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A statement of Weierstrass on meromorphic functions which admit an algebraic addition theorem

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Abstract. A statement of Weierstrass is known for meromorphic functions which admit an algebraic addition theorem. We give its precise formulation and prove it complex analytically. In fact, we show that if K is a non-degenerate algebraic function field in n variables over $\boldsymbol{C}$ which admits an algebraic addition theorem, then any $f \in K$ is a rational function of some coordinate functions and abelian functions of other variables.

1. Introduction and history.

Weierstrass frequently stated the following in his lectures at Berlin: Every system of n (independent) functions in n variables which admits an addition theorem is an algebraic combination of n abelian (or degenerate) functions with the same periods. But his proof was never published (see [**12**], other episodes are also stated there).

The first attempt to prove Weierstrass' statement was done by Painlevé in his Stockholm lectures ([**13**]). He simplified it in [**12**]. However, Painlevé's argument is not clear at least to us. Later, Severi studied meromorphic functions on $\boldsymbol{C}^n$ with μ ($< 2n$) independent periods admitting an algebraic addition theorem, which are called quasi-abelian functions (see [**14**] and [**17**]). It seems that Severi's work also does not explain the problem clearly. Unfortunately, we could not find any paper which studies this problem from complex analytic viewpoints. The referee kindly taught him works by Rosenlicht and Weil ([**15**], [**21**], [**22**]; see also Capocasa and Catanese [**7**]) done from viewpoints of algebraic geometry.

The purpose of this paper is to deal explicitly with the statement of Weierstrass and to give a complex analytic proof.

Let us consider the statement of Weierstrass in the case $n = 1$. A degenerate elliptic function is a rational function or a rational function of an exponential function (see [**18**]), and the statement is true. We can see its proof due to Osgood in [**4**] (see [**2**] for another proof).

On the other hand, it is not clear as to what are degenerate abelian functions or quasi-abelian functions when $n \geqq 2$. If we consider the field of meromorphic functions on $\boldsymbol{C}^n$ with period Γ of rank $\Gamma < 2n$, its transcendence degree over $\boldsymbol{C}$ is not always finite even when $\boldsymbol{C}^n/\Gamma$ does not contain $\boldsymbol{C}$ or $\boldsymbol{C}^*$ as a direct summand. Then, we can not

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think of degenerate abelian functions merely as meromorphic functions with degenerate periods.

Let K be a subfield of the field $\mathfrak{M}(\boldsymbol{C}^n)$ of meromorphic functions on $\boldsymbol{C}^n$. We assume that K is a non-degenerate algebraic function field over $\boldsymbol{C}$ which admits an algebraic addition theorem (for a precise definition, see the next section). The problem is to determine such a subfield K.

The referee suggested to us that there is a way to study it as follows. Weil's result ([**21**]) says that K is regarded as the function field of an algebraic group G. By the structure theorem for algebraic groups ([**15**]), there exists a linear algebraic subgroup H of G such that G/H is an abelian variety.

Our argument is completely different from this. We treat the problem in a more analytic way. First in [**2**] we studied it with an additional condition that K is algebraically degenerate with respect to some coordinates. In this paper, we discuss it without any additional assumption by clarifying Weierstrass' statement.

Our main theorem is the following.

Theorem 1.1. *Let $K \subset \mathfrak{M}(\boldsymbol{C}^n)$ be a non-degenerate algebraic function field of n variables over $\boldsymbol{C}$ which admits an algebraic addition theorem. Then K is considered as a subfield of $\boldsymbol{C}(z_1, \dots, z_p, w_1, \dots, w_q, g_0, \dots, g_r)$, where $z_1, \dots, z_p$ are coordinate functions of $\boldsymbol{C}^p$, $w_1, \dots, w_q$ are those of $(\boldsymbol{C}^*)^q$ and $g_0, \dots, g_r$ are generators of an abelian function field of dimension r, $p + q + r = n$.*

The proof follows the same line of the previous paper [**2**]. However, we need several steps to drop the above additional condition. Our argument is as follows. We showed in [**2**] that K is regarded as a subfield of the meromorphic function field $\mathfrak{M}(\boldsymbol{C}^n/\Gamma)$ on $\boldsymbol{C}^n/\Gamma$, where Γ is a discrete subgroup of $\boldsymbol{C}^n$. There exist holomorphic functions $\varphi_0, \dots, \varphi_N$ on $\boldsymbol{C}^n$ which give a Lie group isomorphism $\overline{\Phi} = (\varphi_0, \dots, \varphi_N) : G := \boldsymbol{C}^n/\Gamma \longrightarrow \Omega$, where Ω is a set in the N-dimensional complex projective space $\boldsymbol{P}^N$. The abelian Lie group Ω has the Zariski closure Y with $K \cong \boldsymbol{C}(Y)$. By Remmert-Morimoto's theorem we have

$$G \cong \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times X,$$

where X is a toroidal group. In particular, X is a quasi-abelian variety because there exists a non-degenerate function in K (see Section 7 for toroidal groups and quasi-abelian varieties). Any toroidal group has a bundle structure on a complex torus. Compactifying the fibers, we get a compactification $\overline{G}$ of G. With the help of one-dimensional case, we show that every $f \in K$ meromorphically extends to $\overline{G}$. The additional condition made this part easy in [**2**]. Without it we first consider the extension of a closed Lie subgroup of Ω to Y. We show that it has the Zariski closure with the same dimension, by investigating Lie algebras and Pfaffian equations. The result in the one-dimensional case guarantees that any $f \in K$ is separately extendable to the compactification of fibers. In [**2**], the author quoted Sakai's paper [**16**] in order to show that the separate extendability implies the extendability as multidimensional functions. Although Sakai's argument works also in the present case, we give a direct proof in Section 6. We see that X is an abelian variety by the extendability of f to $\overline{G}$ through a result about extendable line bundles on toroidal groups. Finally, Theorem 1.1 will be proved in Section 9.

2. Definitions.

Let $\mathfrak{M}(\boldsymbol{C}^n)$ be the field of meromorphic functions on $\boldsymbol{C}^n$. We consider a subfield K of $\mathfrak{M}(\boldsymbol{C}^n)$. We assume that K is finitely generated over $\boldsymbol{C}$ and $\operatorname{Trans} K = n$, where $\operatorname{Trans} K$ is the transcendence degree of K over $\boldsymbol{C}$. Such a field K is called an algebraic function field in n variables over $\boldsymbol{C}$. Let $f_0, \ldots, f_n$ be generators of K.

Definition 2.1. We say that $f_0, \ldots, f_n$ admit an algebraic addition theorem (we write it shortly as (AAT)) if for any $j = 0, \ldots, n$ there exists a rational function R_j such that

$$f_j(x+y) = R_j(f_0(x), \ldots, f_n(x), f_0(y), \ldots, f_n(y)) \tag{1}$$

for all $x, y \in \boldsymbol{C}^n$. An algebraic function field K in n variables over $\boldsymbol{C}$ admits (AAT) if it has generators $f_0, \ldots, f_n$ which admit (AAT).

We note that if K admits (AAT), then any generators $g_0, \ldots, g_n$ of K admit (AAT).

Definition 2.2. An algebraic function field K of n variables over $\boldsymbol{C}$ admits another addition theorem (AAT*) if there exist algebraically independent $f_1, \ldots, f_n \in K$ such that for any $j = 1, \ldots, n$ we have a non-zero polynomial P_j with

$$P_j(f_j(x+y), f_1(x), \ldots, f_n(x), f_1(y), \ldots, f_n(y)) = 0 \tag{2}$$

for all $x, y \in \boldsymbol{C}^n$.

By an elementary algebraic argument, we obtain the following lemma.

Lemma 2.3. *Let K be an algebraic function field of n variables over $\boldsymbol{C}$. Then, K admits (AAT) if and only if it admits (AAT*).*

A function $f \in \mathfrak{M}(\boldsymbol{C}^n)$ is degenerate if there exist an invertible linear transformation $\mathscr{L} : \boldsymbol{C}^n \longrightarrow \boldsymbol{C}^n$, $x = \mathscr{L}(y)$ and a non-negative integer r with $r < n$ such that $f(\mathscr{L}(y))$ does not depend on $y_{r+1}, \ldots, y_n$. We say that f is non-degenerate if it is not degenerate.

Definition 2.4. A subfield K of $\mathfrak{M}(\boldsymbol{C}^n)$ is said to be non-degenerate if there exists a non-degenerate function in K.

Throughout the paper, we assume that K is a non-degenerate algebraic function field of n variables over $\boldsymbol{C}$ which admits (AAT).

3. Picard varieties.

We proved in [**2**] the following theorem which is basic to our argument.

Theorem 3.1 ([**2**, Theorem 2.6]). *There exist holomorphic functions $\varphi_0, \ldots, \varphi_N$ on $\boldsymbol{C}^n$, a discrete subgroup Γ of $\boldsymbol{C}^n$, an algebraic subvariety Y of the N-dimensional complex projective space $\boldsymbol{P}^N$ and a connected complex abelian Lie group Ω in Y such that*

(a) $\varphi_0, \ldots, \varphi_N$ *give a Lie group isomorphism*

$$\overline{\Phi} = (\varphi_0, \ldots, \varphi_N) : G := \boldsymbol{C}^n/\Gamma \longrightarrow \Omega,$$

where $(x_0, \ldots, x_N)$ *are homogeneous coordinates of* $\boldsymbol{P}^N$,

(b) $\varphi_1/\varphi_0, \ldots, \varphi_N/\varphi_0$ *generate* K *and* K *is considered as a subfield of* $\mathfrak{M}(G)$, *where* $\mathfrak{M}(G)$ *is the field of meromorphic functions on* G,

(c) Y *is the Zariski closure of* Ω *and*

$$\overline{\Phi}^* : \boldsymbol{C}(Y) \longrightarrow K,\ R \longmapsto R \circ \overline{\Phi}$$

is an isomorphism, then $\dim_{\boldsymbol{C}} Y = \dim_{\boldsymbol{C}} \Omega = n$, *where* $\boldsymbol{C}(Y)$ *is the rational function field of* Y.

We note that the group operation $\cdot$ on Ω is rational (see the proof of Proposition 2.4 in [**2**]). We call Y a Picard variety of K. Let K' be a subfield of $\mathfrak{M}(\boldsymbol{C}^n)$ satisfying the same assumptions, and let Y' be a Picard variety of K'. Then, K and K' are isomorphic if and only if Y and Y' are birationally equivalent.

Applying the above theorem, we gave a short proof of the statement of Weierstrass when $n = 1$ (the proof of Theorem 2.7 in [**2**]).

Theorem 3.2 (Weierstrass). *A function* $f \in \mathfrak{M}(\boldsymbol{C})$ *admits* (AAT^*) *if and only if it is an elliptic function or a rational function or a rational of* e^{az}.

We later use this theorem in our argument.

4. Continuation of closed subgroups.

We assume the situation in Theorem 3.1. Let Ω be the connected complex abelian Lie group embedded in $\boldsymbol{P}^N$. It has the Zariski closure Y with $\dim_{\boldsymbol{C}} Y = \dim_{\boldsymbol{C}} \Omega = n$. Let $(x_0, \ldots, x_N)$ be homogeneous coordinates of $\boldsymbol{P}^N$. We write $x(p) = (x_0(p), \ldots, x_N(p))$ for any $p \in \Omega$. For any $p, q \in \Omega$, we have

$$x_i(p \cdot q) = R_i(x(p), x(q)), \quad i = 0, \ldots, N, \tag{3}$$

where R_i is a rational function. Let $e \in \Omega$ be the unit element of Ω. We denote by $T_e(\Omega)$ the complex tangent space of Ω at e. For any $v \in T_e(\Omega)$ we can construct a left invariant holomorphic vector field $X(v)$ on Ω by

$$X(v)_p := (L_p)_* v \quad \text{for} \ \ p \in \Omega,$$

where $L_p : \Omega \longrightarrow \Omega$, $q \longmapsto p \cdot q$ is the left translation defined by p. Let $\mathfrak{g}$ be the Lie algebra of Ω. Then we have

$$\mathfrak{g} = \{X(v); v \in T_e(\Omega)\}.$$

The following lemma is an immediate consequence of (3).

LEMMA 4.1. *For any $X \in \mathfrak{g}$ and any $p \in \Omega$, there exists a neighborhood U of p in $\boldsymbol{P}^N$ such that*

$$X = \sum_{j=1}^{N} \tilde{R}_j(t) \frac{\partial}{\partial t_j} \quad \text{on} \ \ U \cap \Omega,$$

where $t = (t_1, \ldots, t_N)$ are affine coordinates on U and $\tilde{R}_j(t)$ is a rational function of t.

Let M be an n-dimensional complex manifold, and let $T_p(M)$ be the complex tangent space of M at p. An assignment $\mathscr{D} : M \ni p \longmapsto \mathscr{D}_p \subset T_p(M)$ is called an r-dimensional complex differential system on M if for any $p \in M$ there exist a neighborhood U of p and holomorphic 1-forms $\omega_1, \ldots, \omega_{n-r}$ on U such that

$$\mathscr{D}_q = \{v \in T_q(M); (\omega_1)_q(v) = \cdots = (\omega_{n-r})_q(v) = 0\} \tag{4}$$

for $q \in U$, and $\mathscr{D}_q$ is an r-dimensional complex linear subspace of $T_q(M)$. In this case, we say that $\mathscr{D}$ is defined by the Pfaffian equations

$$\omega_1 = \cdots = \omega_{n-r} = 0 \tag{5}$$

on U and that (5) is the local equations of $\mathscr{D}$ on U. A complex differential system $\mathscr{D}$ is completely integrable if for any $p \in M$ there exist a neighborhood U of p and holomorphic functions $f_1, \ldots, f_{n-r}$ on U such that

$$df_1 = \cdots = df_{n-r} = 0$$

is the local equations of $\mathscr{D}$ on U. If $\mathscr{D}$ is completely integrable, there is an r-dimensional integral manifold of $\mathscr{D}$ passing through p for any point $p \in M$. We say that $\mathscr{D}$ satisfies the integrability condition if for any $p \in M$ there exist a neighborhood U of p and the local equations

$$\omega_1 = \cdots = \omega_{n-r} = 0$$

of $\mathscr{D}$ on U with

$$d\omega_i = \sum_{j=1}^{n-r} \tilde{\omega}_{ij} \wedge \omega_j, \quad i = 1, \ldots, n-r, \tag{6}$$

where $\tilde{\omega}_{ij}$ are holomorphic 1-forms on U. It is the complex version of Frobenius' theorem that a complex differential system $\mathscr{D}$ is completely integrable if and only if it satisfies the integrability condition (see, e.g., p. 165 of [**9**]).

We now consider a connected closed complex Lie subgroup H of Ω. Let $\mathfrak{h}$ be the Lie algebra of H. We can take a basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_r\}$ of $\mathfrak{g}$ such that $\{Y_1, \ldots, Y_r\}$

is a basis of $\mathfrak{h}$, where $r = \dim_{\boldsymbol{C}} H$. Let $\{\omega_1, \dots, \omega_m, \eta_1, \dots, \eta_r\}$ be a set of holomorphic 1-forms on Ω which forms the dual system of $\{X_1, \dots, X_m, Y_1, \dots, Y_r\}$. For any $p \in \Omega$ we assign

$$\mathscr{D}_p := \{v \in T_p(\Omega); (\omega_1)_p(v) = \cdots = (\omega_m)_p(v) = 0\}.$$

Then $\mathscr{D} : \Omega \ni p \longmapsto \mathscr{D}_p \subset T_p(\Omega)$ is an r-dimensional complex differential system. Since Ω is abelian, $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Then we obtain the following lemma.

Lemma 4.2. *All holomorphic* 1*-forms* ω_i $(i = 1, \dots, m)$ *and* η_j $(j = 1, \dots, r)$ *are* d*-closed.*

We may assume by the resolution of singularities that Y is non-singular. By Lemma 4.1 each $X_i, Y_j, \omega_i, \eta_j$ is meromorphically extendable to Y. We use the same notations $X_i, Y_j, \omega_i, \eta_j$ for their extensions to Y, without confusion. It follows from Lemma 4.2 that $d\omega_i = 0$ on Y for $i = 1, \dots, m$.

Lemma 4.3. *Let* U *be an open set in* $\boldsymbol{C}^n$*, and let* ω *be a meromorphic* 1*-form on* U *with* $d\omega = 0$*. Assume that there exists a holomorphic function* f *on* U *such that* $\tilde{\omega} := f\omega$ *is a holomorphic 1-form on* U *and* $\tilde{\omega}_q \neq 0$ *for any* $q \in U$*. Then the complex differential system given by* $\tilde{\omega}$ *on* U *satisfies the integrability condition on* U*.*

Proof. For any $p \in U$ there exists a neighborhood V of p on which we have the unique representation

$$d\tilde{\omega} = \tau \wedge \tilde{\omega} + \sigma,$$

where τ is a holomorphic 1-form and σ is a holomorphic 2-form without terms including $\tilde{\omega}$. On the other hand, we have

$$d\tilde{\omega} = \frac{df}{f} \wedge \tilde{\omega}$$

on $V' := \{q \in V; f(q) \neq 0\}$ for ω is d-closed. Then $\sigma = 0$ on V', and on V. $\square$

Let $\mathscr{D}_i$ be the $(n-1)$-dimensional complex differential system on Ω defined by the local equation $\omega_i = 0$, for $i = 1, \dots, m$. Since $\mathscr{D}_i$ is completely integrable on Ω (Lemma 4.2), there exists an $(n-1)$-dimensional integral manifold Z_i of $\mathscr{D}_i$ such that

$$H = \bigcap_{i=1}^{m} Z_i.$$

Proposition 4.4. *For any* $i = 1, \dots, m$*, there exists an irreducible analytic subset* $\widetilde{Z}_i$ *of* Y *of pure codimension* 1 *such that*

$$Z_i = \widetilde{Z}_i \cap \Omega.$$

PROOF. Let $p \in Y$. There exist a neighborhood U of p and holomorphic functions f_j, g_j which are mutually prime at every point of U such that

$$\omega_i = \sum_{j=1}^{n} \frac{g_j}{f_j} dt_j,$$

where $(t_1, \ldots, t_n)$ are coordinates on U. Take any k, ℓ with $k \neq \ell$, $1 \leqq k, \ell \leqq n$. Let f_0 be the common factor of f_k and f_ℓ. Then we have $f_k = f_0 \tilde{f}_k$ and $f_\ell = f_0 \tilde{f}_\ell$, where $\tilde{f}_k$ and $\tilde{f}_\ell$ are mutually prime. We set

$$N_{k,\ell} := \{q \in U; \tilde{f}_k(q) = 0, \tilde{f}_\ell(q) = 0\},$$
$$I_j := \{q \in U; f_j(q) = 0, g_j(q) = 0\}$$

and

$$N := \Bigl(\bigcup_{k \neq \ell} N_{k,\ell}\Bigr) \cup \Bigl(\bigcup_{j=1}^{n} I_j\Bigr).$$

Then N is an analytic set of U with $\dim_{\boldsymbol{C}} N = n-2$. Therefore, it suffices to show that Z_i continues to $U \setminus N$, by a classical continuation theorem of analytic sets.

Let $p \in U \setminus N$. If p is a regular point of ω_i, then there exist a neighborhood V of p and a holomorphic function f on V such that $\omega_i = df$ on V, for $d\omega_i = 0$. Then Z_i extends to V.

Suppose that p is a singular point of ω_i. Since $p \notin N$, there exist a neighborhood W of p and a holomorphic function f on W such that $\tilde{\omega}_i := f\omega_i$ is a holomorphic 1-form and $(\tilde{\omega}_i)_q \neq 0$ for any $q \in W$. By Lemma 4.3, $\tilde{\omega}_i$ satisfies the integrability condition on W. Then Z_i is also extendable to W. □

We set

$$Z := \bigcap_{i=1}^{m} \widetilde{Z}_i.$$

Then Z is an r-dimensional irreducible analytic subset of Y with $Z \cap \Omega = H$. Therefore, it is the Zariski closure of H. We summarize the above results in the following theorem for the later use.

THEOREM 4.5. *Let H be a connected closed complex Lie subgroup of Ω. Then the Zariski closure Z of H has the same dimension as H.*

5. Restriction to a closed subgroup.

Let H be a connected closed complex Lie subgroup of $G = \boldsymbol{C}^n/\Gamma$. We consider the restriction of K to H. For any $f \in \mathfrak{M}(G)$ we denote by $P(f)$ the polar set of f. We define the restriction f_H of f to H by

$$f_H := \begin{cases} 0, & \text{if } H \subset P(f) \\ f|_H, & \text{otherwise.} \end{cases}$$

Let $K_H := \{f_H; f \in K\}$ be the restriction of K to H. If $f_0, \dots, f_n$ are generators of K, then $K_H = \boldsymbol{C}((f_0)_H, \dots, (f_n)_H)$. It is obvious that K_H is non-degenerate and admits (AAT).

Proposition 5.1. *It holds that* $\operatorname{Trans} K_H = \dim_{\boldsymbol{C}} H$.

Proof. Let $\overline{\varPhi} : G \longrightarrow \varOmega$ be the isomorphism in Theorem 3.1. Then $\widetilde{H} := \overline{\varPhi}(H)$ is a connected closed complex Lie subgroup of $\varOmega$. Let Z be the Zariski closure of $\widetilde{H}$. By Theorem 4.5 we have $\dim_{\boldsymbol{C}} Z = \dim_{\boldsymbol{C}} \widetilde{H} = \dim_{\boldsymbol{C}} H$. Since

$$K_H \cong \{f_{\widetilde{H}}; f \in \boldsymbol{C}(Y)\} \cong \boldsymbol{C}(Z)$$

and $\operatorname{Trans} \boldsymbol{C}(Z) = \dim_{\boldsymbol{C}} Z$, we obtain the conclusion. $\square$

6. Separately extendable meromorphic functions.

In this section, we discuss the extendability of separately extendable meromorphic functions improving the arguments in [**6**].

Let D and E be domains in $\boldsymbol{C}^n$ and $\boldsymbol{C}^m$ respectively. We consider meromorphic functions $F_1(z,w), \dots, F_N(z,w)$ on $D \times E$, which are not all identically zero. Let P_i be the polar set of F_i. We set

$$P := \bigcup_{i=1}^{N} P_i.$$

Then P is an analytic subset of $D \times E$ with $\operatorname{codim}_{\boldsymbol{C}} P = 1$. There exist subdomains $D_0 \subset D$ and $E_0 \subset E$ such that

$$D_0 \times E_0 \subset (D \times E) \setminus P.$$

We obtain the following lemma along an argument in [**6**] (Chapter IX, Section 5, Lemma 6).

Lemma 6.1. *Assume that there exist functions $c_1(w), \dots, c_N(w)$ on E_0 such that*

$$c_1(w)F_1(z,w) + \cdots + c_N(w)F_N(z,w) \equiv 0 \quad \text{on} \quad D_0 \times E_0,$$

where $c_1(w), \dots, c_N(w)$ are not all zero for any $w \in E_0$. Then, there exist meromorphic functions $C_1(w), \dots, C_N(w)$ on E, which are not all identically zero, such that

$$C_1(w)F_1(z,w)+\cdots+C_N(w)F_N(z,w)\equiv 0 \quad \textit{on} \quad D\times E.$$

Proof. By the assumption we have the following equalities

$$\begin{cases} c_1(w)F_1(z^{(1)},w)+\cdots+c_N(w)F_N(z^{(1)},w)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1(w)F_1(z^{(N)},w)+\cdots+c_N(w)F_N(z^{(N)},w)=0 \end{cases}$$

for any $(z^{(1)},\dots,z^{(N)};w)\in D_0^N\times E_0$. Since $(c_1(w),\dots,c_N(w))\neq(0,\dots,0)$, we obtain

$$\det\left(F_i(z^{(j)},w)\right)_{i,j=1,\dots,N}\equiv 0 \quad \text{on} \quad D_0^N\times E_0.$$

Let k be the positive integer such that determinants of all submatrices of order k are zero and that there exists a submatrix of order $k-1$ with non-zero determinant. Trivially $k\geqq 2$. Renumbering $F_i(z^{(j)},w)$ if necessary, we may assume that

$$\begin{vmatrix} F_1(z^{(1)},w)\dots F_k(z^{(1)},w) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ F_1(z^{(k)},w)\dots F_k(z^{(k)},w) \end{vmatrix}\equiv 0 \tag{7}$$

and

$$\begin{vmatrix} F_1(z^{(1)},w) & \dots & F_{k-1}(z^{(1)},w) \\ \cdots & \cdots & \cdots \\ F_1(z^{(k-1)},w) & \dots & F_{k-1}(z^{(k-1)},w) \end{vmatrix} \tag{8}$$

is not identically zero. Letting $z^{(k)}=z$, we expand the determinant (7) according to the last row. Then we obtain

$$\sum_{i=1}^{k} C_i(z^{(1)},\dots,z^{(k-1)};w)F_i(z,w)\equiv 0, \tag{9}$$

where $C_i(z^{(1)},\dots,z^{(k-1)};w)$ is the cofactor of the (k,i)-element of (7). There exist $a^{(1)},\dots,a^{(k-1)}\in D_0$ such that at least a $C_i(a^{(1)},\dots,a^{(k-1)};w)$ is not identically zero. We fix such $a^{(1)},\dots,a^{(k-1)}$. Let $C_i(w):=C_i(a^{(1)},\dots,a^{(k-1)};w)$ for $i=1,\dots,k$ and $C_j(w):=0$ for $j=k+1,\dots,N$. Then we get by (9)

$$\sum_{i=1}^{N} C_i(w)F_i(z,w)\equiv 0 \tag{10}$$

on $D_0\times E_0$. By the definition of $C_i(w)$, it is a meromorphic function on E. Hence, (10) holds on $D\times E$ by the uniqueness theorem. □

Remark 6.2. If $F_i(z,w)$ $(i=1,\dots,N)$ is a rational function of w for any fixed $z \in D$ in addition to the assumption in Lemma 6.1, then we can take $C_i(w)$ $(i=1,\dots,N)$ as a rational function.

Using this fact, we can prove the following proposition.

Proposition 6.3 ([**6**, Chapter IX, Section 5, Theorem 5]). *Let $D \times E \subset \boldsymbol{C}^n \times \boldsymbol{C}^m$ be a domain, and let $f(z,w)$ be a holomorphic function on $D \times E$. If $f(z,w)$ is rational in w for any $z \in D$ and rational in z for any $w \in E$, then it is a rational function of (z,w).*

Let $f(z)$ be a meromorphic function on $D := \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q$, $p+q=n$, and let P be its polar set. We define

$$P_i := \{(z',z'') = (z_1,\dots,z_{i-1};z_{i+1},\dots,z_n); \{z'\} \times \boldsymbol{C} \times \{z''\} \subset P\}$$

for $i = 1,\dots,p$ and

$$P_i := \{(z',z'') = (z_1,\dots,z_{i-1};z_{i+1},\dots,z_n); \{z'\} \times \boldsymbol{C}^* \times \{z''\} \subset P\}$$

for $i = p+1,\dots,n$, where $z = (z_1,\dots,z_n)$ are coordinates of D. We set $D_i = \{(z',z'') = (z_1,\dots,z_{i-1};z_{i+1},\dots,z_n)\}$ for $i=1,\dots,n$.

Proposition 6.4. *If for any $i = 1,\dots,n$ and any $(z',z'') \in D_i \setminus P_i$, $f(z',z_i,z'')$ is rational in z_i, then $f(z)$ is a rational function of z.*

Proof. We can take a polydisc Δ in $D \setminus P$. By Proposition 6.3, $f(z)$ is a rational function on Δ. Then, it is rational on D by the uniqueness theorem. □

Let D be a domain in $\boldsymbol{C}^k$, and let $E := \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q$, $p+q=\ell$, $k+\ell=n$. We consider a meromorphic function $f(z,w)$ on $D \times E$. Let P be the polar set of f. If we set

$$P_z := \{z \in D; \{z\} \times E \subset P\},$$

then P_z is an analytic subset of D. Let $\{p_\nu(w)\}_{\nu=1}^\infty$ be the sequence of all monomials $\{w_1^{\alpha_1} \cdots w_\ell^{\alpha_\ell}\}$.

Theorem 6.5. *Assume that $f(z_0,w)$ is rational in w for any $z_0 \in D \setminus P_z$. Then, there exist meromorphic functions $a_1(z),\dots,a_M(z)$, $b_1(z),\dots,b_N(z)$ on D such that*

$$f(z,w) = \frac{Q(z,w)}{P(z,w)} \quad \text{on} \quad D \times E,$$

where $P(z,w) = \sum_{\mu=1}^M a_\mu(z)p_\mu(w)$ and $Q(z,w) = \sum_{\nu=1}^N b_\nu(z)p_\nu(w)$. Therefore, $f(z,w)$ meromorphically extends to $D \times (\boldsymbol{P}^1)^\ell$.

PROOF. We follow an argument in the proof of [**6**, Chapter IX, Section 5, Theorem 5].

There exist domains $U \subset D$ and $V \subset E$ such that $U \times V \subset (D \times E) \setminus P$. For any $z \in U$, there exist positive integers M_z and N_z, complex numbers $a_\mu(z)$ ($\mu = 1, \ldots, M_z$) and $b_\nu(z)$ ($\nu = 1, \ldots, N_z$) such that

$$f(z, w) = \frac{-\sum_{\nu=1}^{N_z} b_\nu(z) p_\nu(w)}{\sum_{\mu=1}^{M_z} a_\mu(z) p_\mu(w)}$$

for $w \in V$, where $(a_1(z), \ldots, a_{M_z}(z)) \neq (0, \ldots, 0)$. Then we have

$$\left(\sum_{\mu=1}^{M_z} a_\mu(z) p_\mu(w)\right) f(z, w) + \sum_{\nu=1}^{N_z} b_\nu(z) p_\nu(w) = 0 \tag{11}$$

for $w \in V$. We can normalize $a_\mu(z)$ and $b_\nu(z)$ as follows

$$\sum_{\mu=1}^{M_z} |a_\mu(z)|^2 + \sum_{\nu=1}^{N_z} |b_\nu(z)|^2 = 1. \tag{12}$$

For any $(M, N) \in \boldsymbol{N}^2$, we define

$$U_{M,N} := \{z \in U;\ \text{there exist } a_\mu(z)\text{'s and } b_\nu(z)\text{'s with (11) and (12) for } (M, N)\}.$$

Then $U = \bigcup_{(M,N)} U_{M,N}$. Take any sequence $\{z^{(j)}\}$ in $U_{M,N}$ with $z^{(j)} \to z^{(0)}$, where $z^{(0)} \in U$. Without loss of generality, we may assume that $a_\mu(z^{(j)}) \to a_\mu$, $b_\nu(z^{(j)}) \to b_\nu$ and $\sum_{\mu=1}^{M} |a_\mu|^2 + \sum_{\nu=1}^{N} |b_\nu|^2 = 1$. By the continuity of $f(z, w)$ we obtain

$$\left(\sum_{\mu=1}^{M} a_\mu p_\mu(w)\right) f(z^{(0)}, w) + \sum_{\nu=1}^{N} b_\nu p_\nu(w) = 0$$

for all $w \in V$. This means that $U_{M,N}$ is a closed set. By Baire's category theorem, at least one of $U_{M,N}$'s has an interior point. Take such a $U_{M,N}$ and let U_0 be its open kernel. We define

$$F_\mu(z, w) := p_\mu(w) f(z, w) \quad \text{for} \quad \mu = 1, \ldots, M,$$
$$F_{M+\nu}(z, w) := p_\nu(w) \quad \text{for} \quad \nu = 1, \ldots, N.$$

Then $F_1(z, w), \ldots, F_{M+N}(z, w)$ are meromorphic on $D \times E$ and holomorphic on $U_0 \times V$. It follows from (11) that

$$\sum_{\mu=1}^{M} a_\mu(z) F_\mu(z, w) + \sum_{\nu=1}^{N} b_\nu(z) F_{M+\nu}(z, w) \equiv 0$$

on $U_0 \times V$. By Lemma 6.1 we can take meromorphic functions $A_\mu(z)$ ($\mu = 1, \dots, M$) and $B_\nu(z)$ ($\nu = 1, \dots, N$) on D such that

$$\sum_{\mu=1}^{M} A_\mu(z) F_\mu(z, w) + \sum_{\nu=1}^{N} B_\nu(z) F_{M+\nu}(z, w) \equiv 0$$

on $D \times E$. Thus we obtain the conclusion. □

7. Extendable line bundles on toroidal groups.

A connected complex Lie group G_0 is called a toroidal group if $H^0(G_0, \mathscr{O}) = \boldsymbol{C}$. Every toroidal group is abelian ([**10**]). Then we can write $G_0 = \boldsymbol{C}^r/\Gamma^*$, where Γ^* is a discrete subgroup of $\boldsymbol{C}^r$ with rank $\Gamma^* = r + m$ ($1 \leqq m \leqq r$). We denote by $\Gamma^*_{\boldsymbol{R}}$ the real linear subspace of $\boldsymbol{C}^r$ spanned by Γ^*. Let $\Gamma^*_{\boldsymbol{C}} := \Gamma^*_{\boldsymbol{R}} \cap \sqrt{-1}\Gamma^*_{\boldsymbol{R}}$ be the maximal complex linear subspace contained in $\Gamma^*_{\boldsymbol{R}}$. It is easy to see that $\dim_{\boldsymbol{C}} \Gamma^*_{\boldsymbol{C}} = m$.

The following definition is due to Andreotti-Gherardelli [**5**].

Definition 7.1. A toroidal group $G_0 = \boldsymbol{C}^r/\Gamma^*$ is said to be a quasi-abelian variety if there exists a Hermitian form $\mathscr{H}$ on $\boldsymbol{C}^r$ such that
(a) $\mathscr{H}$ is positive definite on $\Gamma^*_{\boldsymbol{C}}$,
(b) the imaginary part $\mathscr{A} := \operatorname{Im} \mathscr{H}$ of $\mathscr{H}$ is $\boldsymbol{Z}$-valued on $\Gamma^* \times \Gamma^*$.
We call such a Hermitian form $\mathscr{H}$ an ample Riemann form for Γ^* or G_0.

From the projection $\mathrm{pr} : \boldsymbol{C}^r \longrightarrow \Gamma^*_{\boldsymbol{C}}$ we obtain a fiber bundle structure $\sigma : G_0 \longrightarrow \boldsymbol{T}$ on an m-dimensional complex torus $\boldsymbol{T}$ with fibers $(\boldsymbol{C}^*)^\ell$, $\ell = r - m$ ([**20**, Proposition 3]). Replacing fibers $(\boldsymbol{C}^*)^\ell$ with $(\boldsymbol{P}^1)^\ell$, we obtain the associated $(\boldsymbol{P}^1)^\ell$-bundle $\overline{\sigma} : \overline{G}_0 \longrightarrow \boldsymbol{T}$.

Proposition 7.2 ([**19**, Satz 3.2.8]). *Let $L \longrightarrow G_0$ be a holomorphic line bundle which is holomorphically extendable to $\overline{G}_0$. Then there exists a theta bundle $L_\theta \longrightarrow \boldsymbol{T}$ such that*

$$L \cong \sigma^* L_\theta.$$

8. Extension to a compactification of G.

We return to our situation. By the theorem of Remmert-Morimoto ([**8**] and [**11**]), we have

$$G \cong \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times X,$$

where $X = \boldsymbol{C}^r/\Gamma^*$ is a toroidal group of rank $\Gamma^* = r + m$ ($1 \leqq m \leqq r$) and $p + q + r = n$. Since there exists a non-degenerate meromorphic function on X, X is a quasi-abelian variety ([**1**] and [**7**]). We have a $(\boldsymbol{C}^*)^s$-bundle $\sigma : X \longrightarrow \boldsymbol{T}$ on an m-dimensional complex torus $\boldsymbol{T}$, where $s = r - m$. Let $\overline{\sigma} : \overline{X} \longrightarrow \boldsymbol{T}$ be the associated $(\boldsymbol{P}^1)^s$-bundle. These bundles give fiber bundles $\tau : G \longrightarrow \boldsymbol{T}$ with fibers $\boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times (\boldsymbol{C}^*)^s$ and $\overline{\tau} : \overline{G} \longrightarrow \boldsymbol{T}$

with fibers $(\boldsymbol{P}^1)^\ell$, $\ell = p+q+s$, where $\overline{G} = (\boldsymbol{P}^1)^{p+q} \times \overline{X}$. For any $a \in \boldsymbol{T}$ we set

$$F_a := \tau^{-1}(a) \cong \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times (\boldsymbol{C}^*)^s,$$
$$\overline{F}_a := \overline{\tau}^{-1}(a) \cong (\boldsymbol{P}^1)^\ell.$$

THEOREM 8.1. *Every $f \in K$ meromorphically extends to $\overline{G}$.*

PROOF. Let $e \in \boldsymbol{T}$ be the unit element of $\boldsymbol{T}$. We take coordinates $(z_1, \ldots, z_\ell)$ on F_e. For any $i = 1, \ldots, \ell$ we define

$$L_i := \{(0, z_i, 0) \in F_e\}.$$

Then L_i is a connected closed complex Lie subgroup of G with $\dim_{\boldsymbol{C}} L_i = 1$. It follows from Proposition 5.1 that $\operatorname{Trans} K_{L_i} = 1$. K_{L_i} is non-degenerate, admits (AAT) and is not periodic. Then, any $g \in K_{L_i}$ is a rational function of z_i by Theorem 3.2. Therefore, f_{F_e} is rational for any $f \in K$ by Proposition 6.4.

Let f be any function in K. Take any point $a \in \boldsymbol{T}$. We define $g(x) := f(x + \tilde{a})$ for some $\tilde{a} \in G$ with $\tau(\tilde{a}) = a$. By (AAT), we can verify that $g \in K$. From the above observation, we know that g_{F_e} is rational. Since $F_a = F_e + \tilde{a}$ and $f_{F_a} = g_{F_e}$, f_{F_a} is rational. Furthermore, there exists an open set $U \subset \boldsymbol{T}$ such that

$$\tau^{-1}(U) \cong U \times (\boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times (\boldsymbol{C}^*)^s).$$

As we have seen in the above, $f_{\tau^{-1}(U)}$ satisfies the assumption in Theorem 6.5. Then $f_{\tau^{-1}(U)}$ meromorphically extends to $\overline{\tau}^{-1}(U) \cong U \times (\boldsymbol{P}^1)^\ell$. This completes the proof. □

9. Proof of Theorem 1.1.

We write the situation again, in order to have a clear picture of the problem.

Let $K \subset \mathfrak{M}(\boldsymbol{C}^n)$ be a non-degenerate algebraic function field of n variables over $\boldsymbol{C}$. We assume that K admits (AAT). It is considered as a subfield of $\mathfrak{M}(G)$, where $G = \boldsymbol{C}^n/\Gamma$. We have the decomposition

$$G \cong \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times X,$$

where $X = \boldsymbol{C}^r/\Gamma^*$ is a quasi-abelian variety. And X has the fibration in the previous section.

PROPOSITION 9.1. *The quasi-abelian variety X is an abelian variety.*

PROOF. There exists a function $f \in K$ such that $g := f_X$ is non-degenerate. Let $\tilde{L}$ be the holomorphic line bundle on G given by the zero-divisor of f. Since f is meromorphically extendable to $\overline{G}$ (Theorem 8.1), $\tilde{L}$ has the holomorphic extension to $\overline{G}$. Then, $L := \tilde{L}|_X$ extends to $\overline{X}$.

Let $\operatorname{rank} \Gamma^* = r + s$. Suppose that $1 \leqq s < r$. Then there exists a $(\boldsymbol{C}^*)^{r-s}$-bundle $\sigma : X \longrightarrow \boldsymbol{T}$ on a complex torus $\boldsymbol{T}$ with $\dim_{\boldsymbol{C}} \boldsymbol{T} = s < r$. By Proposition 7.2 we can take a theta bundle $L_\theta \longrightarrow \boldsymbol{T}$ such that $L \cong \sigma^* L_\theta$. Let $\overline{\sigma} : \overline{X} \longrightarrow \boldsymbol{T}$ be the associated $(\boldsymbol{P}^1)^{r-s}$-bundle. We denote by $\overline{g}$ and $\overline{L}$ the extensions of g and L, respectively. Then there exist $\overline{\varphi}, \overline{\psi} \in H^0(\overline{X}, \mathscr{O}(\overline{L}))$ such that $\overline{g} = \overline{\psi}/\overline{\varphi}$. Since

$$H^0(\overline{X}, \mathscr{O}(\overline{L})) = \overline{\sigma}^* H^0(\boldsymbol{T}, \mathscr{O}(L_\theta)),$$

$\overline{g}$ is constant on the fibers. This contradicts the assumption that g is non-degenerate. □

Proof of Theorem 1.1. It follows from Proposition 9.1 that

$$G \cong \boldsymbol{C}^p \times (\boldsymbol{C}^*)^q \times A,$$

where $A = \boldsymbol{C}^r/\Gamma^*$ is an abelian variety. By Theorem 8.1, any $f \in K$ meromorphically extends to $\overline{G} \cong (\boldsymbol{P}^1)^{p+q} \times A$. Then we obtain the conclusion. □

References

[1] Y. Abe, Lefschetz type theorem, Ann. Mat. Pura Appl., **169** (1995), 1–33.

[2] Y. Abe, Meromorphic functions admitting an algebraic addition theorem, Osaka J. Math., **36** (1999), 343–363.

[3] Y. Abe and K. Kopfermann, Toroidal Groups, Lecture Notes in Math., **1759**, Springer, Berlin, 2001.

[4] N. I. Ahiezer, Elements of the Theory of Elliptic Functions, Transl. Math. Monogr., **79** (1990).

[5] A. Andreotti and F. Gherardelli, Some remarks on quasi-abelian manifolds, in Global Analysis and Its Applications, vol. II, Intern. Atomic Energy Agency, **203–206**, Vienna, 1974.

[6] S. Bochner and W. T. Martin, Several Complex Variables, Princeton Univ. Press, Princeton, 1948.

[7] F. Capocasa and F. Catanese, Periodic meromorphic functions, Acta Math., **166** (1991), 27–68.

[8] K. Kopfermann, Maximale Untergruppen Abelscher Komplexer Liescher Gruppen, Schriftenreihe Math. Inst. Univ. Münster, **29**, Münster, 1964.

[9] Y. Matsushima, Differentiable Manifolds, Marcel Dekker, Inc., New York, 1972.

[10] A. Morimoto, Non-compact Lie groups without non-constant holomorphic functions, In: Proceedings of Conference on Complex Analysis, Minneapolis, Springer, Berlin, 1965, 256–272.

[11] A. Morimoto, On the classification of noncompact complex abelian Lie groups, Trans. Amer. Math. Soc., **123** (1966), 200–228.

[12] P. Painlevé, Sur les fonctions qui admettent un théorème d'addition, Acta Math., **27** (1903), 1–54.

[13] P. Painlevé, Œuvres de Paul Painlevé, Tome I, Centre National de la Recherche Scientifique, Paris, 1973.

[14] M. Rosati, Le Funzioni e le Varietà Quasi Abeliane dalla Teoria del Severi ad Oggi, Pont. Acad. Scient. Scripta Varia, **23**, Vatican, 1962.

[15] M. Rosenlicht, Some basic theorems on algebraic groups, Amer. J. Math., **78** (1956), 401–443.

[16] E. Sakai, A note on meromorphic functions in several complex variables, Mem. Fac. Sci., Kyushu Univ. Ser. A, **11** (1957), 75–80.

[17] F. Severi, Funzioni Quasi Abeliane, Seconda Edizione Ampliata, Pont. Acad. Scient. Scripta Varia, **20**, Vatican, 1961.

[18] C. L. Siegel, Topics in Complex Function Theory, Vol. 1, Wiley-Interscience, New York, 1969.

[19] M. Stein, Abgeschlossene Untergruppen komplexer abelscher Liescher Gruppen, Dissertation, Univ. Hannover, 1994.

[20] Ch. Vogt, Line bundles on toroidal groups, J. Reine Angew. Math., **335** (1982), 197–215.

[21] A. Weil, On algebraic groups of transformations, Amer. J. Math., **77** (1955), 355–391.

[22] A. Weil, On algebraic groups and homogeneous spaces, Amer. J. Math., **77** (1955), 493–512.

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