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Divisorial contractions to 3-dimensional terminal singularities with discrepancy one

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Abstract. We study a divisorial contraction $\pi : Y \to X$ such that π contracts an irreducible divisor E to a point P and that the discrepancy of E is 1 when $P \in X$ is a 3-dimensional terminal singularity of type (cD/2) and (cE/2).

1. Introduction.

Let Y be a normal projective 3-fold with only $\boldsymbol{Q}$-factorial terminal singularities. If K_Y is not nef, then there is an extremal contraction $\pi : Y \to X$ which is either a birational morphism or of fiber type. Extremal contractions of fiber type are called Mori fiber spaces, and birational extremal contractions are divided into divisorial contractions and flipping contractions. The aim of this article is to study divisorial contractions. Since Mori completed the minimal model program for 3-folds in [**10**], it has been recognized that one has to know divisorial contractions explicitly for further study of 3-folds. Recent progress of Sarkisov program, which studies birational maps between Mori fiber spaces, shows that it is indispensable to know divisorial contractions explicitly ([**1**], [**2**], [**3**]).

We study divisorial contractions $\pi : Y \to X$ starting from X and we regard π as extractions. Since X has only terminal singularities, we define divisorial contractions as follows: Let $P \in X$ be a germ of a 3-dimensional terminal singularity. A projective birational morphism $\pi : Y \to X$ is called a *divisorial contraction* if (i) Y has only terminal singularities, (ii) $-K_Y$ is π-ample and (iii) the exceptional set E of π is an irreducible divisor. In this situation, we write $K_Y = \pi^* K_X + a(E,X)E$ with $a(E,X) \in \boldsymbol{Q}$. The coefficient $a(E,X)$ of E is called the *discrepancy* of E over X. If we further assume that $\pi(E) = \{P\}$, i.e., $\pi_{|Y \setminus E} : Y \setminus E \to X \setminus P$ is isomorphic, then we write $\pi : (E \subset Y) \to (P \in X)$ for the divisorial contraction π.

Let $\pi : (E \subset Y) \to (P \in X)$ be a divisorial contraction. Then Kawakita ([**7**], [**8**]) showed that a general member of $|-K_Y|$ has only rational double points as singularities. If we further assume that $P \in X$ is of index $m \geq 2$, then we know the explicit description of divisorial contractions $\pi : (E \subset Y) \to (P \in X)$ when $P \in X$ is of type (cA/m) ([**8**]) and when $a(E,X) = 1/m$ ([**4**], [**5**]). Moreover if $P \in X$ is not of type (cA/m) and if $a(E,X) \geq 2/m$, then $\pi : (E \subset Y) \to (P \in X)$ satisfies one of the following Table 1. (We reproduce the table in [**8**].)

Kawakita ([**8**]) also provided the explicit description of π in cases (d) and (e) of Table 1. Our purpose here is to study divisorial contractions in cases (a), (b) and (f) of Table 1, all of which satisfy $a(E,X) = 1$. We shall not treat the case (c) in Table 1 here.

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Table 1.

	$P \in X$	$a(E,X)$	singularities on Y at which E is not Cartier
(a)	(cD/2)	1	$\frac{1}{4l+2}(1,-1,2l-1)$, $l \geq 1$
(b)	(cD/2)	1	$\{xy+z^{4lk}+u^2=0\}/\frac{1}{4l}(1,-1,2l-1,0)$, k, $l \geq 1$
(c)	(cD/2)	2	$\frac{1}{4l+2}(1,-1,2l-1)$, $l \geq 1$, $l \equiv 0$ or $3 \pmod 4$
(d)	(cD/2)	$a/2$	$\frac{1}{r}\left(1,-1,\frac{a+r}{2}\right)$, $\frac{1}{r+2}\left(1,-1,\frac{a+r+2}{2}\right)$, a, r odd
(e)	(cD/2)	$a/2$	$\frac{1}{r}\left(1,-1,\frac{a+r}{2}\right)$, $\frac{1}{r+4}\left(1,-1,\frac{a+r+4}{2}\right)$, $a+r$ even
(f)	(cE/2)	1	$\frac{1}{6}(1,5,5)$, $\frac{1}{2}(1,1,1)$

Our strategy for classifying divisorial contractions with discrepancy one is very simple. In [**6**], we determined the number of divisors E over $P(\in X)$ with $a(E,X)=1$ for each terminal singularity $P \in X$, which is always finite. Hence we shall extract these divisors E one by one. As in [**4**] and [**5**], we can do this by embedding X into 4 or 5-dimensional cyclic quotient singularities and constructing certain weighted blow ups of X. Let $\pi : Y \to X$ be one of these weighted blow ups. Then the remaining task is to study singularities of the variety Y. In general Y has only canonical singularities, and we shall determine when Y has only terminal singularities. Since there are infinitely many divisors with discrepancy two over $P(\in X)$, our method can not be applied to classify divisorial contractions in case (c) of Table 1.

Our result shows that there are only a few divisorial contractions $\pi : (E \subset Y) \to (P \in X)$ with $a(E,X)=1$. In fact, as shown in [**8**] by examples, $P \in X$ has such divisorial contractions only when the defining equation of X has some special terms. We shall state our main results as follows:

Theorem 1.1. *Let $P \in X$ be a germ of a 3-dimensional terminal singularity of type* (cD/2). *Then there is a divisorial contraction $\pi : (E \subset Y) \to (P \in X)$ with $a(E,X)=1$ if and only if one of the following* (i), (i′), (ii), (ii′) *or* (iii) *holds*:

(i) *There is a positive integer l and an embedding of X into a 4-dimensional cyclic quotient singularity $(x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that*

$$X = \{x^2+y^2u+s(z,u)yzu+r(z)y+p(z,u)=0\}/\frac{1}{2}(1,1,1,0),$$

where $s(z,u) = \sum_{2i+j=2l-2} s_{ij}z^{2i}u^j$, $r(z) = \sum_{i \geq l} r_i z^{2i+1}$ and $p(z,u) = \sum_{2i+j \geq 4l} p_{ij}z^{2i}u^j$ with $p_{2l,0} \neq 0$.

(i′) *There is an embedding of X into $(x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that*

$$X = \{x^2 + yzu + y^4 + z^{2b} + u^c = 0\}/\frac{1}{2}(1,1,1,0)$$

with $b \geq 2$, $c \geq 4$.

(ii) *There is a positive integer l and an embedding of X into a 5-dimensional cyclic quotient singularity $(x,y,z,u,t)/\frac{1}{2}(1,1,1,0,0)$ such that*

$$X = \left\{ \begin{array}{l} x^2 + ut + r(z)y + p(z,u) = 0, \\ y^2 + s(z,u)zx + q(z,u) - t = 0 \end{array} \right\} \Big/ \frac{1}{2}(1,1,1,0,0),$$

where $r(z) = \sum_{i \geq (l+1)/2} r_i z^{2i+1}$, $p(z,u) = \sum_{2i+j \geq 2l+2} p_{ij} z^{2i} u^j$, $s(z,u) = \sum_{2i+j=l-2} s_{ij} z^{2i} u^j$ and $q(z,u) = \sum_{2i+j=2l} q_{ij} z^{2i} u^j$ with either

(a) *$l \in 2\boldsymbol{Z}$, ${q_{l,0}}^2 + {s_{l/2-1,0}}^2 p_{l+1,0} \neq 0$,*

(b) *$l \in 2\boldsymbol{Z} + 1$, $s(z,u) \neq 0$, ${p_{l+1,0}}^2 + q_{l,0}{r_{(l+1)/2}}^2 \neq 0$, or*

(c) *$l \in 2\boldsymbol{Z} + 1$, $s(z,u) = 0$, ${p_{l+1,0}}^2 + q_{l,0}{r_{(l+1)/2}}^2 \neq 0$, and $q(z,u)$ is nonzero and is not a square of a polynomial in z, u with only odd degree terms in z.*

(ii′) *There is an embedding of X into $(x,y,z,u,t)/\frac{1}{2}(1,1,1,0,0)$ such that*

$$X = \{x^2 + ut + y^4 + z^4 = 0,\ yz + u^2 - t = 0\}/\frac{1}{2}(1,1,1,0,0).$$

(iii) *There is a positive integer l and an embedding of X into $(x,y,z,u,t)/\frac{1}{2}(1,1,1,0,1)$ such that*

$$X = \{x^2 + yt + p(z,u) = 0,\ yu + s(z,u)zu + r(z) - t = 0\}/\frac{1}{2}(1,1,1,0,1),$$

where $p(z,u) = \sum_{2i+j \geq 4l+2} p_{ij} z^{2i} u^j$, $s(z,u) = \sum_{2i+j=2l-1} s_{ij} z^{2i} u^j$ and $r(z) = \sum_{i \geq l} r_i z^{2i+1}$ with $r_l \neq 0$.

In case (1.1)(i), the blow up $\pi : Y \to X$ with weight $(2l, 2l, 1, 1)$ is a divisorial contraction with discrepancy 1. There is one non-Gorenstein point on Y at which the exceptional divisor is not Cartier. The point is isomorphic to $\{xy + z^{4l} + u^2 = 0\}/\frac{1}{4l}(1, -1, 2l-1, 0)$, which is a terminal singularity of type (cA$/4l$) and deforms to two cyclic quotient terminal singularities $\frac{1}{4l}(1,-1,2l-1)$.

Similarly, in case (1.1)(i′), the blow up $\pi : Y \to X$ with weight $(2,2,1,1)$ is a divisorial contraction with discrepancy 1. There is one non-Gorenstein point on Y at which the exceptional divisor is not Cartier, which is isomorphic to $\{xy + z^{4(b-1)} + u^2 = 0\}/\frac{1}{4}(1,-1,1,0)$. A part of this case is contained in case (i).

In case (1.1)(ii), the blow up $\pi : Y \to X$ with weight $(l+1, l, 1, 1, 2l+1)$ is a divisorial contraction with discrepancy 1. There is one non-Gorenstein point on Y at which the exceptional divisor is not Cartier. The point is a cyclic quotient terminal singularity $\frac{1}{4l+2}(1,-1,2l-1)$. The case (1.1)(ii′) is a special case of (ii)(c) with $l = 1$.

In case (1.1)(iii), the blow up $\pi : Y \to X$ with weight $(2l+1, 2l, 1, 1, 2l+2)$ is a divisorial contraction with discrepancy 1. There are exactly two non-Gorenstein points on Y at which the exceptional divisor is not Cartier. These points are cyclic quotient terminal singularities $\frac{1}{4l}(1,-1,2l-1)$ and $\frac{1}{4l+4}(1,-1,2l+1)$.

Theorem 1.2. *Let $P \in X$ be a germ of a 3-dimensional terminal singularity of type* (cE/2). *Then there is a divisorial contraction $\pi : (E \subset Y) \to (P \in X)$ with $a(E,X) = 1$ if and only if there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(0,1,1,1)$ such that*

$$X = \left\{ \begin{matrix} u^2 + x^3 + 3\nu x^2 z^2 + \sum_{i+j\geq 4, even} a_{ij} y^i z^j x \\ + \sum_{i+j\geq 4, even} b_{ij} y^i z^j = 0 \end{matrix} \right\} \Big/ \frac{1}{2}(0,1,1,1),$$

with $\nu \in \boldsymbol{C}$, $a_{ij} = 0$ if $3i + j \leq 4$, $b_{ij} = 0$ if $3i + j \leq 8$ and $b_{40} \neq 0$, $b_{08} \neq 0$.

In (1.2), we embed X as in the theorem, then the blow up $\pi : Y \to X$ with weight $(3,2,1,4)$ is in fact a divisorial contraction with discrepancy 1. There are two non-Gorenstein points on Y, which are isomorphic to $\frac{1}{6}(1,1,5)$ and $\frac{1}{2}(1,1,1)$. The exceptional divisor is not Cartier at these points.

As we mentioned above, we study divisorial contractions $\pi : (E \subset Y) \to (P \in X)$ with $a(E,X) = 1$ by extracting divisors from $P \in X$, and we shall use classification of terminal singularities due to [**9**]. In sections 2 and 3, we shall deal with the case $P \in X$ is of type (cD/2). The case $P \in X$ is of type (cE/2) will be treated in section 4.

2. Terminal singularities of type (cD/2-1).

Let $P \in X$ be a germ of a 3-dimensional terminal singularity of type (cD/2). By [**9**], there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that

$$X = \begin{cases} \{x^2 + yzu + y^{2a} + z^{2b} + u^c = 0\}/\dfrac{1}{2}(1,1,1,0) \quad \text{or} \\ \{x^2 + y^2 u + h(z) y + g(z,u) = 0\}/\dfrac{1}{2}(1,1,1,0), \end{cases}$$

where $a,\ b \geq 2,\ c \geq 3,\ h(z) = \sum_{i\geq 1} b_i z^{2i+1} \in \boldsymbol{C}\{z\},\ g(z,u) = \sum_{i,j} a_{ij} z^{2i} u^j \in (z^4, z^2u^2, u^3)\boldsymbol{C}\{z^2, u\}$. In this section, we shall study the first half of terminal singularities of type (cD/2). The remaining (cD/2) case will be treated in section 3.

Theorem 2.1. *Let $P \in X$ be a germ of a 3-dimensional terminal singularity of type* (cD/2), *and assume that there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that*

$$X = \{x^2 + yzu + y^{2a} + z^{2b} + u^c = 0\}/\frac{1}{2}(1,1,1,0) \tag{2.1.1}$$

with $a, b \geq 2$ and $c \geq 3$. Then there is a divisorial contraction $\pi : (E \subset Y) \to (P \in X)$ with $a(E,X) = 1$ if and only if one of the following holds:

(i) $c \geq 4$, $a = 2$. (ii) $c \geq 4$, $b = 2$. (iii) $c = 3$, $a = b = 2$.

Proof. We embed X as in (2.1.1) and let $\pi_1 : Y_1 \to X$ be the blow up with weight $(2,1,1,2)$. Then the exceptional set E_1 of π_1 is a $\boldsymbol{Z}_2$-quotient of

$$\tilde{E}_1 = \{x^2 + yzu + \delta_{a,2}y^{2a} + \delta_{b,2}z^{2b} = 0\} \subseteq \boldsymbol{P}(2,1,1,2),$$

hence E_1 is an irreducible divisor over $P(\in X)$. (Here δ_{ij} means the Kronecker's symbol, so that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.) It also satisfies $a(E_1, X) = 1$. The y-chart of Y_1 is isomorphic to

$$\{x^2 + zu + y^{2a-4} + z^{2b}y^{2b-4} + u^c y^{2c-4} = 0\}/\frac{1}{2}(1,1,0,0),$$

which has one dimensional singular locus, hence Y_1 is not terminal.

If $c \geq 4$, then there are exactly three divisors with discrepancy 1 by [**6**, 8.3, 8.6, 8.7]. We again embed X as in (2.1.1) and let $\pi_2 : Y_2 \to X$ be the blow up with weight $(2,2,1,1)$. The exceptional set E_2 of π_2 is a $\boldsymbol{Z}_2$-quotient of

$$\tilde{E}_2 = \{x^2 + yzu + \delta_{b,2}z^{2b} + \delta_{c,4}u^c = 0\} \subseteq \boldsymbol{P}(2,2,1,1),$$

hence E_2 is an irreducible divisor over $P(\in X)$. It also satisfies $a(E_2, X) = 1$. Since $(1:0:0:0) \notin \tilde{E}_2$, we see that Y_2 is covered by the y-chart U_2, the z-chart U_3 and the u-chart U_4 as follows:

$$U_2 = \{x^2 + zu + y^{4a-4} + z^{2b}y^{2b-4} + u^c y^{c-4} = 0\}/\frac{1}{4}(0,1,1,3),$$

$$U_3 = \{x^2 + yu + y^{2a}z^{4a-4} + z^{2b-4} + u^c z^{c-4} = 0\}/\frac{1}{2}(1,1,1,1),$$

$$U_4 = \{x^2 + yz + y^{2a}u^{4a-4} + z^{2b}u^{2b-4} + u^{c-4} = 0\}/\frac{1}{2}(1,1,1,0).$$

Hence Y_2 is terminal if and only if $b = 2$. Similarly the blow up $\pi_2' : Y_2' \to X$ with weight $(2,1,2,1)$ has an irreducible exceptional divisor E_2' over $P(\in X)$ with $a(E_2', X) = 1$, and Y_2' is terminal if and only if $a = 2$. Since Y_1, Y_2 and Y_2' are not mutually isomorphic over X, we get the desired result when $c \geq 4$.

If $c = 3$, then there are exactly two divisors with discrepancy 1 by [**6**, 8.3, 8.6, 8.7]. In this case, we can embed X into $(x,y,z,u,t)/\frac{1}{2}(1,1,1,0,0)$ such that

$$X = \{x^2 + ut + y^{2a} + z^{2b} = 0,\ yz + u^2 - t = 0\}/\frac{1}{2}(1,1,1,0,0). \tag{2.1.2}$$

Using this embedding, we denote the blow up with weight $(2,1,1,1,3)$ by $\pi_3 : Y_3 \to X$. The exceptional set E_3 of π_3 is a $\boldsymbol{Z}_2$-quotient of

$$\tilde{E}_3 = \{x^2 + ut + \delta_{a,2}y^{2a} + \delta_{b,2}z^{2b} = 0,\ yz + u^2 = 0\} \subseteq \boldsymbol{P}(2,1,1,1,3),$$

hence E_3 is an irreducible divisor over $P(\in X)$. We also see that $a(E_3, X) = 1$. Since $\tilde{E}_3 \cap \{(* : 0 : 0 : * : 0)\} = \varnothing$, we see that Y_3 is covered by the y-chart V_2, the z-chart V_3 and the t-chart V_5 as follows:

$$V_2 = \{x^2 + ut + y^{2a-4} + z^{2b}y^{2b-4} = 0,\ z + u^2 - ty = 0\}/\frac{1}{2}(1, 1, 0, 1, 1),$$

$$V_3 = \{x^2 + ut + y^{2a}z^{2a-4} + z^{2b-4} = 0,\ y + u^2 - tz = 0\}/\frac{1}{2}(1, 0, 1, 1, 1),$$

$$V_5 = \{x^2 + u + y^{2a}u^{2a-4} + z^{2b}u^{2b-4} = 0,\ yz + u^2 - t = 0\}/\frac{1}{6}(5, 1, 1, 4, 2).$$

Hence Y_3 is terminal if and only if $a = b = 2$. Since Y_1 and Y_3 are not isomorphic over X, we get the desired result when $c = 3$. □

Remark 2.2. (1) If $c \geq 4$ and $b = 2$, then under the embedding of X as in (2.1.1), the blow up $\pi_2 : (E_2 \subset Y_2) \to (P \in X)$ with weight $(2, 2, 1, 1)$ gives a divisorial contraction with $a(E_2, X) = 1$. In this case, the non-Gorenstein singularity on Y_2 at which E_2 is not Cartier is unique and is isomorphic to the origin of $\{xy + z^{4a-4} + u^2 = 0\}/\frac{1}{4}(1, 3, 1, 0)$, which is a terminal singularitiy of type (cA/4) and deforms to two cyclic quotient terminal singularities $\frac{1}{4}(1, 3, 1)$. Similarly, if $c \geq 4$ and $a = 2$, then the blow up $\pi_2' : (E_2' \subset Y_2') \to (P \in X)$ with weight $(2, 1, 2, 1)$ is also divisorial with $a(E_2', X) = 1$. We also see that these are all divisorial contractions with discrepancy 1 if $c \geq 4$. In particular, there are exactly two divisorial contractions with discrepancy 1 when $c \geq 4$ and $a = b = 2$. Both of these are the case (i′) of (1.1).

(2) In the case $c = 3$, $a = b = 2$, we embed X as in (2.1.2), then the blow up $\pi_3 : (E_3 \subset Y_3) \to (P \in X)$ with weight $(2, 1, 1, 1, 3)$ gives a divisorial contraction with $a(E_3, X) = 1$. The unique non-Gorenstein singularity on Y_3 is a cyclic quotient terminal singularity $\frac{1}{6}(1, 5, 1)$. The exceptional divisor E_3 is not Cartier at this point. We see that this is the unique divisorial contraction with discrepancy 1 when $c = 3$. This is the case (ii′) of (1.1).

3. Terminal singularities of type (cD/2-2).

In this section, we continue the study of terminal singularities of type (cD/2). We shall complete the analysis of type (cD/2) in this section. Results in this section are summarized in (3.9) and (3.22).

3.1. Let $P \in X$ be a germ of a 3-dimensional terminal singularity of type (cD/2). Here we shall assume that there is an embedding of X into $(x, y, z, u)/\frac{1}{2}(1, 1, 1, 0)$ such that

$$X = \{x^2 + y^2u + h(z)y + g(z, u) = 0\}/\frac{1}{2}(1, 1, 1, 0), \tag{3.1.1}$$

where $h(z) = \sum_{i \geq 1} b_i z^{2i+1} \in \boldsymbol{C}\{z\}$ and $g(z, u) = \sum_{i,j} a_{ij} z^{2i}u^j \in (z^4, z^2u^2, u^3)\boldsymbol{C}\{z^2, u\}$.

In order to study blow ups of X, we introduce some invariants of X and a condition on X. Let

$$w = \min(\{2i \mid b_i \neq 0\} \cup \{i+j \mid a_{ij} \neq 0\}),$$
$$w'_1 = \min\{4i+1 \mid b_i \neq 0\} \text{ and } w'_2 = \min\{2i+j \mid a_{ij} \neq 0\}.$$

Furthermore we set $w' = \min\{w'_1, w'_2\}$.

Let $F(z,u)$ be the lowest degree part of $\frac{1}{4}h(z)^2 - g(z,u)u$, and let $\chi(z) = F(z,1)$. We consider the following condition:

$$\chi(z) = \left(\textstyle\sum_{i\geq 0} \beta_i z^{2i+1}\right)^2 \text{ for some } \beta_i \in \boldsymbol{C}. \tag{†}$$

If w' is even, then this condition is equivalent to saying that the lowest degree part of $g(z,u)$ is a square of a polynomial in z and u with only odd degree terms in z. If w' is odd, then the condition (†) says that the lowest degree part of $\frac{1}{4}h(z)^2 - g(z,u)u$ is a square of a polynomial in z and u with only odd degree terms in z.

We shall keep these notations throughout this section.

Proposition 3.2. *Let k be a positive integer with $k \leq w-1$. Under the embedding of X as in* (3.1.1)*, let $\pi_k : Y_k \to X$ be the blow up with weight $(k+1,k,1,2)$. Then the exceptional set E_k of π_k is an irreducible divisor over $P(\in X)$ with $a(E_k,X)=1$ and Y_k is canonical. However $\pi_k : (E_k \subset Y_k) \to (P \in X)$ is not a divisorial contraction. These Y_k are not mutually isomorphic over X.*

Proof. Since E_k is a $\boldsymbol{Z}_2$-quotient of

$$\tilde{E}_k = \{x^2 + y^2u + b_{(k+1)/2}z^{k+2}y + \textstyle\sum_{i+j=k+1} a_{ij}z^{2i}u^j = 0\} \subseteq \boldsymbol{P}(k+1,k,1,2),$$

we see that E_k is an irreducible divisor over $P(\in X)$. (Here the third term is assumed to be zero if $(k+1)/2$ is not an integer.) We also see that $a(E_k,X)=1$. The z-chart of Y_k is isomorphic to

$$\{x^2 + y^2u + h(z)/z^{k+2}\cdot y + g(z,z^2u)/z^{2k+2} = 0\}/\frac{1}{2}(k,k+1,1,0),$$

which has one dimensional singular locus, hence Y_k is not terminal. □

If the condition (†) does not hold, then there are exactly w divisors with discrepancy 1 over $P \in X$ by [**6**, 9.21]. Hence we have to find one more blowing ups of X whose exceptional divisor is irreducible and has discrepancy 1. This will be done by dividing into several cases. First we study the case where w' is even.

Proposition 3.3. *Assume that the condition* (†) *does not hold and that w' is even. Under the embedding of X as in* (3.1.1)*, let $\pi : Y \to X$ be the blow up with weight $(w'/2, w'/2, 1, 1)$. Then,*

(1) *The exceptional set E of π is an irreducible divisor over $P(\in X)$ with $a(E,X) = 1$ and Y is canonical.*

(2) *The birational morphism* $\pi : (E \subset Y) \to (P \in X)$ *is a divisorial contraction if and only if* $w' \in 4\boldsymbol{Z}$ *and* $a_{w'/2,0} \neq 0$.

Proof. Since the condition (†) does not hold and since E is a $\boldsymbol{Z}_2$-quotient of

$$\tilde{E} = \{x^2 + \textstyle\sum_{2i+j=w'} a_{ij} z^{2i} u^j = 0\} \subseteq \boldsymbol{P}(w'/2, w'/2, 1, 1),$$

we see that E is an irreducible divisor over $P(\in X)$. We also see that $a(E, X) = 1$. Since $(1:0:0:0) \notin \tilde{E}$, Y is covered by the y-chart U_2, the z-chart U_3 and the u-chart U_4 as follows:

$$U_2 = \{x^2 + yu + h(yz)/y^{w'/2} + g(yz, yu)/y^{w'} = 0\} / \frac{1}{w'}(0, 1, w'/2 - 1, -1),$$

$$U_3 = \{x^2 + y^2 zu + h(z)/z^{w'/2} \cdot y + g(z, zu)/z^{w'} = 0\} / \frac{1}{2}(w'/2 - 1, w'/2 - 1, 1, 1),$$

$$U_4 = \{x^2 + y^2 u + h(zu)/u^{w'/2} \cdot y + g(zu, u)/u^{w'} = 0\} / \frac{1}{2}(1, 1, 1, 0).$$

If $w' \in 4\boldsymbol{Z} + 2$, then U_3 has one dimensional singular locus. Hence, if Y is terminal, then we have $w' \in 4\boldsymbol{Z}$ and $a_{w'/2,0} \neq 0$. Conversely, if $w' \in 4\boldsymbol{Z}$ and $a_{w'/2,0} \neq 0$, then w' and $w'/2 - 1$ are coprime, hence Y is terminal. □

Remark 3.4. If $\pi : (E \subset Y) \to (P \in X)$ is a divisorial contraction in (3.3), then the non-Gorenstein singularity on Y at which E is not Cartier is unique and is isomorphic to the origin of $\{xy + z^{w'} + u^2 = 0\} / \frac{1}{w'}(1, -1, w'/2 - 1, 0)$, which deforms to two cyclic quotient terminal singularities $\frac{1}{w'}(1, -1, w'/2 - 1)$.

3.5. Next we consider the case where the condition (†) does not hold and w' is odd. In this case, either $w'_1 > w'_2$ or $w'_1 = w'_2$ occurs. In the case $w'_1 > w'_2$, we have $u \mid \sum_{2i+j=w'} a_{ij} z^{2i} u^j$ since w' is odd. Therefore there is an embedding of X into $(x, y, z, u, t)/\frac{1}{2}(1, 1, 1, 0, 0)$ such that

$$X = \left\{ \begin{array}{c} x^2 + ut + h(z)y + g_1(z, u) = 0, \\ y^2 + g'_0(z, u) - t = 0 \end{array} \right\} \Big/ \frac{1}{2}(1, 1, 1, 0, 0), \tag{3.5.1}$$

where $g'_0(z, u) = \sum_{2i+j=w'} a_{ij} z^{2i} u^{j-1}$ and $g_1(z, u) = \sum_{2i+j \geq w'+1} a_{ij} z^{2i} u^j$.

Proposition 3.6. *Assume that the condition* (†) *does not hold,* w' *is odd and* $w'_1 > w'_2$. *Under the embedding of* X *as in* (3.5.1), *let* $\pi : Y \to X$ *be the blow up with weight* $((w'+1)/2, (w'-1)/2, 1, 1, w')$. *Then we have the following*:

(1) *The exceptional set* E *of* π *is an irreducible divisor over* $P(\in X)$ *with* $a(E, X) = 1$ *and* Y *is canonical.*

(2) *The birational morphism* $\pi : (E \subset Y) \to (P \in X)$ *is a divisorial contraction if and only if one of the following holds*:

(i) $w' \in 4\boldsymbol{Z} + 1$ *and* $a_{(w'-1)/2,1} \neq 0$.

(ii) $w' \in 4\boldsymbol{Z} + 3$ *and* $a_{(w'-1)/2,1} {b_{(w'+1)/4}}^2 - {a_{(w'+1)/2,0}}^2 \neq 0$.

PROOF. Since $g_0'(z,u)$ is not a square of a polynomial in z and u with only odd degree terms in z and since E is a $\boldsymbol{Z}_2$-quotient of

$$\begin{aligned}\tilde{E} &= \left\{\begin{matrix} x^2+ut+b_{(w'+1)/4}z^{(w'+3)/2}y+\sum_{2i+j=w'+1}a_{ij}z^{2i}u^j=0, \\ y^2+g_0'(z,u)=0 \end{matrix}\right\} \\ &\subseteq \boldsymbol{P}((w'+1)/2,(w'-1)/2,1,1,w'),\end{aligned}$$

we see that E is an irreducible divisor over $P(\in X)$. (Here the third term of the first equation is assumed to be zero if $(w'+1)/4$ is not an integer.) We also see that $a(E,X)=1$. Since $\tilde{E}\cap\{(*:*:0:0:0)\}=\varnothing$, we see that Y is covered by the z-chart V_3, the u-chart V_4 and the t-chart V_5 as follows:

$$\begin{aligned}V_3 &= \left\{\begin{matrix} x^2+ut+h(z)/z^{(w'+3)/2}\cdot y+g_1(z,zu)/z^{w'+1}=0, \\ y^2+g_0'(1,u)-tz=0 \end{matrix}\right\} \\ &\quad\Big/\frac{1}{2}((w'-1)/2,(w'-3)/2,1,1,1), \\ V_4 &= \left\{\begin{matrix} x^2+t+h(zu)/u^{(w'+3)/2}\cdot y+g_1(zu,u)/u^{w'+1}=0, \\ y^2+g_0'(z,1)-tu=0 \end{matrix}\right\}\Big/\frac{1}{2}(1,1,1,0,0), \\ V_5 &= \left\{\begin{matrix} x^2+u+h(zt)/t^{(w'+3)/2}\cdot y+g_1(zt,ut)/t^{w'+1}=0, \\ y^2+g_0'(z,u)-t=0 \end{matrix}\right\} \\ &\quad\Big/\frac{1}{2w'}(-1,1,w'-2,-2,2).\end{aligned}$$

If $w'\in 4\boldsymbol{Z}+1$, then the fixed points on V_3 of the action of the cyclic group satisfy $y=z=u=0$ and $x^2+a_{(w'+1)/2,0}=0$, $a_{(w'-1)/2,1}=0$. Hence the condition $a_{(w'-1)/2,1}\neq 0$ is necessary for Y to be terminal. Similarly if $w'\in 4\boldsymbol{Z}+3$, then the fixed points on V_3 of the action of the cyclic group satisfy $x=z=u=0$ and $y^2+a_{(w'-1)/2,1}=0$, $b_{(w'+1)/4}y+a_{(w'+1)/2,0}=0$. Therefore we need the condition $a_{(w'-1)/2,1}{b_{(w'+1)/4}}^2-{a_{(w'+1)/2,0}}^2\neq 0$ for Y to be terminal. Conversely each of these conditions (i) or (ii) is sufficient for Y to be terminal. $\square$

REMARK 3.7. If $\pi:(E\subset Y)\to(P\in X)$ is a divisorial contraction in (3.6), then the non-Gorenstein singularity on Y at which E is not Cartier is unique and the point is isomorphic to the cyclic quotient terminal singularity $\frac{1}{2w'}(1,-1,w'-2)$.

PROPOSITION 3.8. *Assume that the condition* (†) *does not hold, w' is odd and $w_1'=w_2'$. Under the embedding of X as in* (3.1.1), *let $\pi:Y\to X$ be the blow up with weight $((w'+1)/2,(w'-1)/2,1,1)$. Then the exceptional set E of π is an irreducible divisor over $P(\in X)$ with $a(E,X)=1$ and Y is canonical. However $\pi:(E\subset Y)\to(P\in X)$ is not a divisorial contraction.*

Proof. There is a positive integer i_0 such that $4i_0+1=w'$ and $b_{i_0}\neq 0$. Since the condition (†) does not hold and since E is a $\boldsymbol{Z}_2$-quotient of

$$\begin{aligned}\tilde{E} &= \{y^2u + b_{i_0}z^{2i_0+1}y + \textstyle\sum_{2i+j=w'} a_{ij}z^{2i}u^j = 0\}\\ &\subseteq \boldsymbol{P}((w'+1)/2,(w'-1)/2,1,1),\end{aligned}$$

we see that E is an irreducible divisor over $P(\in X)$. We also see that $a(E,X)=1$. Since $w'\equiv 1 \pmod 4$, the z-chart of Y is expressed as

$$\{x^2z + y^2u + h(z)/z^{(w'+1)/2}\cdot y + g(z,zu)/z^{w'} = 0\}/\frac{1}{2}(0,1,1,1),$$

which is singular along x-axis. In particular Y is not terminal. $\square$

Theorem 3.9. *Let $P\in X$ be as in* (3.1) *and assume that the condition* (†) *does not hold. Then there is a divisorial contraction $\pi:(E\subset Y)\to(P\in X)$ with $a(E,X)=1$ if and only if one of the following holds*:

(i) $w_1'>w_2'=w'\in 4\boldsymbol{Z}$ *and* $a_{w'/2}\neq 0$.

(ii) $w_1'>w_2'=w'\in 4\boldsymbol{Z}+1$ *and* $a_{(w'-1)/2,1}\neq 0$.

(iii) $w_1'>w_2'=w'\in 4\boldsymbol{Z}+3$ *and* $a_{(w'-1)/2,1}{b_{(w'+1)/4}}^2-{a_{(w'+1)/2,0}}^2\neq 0$.

Proof. By [**6**, 9.21], there are exactly w divisors with discrepancy 1 over X. Among these divisors, we found $w-1$ in (3.2). Since Y in (3.3), (3.6) or (3.8) is not isomorphic to Y_k in (3.2), we complete the proof. $\square$

Remark 3.10. In each case of (3.9), there is exactly one divisorial contraction with discrepancy 1. We see that (3.9)(i) is the case $s(z,u)=0$ in (1.1)(i) (by setting $l=w'/4$), and that (3.9)(ii), (iii) are contained in the case $s(z,u)=0$ in (1.1)(ii)(a), (ii)(c) respectively (by setting $l=(w'-1)/2$).

Now we turn to the case where the condition (†) holds. In this case we know that there are exactly $w+1$ divisors with discrepancy 1 over $P(\in X)$ by [**6**, 9.21]. Among these divisors, we found $w-1$ in (3.2), therefore we have to find two more divisors with discrepancy 1.

3.11. If (†) holds and w' is even, then we can write $\sum_{2i+j=w'}a_{ij}z^{2i}u^j=-s(z,u)^2$ for some $s(z,u)\in\boldsymbol{C}[z,u]$ with only odd degree terms in z. There are two embeddings $X\simeq X_\pm\subseteq(x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that

$$X_\pm = \{x^2 \pm 2s(z,u)x + y^2u + h(z)y + g_1(z,u) = 0\}/\frac{1}{2}(1,1,1,0), \tag{3.11.1}$$

where $g_1(z,u)=\sum_{2i+j\geq w'+1}a_{ij}z^{2i}u^j$. We first study the case where (a) $w'\in 4\boldsymbol{Z}$, $b_{w'/4}\neq 0$ or (b) $w'\in 4\boldsymbol{Z}+2$, $a_{w'/2,0}\neq 0$.

Proposition 3.12. *Assume that the condition* (†) *holds and that w' is even. We further assume that* (a) *$w'\in 4\boldsymbol{Z}$, $b_{w'/4}\neq 0$ or* (b) *$w'\in 4\boldsymbol{Z}+2$, $a_{w'/2,0}\neq 0$. Under* *the embeddings of X as in* (3.11.1), *let $\pi_\pm : Y_\pm \to X_\pm \simeq X$ be the blow up with weight $(w'/2+1, w'/2, 1, 1)$. Then the exceptional set $E_\pm$ of $\pi_\pm$ is an irreducible divisor over $P(\in X)$ with $a(E_\pm, X) = 1$ and $Y_\pm$ is canonical. However $\pi_\pm : (E_\pm \subset Y_\pm) \to (P \in X)$ is not a divisorial contraction.*

Proof. If $a_{w'/2,0} \neq 0$, then we see that $u \not| \, s(z,u)$. By conditions (a) or (b), $E_\pm$ is a $\boldsymbol{Z}_2$-quotient of

$$\begin{aligned}\tilde{E}_\pm &= \{\pm 2s(z,u)x + y^2u + b_{w'/4}z^{w'/2+1}y + \textstyle\sum_{2i+j=w'+1} a_{ij}z^{2i}u^j = 0\} \\ &\subseteq \boldsymbol{P}(w'/2+1, w'/2, 1, 1).\end{aligned}$$

(Here we assume that the third term is zero if $w'/4$ is not an integer.) Hence $E_\pm$ is an irreducible divisor over $P(\in X)$. We also see that $a(E_\pm, X) = 1$. The x-chart U_1 and the y-chart U_2 of $Y_\pm$ are described as

$$\begin{aligned}U_1 &= \left\{x \pm 2s(z,u) + y^2u + h(xz)/x^{w'/2+1}\cdot y + g_1(xz,xu)/x^{w'+1} = 0\right\} \\ &\quad \Big/ \frac{1}{w'+2}(1,1,w'/2,-1), \\ U_2 &= \left\{x^2y \pm 2s(z,u)x + u + h(yz)/y^{w'/2+1} + g_1(yz,yu)/y^{w'+1} = 0\right\} \\ &\quad \Big/ \frac{1}{w'}(-1,1,w'/2-1,-1)\end{aligned}$$

respectively. Thus the origin of U_1 is isomorphic to $\frac{1}{w'+2}(1,-1,w'/2)$ and the origin of U_2 is isomorphic to $\frac{1}{w'}(1,-1,w'/2-1)$. If $Y_\pm$ is terminal, then $w'+2$ and $w'/2$ are coprime, w' and $w'/2-1$ are coprime. Hence $w' \in 4\boldsymbol{Z}+2$ and $w' \in 4\boldsymbol{Z}$, which is a contradiction. $\square$

3.13. We continue the study of the case that the condition (†) holds and w' is even. Here we further assume that (a) $w' \in 4\boldsymbol{Z}$, $b_{w'/4} = 0$ or (b) $w' \in 4\boldsymbol{Z}+2$, $a_{w'/2,0} = 0$. In both of these cases, we have $u \mid s(z,u)$ and $u \mid \sum_{2i+j=w'+1} a_{ij}z^{2i}u^j$ since w' is even. Hence there are two embeddings $X \simeq X_\pm \subseteq (x,y,z,u,t)/\frac{1}{2}(1,1,1,0,0)$ such that

$$X_\pm = \left\{\begin{array}{c} x^2 + ut + h(z)y + g_2(z,u) = 0, \\ y^2 \pm 2s'(z,u)x + g_1'(z,u) - t = 0 \end{array}\right\} \Big/ \frac{1}{2}(1,1,1,0,0), \tag{3.13.1}$$

where $s'(z,u) = s(z,u)/u$, $g_1'(z,u) = \sum_{2i+j=w'+1} a_{ij}z^{2i}u^{j-1}$ and $g_2(z,u) = \sum_{2i+j\geq w'+2} a_{ij}z^{2i}u^j$.

Proposition 3.14. *Assume that the condition (†) holds and that w' is even. We further assume that* (a) *$w' \in 4\boldsymbol{Z}$, $b_{w'/4} = 0$ or* (b) *$w' \in 4\boldsymbol{Z}+2$, $a_{w'/2,0} = 0$. Under the embeddings of X as in* (3.13.1), *let $\pi_\pm : Y_\pm \to X_\pm \simeq X$ be the blow up with weight $(w'/2+1, w'/2, 1, 1, w'+1)$. Then the following holds*:

(1) *The exceptional set* $E_\pm$ *of* $\pi_\pm$ *is an irreducible divisor over* $P(\in X)$ *with* $a(E_\pm, X) = 1$ *and* $Y_\pm$ *is canonical.*

(2) *The birational morphism* $\pi_\pm : (E_\pm \subset Y_\pm) \to (P \in X)$ *is a divisorial contraction if and only if one of the following holds*:

(i) $w' \in 4\boldsymbol{Z}$ *and* ${a_{w'/2,1}}^2 - 4a_{w'/2+1,0}a_{w'/2-1,2} \neq 0$.

(ii) $w' \in 4\boldsymbol{Z} + 2$ *and* $a_{w'/2,1}{b_{(w'+2)/4}}^2 + {a_{w'/2+1,0}}^2 \neq 0$.

Proof. We can prove this as in the proof of (3.6). Since $s'(z,u) \neq 0$ and since $E_\pm$ is a $\boldsymbol{Z}_2$-quotient of

$$\begin{aligned}\tilde{E}_\pm &= \left\{\begin{matrix} x^2 + ut + b_{(w'+2)/4}z^{w'/2+1}y + \sum_{2i+j=w'+2} a_{ij}z^{2i}u^j = 0, \\ y^2 \pm 2s'(z,u)x + g_1'(z,u) = 0 \end{matrix}\right\} \\ &\subseteq \boldsymbol{P}(w'/2+1, w'/2, 1, 1, w'+1),\end{aligned}$$

we see that $E_\pm$ is an irreducible divisor over $P(\in X)$. (Here the third term of the first equation is assumed to be zero if $(w'+2)/4$ is not an integer.) We also see that $a(E_\pm, X) = 1$. Since $\tilde{E}_\pm \cap \{(* : * : 0 : 0 : 0)\} = \varnothing$, we see that $Y_\pm$ is covered by the z-chart V_3, the u-chart V_4 and the t-chart V_5 as follows:

$$\begin{aligned}V_3 &= \left\{\begin{matrix} x^2 + ut + h(z)/z^{w'/2+2} \cdot y + g_2(z, zu)/z^{w'+2} = 0, \\ y^2 \pm 2s'(1,u)x + g_1'(1,u) - tz = 0 \end{matrix}\right\} \\ &\Big/ \frac{1}{2}(w'/2, w'/2-1, 1, 1, 1), \\ V_4 &= \left\{\begin{matrix} x^2 + t + h(zu)/u^{w'/2+2} \cdot y + g_1(zu, u)/u^{w'+2} = 0, \\ y^2 \pm 2s'(z,1)x + g_1'(z,1) - tu = 0 \end{matrix}\right\} \Big/ \frac{1}{2}(1,1,1,0,0), \\ V_5 &= \left\{\begin{matrix} x^2 + u + h(zt)/t^{w'/2+2} \cdot y + g_1(zt, ut)/t^{w'+2} = 0, \\ y^2 \pm 2s'(z,u)x + g_1'(z,u) - t = 0 \end{matrix}\right\} \\ &\Big/ \frac{1}{2(w'+1)}(-1, 1, w'-1, -2, 2).\end{aligned}$$

If $w' \in 4\boldsymbol{Z}$, then the fixed points on V_3 of the action of the cyclic group satisfy $y = z = u = 0$ and $x^2 + a_{w'/2+1,0} = 0$, $\pm 2s'(1,0)x + a_{w'/2,1} = 0$. Since $s'(1,0)^2 = -a_{w'/2-1,2}$, we need the condition ${a_{w'/2,1}}^2 - 4a_{w'/2+1,0}a_{w'/2-1,2} \neq 0$ in order that $Y_\pm$ is terminal. Similarly if $w' \in 4\boldsymbol{Z}+2$, then the fixed points on V_3 of the action of the cyclic group satisfy $x = z = u = 0$ and $y^2 + a_{w'/2,1} = 0$, $b_{(w'+2)/4}y + a_{w'/2+1,0} = 0$. Therefore we need the condition $a_{w'/2,1}{b_{(w'+2)/4}}^2 + {a_{w'/2+1,0}}^2 \neq 0$ for $Y_\pm$ to be terminal. Conversely each of these conditions (i) or (ii) is sufficient for $Y_\pm$ to be terminal. □

Remark 3.15. If $\pi_\pm : (E_\pm \subset Y_\pm) \to (P \in X)$ is a divisorial contraction in (3.14), then the non-Gorenstein singularity on $Y_\pm$ at which $E_\pm$ is not Cartier is unique and the point is isomorphic to the cyclic quotient terminal singularity $\frac{1}{2(w'+1)}(1, -1, w'-1)$.

3.16. If the condition (†) holds, w' is odd and $w'_1 > w'_2$, then we can write $\sum_{2i+j=w'} a_{ij} z^{2i} u^j = -s'(z,u)^2 u$ for some $s'(z,u) \in \boldsymbol{C}[z,u]$ with only odd degree terms in z. There are two embeddings $X \simeq X_\pm \subseteq (x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that

$$X_\pm = \{x^2 + y^2 u \pm 2s'(z,u)yu + h(z)y + g_1(z,u) = 0\}/\frac{1}{2}(1,1,1,0), \qquad (3.16.1)$$

where $g_1(z,u) = \sum_{2i+j \geq w'+1} a_{ij} z^{2i} u^j \pm h(z)s'(z,u)$.

Proposition 3.17. *Assume that the condition* (†) *holds, w' is odd and $w'_1 > w'_2$. Then under the embeddings of X as in* (3.16.1), *let $\pi_\pm : Y_\pm \to X_\pm \simeq X$ be the blow up with weight $((w'+1)/2, (w'+1)/2, 1, 1)$. Then the following holds*:

(1) *The exceptional set $E_\pm$ of $\pi_\pm$ is an irreducible divisor over $P(\in X)$ with $a(E_\pm, X) = 1$ and $Y_\pm$ is canonical.*

(2) *The birational morphism $\pi_\pm : (E_\pm \subset Y_\pm) \to (P \in X)$ is a divisorial contraction if and only if $w' \in 4\boldsymbol{Z}+3$ and $a_{(w'+1)/2,0} \neq 0$.*

Proof. Since $s'(z,u) \neq 0$ and since $E_\pm$ is a $\boldsymbol{Z}_2$-quotient of

$$\tilde{E}_\pm = \left\{ \begin{array}{c} x^2 \pm 2s'(z,u)yu + \sum_{i+j=w+1} a_{ij} z^{2i} u^j \\ \pm b_{(w'+1)/4} z^{(w'+3)/2} s'(z,u) = 0 \end{array} \right\} \subseteq \boldsymbol{P}((w'+1)/2, (w'+1)/2, 1, 1),$$

we see that $E_\pm$ is an irreducible divisor over $P(\in X)$. (Here the last term is assumed to be zero if $(w'+1)/4$ is not an integer.) We also see that $a(E_\pm, X) = 1$. Since $(1:0:0:0) \notin \tilde{E}_\pm$, we see that $Y_\pm$ is covered by the y-chart U_2, the z-chart U_3 and the u-chart U_4 as follows:

$$\begin{aligned} U_2 &= \{x^2 + yu \pm 2s'(z,u)u + h(yz)/y^{(w'+1)/2} + g_1(yz, yu)/y^{w'+1} = 0\} \\ &\quad \Big/ \frac{1}{w'+1}(0, 1, (w'-1)/2, -1), \\ U_3 &= \{x^2 + y^2 zu \pm 2s'(1,u)yu + h(z)/z^{(w'+1)/2} \cdot y + g_1(z, zu)/z^{w'+1} = 0\} \\ &\quad \Big/ \frac{1}{2}((w'-1)/2, (w'-1)/2, 1, 1), \\ U_4 &= \{x^2 + y^2 u \pm 2s'(z,1)y + h(zu)/u^{(w'+1)/2} \cdot y + g_1(zu, u)/u^{w'+1} = 0\} \\ &\quad \Big/ \frac{1}{2}(1,1,1,0). \end{aligned}$$

If $Y_\pm$ is terminal, then it follows from the expression of U_3 that $(w'-1)/2$ is odd and $a_{(w'+1)/2,0} \neq 0$. Conversely these conditions assure that $w'+1$ and $(w'-1)/2$ are coprime, and are sufficient for $Y_\pm$ to be terminal. $\square$

Remark 3.18. If $\pi_\pm : (E_\pm \subset Y_\pm) \to (P \in X)$ are divisorial contractions in (3.17), then the non-Gorenstein singularity on $Y_\pm$ at which $E_\pm$ is not Cartier is unique and is isomorphic to the origin of $\{xy + z^{w'+1} + u^2 = 0\}/\frac{1}{w'+1}(1, -1, (w'-1)/2, 0)$, which is a terminal singularity of type $(\mathrm{cA}/w'+1)$ and deforms to two cyclic quotient terminal singularities $\frac{1}{w'+1}(1, -1, (w'-1)/2)$.

3.19. Lastly we consider the case where the condition (†) holds, w' is odd and $w'_1 \leq w'_2$. In this case, there is a positive integer i_0 such that $4i_0 + 1 = w'$ and $b_{i_0} \neq 0$. Since the condition (†) holds, we can write

$$\frac{1}{4}b_{i_0}^2 z^{4i_0+2} - \textstyle\sum_{2i+j=w'} a_{ij} z^{2i} u^{j+1} = \left(\frac{1}{2}b_{i_0} z^{2i_0+1} - s'(z,u)u\right)^2$$

for some polynomial $s'(z,u)$ in z and u with only odd degree terms in z. We have $\sum_{2i+j=w'} a_{ij} z^{2i} u^j = b_{i_0} z^{2i_0+1} s'(z,u) - s'(z,u)^2 u$, and hence

$$X = \left\{ \begin{matrix} x^2 + (y + s'(z,u))((y - s'(z,u))u + b_{i_0} z^{2i_0+1}) \\ +h_1(z)y + g_1(z,u) = 0 \end{matrix} \right\} \Big/ \frac{1}{2}(1,1,1,0),$$

where $h_1(z) = h(z) - b_{i_0} z^{2i_0+1}$, $g_1(z,u) = \sum_{2i+j \geq w'+1} z^{2i} u^j$. Thus we can construct two embeddings of X. One is the embedding of X into $(x,y,z,u)/\frac{1}{2}(1,1,1,0)$ such that

$$X = \{x^2 + y^2 u - 2s'(z,u)yu + h(z)y + g'_1(z,u) = 0\}/\frac{1}{2}(1,1,1,0), \tag{3.19.1}$$

where $g'_1(z,u) = g_1(z,u) - h_1(z)s'(z,u)$. The other one is the embedding of X into $(x,y,z,u,t)/\frac{1}{2}(1,1,1,0,1)$ such that

$$X = \left\{ \begin{matrix} x^2 + yt + g'_1(z,u) = 0, \\ (y - 2s'(z,u))u + h(z) - t = 0 \end{matrix} \right\} \Big/ \frac{1}{2}(1,1,1,0,1). \tag{3.19.2}$$

Proposition 3.20. *Assume that the condition* (†) *holds, w' is odd and $w'_1 \leq w'_2$. Then the following holds*:

(1) *Under the embedding of X as in* (3.19.1), *the blow up $\pi : Y \to X$ with weight $((w'+1)/2, (w'+1)/2, 1, 1)$ has an irreducible exceptional divisor E over $P(\in X)$ with $a(E,X) = 1$ and Y is canonical. However $\pi : (E \subset Y) \to (P \in X)$ is not a divisorial contraction.*

(2) *Under the embedding of X as in* (3.19.2), *the blow up $\pi' : Y' \to X$ with weight $((w'+1)/2, (w'-1)/2, 1, 1, (w'+3)/2)$ has an irreducible exceptional divisor E' over $P(\in X)$ with $a(E',X) = 1$ and Y' is terminal. In particular $\pi' : (E' \subset Y') \to (P \in X)$ is a divisorial contraction.*

Proof. (1) We see that E is a $\boldsymbol{Z}_2$-quotient of

$$\begin{aligned} \tilde{E} &= \{x^2 - 2s'(z,u)yu + b_{i_0} z^{2i_0+1} y + \textstyle\sum_{2i+j=w'+1} a_{ij} z^{2i} u^j = 0\} \\ &\subseteq \boldsymbol{P}((w'+1)/2, (w'+1)/2, 1, 1), \end{aligned}$$

hence E is an irreducible divisor over $P(\in X)$. We also see that $a(E,X)=1$. Since $w'\in 4\boldsymbol{Z}+1$, the z-chart of Y is described as

$$\left\{\begin{matrix} x^2+y^2zu-2s'(1,u)yu+h(z)/z^{(w'+1)/2}\cdot y \\ +g_1'(z,zu)/z^{w'+1}=0 \end{matrix}\right\} \Big/ \frac{1}{2}(0,0,1,1),$$

hence Y has one dimensional singular locus. In particular Y is not terminal.

(2) This is the case treated in [**8**, 1.2(ii)]. □

REMARK 3.21. Let $\pi' : (E' \subset Y') \to (P \in X)$ be a divisorial contraction in (3.20)(2). Then there are two non-Gorenstein singularities on Y' at which E' is not Cartier. These points are isomorphic to the cyclic quotient terminal singularities $\frac{1}{w'-1}(1,-1,(w'-3)/2)$ and $\frac{1}{w'+3}(1,-1,(w'+1)/2)$.

THEOREM 3.22. *Let $P\in X$ be as in* (3.1) *and assume that the condition* (†) *holds. Then there is a divisorial contraction $\pi:(E\subset Y)\to(P\in X)$ with $a(E,X)=1$ if and only if one of the following holds*:

(i) $w_1'>w_2'=w'\in 4\boldsymbol{Z}+3$ *and* $a_{(w'+1)/2,0}\neq 0$.

(ii) $w_1'>w_2'=w'\in 4\boldsymbol{Z}$ *and* $b_{w'/4}=0$, ${a_{w'/2,1}}^2-4a_{w'/2+1,0}a_{w'/2-1,2}\neq 0$.

(iii) $w_1'>w_2'=w'\in 4\boldsymbol{Z}+2$ *and* $a_{w'/2,0}=0$, $a_{w'/2,1}{b_{(w'+2)/4}}^2+{a_{w'/2+1,0}}^2\neq 0$.

(iv) $w_2'\geq w_1'=w'$ *and w' is odd.*

PROOF. By [**6**, 9.21], there are exactly $w+1$ divisors with discrepancy 1 over X. Among these divisors, we found $w-1$ in (3.2). We obtain the remaining two divisors as exceptional divisors of some blow ups in (3.12), (3.14), (3.17) and (3.20). We also see that these two divisors are different as valuations on the function field of X and are not the same as E_k in (3.2). □

REMARK 3.23. In cases (i), (ii) and (iii) of (3.22), there are exactly two divisorial contractions with discrepancy 1, while in case (iv) of (3.22), there is only one divisorial contraction with discrepancy 1. We see that (3.22)(i) is contained in the case $s(z,u)\neq 0$ in (1.1)(i) (by setting $l=(w'+1)/4$), and that (3.22)(ii), (iii) are the cases $s(z,u)\neq 0$ in (1.1)(ii)(a), (ii)(b) respectively (by setting $l=w'/2$). We also see that (3.22)(iv) is the case (1.1)(iii).

By (2.1), (3.9) and (3.22) (see also (2.2), (3.10) and (3.23)), we know all divisorial contractions to (cD/2) type terminal singularities which have discrepancy 1, and we complete the proof of (1.1). □

4. Terminal singularities of type (cE/2).

In this section, we shall study terminal singularities of type (cE/2). We shall show that there are only a few divisorial contractions with discrepancy 1 in the (cE/2) case.

THEOREM 4.1. *Let $P\in X$ be a germ of a 3-dimensional terminal singularity of type* (cE/2). *Then there is a divisorial contraction $\pi:(E\subset Y)\to(P\in X)$ with $a(E,X)=1$ if and only if there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(0,1,1,1)$ such that*

$$X = \{u^2 + x^3 + 3\nu x^2 z^2 + g(y,z)x + h(y,z) = 0\}/\frac{1}{2}(0,1,1,1),$$

where $\nu \in \boldsymbol{C}$, $g(y,z) = \sum_{i+j\geq 4, even} a_{ij} y^i z^j$, $h(y,z) = \sum_{i+j\geq 4, even} b_{ij} y^i z^j \in \boldsymbol{C}\{y,z\}$ *with* $a_{ij} = 0$ *if* $3i + j \leq 4$, $b_{ij} = 0$ *if* $3i + j \leq 8$ *and* $b_{40} \neq 0$, $b_{08} \neq 0$.

Proof. By [**9**], there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(0,1,1,1)$ such that

$$X = \{u^2 + x^3 + g(y,z)x + h(y,z) = 0\}/\frac{1}{2}(0,1,1,1), \tag{4.1.1}$$

where $g(y,z) = \sum_{i+j\geq 4, even} a_{ij} y^i z^j$, $h(y,z) = \sum_{i+j\geq 4, even} b_{ij} y^i z^j \in \boldsymbol{C}\{y,z\}$ and the degree four part $h_4(y,z)$ of $h(y,z)$ is nonzero.

Under the embedding of X as in (4.1.1), let $\pi_1 : Y_1 \to X$ be the blow up with weight $(2,1,1,2)$. Then the y-chart of Y_1 is isomorphic to

$$\{u^2 + x^3 y^2 + g(y,yz)/y^2 \cdot x + h(y,yz)/y^4 = 0\}/\frac{1}{2}(0,1,0,1).$$

Hence the y-chart or the z-chart of Y_1 has singularities which are not terminal. The exceptional set E_1 of π_1 is an irreducible divisor over $P(\in X)$ and satisfies $a(E_1, X) = 1$. If $h_4(y,z)$ has four distinct factors, then it follows from [**6**, 7.3] that there is only one divisor with discrepancy 1. Hence we can conclude that there are no divisorial contractions with discrepancy 1 in this case.

If $h_4(y,z)$ has multiple factors, then we may assume that $y^2 \mid h_4(y,z)$ by a linear transformation in y and z. In this situation, we embed X as in (4.1.1) and take the blow up $\pi_2 : Y_2 \to X$ with weight $(2,2,1,3)$. The z-chart of Y_2 is isomorphic to

$$\{u^2 + x^3 + g(yz^2,z)/z^4 \cdot x + h(yz^2,z)/z^6 = 0\}/\frac{1}{2}(0,1,1,0),$$

which has one dimensional singular locus, hence Y_2 is not terminal. The exceptional set E_2 of π_2 is an irreducible divisor over $P(\in X)$ and satisfies $a(E_2, X) = 1$. Since Y_1 and Y_2 are not isomorphic over X, we see that there are no divisorial contractions with discrepancy 1 if the number of divisors with discrepancy 1 is not greater than two.

If $h_4(y,z)$ has two double factors, then we may assume that $h_4(y,z) = y^2 z^2$. In this case we can take the blow up $\pi_2' : Y_2' \to X$ with weight $(2,1,2,3)$. By [**6**, 7.5], there are exactly three divisors with discrepancy 1. Since Y_1, Y_2 and Y_2' are not mutually isomorphic over X, we see that there are no divisorial contractions if $h_4(y,z)$ has two double factors.

Thus, by [**6**, 7.4, 7.7, 7.9], we know that there are no divisorial contractions with discrepancy 1 if $h_4(y,z)$ does not have a quadruple factor or if there are at most two divisors with discrepancy 1.

In the following we assume that $h_4(y,z) = b_{40} y^4$ with $b_{40} \neq 0$ and that X has more than two divisors with discrepancy 1. By [**6**, 7.9], there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(0,1,1,1)$ such that

$$X = \{u^2 + x^3 + 3\nu x^2 z^2 + g(y,z)x + h(y,z) = 0\}/\frac{1}{2}(0,1,1,1), \tag{4.1.2}$$

where $\nu \in \boldsymbol{C}$, $g(y,z) = \sum_{i+j\geq 4, even} a_{ij}y^i z^j$, $h(y,z) = \sum_{i+j\geq 4, even} b_{ij}y^i z^j \in \boldsymbol{C}\{z,u\}$ with $a_{ij} = 0$ if $3i + j \leq 4$, $b_{ij} = 0$ if $3i + j \leq 8$ and $b_{40} \neq 0$.

Under the embedding of X as in (4.1.2), let $\pi_3 : Y_3 \to X$ be the blow up with weight $(x,y,z,u) = (3,2,1,4)$. Then Y_3 is covered by the x-chart U_1, the y-chart U_2 and the z-chart U_3 as follows:

$$U_1 = \{u^2 + x + 3\nu z^2 + g(yx^2, zx)/x^5 + h(yx^2, zx)/x^8 = 0\}/\frac{1}{6}(2,5,1,1),$$

$$U_2 = \{u^2 + x^3 y + 3\nu x^2 z^2 + g(y^2, zy)/y^5 \cdot x + h(y^2, zy)/y^8 = 0\}/\frac{1}{4}(1,1,1,2),$$

$$U_3 = \{u^2 + x^3 z + 3\nu x^2 + g(yz^2, z)/z^5 \cdot x + h(yz^2, z)/z^8 = 0\}/\frac{1}{2}(1,1,1,1).$$

Thus we see that Y_3 is terminal if and only if $b_{08} \neq 0$. The exceptional set E_3 of π_3 is an irreducible divisor and satisfies $a(E_3, X) = 1$. Since Y_1, Y_2 and Y_3 are not mutually isomorphic over X, we get the desired result when X has three divisors with discrepancy 1.

Lastly we assume that X has more than three divisors with discrepancy 1. It follows from [**6**, 7.9] that there is an embedding of X into $(x,y,z,u)/\frac{1}{2}(0,1,1,1)$ such that

$$X = \{u^2 + x^3 + g(y,z)x + h(y,z) = 0\}/\frac{1}{2}(0,1,1,1), \tag{4.1.3}$$

where $g(y,z) = \sum_{i+j\geq 4, even} a_{ij}y^i z^j$, $h(y,z) = \sum_{i+j\geq 4, even} b_{ij}y^i z^j \in \boldsymbol{C}\{z,u\}$ with $a_{ij} = 0$ if $3i + j \leq 6$, $b_{ij} = 0$ if $3i + j \leq 10$ and $b_{40} \neq 0$. In this case, under the embedding of X as in (4.1.3), let $\pi_4 : Y_4 \to X$ be the blow up with weight $(4,3,1,6)$. Then the z-chart of Y_4 is isomorphic to

$$\{u^2 + x^3 + g(yz^3, z)/z^8 \cdot x + h(yz^3, z)/z^{12} = 0\}/\frac{1}{2}(0,0,1,1),$$

which has one dimensional singular locus, hence Y_4 is not terminal. The exceptional set E_4 of π_4 is an irreducible divisor over $P(\in X)$ and satisfies $a(E_4, X) = 1$. Since Y_i $(i = 1, \ldots, 4)$ are not mutually isomorphic over X and since none of these are terminal, we know that there are no divisorial contractions with discrepancy 1 if X has more than three divisors with discrepancy 1. □

REMARK 4.2. In (4.1), using the embedding of X as in (4.1.2), the blow up $\pi_3 : (E_3 \subset Y_3) \to (P \in X)$ with weight $(3,2,1,4)$ is the unique divisorial contraction with discrepancy 1. We see that Y_3 has two non-Gorenstein singularities which are isomorphic to $\frac{1}{6}(1,1,5)$ and $\frac{1}{2}(1,1,1)$. The exceptional divisor E_3 is not Cartier at each of these points.

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