\left(\left\|\int_{]S_n,T_n]}K^n_{s-}dM^n_s\right\|\ge\epsilon_1\right)\le\frac{\delta}{\epsilon_1^2}+2\epsilon. \tag{3.7}$$

By Aldous' Theorem (Theorem 2.2.2 in [**6**]), we have that $\{\int_0^\cdot K^n_{s-}dM^n_s\}_{n\ge1}$ satisfies Theorem 3.3 (ii) of Xie in [**14**] by (3.7).

By (a), (b) and the hypothesis (3.1), we deduce that $\{\int_0^\cdot K^n_{s-}dM^n_s\}_{n\ge1}$ satisfies the conditions of Theorem 3.3 of Xie ([**14**]). Hence $\{\int_0^\cdot K^n_{s-}dM^n_s\}_{n\ge1}$ is tight.

(c) $\sup_{t\le N}\|\int_0^t K^n_{s-}dA^n_s\|\le\int_0^{N\wedge T^{n,L}}\|K^n_{s-}\|d(\mathrm{Var}(A^n)_s)$ and $\int_0^\cdot\|K^n_{s-}\|d(\mathrm{Var}(A^n)_s)$ being strongly majoration $\int_0^\cdot K^n_{s-}dA^n_s$ yield $w_N(\int_0^\cdot K^n_{s-}dA^n_s,\theta)\le w_N(\int_0^\cdot\|K^n_{s-}\|d(\mathrm{Var}(A^n)_s),\theta)$. As the proof of (a) and (b) we can deduce that $\{\int_0^\cdot K^n_{s-}dA^n_s\}_{n\ge1}$ is C-tight by (3.2).

Since $\int_0^\cdot K^n_{s-}dX^n_s=\int_0^\cdot K^n_{s-}dM^n_s+\int_0^\cdot K^n_{s-}dA^n_s$, we obtain that $\{\int_0^\cdot K^n_{s-}dX^n_s\}_{n\ge1}$ is tight by Lemma 3.5 of Xie ([**14**]). □

Theorem 3.4. *Let A^n be an $\boldsymbol{H}$-valued finite variation adapted process and let K^n be an $\mathscr{L}(\boldsymbol{H},\boldsymbol{G})$-valued càdlàg adapted process. Suppose that $\{\mathrm{Var}(A^n)\}_{n\geq 1}$ is tight, $(K^n, A^n) \xrightarrow{\mathscr{L}} (K, A)$ and that A satisfies the condition (C). We then have the followings*:

(i) *A is an $\boldsymbol{H}$-valued finite variation adapted process*;

(ii) *Suppose that $\{K^n\}_{n\geq 1}$ and $\{A^n\}_{n\geq 1}$ satisfy* (3.2). *We then have $\int_0^\cdot K^n_{s-}dA^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot K_{s-}dA_s$*;

(iii) *Choose $\boldsymbol{G} = \boldsymbol{R}$. Put $[K^n]^a = \sum_{s\leq\cdot} \Delta K^n_s I_{\{\|\Delta K^n_s\|>a\}}$. Suppose $\{[K^n]^a\}_{n\geq 1}$ satisfy the condition (UT) for all $a > 0$. Then we have*

$$\left(K^n, A^n, \sum_{s\leq\cdot}(\Delta K^n_s)\cdot(\Delta A^n_s)\right) \xrightarrow{\mathscr{L}} \left(K, A, \sum_{s\leq\cdot}(\Delta K_s)\cdot(\Delta A_s)\right); \tag{3.8}$$

(iv) *Let $\|\Delta K^n\| \leq C$ for some constant $C > 0$. Let $\{[(K^n)^*(g)]^a\}_{n\geq 1}$ meet the condition (UT) for all $g \in \boldsymbol{G}$ and $a > 0$. Suppose that, for all $\epsilon > 0$, $\delta > 0$ and $N > 0$, there is an $n_0 \in \boldsymbol{N}$ such that*

$$P^n\left(\sup_{t\leq N}\left\|\int_0^t K^n_s dA^n_s - \mathrm{V}_m\Pi_m\int_0^t K^n_s dA^n_s\right\| \geq \delta\right) < \epsilon, \quad \forall m,\ n \geq n_0. \tag{3.9}$$

We then have $\int_0^\cdot K^n_s dA^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot K_s dA_s$.

Proof. (i) By the hypothesis we deduce that $\{A^n\}_{n\geq 1}$ satisfies the condition (UT). Theorem 2.3 implies that A is a semimartingale. Moreover, Proposition 2.6(a) of Jakubowshi and Mémin and Pages ([**5**]) implies that every component of A is a real finite variation adapted process. This yields that the martingale part of A is zero. Hence A is a finite variation adapted process.

(ii) This is the corollary of Theorem 3.3.

(iii) Choose $a > 0$ such that $\Delta K \neq a$, $P-a.s.$. Hence $(K^n, A^n) \xrightarrow{\mathscr{L}} (K, A)$ implies $([K^n]^a, A) \xrightarrow{\mathscr{L}} ([K]^a, A)$. Since $\{[K^n]^a\}_{n\geq 1}$ satisfies the condition (UT), by Theorem 3.1 we have $\int_0^\cdot A^n_{s-}\cdot d[K^n]^a_s \xrightarrow{\mathscr{L}} \int_0^\cdot A_{s-}\cdot d[K]^a_s$. By the hypotheses, it is easy to prove

$$\begin{aligned}&\left([K^n]^a, A^n, \int_0^\cdot [K^n]^a_{s-}\cdot dA^n_s, \int_0^\cdot A^n_{s-}\cdot d[K^n]^a_s\right)\\ &\quad\xrightarrow{\mathscr{L}} \left([K]^a, A, \int_0^\cdot [K]^a_{s-}\cdot dA_s, \int_0^\cdot A_{s-}\cdot d[K]^a_s\right).\end{aligned}$$

This yields that

$$\begin{aligned}&[K^n]^a\cdot A^n - \int_0^\cdot [K^n]^a_{s-}\cdot dA^n_s - \int_0^\cdot A^n_{s-}\cdot d[K^n]^a_s\\ &\quad\xrightarrow{\mathscr{L}} [K]^a\cdot A - \int_0^\cdot [K]^a_{s-}\cdot dA_s - \int_0^\cdot A_{s-}\cdot d[K]^a_s.\end{aligned} \tag{3.10}$$

Theorem 26.11 of Métivier ([**7**]) and (3.10) imply that

$$\sum_{s\leq \cdot}(\Delta[K^n]^a_s)\cdot(\Delta A^n_s)\xrightarrow{\mathscr{L}}\sum_{s\leq \cdot}(\Delta[K]^a_s)\cdot(\Delta A_s). \tag{3.11}$$

For any $N>0$, since

$$\sup_{t\leq N}\left\|\sum_{s\leq t}(\Delta K^n_s)\cdot(\Delta A^n_s)-\sum_{s\leq t}(\Delta[K^n]^a_s)\cdot(\Delta A^n_s)\right\|\leq a\cdot \mathrm{Var}(A^n)_N \tag{3.12}$$

and

$$\sup_{t\leq N}\left\|\sum_{s\leq t}(\Delta K_s)\cdot(\Delta A_s)-\sum_{s\leq t}(\Delta[K]^a_s)\cdot(\Delta A_s)\right\|\leq a\cdot \mathrm{Var}(A)_N, \tag{3.13}$$

we deduce that (3.8) is true by (3.11), (3.12) and (3.13).

(iv) Since $\|\Delta K^n\|\leq C$, it is easy to prove that (3.2) is true from (3.9). Hence, Theorem 3.3 implies that

$$\left(K^n, A^n, \int_0^\cdot K^n_{s-}dA^n_s\right)\xrightarrow{\mathscr{L}}\left(K, A, \int_0^\cdot K_{s-}dA_s\right). \tag{3.14}$$

By (iii) and (3.14) we can deduce, for all $g\in \boldsymbol{G}$,

$$\left(\int_0^\cdot (K^n_{s-})^*(g)\cdot dA^n_s,\ \sum_{s\leq\cdot}(\Delta(K^n_s)^*(g))\cdot(\Delta A^n_s)\right)$$
$$\xrightarrow{\mathscr{L}}\left(\int_0^\cdot K^*_{s-}(g)\cdot dA_s,\ \sum_{s\leq\cdot}(\Delta(K_s)^*(g))\cdot(\Delta A_s)\right).$$

This implies that, for all $g\in\boldsymbol{G}$,

$$\int_0^\cdot (K^n_{s-})^*(g)\cdot dA^n_s+\sum_{s\leq\cdot}(\Delta(K^n_s)^*(g))\cdot(\Delta A^n_s)$$
$$\xrightarrow{\mathscr{L}}\int_0^\cdot K^*_{s-}(g)\cdot dA_s+\sum_{s\leq\cdot}(\Delta(K_s)^*(g))\cdot(\Delta A_s).$$

That is,

$$\left(\int_0^\cdot K^n_s dA^n_s\right)\cdot g\xrightarrow{\mathscr{L}}\left(\int_0^\cdot K_s dA_s\right)\cdot g,\quad \forall g\in\boldsymbol{G}. \tag{3.15}$$

(3.15) means that $\int_0^\cdot K_s dA_s$ is the only possible limit point of the sequence $\{\int_0^\cdot K_s^n dA_s^n\}_{n\geq 1}$ in the Skorohod topology. As the proof (c) in Theorem 3.3 we can prove that the sequence $\{\int_0^\cdot K_s^n dA_s^n\}_{n\geq 1}$ is tight. □

COROLLARY 3.2. *Let X^n satisfy the conditions of Theorem* 3.3, $X^n \xrightarrow{\mathscr{L}} X$ *and X meet the condition* (C).

(i) *We replace K^n by X^n and suppose that* (3.1) *and* (3.2) *hold for* $\{X^n\}_{n\geq 1}$, $\{M^n\}_{n\geq 1}$ *and* $\{A^n\}_{n\geq 1}$. *We then have* $(X^n, [\![X^n]\!]) \xrightarrow{\mathscr{L}} (X, [\![X]\!])$;

(ii) *Let Z^n be an $\mathscr{L}(\boldsymbol{H}\hat{\otimes}_2\boldsymbol{H}, \boldsymbol{G})$-valued càdlàg adapted process, $\|\Delta Z^n\| \leq C$ for some constant $C > 0$ and $(Z^n, X^n) \xrightarrow{\mathscr{L}} (Z, X)$. Suppose that, for above $\epsilon > 0$, $\eta > 0$ and $N > 0$, there exists an $n_0 \in \boldsymbol{N}$ such that*

$$P^n\left(\sup_{t\leq N}\left\|\int_0^t Z_{s-}^n d[\![X^n]\!]_s - \mathrm{V}_{m\times m}\Pi_{m\times m}\int_0^t Z_{s-}^n d[\![X^n]\!]_s\right\| \geq \delta\right) < \epsilon, \quad \forall n, m \geq n_0, \tag{3.16}$$

under the hypothesis (i) *we have*

$$\left(X^n,\ [\![X^n]\!],\ \int_0^\cdot Z_s^n d[\![X^n]\!]_s\right) \xrightarrow{\mathscr{L}} \left(X,\ [\![X]\!],\ \int_0^\cdot Z_s d[\![X]\!]_s\right).$$

PROOF. (i) By Theorem 3.3 and the hypotheses for X^n, M^n and A^n we have

$$\left(X^n, \int_0^\cdot X_{s-}^n \otimes dX_s^n\right) \xrightarrow{\mathscr{L}} \left(X, \int_0^\cdot X_{s-} \otimes dX_s\right). \tag{3.17}$$

Since $\|x\otimes y - \mathrm{V}_{m\times m}\Pi_{m\times m}x\otimes y\|_1 = \|y\otimes x - \mathrm{V}_{m\times m}\Pi_{m\times m}y\otimes x\|_1$, the hypotheses imply that $\{\int_0^\cdot dM_s^n \otimes X_{s-}^n\}_{n\geq 1}$ and $\{\int_0^\cdot dA_s^n \otimes X_{s-}^n\}_{n\geq 1}$ satisfy Theorem 3.3 (iii) of Xie in [**14**]. As the proof of Theorem 3.3 we may prove that $\{\int_0^\cdot dM_s^n \otimes X_{s-}^n\}_{n\geq 1}$ and $\{\int_0^\cdot dA_s^n \otimes X_{s-}^n\}_{n\geq 1}$ are tight by Theorem 3.3 of Xie in [**14**] and that $\int_0^\cdot dX_s \otimes X_{s-}$ is the only possible limit point of the sequence $\{\int_0^\cdot dX_s^n \otimes X_{s-}^n\}_{n\geq 1}$. It is easy to prove

$$\left(X^n, \int_0^\cdot dX_s^n \otimes X_{s-}^n\right) \xrightarrow{\mathscr{L}} \left(X, \int_0^\cdot dX_s \otimes X_{s-}\right). \tag{3.18}$$

By Theorem 26.11 of Métivier ([**7**]), the continuity of tensor product, (3.17) and (3.18), we deduce that

$$\begin{aligned}[\![X^n]\!] &= (X^n)^{\otimes_2} - (X_0^n)^{\otimes_2} - \int_0^\cdot X_{s-}^n \otimes dX_s^n - \int_0^\cdot dX_s^n \otimes X_{s-}^n \\ &\xrightarrow{\mathscr{L}} X^{\otimes_2} - X_0^{\otimes_2} - \int_0^\cdot X_{s-} \otimes dX_s - \int_0^\cdot dX_s \otimes X_{s-} = [\![X]\!].\end{aligned}$$

(ii) This is the corollary of (i) and Theorem 3.4(iv). □

THEOREM 3.5. *Let X^n and K^n be an $\boldsymbol{H}$-valued semimartingale and an $\mathscr{L}(\boldsymbol{H},\boldsymbol{G})$-valued càdlàg adapted process, respectively. K^n is a predictable process with respect to the natural filtration of (K^n, X^n) with $\|\Delta K^n\| \le C$ for some $C>0$, and $\{[(K^n)^*(g)]^a\}_{n\ge1}$ satisfies the condition (UT) for all $g\in\boldsymbol{G}$ and $a>0$. $[(K^n)^*(g)]^a$ and $[K^*(g)]^a$ are defined as Theorem 3.4(iii). Suppose that X^n satisfies the conditions in Theorem 3.3 and for every $\epsilon>0$, $\delta>0$ and $N>0$, there exists an $n_0\in\boldsymbol{N}$ such that, for $n,m\ge n_0$,*

$$P^n\left(\sup_{t\le N}\left\|\int_0^t \Delta K^n_s dM^n_s - \mathrm{V}_m\Pi_m\int_0^t \Delta K^n_s dM^n_s\right\|\ge\delta\right)<\epsilon, \tag{3.19}$$

$$P^n\left(\sup_{t\le N}\left\|\int_0^t \Delta K^n_s dA^n_s - \mathrm{V}_m\Pi_m\int_0^t \Delta K^n_s dA^n_s\right\|\ge\delta\right)<\epsilon. \tag{3.20}$$

If $(K^n,X^n)\xrightarrow{\mathscr{L}}(K,X)$ and X is a semimartingale, then $\int_0^\cdot K^n_s dX^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot K_s dX_s$.

PROOF. By the hypotheses, Theorem 3.3 implies that $\int_0^\cdot K^n_{s-}dX^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot K_{s-}dX_s$. Hence we only prove $\int_0^\cdot \Delta K^n_s dX^n_s \xrightarrow{\mathscr{L}} \int_0^\cdot \Delta K_s dX_s$.

For every fixed $a>0$ such that $\Delta K\neq a, P-a.s.$. $[(K^n)^*(g)]^a$ and $[K^*(g)]^a$ are $\boldsymbol{H}$-valued finite variation processes. We deduce

$$\left((K^n)^*,[(K^n)^*(g)]^a,X^n\right)\longrightarrow\left(K^*,[K^*(g)]^a,X\right),\quad \forall g\in\boldsymbol{G}$$

by the Lemma 3.1 and the hypotheses. As the proof of Theorem 3.4(iii), we have

$$\begin{aligned}&\left([(K^n)^*(g)]^a,X^n,\int_0^\cdot [(K^n)^*(g)]^a_{s-}\cdot dX^n_s,\int_0^\cdot X^n_{s-}\cdot d\,[(K^n)^*(g)]^a_s\right)\\ &\quad\xrightarrow{\mathscr{L}}\left([K^*(g)]^a,X,\int_0^\cdot [K^*(g)]^a_{s-}\cdot dX_s,\int_0^\cdot X_{s-}\cdot d\,[K^*(g)]^a_s\right).\end{aligned}$$

This implies that

$$\begin{aligned}\left[[(K^n)^*(g)]^a,X^n\right]&=[(K^n)^*(g)]^a\cdot X^n-\int_0^\cdot [(K^n)^*(g)]^a_{s-}\cdot dX^n_s-\int_0^\cdot X^n_{s-}\cdot d\,[(K^n)^*(g)]^a_s\\ &\xrightarrow{\mathscr{L}}[K^*(g)]^a\cdot X-\int_0^\cdot [K^*(g)]^a_{s-}\cdot dX_s-\int_0^\cdot X_{s-}\cdot d\,[K^*(g)]^a_s\\ &=\left[[K^*(g)]^a,X\right].\end{aligned}\tag{3.21}$$

Since $[(K^n)^*(g)]^a$ and $[K^*(g)]^a$ are $\boldsymbol{H}$-valued predictable processes, by Yœurp's lemma and (3.21) we deduce

$$\int_0^\cdot \Delta\,[(K^n)^*(g)]^a_s\cdot dX^n_s\xrightarrow{\mathscr{L}}\int_0^\cdot \Delta\,[K^*(g)]^a_s\cdot dX_s. \tag{3.22}$$

For every $N > 0$, since

$$\begin{aligned}\sup_{t\le N}\left|\int_0^t \Delta(K_s^n)^*(g)\cdot dX_s^n - \int_0^t \Delta\left[(K^n)^*(g)\right]_s^a \cdot dX_s^n\right| \\ \le \sup_{t\le N}\left|\int_0^t \Delta(K_s^n)^*(g) I_{\{\|\Delta(K_s^n)^*(g)\|\le a\}}\cdot dX_s^n\right|\end{aligned}$$

and $\{X^n\}_{n\ge1}$ satisfies the condition (UT), we have the following: for all $\epsilon > 0$, there exists a $\delta_0 > 0$ such that

$$\sup_n P^n\left(\sup_{t\le N}\left|\int_0^t \Delta(K_s^n)^*(g) I_{\{\|\Delta(K_s^n)^*(g)\|\le a\}}\cdot dX_s^n\right| \ge \epsilon\right) < \epsilon_0, \quad \forall a < \delta_0. \tag{3.23}$$

For $(K^*(g), X)$, Theorem 2.1 implies that, for above $\epsilon > 0$, there exists a $\delta_1 > 0$ ($\delta_1 < \delta_0$) such that

$$P\left(\sup_{t\le N}\left|\int_0^t \Delta K_s^*(g) I_{\{\|\Delta K_s^*(g)\|\le a\}}\cdot dX_s\right| \ge \epsilon\right) < \epsilon_1, \quad \forall a < \delta_0. \tag{3.24}$$

(3.22)–(3.24) imply that $\int_0^\cdot \Delta(K_s^n)^*(g)\cdot dX_s^n \xrightarrow{\mathscr{L}} \int_0^\cdot \Delta K_s^*(g)\cdot dX_s$, that is,

$$\left(\int_0^\cdot (\Delta K_s^n) dX_s^n\right)\cdot g \xrightarrow{\mathscr{L}} \left(\int_0^\cdot (\Delta K_s) dX_s\right)\cdot g. \tag{3.25}$$

For $g \in \boldsymbol{G}$ being arbitrary, (3.25) means that $\int_0^\cdot (\Delta K_s) dX_s$ is the only possible limit point of sequence $\left\{\int_0^\cdot (\Delta K_s^n) dX_s^n\right\}_{n\ge1}$. As the proof of Theorem 3.3, we can prove that $\left\{\int_0^\cdot (\Delta K_s^n) dX_s^n\right\}_{n\ge1}$ is tight by the hypotheses. □

COROLLARY 3.3. *Let K^n and X^n satisfy the conditions of Theorem* 3.5. *We have that $\left\{\int_0^\cdot K_s^n dX_s^n\right\}_{n\ge1}$ satisfies the condition (UT).*

PROOF. It is easy to prove by Lenglart's inequality. □

4. Stability of strong solutions of stochastic differential equations.

THEOREM 4.1. *Let H^n be a $\boldsymbol{G}$-valued càdlàg adapted process and let Y^n be an $\boldsymbol{H}$-valued special semimartingale with the canonical decomposition $Y^n = M^n + A^n$ and ν^n be the compensator of the random measure μ^n associated to the jumps of Y^n. Assume that X^n and X are the strong solutions of* (SDE) (1.1) *and* (1.2), *respectively. Suppose that the following conditions are fulfilled*:

(i) $\{\langle M^n\rangle\}_{n\ge1}$ *and* $\{\mathrm{Var}(A^n)\}_{n\ge1}$ *are tight*;

(ii) $\{\|x\| \vee \|x\|^2.\nu^n\}_{n\ge1}$ *is tight*;

(iii) *For all $K > 0$ and $\epsilon > 0$, there exists an $m_0 \in \boldsymbol{N}$ such that, for all $m \ge m_0$, $\sup_{\|y\|\le K}\|(I - \mathrm{V}_m\Pi_m)f(x)y\| < \epsilon$, where I is the identical operator on $\boldsymbol{H}$.*

If $(H^n, Y^n) \xrightarrow{\mathscr{L}} (H, Y)$, *then* $X^n \xrightarrow{\mathscr{L}} X$ *and* $(H^n, Y^n, X^n) \xrightarrow{\mathscr{L}} (H, Y, X)$.

Proof. Let $h : \boldsymbol{H} \to \boldsymbol{H}$ be a truncation function and satisfy $h(x) = x$ for $\|x\| \leq 1$ and $h(x) = 0$ for $x > 2$. Put

$$\check{Y}^n(h) = \sum_{s\leq \cdot}[\Delta Y^n_s - h(\Delta Y^n_s)], \quad Y^n(h) = Y^n - \check{Y}^n(h).$$

Then $Y^n(h)$ is a special semimartingale. We denote its canonical decomposition $Y^n(h) = M^n(h) + A^n(h)$. By Proposition 1.4.10 of Xie ([**15**]), we have

$$A^n(h) = A^n - \int_0^\cdot \int_{\boldsymbol{H}} (x - h(x))\nu^n(ds, dx), \tag{4.1}$$

$$\begin{aligned}\langle\!\langle M^n(h)\rangle\!\rangle = \langle\!\langle M^n\rangle\!\rangle &+ \int_0^\cdot \int_{\boldsymbol{H}} \left[h^{\otimes_2}(x) - x^{\otimes_2}\right]\nu^n(ds, dx) \\ &- \sum_{s\leq\cdot}\left[\int_{\boldsymbol{H}} h(x)\nu^n(\{s\}\times dx)\right]^{\otimes_2} + \sum_{s\leq\cdot}\left[\int_{\boldsymbol{H}} x\nu^n(\{s\}\times dx)\right]^{\otimes_2}. \end{aligned}\tag{4.2}$$

Moreover, (4.1) and (4.2) imply that

$$\mathrm{Var}(A^n(h)) \leq \mathrm{Var}(A^n) + C\|x\|.\nu^n \tag{4.3}$$

and

$$\langle M^n(h)\rangle \leq \langle M^n\rangle + C\|x\| \vee \|x\|^2.\nu^n \tag{4.4}$$

for some constant $C > 0$. Conditions (i) and (ii) imply the tightness of $\{\langle M^n(h)\rangle\}_{n\geq 1}$ and $\{\mathrm{Var}(A^n(h))\}_{n\geq 1}$ by (4.3) and (4.4). Since $\langle M^n(h)\rangle$ and $[M^n(h)]$ strongly majorize each other, the tightness of $\{\langle M^n(h)\rangle\}_{n\geq 1}$ yields the tightness of $\{[M^n(h)]\}_{n\geq 1}$.

For every $N > 0$, put

$$T^n_N = \inf\{t > 0 : \|X^n_t\| + \|H^n_t\| \geq N \ \text{ or } \ \|X^n_{t-}\| + \|H^n_{t-}\| \geq N\}, \tag{4.5}$$

$$T_N = \inf\{t > 0 : \|X_t\| + \|H_t\| \geq N \ \text{ or } \ \|X_{t-}\| + \|H_{t-}\| \geq N\}, \tag{4.6}$$

then T^n_N and T_N are stopping times. If for simplicity we denote $X^{n,N} = (X^n)^{T^n_N}$, $H^{n,N} = (H^n)^{T^n_N}$, $Y^{n,N} = (Y^n)^{T^n_N}$, $X^{\cdot,N} = X^{T_N}$, $H^{\cdot,N} = H^{T_N}$, $Y^{\cdot,N} = Y^{T_N}$, then $X^{n,N}$ and $X^{\cdot,N}$ are the strong solutions of (SDE):

$$X^{n,N} = H^{n,N} + \int_0^\cdot f(X^{n,N}_{s-})dY^{n,N}_s$$

and

$$X^{\cdot,N} = H^{\cdot,N} + \int_0^{\cdot} f(X_{s-}^{\cdot,N}) dY_s^{\cdot,N}.$$

As the proof of Theorem 1 of Slominski in [**11**], we can prove that $\{X^{n,N}\}_{n\geq 1}$ satisfies the following conditions:

(a) for all $\epsilon > 0$, $N > 0$, there exist an $n_0 \in \boldsymbol{N}$ and a $K > 0$ such that

$$P^n\left(\sup_{t\leq N} \|X_t^{n,N}\| \geq K\right) < \epsilon, \quad n \geq n_0, \tag{4.7}$$

(b) for all $\epsilon > 0$, $\delta > 0$, $N > 0$, there exist an $n_0 \in \boldsymbol{N}$ and a $\theta > 0$ such that

$$P^n\left(w'_N(X^{n,N}, \theta) \geq \delta\right) < \epsilon, \quad n \geq n_0. \tag{4.8}$$

From the hypothesis (iii) and $Y^n \xrightarrow{\mathscr{L}} Y$, by using stopping skill we can prove that, for all $\epsilon > 0$, $\delta > 0$ and $N > 0$, there exists $n_0 \in \boldsymbol{N}$ such that,

$$P^n\left(\sup_{t\leq N} \left\| \int_0^t (I - \mathrm{V}_m \Pi_m) f(X_{s-}^n) dY_s^n \right\| \geq \delta\right) < \frac{\epsilon}{2}, \quad \forall m,\ n \geq n_0. \tag{4.9}$$

Since $H^n \xrightarrow{\mathscr{L}} H$, we have that there exists $n_1 > n_0$ such that,

$$P^n\left(\sup_{t\leq N} \|X_t^n - \mathrm{V}_m \Pi_m X_t^n\| \geq \delta\right) < \epsilon, \quad \forall n,\ m \geq n_1. \tag{4.10}$$

From (4.7) to (4.10), we have that $\{X^n\}_{n\geq 1}$ satisfies the conditions of Theorem 3.3 of Xie ([**14**]). Hence $\{X^{n,N}\}_{n\geq 1}$ is tight. Moreover, the hypotheses yield that $\left\{(X^{n,N}, H^{n,N}, Y^{n,N})\right\}_{n\geq 1}$ is tight. As the proof of Theorem 1 of Slominski in [**11**] we can prove this theorem by using Theorem 3.3. □

Remark. If $\|\Delta Y^n\| \leq a$ for some constant $a > 0$, then the condition (ii) in Theorem 4.1 may be omitted.

Corollary 4.1. *Let M^n and A^n be an $\boldsymbol{H}$-valued local square integrable martingale and an $\boldsymbol{H}$-valued finite variation process with $\|\Delta M^n\| \leq a$, respectively. Where $a > 0$ is a constant. And let f and g be continuous $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$-valued mappings and satisfy a Lipschitz condition, i.e. there exists a constant $L > 0$ such that*

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \|g(x) - g(y)\| \leq L\|x - y\|.$$

Assume that X^n and X are the strong solutions of the following (SDE):

$$X^n = H^n + \int_0^{\cdot} f(X_{s-}^n) dM_s^n + \int_0^{\cdot} g(X_{s-}^n) dA_s^n$$

$$X = H + \int_0^{\cdot} f(X_{s-})dM_s + \int_0^{\cdot} g(X_{s-})dA_s,$$

where M and A be an $\boldsymbol{H}$-valued local square integrable martingale and an $\boldsymbol{H}$-valued finite variation adapted process, respectively.

If Theorem 4.1 (iii) *holds for f and g and if $(H^n, M^n, A^n) \xrightarrow{\mathscr{L}} (H, M, A)$, then $X^n \xrightarrow{\mathscr{L}} X$ and $(H^n, M^n, A^n, X^n) \xrightarrow{\mathscr{L}} (H, M, A, X)$.*

PROOF. For every $N > 0$, we define T_N^n and T_N as (4.5) and (4.6). As the proof of Corollary 1 of Slominski ([**11**]) and Theorem 4.1, we can prove that $\{(X^n, H^n, M^n, A^n)^{T_T^n}\}_{n\geq 1}$ is tight. Using Theorem 3.3 and Theorem 3.4, this theorem can be proved as the proof of Corollary 1 of Slominski ([**11**]). □

DEFINITION 4.1. Let f and f^n $(n \geq 1)$ be $\mathscr{L}(\boldsymbol{H}, \boldsymbol{G})$-valued functions on $\boldsymbol{R}_+ \times \boldsymbol{G}$. We say that $\{f,\ f^n, n \geq 1\}$ satisfies the condition (L) if the three conditions below are fulfilled:

(i) for all $t > 0$, $f(t, \cdot)$ and $f^n(t, \cdot)$ satisfy a Lipschitz condition;

(ii) for all $y \in \boldsymbol{G}$, $f(\cdot, y)$ and $f^n(\cdot, y)$ are càg functions on $\boldsymbol{R}_+$;

(iii) for every sequence $\{\alpha, \alpha_n, n \geq 1\} \subset \boldsymbol{D}(\boldsymbol{G})$, if $\alpha_n \to \alpha$ in the Skorohod topology, then $(\alpha_n, \beta_n) \to (\alpha, \beta)$ in the Skorohod topology on $\boldsymbol{D}(\boldsymbol{G} \times \mathscr{L}(\boldsymbol{H}, \boldsymbol{G}))$, where $\beta(t) = f(t+, \alpha(t))$, $\beta_n(t) = f^n(t+, \alpha_n(t))$, $\boldsymbol{D}(\boldsymbol{E})$ is the space of all càdlàg function $\alpha : \boldsymbol{R}_+ \to \boldsymbol{E}$ with the Skorohod topology, $\boldsymbol{E}$ is a Polish space.

COROLLARY 4.2. *Let $\{f,\ f^n, n \geq 1\}$ satisfy the condition* (L). *Assume that X^n and X are the strong solutions of the following* (SED):

$$X^n = H^n + \int_0^{\cdot} f^n(s, X^n_{s-})dY^n_s,$$

$$X = H + \int_0^{\cdot} f(s, X_{s-})dY_s.$$

Suppose that the conditions (i) *and* (ii) *of Theorem* 4.1 *are fulfilled and that, for all $K > 0$ and $\epsilon > 0$, there exists an $n_0 \in \boldsymbol{N}$ such that $\sup_{\|y\|\leq K} \|(I - \mathrm{V}_m\Pi_m)f^n(x)y\| < \epsilon$ for all $n, m \geq n_0$. If $(H^n, Y^n) \xrightarrow{\mathscr{L}} (H, Y)$, then $X^n \xrightarrow{\mathscr{L}} X$ and $(H^n, Y^n, X^n) \xrightarrow{\mathscr{L}} (H, Y, X)$.*

PROOF. As the Proof of Theorem 4.1 and Corollary 3 of Slominski ([**11**]), we can prove that $\{X^{n,N}\}_{n\geq 1}$ satisfies the conditions of Theorem 3.3 of Xie ([**14**]). This implies that $\{X^{n,N}, H^{n,N}, Y^{n,N}\}_{n\geq 1}$ is tight by the hypotheses. Using Theorem 3.3, this theorem can be proved as that in Corollary 3 of Slominski ([**11**]). □

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