J. Math. Soc. Japan
Vol. 57, No. 4, 2005

Regularizations and finite ladders in multiple trigonometry

By Nobushige Kurokawa and Masato Wakayama

(Received Jul. 4, 2003)
(Revised Jan. 7, 2005)

Abstract. We provide an extended interpretation of the zeta regularized product in [**D**]. This allows us to get regularized product expressions of Hölder's double sine function and its companion, i.e. the double and triple trigonometric functions. The expressions may reasonably explain the ladder structure among these multiple trigonometric functions. We also introduce the notion of finite ladders of functions which helps us understand the meaning behind these regularizations.

1. Introduction.

There exist various couples of important functions satisfying the "extension" property:

$$\frac{F(x+1)}{F(x)} = G(x). \tag{E}$$

The multiple gamma functions $\Gamma_m(x)$ due to Barnes [**B**] and the multiple trigonometric functions $\mathscr{S}_r(x)$, $\mathscr{C}_r(x)$ which are generalizations of usual trigonometric functions provide non-trivial solutions to this problem (see [**KW4**]). Actually, one knows $\Gamma_{m+1}(x+1)/\Gamma_{m+1}(x) = \Gamma_m(x)^{-1}$ and $\mathscr{S}_2(x+1)/\mathscr{S}_2(x) = -\mathscr{S}_1(x)$, $\mathscr{S}_3(x+1)/\mathscr{S}_3(x) = -\mathscr{S}_2(x)^2\mathscr{S}_1(x)$, etc. The higher Riemann zeta function $\zeta_\infty(s)$ [**KMW**] (see also [**CL**]) defined as $\zeta_\infty(s) := \prod_{n=0}^{\infty}\zeta(s+n)$, $\zeta(s)$ being the Riemann zeta function, and the higher Selberg zeta function $z_\Gamma(s)$ [**KW2**] defined similarly from (a shifted product of) the Selberg zeta function $Z_\Gamma(s)$ for a Riemann surface $\Gamma\backslash\boldsymbol{H}$ are also important examples.

Since it is well known that the multiple gamma function $\Gamma_m(x)$ has the zeta regularized product expression

$$\Gamma_m(x)^{-1} = \prod_{n_1,\dots,n_m=0}^{\infty}(n_1+\cdots+n_m+x) := \exp(-\zeta_m'(0,x)),$$

it is immediate to see from a general property of the zeta regularized product; $\prod_{n\in I\sqcup J}a_n = \prod_{n\in I}a_n\prod_{n\in J}a_n$ (see, e.g. [**KiW**]), the pair $(F(x),G(x)) = (\Gamma_{m+1}(x), \Gamma_m(x)^{-1})$ gives a solution of (E). Here $\zeta_m(s,x) := \sum_{n_1,\dots,n_m=0}^{\infty}(n_1+\cdots+n_m+x)^{-s}$ is

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2000 *Mathematics Subject Classification.* 11M06, 11M36.

Key Words and Phrases. Riemann zeta function, multiple trigonometric function, zeta regularized product, Euler-Maclaurin formula, Weierstrass canonical form.

Partially supported by Grant-in-Aid for Scientific Research (B) (No. 15340012), and by Grant-in-Aid for Exploratory Research (No. 15654003).

the multiple Hurwitz zeta function.

Let us recall next the multiple trigonometric functions. For $r = 2, 3, 4, \ldots$, the multiple sine $\mathscr{S}_r(x)$ and cosine $\mathscr{C}_r(x)$ functions of order r are given by the Weierstrass products:

$$\mathscr{S}_r(x) := e^{\frac{x^{r-1}}{r-1}} \prod_{n=-\infty, n\neq 0}^{\infty} P_r\left(\frac{x}{n}\right)^{n^{r-1}} = e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \left\{ P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right\}^{n^{r-1}}$$

and

$$\mathscr{C}_r(x) := \prod_{n=-\infty, n:\text{odd}}^{\infty} P_r\left(\frac{x}{\frac{n}{2}}\right)^{(\frac{n}{2})^{r-1}} = \prod_{n=1, n:\text{odd}}^{\infty} \left\{ P_r\left(\frac{x}{\frac{n}{2}}\right) P_r\left(-\frac{x}{\frac{n}{2}}\right)^{(-1)^{r-1}} \right\}^{(\frac{n}{2})^{r-1}},$$

where

$$P_r(u) := (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right).$$

The study of multiple trigonometric functions was started by Hölder [**Hö**] in 1886, where he discovered the first non-trivial example $\mathscr{S}_2(x)$. This double trigonometric function is also used for describing the functional equation (and the calculation of the gamma factor) of the Selberg zeta function [**K**] attached to a Riemann surface (see also [**KKo**], [**KW2**]). Moreover, the multiple sine functions $\mathscr{S}_r(x)$ of order r provides the expression of the value $L(r, \chi)$ of the Dirichlet L-function for a Dirichlet character χ satisfying $\chi(-1) = (-1)^{r-1}$ (see [**KW1**], [**KOW**]). Among basic properties of multiple trigonometric functions, the most characteristic one is the periodicity such as

$$\begin{aligned}
\mathscr{S}_2(x+1) &= -\mathscr{S}_2(x)\mathscr{S}_1(x), \quad \mathscr{S}_3(x+1) = -\mathscr{S}_3(x)\mathscr{S}_2(x)^2\mathscr{S}_1(x), \ldots \\
\mathscr{C}_2(x+1)^2 &= -\mathscr{C}_2(x)^2\mathscr{C}_1(x)^2, \quad \mathscr{C}_3(x+1)^4 = -\mathscr{C}_3(x)^4\mathscr{C}_2(x)^8\mathscr{C}_1(x)^4, \ldots,
\end{aligned}$$

which are considered as generalization of the usual periodicity of $\mathscr{S}_1(x) := 2\sin \pi x$ and $\mathscr{C}_1(x) := 2\cos \pi x$. In other words, these examples give the solutions to the problem (E):

$$\begin{aligned}
(F(x), G(x)) = {} & (\mathscr{S}_1(x), -1),\ (\mathscr{S}_2(x), -\mathscr{S}_1(x)),\ (\mathscr{S}_3(x), -\mathscr{S}_2(x)^2\mathscr{S}_1(x)), \\
& (\mathscr{C}_1(x), -1),\ (\mathscr{C}_2(x)^2, -\mathscr{C}_1(x)^2),\ (\mathscr{C}_3(x)^4, -\mathscr{C}_2(x)^8\mathscr{C}_1(x)^4).
\end{aligned}$$

In view of these ladder structure, one may expect the existence of regularized product expressions of $\mathscr{S}_r(x)$ and $\mathscr{C}_r(x)$. Such expressions, however, have not been known so far. Thus, in this paper, we first investigate zeta regularized product expressions of these functions (of small order). When we try to have such expressions, however, it is necessary to introduce a certain extended interpretation of the zeta regularization (see §2).

The second aim of this paper is to study "finite" companions

$$\frac{F_N(x+1)}{F_N(x)} = G_N(x) \qquad (E_N)$$

of multiple trigonometric functions. We call solutions of (E_N) finite ladders in multiple trigonometry. The pair (F_N, G_N) is expected to give a solution to the original problem (E) as $N \to \infty$. With this, we study also the divergent structure behind the regularized product expressions of multiple trigonometric functions via the Euler-Maclaurin formula. The present study may be considered as a refinement of a part of the work of Hardy [**H**] from the view point of the zeta regularized product (see §5).

2. Regularizations.

We have

$$\mathscr{S}_2(x) = e^x \prod_{n=1}^{\infty} \left\{ \left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^n e^{2x} \right\}, \qquad \mathscr{C}_2(x) = \prod_{n=1, n:\mathrm{odd}}^{\infty} \left\{ \left(\frac{1-\frac{x}{(\frac{n}{2})}}{1+\frac{x}{(\frac{n}{2})}} \right)^{\frac{n}{2}} e^{2x} \right\},$$

$$\mathscr{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n=1}^{\infty} \left\{ \left(1-\frac{x^2}{n^2} \right)^{n^2} e^{x^2} \right\}, \quad \mathscr{C}_3(x) = \prod_{n=1, n:\mathrm{odd}}^{\infty} \left\{ \left(1-\frac{x^2}{(\frac{n}{2})^2} \right)^{(\frac{n}{2})^2} e^{x^2} \right\}.$$

We write $\tilde{\mathscr{C}_r}(x) = \mathscr{C}_r(x)^{2^{r-1}}$. Note that $\mathscr{C}_r(x)$ is a 2^{r-1}-multi-valued function and hence $\tilde{\mathscr{C}_r}(x)$ defines a single valued function. Note that

$$\mathscr{S}_1(x) = 2\sin(\pi x) = 2\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right),$$

$$\mathscr{C}_1(x) = 2\cos(\pi x) = 2 \prod_{n=1, n:\mathrm{odd}}^{\infty} \left(1 - \frac{x^2}{(\frac{n}{2})^2} \right).$$

We introduce the following notation:

$$\prod_{n=1}^{\infty} (\!(a_n)\!)^{b_n} := \exp\big(-\phi'_{\boldsymbol{a},\boldsymbol{b}}(0) \big).$$

Here we put $\phi_{\boldsymbol{a},\boldsymbol{b}}(s) := \sum_{n=1}^{\infty} b_n \cdot a_n^{-s}$ and $\phi_{\boldsymbol{a},\boldsymbol{b}}(s)$ is assumed to be holomorphic around $s = 0$. Hereafter, we always assume that $-\pi \le \arg(a_n) < \pi$.

Obviously, $\prod_{n=1}^{\infty} (\!(a_n)\!)^1$ (i.e. when $b_n \equiv 1$) gives the usual zeta regularized product $\coprod_{n=1}^{\infty} a_n$ in [**D**]. If $b_n \in \boldsymbol{Z}_{>0}$, each b_n can be interpreted as the multiplicity of a_n^{-s} in the Dirichlet series $\phi_{\boldsymbol{a},\boldsymbol{b}}(s)$. Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function. Notice that $\prod_{n=1}^{\infty} (\!(n)\!)^n = \exp(-\zeta'(-1))$ but we do not know the existence of $\coprod_{n=1}^{\infty} n^n$, (even if we employ the dotted products $\boxdot$, $\boxdot$ developed in [**KW3**], [**KiW**]) where we need to look at the behavior of $\sum_{n=1}^{\infty} (n^n)^{-s} = \sum_{n=1}^{\infty} n^{-ns}$ around $s = 0$. This new regularized product, however, allows us to give zeta regularized expression of the multiple trigonometric functions of small order.

THEOREM 2.1. *We have*

(1) $$\mathscr{S}_1(x) = x\prod_{n=1}^{\infty}(n-x)\prod_{n=1}^{\infty}(n+x).$$

(2) $$\mathscr{C}_1(x) = \prod_{n=1}^{\infty}\left(n-\frac{1}{2}-x\right)\prod_{n=1}^{\infty}\left(n-\frac{1}{2}+x\right).$$

(3) $$\mathscr{S}_2(x) = \frac{\prod_{n=1}^{\infty}(\!(n-x)\!)^n}{\prod_{n=1}^{\infty}(\!(n+x)\!)^n}.$$

(4) $$\tilde{\mathscr{C}}_2(x) = \frac{\prod_{n=1}^{\infty}(\!(n-\frac{1}{2}-x)\!)^{2n-1}}{\prod_{n=1}^{\infty}(\!(n-\frac{1}{2}+x)\!)^{2n-1}}.$$

(5) $$\mathscr{S}_3(x) = \left(\prod_{n=1}^{\infty}(\!(n)\!)^{n^2}\right)^{-2}\times\prod_{n=1}^{\infty}(\!(n-x)\!)^{n^2}\cdot\prod_{n=1}^{\infty}(\!(n+x)\!)^{n^2}$$

$$\textit{with}\quad \prod_{n=1}^{\infty}(\!(n)\!)^{n^2} = \exp\big(-\zeta'(-2)\big) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

(6) $$\tilde{\mathscr{C}}_3(x) = \left(\prod_{n=1}^{\infty}\left(\!\!\left(n-\frac{1}{2}\right)\!\!\right)^{(2n-1)^2}\right)^{-2}$$

$$\times\prod_{n=1}^{\infty}\left(\!\!\left(n-\frac{1}{2}-x\right)\!\!\right)^{(2n-1)^2}\cdot\prod_{n=1}^{\infty}\left(\!\!\left(n-\frac{1}{2}+x\right)\!\!\right)^{(2n-1)^2}$$

$$\textit{with}\quad \prod_{n=1}^{\infty}\left(\!\!\left(n-\frac{1}{2}\right)\!\!\right)^{(2n-1)^2} = \exp\big(3\zeta'(-2)\big) = \exp\left(-\frac{3\zeta(3)}{4\pi^2}\right).$$

The first two are simple. Actually, since $\mathscr{S}_1(x) = 2\sin(\pi x) = \frac{2\pi}{\Gamma(x)\Gamma(1-x)}$, the assertion (1) follows immediately from the formula due to Lerch ([**L**]) (concerning the first derivative of the Hurwitz zeta function at $s=0$); $\sqrt{2\pi}\Gamma(x)^{-1} = \prod_{n=0}^{\infty}(n+x)$. The claim (2) follows easily from the fact $\mathscr{C}_1(x) = \mathscr{S}_1(x+\frac{1}{2})$. We may prove (3), (4), (5) and (6) by using the explicit expressions of those multiple trigonometric functions in terms of the normalized multiple trigonometric functions because they are defined via the zeta regularized products (see [**KKo**]). To do so, however, it is necessary to make rather tedious calculations. Thus we give here a direct proof based on the general property of this new zeta regularized product which we study now. In other words, the expressions for $\mathscr{S}_r(x)$ and $\tilde{\mathscr{C}}_r(x)$ obtained here can reprove the explicit relations between these and the normalized multiple trigonometric functions.

In order to develop the theory nicely, we first establish a certain general result concerning the zeros (or poles) of the function defined by $\prod_{n=1}^{\infty}(\!(a_n-x)\!)^{b_n}$, where all b_n are the integers.

Let $\boldsymbol{a} = \{a_n\}_{n=1,2,\ldots}$ be a divergent sequence of nonzero complex numbers and $\boldsymbol{b} = \{b_n\}_{n=1,2,\ldots}$ be a sequence of integers. Denote by μ the exponent of convergence of the series

$$\sum_{n=1}^{\infty} |b_n| \cdot |a_n|^{-t},$$

that is, the series converges for $\mathrm{Re}(t) = \mu + \epsilon$ and diverges for $\mathrm{Re}(t) = \mu - \epsilon$ for any $\epsilon > 0$. Let p be the integer part of μ. Define a Dirichlet series attached to the data $(\boldsymbol{a}, \boldsymbol{b})$ by

$$\phi_{\boldsymbol{a},\boldsymbol{b}}(s, x) := \sum_{n=1}^{\infty} b_n \cdot (a_n - x)^{-s}.$$

This Dirichlet series converges absolutely in the region $\mathrm{Re}(s) > \mu$ and uniformly for each compact subset in x-space $\boldsymbol{C}$ which does not meet any a_n. We assume that $\phi_{\boldsymbol{a},\boldsymbol{b}}(s, x)$ can be holomorphically extended to a region containing $s = 0$. Then $\prod_{n=1}^{\infty} (\!(a_n - x)\!)^{b_n} = \exp\big(-\frac{\partial}{\partial s}\phi_{\boldsymbol{a},\boldsymbol{b}}(0, x)\big)$. We now show the relation between this regularized product and the Weierstarss canonical product which is a generalization of the result established in [**V**] (see also [**I**], [**KiW**]). In other words, it can describe the location of zeroes of the function expressed by this regularized product.

THEOREM 2.2. *Retain the notation above. Suppose that b_n are all positive integers. Then the function $\prod_{n=1}^{\infty} (\!(a_n - x)\!)^{b_n}$ is analytically extended to the whole complex plane as an entire function whose zeros are exactly given by $x = a_n$ with multiplicity b_n. More precisely, there exists a polynomial $P(x)$ of degree at most $p = [\mu]$ such that*

$$\prod_{n=1}^{\infty} (\!(a_n - x)\!)^{b_n} = e^{P(x)} \prod_{n=1}^{\infty} \left(1 - \frac{x}{a_n}\right)^{b_n} \exp\left(b_n \sum_{\ell=1}^{p} \frac{1}{\ell}\left(\frac{x}{a_n}\right)^{\ell}\right).$$

SKETCH OF THE PROOF. The proof is similar to the one given in [**V**]. Write the infinite product in the right hand side by $\Delta_{\boldsymbol{a},\boldsymbol{b}}(x)$. Put

$$\eta_{\boldsymbol{a},\boldsymbol{b}}(s, x) := \Gamma(s)\phi_{\boldsymbol{a},\boldsymbol{b}}(s, x).$$

Then it is easy to verify that

$$\frac{d^{p+1}}{dx^{p+1}} \log \Delta_{\boldsymbol{a},\boldsymbol{b}}(x) = -\eta_{\boldsymbol{a},\boldsymbol{b}}(p + 1, x).$$

Note that $\eta_{\boldsymbol{a},\boldsymbol{b}}(s, x)$ may have poles at $s = 0, 1, 2, \ldots, p$. For a meromorphic function $f(s)$, the finite part $\mathrm{FP}f(a)$ at $s = a$ is in general defined by

$$\mathrm{FP}f(a) := \begin{cases} f(a) & \text{if } f(s) \text{ is holomorphic at } s = a, \\ \lim_{s\to a}\{f(s) - (\text{the principal part of } f(s)) & \text{if } f(s) \text{ has a pole at } s = a. \end{cases}$$

Then, since the operations taking FP and $\frac{d}{dx}$ are compatible, we find the function defined by

$$F_{\boldsymbol{a},\boldsymbol{b}}(x) := \exp\left(-\mathrm{FP}\eta_{\boldsymbol{a},\boldsymbol{b}}(0,x) + \sum_{k=0}^{p} \mathrm{FP}\eta_{\boldsymbol{a},\boldsymbol{b}}(k,0)\frac{x^k}{k!} \right)$$

satisfies

$$\frac{d^{p+1}}{dx^{p+1}} \log F_{\boldsymbol{a},\boldsymbol{b}}(x) = -\eta_{\boldsymbol{a},\boldsymbol{b}}(p+1,x).$$

Hence we see that the difference of the functions $\log \varDelta_{\boldsymbol{a},\boldsymbol{b}}(x) - \log F_{\boldsymbol{a},\boldsymbol{b}}(x)$ is actually given by a polynomial function of degree at most p.

Note here that

$$\frac{\partial^{p+1}}{\partial x^{p+1}}\phi_{\boldsymbol{a},\boldsymbol{b}}(s,x) = s(s+1)\cdots(s+p)\phi_{\boldsymbol{a},\boldsymbol{b}}(s+p+1,x).$$

Since $\phi_{\boldsymbol{a},\boldsymbol{b}}(s,x)$ is holomorphic at $s = p+1$, $\frac{\partial^{p+1}}{\partial x^{p+1}}\phi_{\boldsymbol{a},\boldsymbol{b}}(s,x)$ vanishes at $s = 0$. Since $\Gamma(s) = s^{-1} + \gamma_0 + \gamma_1 s + \cdots$ and $\phi_{\boldsymbol{a},\boldsymbol{b}}(s,x)$ is holomorphic around $s = 0$ (by assumption) it follows hence that

$$\frac{\partial^{p+1}}{\partial x^{p+1}}\mathrm{FP}\eta_{\boldsymbol{a},\boldsymbol{b}}(0,x) = \frac{\partial^{p+1}}{\partial x^{p+1}}\frac{\partial}{\partial s}\phi_{\boldsymbol{a},\boldsymbol{b}}(0,x)$$

This implies that $\log \varDelta_{\boldsymbol{a},\boldsymbol{b}}(x) - \log \frac{\partial}{\partial s}\phi_{\boldsymbol{a},\boldsymbol{b}}(0,x)$ is also a polynomial of degree at most p. This completes the proof of the theorem. □

Before going to the actual proofs of (3), (4), (5) and (6) in Theorem 2.1 we note here the following three useful properties of this new zeta regularized product.

We see easily that

$$\prod_{n=1}^{\infty}(\!(a_n)\!)^{b_n} = \left\{ \prod_{n=1}^{\infty}(\!(a_n)\!)^{-b_n} \right\}^{-1} \quad \text{and} \quad \prod_{n=1}^{\infty}(\!(a_n)\!)^{0} = 1.$$

Using this, we have in general

$$\prod_{n=1}^{\infty}(\!(a_n)\!)^{kb_n} = \left\{ \prod_{n=1}^{\infty}(\!(a_n)\!)^{b_n} \right\}^{k} \quad \text{for any} \quad k \in \boldsymbol{C}. \tag{A}$$

For any two integral sequences $\boldsymbol{b} = \{b_n\}_{n=1,2,\ldots}$ and $\boldsymbol{c} = \{c_n\}_{n=1,2,\ldots}$, we have

$$\prod_{n=1}^{\infty} (\!(a_n)\!)^{b_n+c_n} = \prod_{n=1}^{\infty} (\!(a_n)\!)^{b_n} \cdot \prod_{n=1}^{\infty} (\!(a_n)\!)^{c_n} \tag{B}$$

whenever all of the regularized products exist. This relation follows immediately from the relation $\phi_{\boldsymbol{a},\boldsymbol{b}+\boldsymbol{c}}(s) = \phi_{\boldsymbol{a},\boldsymbol{b}}(s) + \phi_{\boldsymbol{a},\boldsymbol{c}}(s)$.

Suppose $\lambda > 0$. Then we have

$$\prod_{n=1}^{\infty} (\!(\lambda a_n)\!)^{b_n} = \lambda^{\phi_{\boldsymbol{a},\boldsymbol{b}}(0)} \prod_{n=1}^{\infty} (\!(a_n)\!)^{b_n}. \tag{C}$$

This follows from the Taylor expansion of $\phi_{\lambda\boldsymbol{a},\boldsymbol{b}}(s)$ as

$$\begin{aligned}
\phi_{\lambda\boldsymbol{a},\boldsymbol{b}}(s) &= \sum_{n=1}^{\infty} b_n \cdot (\lambda a_n)^{-s} = \lambda^{-s}\phi_{\boldsymbol{a},\boldsymbol{b}}(s) \\
&= \big(1 - s\log\lambda + O(s^2)\big) \times \big(\phi_{\boldsymbol{a},\boldsymbol{b}}(0) + \phi'_{\boldsymbol{a},\boldsymbol{b}}(0)s + O(s^2)\big) \\
&= \phi_{\boldsymbol{a},\boldsymbol{b}}(0) + \big(-\log\lambda^{\phi_{\boldsymbol{a},\boldsymbol{b}}(0)} + \phi'_{\boldsymbol{a},\boldsymbol{b}}(0)\big)s + O(s^2).
\end{aligned}$$

Proof of (3),(4), (5) and (6). We first show (3). Put $f(x) = \prod_{n=1}^{\infty} (\!(n+x)\!)^n$ and $g(x) = f(-x)$. Then, by the theorem above we see that $f(x)$ defines an entire function with zeros at $x = -n$ of order n. Also, by the properties (A) and (B) we calculate as

$$\begin{aligned}
f(x+1) &= \prod_{n=1}^{\infty} (\!(n+x+1)\!)^n = \prod_{n=2}^{\infty} (\!(n+x)\!)^{n-1} = \prod_{n=1}^{\infty} (\!(n+x)\!)^{n-1} \\
&= \left(\prod_{n=1}^{\infty} (\!(n+x)\!)\right)^{-1} \prod_{n=1}^{\infty} (\!(n+x)\!)^n = \left(\prod_{n=1}^{\infty} (n+x)\right)^{-1} \cdot f(x).
\end{aligned}$$

Similarly we have $g(x+1) = -x\prod_{n=1}^{\infty}(n-x) \cdot g(x)$. It follows that

$$\frac{g(x+1)}{f(x+1)} = -x\prod_{n=1}^{\infty}(n-x)\prod_{n=1}^{\infty}(n+x)\frac{g(x)}{f(x)} = -\mathscr{S}_1(x)\frac{g(x)}{f(x)}$$

by (1). It is known that $\mathscr{S}_2(x)$ has the same periodicity (this follows also from (2) and (4) of Theorem 3.1 below). Since two meromorphic functions $\mathscr{S}_2(x)$ and $\frac{g(x)}{f(x)}$ of order 2 have exactly the same zeros and the poles counting with their multiplicities, there exists a quadratic polynomial $ax^2 + bx + c$ such that

$$\mathscr{S}_2(x) = e^{ax^2+bx+c}\frac{g(x)}{f(x)}.$$

The fact that these functions $\mathscr{S}_2(x)$ and $\frac{g(x)}{f(x)}$ have the same periodicity yields $a = 0$ and $b \in 2\pi i\boldsymbol{Z}$. Here, noting that both $\mathscr{S}_2(x)$ and $\frac{g(x)}{f(x)}$ are real whenever $x \in \boldsymbol{R}$ we conclude that $b = 0$. Comparing further the values at $x = 0$ we have $c = 0$. This proves (3).

We prove (4) quite similarly using the periodicity $\tilde{\mathscr{C}_2}(x+1)/\tilde{\mathscr{C}_2}(x) = -\mathscr{C}_1(x)^2$. Actually, putting $k(x) = \prod_{n=1}^{\infty}((n - \frac{1}{2} + x))^{2n-1}$, we have

$$k(x) = \frac{\left\{\prod_{n=1}^{\infty}((n - \frac{1}{2} + x))^n\right\}^2}{\prod_{n=1}^{\infty}((n - \frac{1}{2} + x))} = \frac{f(x - \frac{1}{2})^2}{\prod_{n=1}^{\infty}(n - \frac{1}{2} + x)}.$$

Here we use the aforementioned properties (A) and (B) of the zeta regularized product. Since $f(x+1) = \left(\prod_{n=1}^{\infty}(n + x)\right)^{-1} \cdot f(x)$, we obtain

$$\begin{aligned} k(x+1) &= \frac{f(x + 1 - \frac{1}{2})^2}{\prod_{n=1}^{\infty}((n - \frac{1}{2} + x + 1))} \\ &= \frac{\prod_{n=1}^{\infty}(n - \frac{1}{2} + x)^{-2} f(x - \frac{1}{2})^2}{(\frac{1}{2} + x)^{-1}\prod_{n=1}^{\infty}(n - \frac{1}{2} + x)} = \frac{(\frac{1}{2} + x)k(x)}{\prod_{n=1}^{\infty}(n - \frac{1}{2} + x)^2}. \end{aligned}$$

Similarly, since

$$k(-x) = \frac{g(x + \frac{1}{2})^2}{\prod_{n=1}^{\infty}(n - \frac{1}{2} - x)}$$

we have

$$k(-x-1) = -\left(x + \frac{1}{2}\right)\prod_{n=1}^{\infty}\left(n - \frac{1}{2} - x\right)^2 k(-x).$$

Hence it follows that

$$\frac{k(-x-1)}{k(x+1)} = -\prod_{n=1}^{\infty}\left(n - \frac{1}{2} - x\right)^2 \prod_{n=1}^{\infty}\left(n - \frac{1}{2} + x\right)^2 \frac{k(-x)}{k(x)} = -\mathscr{C}_1(x)^2 \frac{k(-x)}{k(x)}.$$

This shows the function $\frac{k(-x)}{k(x)}$ has the same periodicity of $\tilde{\mathscr{C}_2}(x)$. Also, since these two meromorphic functions $\frac{k(-x)}{k(x)}$ and $\tilde{\mathscr{C}_2}(x)$ (of order 2) have the same zeros and poles counting with the multiplicity, and both are real valued for $x \in \boldsymbol{R}$, as in the previous proof of (3), we see that these two must coincide. This proves (4).

Since the proof of (5) is essentially the same as what we did for (3), we only give a sketch: Put $h(x) = \prod_{n=1}^{\infty}((n + x))^{n^2}$. Then, using (A) and (B), we observe that

$$h(x+1) = h(x)\left\{\prod_{n=1}^{\infty}((n + x))^n\right\}^{-2}\prod_{n=1}^{\infty}((n + x)).$$

Similarly

$$h(-x-1) = -xh(-x)\left\{\prod_{n=1}^{\infty}((n-x))^n\right\}^2 \prod_{n=1}^{\infty}((n-x)).$$

It follows that

$$\begin{aligned} h(x+1)h(-x-1) &= h(x)h(-x)\times(-x)\left\{\frac{\prod_{n=1}^{\infty}((n-x))^n}{\prod_{n=1}^{\infty}((n+x))^n}\right\}^2 \prod_{n=1}^{\infty}((n-x))\prod_{n=1}^{\infty}((n+x)) \\ &= -h(x)h(-x)\mathscr{S}_2(x)^2\mathscr{S}_1(x). \end{aligned}$$

This shows that $h(x)h(-x)$ has the same periodicity of $\mathscr{S}_3(x)$ by (3) of Theorem 5.1. Also, since the zeros of two entire functions $\mathscr{S}_3(x)$ and $h(x)h(-x)$ coincide they must be the identical function up to the factor $\prod_{n=1}^{\infty}((n))^n$. As to the normalizing constant $\prod_{n=1}^{\infty}((n))^{n^2}$, the value is computed by the definition of the regularized product and the functional equation of the Riemann zeta function; $\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$. This proves (5). Also by using the periodicity of $\tilde{\mathscr{C}}_3(x)$, that is, $\mathscr{C}_3(x+1)^4 = -\mathscr{C}_3(x)^4\mathscr{C}_2(x)^8\mathscr{C}_1(x)^4$, the same discussion of (4) can give a proof the formula (6). This completes the proof of Theorem 2.1. $\square$

3. Finite ladders.

We first give finite ladders of the sine and double sine functions.

THEOREM 3.1. *For each integer $N \geq 1$ define*

$$\mathscr{S}_{1,N}(x) = \left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}}\right)^2 x\prod_{n=1}^{N}\left(1-\frac{x^2}{n^2}\right)$$

and

$$\mathscr{S}_{2,N}(x) = \left(1+\frac{x}{N}\right)^N \prod_{n=1}^{N}\left\{\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}}\right)^n e^{2x}\right\}.$$

Then we have

(1) $\lim_{N\to\infty}\mathscr{S}_{1,N}(x) = \mathscr{S}_1(x)$.

(2) $\lim_{N\to\infty}\mathscr{S}_{2,N}(x) = \mathscr{S}_2(x)$.

(3) $\dfrac{\mathscr{S}_{1,N}(x+1)}{\mathscr{S}_{1,N}(x)} = -\left(1+\dfrac{x+1}{N}\right)\left(1-\dfrac{x}{N}\right)^{-1}$.

$$(4)\quad \frac{\mathscr{S}_{2,N}(x+1)}{\mathscr{S}_{2,N}(x)} = -\mathscr{S}_{1,N}(x) \times \left(1-\frac{x^2}{N^2}\right)^{-N}\left(1-\frac{x}{N}\right)^{-1}.$$

Remark that such a finite level object is considered as a refinement of the multiple sine function since there appear various factors vanishing in the limit $N \to \infty$. For the cosine cases we have similarly the following

Theorem 3.2. *For each integer $N \geq 1$ define*

$$\mathscr{C}_{1,N}(x) = \left(\frac{(2N)!(N!)^{-1}2^{-2N}}{N^N e^{-N}}\right)^2 \prod_{n=1}^{N}\left(1-\frac{x^2}{(n-\frac{1}{2})^2}\right)$$

and

$$\tilde{\mathscr{C}}_{2,N}(x) = \prod_{n=1}^{N}\left\{\left(\frac{1-\frac{x}{n-\frac{1}{2}}}{1+\frac{x}{n-\frac{1}{2}}}\right)^{2n-1} e^{4x}\right\}.$$

Then we have

(1) $\displaystyle\lim_{N\to\infty}\mathscr{C}_{1,N}(x) = \mathscr{C}_1(x).$

(2) $\displaystyle\lim_{N\to\infty}\tilde{\mathscr{C}}_{2,N}(x) = \tilde{\mathscr{C}}_2(x).$

(3) $\displaystyle\frac{\mathscr{C}_{1,N}(x+1)}{\mathscr{C}_{1,N}(x)} = -\left(1+\frac{x+\frac{1}{2}}{N}\right)\left(1-\frac{x+\frac{1}{2}}{N}\right)^{-1}.$

(4) $\displaystyle\frac{\tilde{\mathscr{C}}_{2,N}(x+1)}{\tilde{\mathscr{C}}_{2,N}(x)} = -\mathscr{C}_{1,N}(x)^2 \times \left(1-\frac{(x+\frac{1}{2})^2}{N^2}\right)^{-2N}\left(1+\frac{x+\frac{1}{2}}{N}\right)\left(1-\frac{x+\frac{1}{2}}{N}\right)^{-1}.$

We have also the following duplication formulas for finite ladders in multiple trigonometry.

Theorem 3.3. *Duplications formulas in finite ladders hold*:

$$(1)\quad \mathscr{S}_{1,2N}(2x) = \mathscr{S}_{1,N}(x)\mathscr{C}_{1,N}(x).$$

$$(2)\quad \mathscr{S}_{2,2N}(2x) = \mathscr{S}_{2,N}(x)^2\tilde{\mathscr{C}}_{2,N}(x).$$

The proofs of these theorems are based on the Stirling formula. We study $\mathscr{S}_{3,N}(x)$ and $\mathscr{S}_{4,N}(x)$ in the text below (see Theorem 5.1) together with the discussion about the regularization.

4. Proofs of Theorems 3.1, 3.2 and 3.3.

Proof of Theorem 3.1. By the Stirling formula we have the assertion (1). The claim (2) is clear from the fact $\left(1+\frac{x}{N}\right)^N \to e^x$. For (3) and (4), we calculate directly as follows. Concerning the case (3) we have

$$\begin{aligned}
\frac{\mathscr{S}_{1,N}(x+1)}{\mathscr{S}_{1,N}(x)} &= \frac{x+1}{x}\prod_{n=1}^{N}\frac{n^2-(x+1)^2}{n^2-x^2}\\
&= \frac{x+1}{x}\cdot\frac{(-x)(1-x)\cdots(N-1-x)}{(1-x)(2-x)\cdots(N-x)}\cdot\frac{(2+x)(3+x)\cdots(N+1+x)}{(1+x)(2+x)\cdots(N+x)}\\
&= -\frac{N+1+x}{N-x}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\frac{\mathscr{S}_{2,N}(x+1)}{\mathscr{S}_{2,N}(x)} &= \left(1+\frac{x+1}{N}\right)^{N}\left(1+\frac{x}{N}\right)^{-N}\\
&\quad\times\prod_{n=1}^{N}\left\{\left(\frac{n-1-x}{n+1+x}\right)^{n}e^{2x+2}\right\}\prod_{n=1}^{N}\left\{\left(\frac{n+x}{n-x}\right)^{n}e^{-2x}\right\}\\
&= \left(1+\frac{x+1}{N}\right)^{N}\left(1+\frac{x}{N}\right)^{-N}e^{2N}\\
&\quad\times\frac{(-x)^1(1-x)^2\cdots(N-1-x)^N}{(1-x)^1(2-x)^2\cdots(N-x)^N}\cdot\frac{(1+x)^1(2+x)^2\cdots(N+x)^N}{(2+x)^1(3+x)^2\cdots(N+1+x)^N}\\
&= \left(1+\frac{x+1}{N}\right)^{N}\left(1+\frac{x}{N}\right)^{-N}e^{2N}\times(-x)(1-x)\cdots(N-x)\\
&\quad\times(1+x)(2+x)\cdots(N+x)\times(N-x)^{-N-1}\times(N+1+x)^{-N}\\
&= -\frac{(N!)^2}{N^{2N+1}e^{-2N}}\cdot x\prod_{n=1}^{N}\left(1-\frac{x^2}{n^2}\right)\times\left(1-\frac{x^2}{N^2}\right)^{-N}\left(1-\frac{x}{N}\right)^{-1}\\
&= \mathscr{S}_{1,N}(x)\times\left(1-\frac{x^2}{N^2}\right)^{-N}\left(1-\frac{x}{N}\right)^{-1}.
\end{aligned}$$

This proves (4). □

Proof of Theorem 3.2. Since $(2N)!(N!)^{-1}2^{-2N}/N^Ne^{-N}\to\sqrt{2}$ by Stirling's formula, we easily observe that $\lim_{N\to\infty}\tilde{\mathscr{C}}_{1,N}(x)=2\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n-\frac{1}{2})^2}\right)=\mathscr{C}_1(x)$, which shows (1). The assertion (2) is clear. For (3) we have

$$\begin{aligned}
\frac{\mathscr{C}_{1,N}(x+1)}{\mathscr{C}_{1,N}(x)} &= \prod_{n=1}^{N}\frac{(n-\frac{1}{2})^2-(x+1)^2}{(n-\frac{1}{2})^2-x^2}=\prod_{n=1}^{N}\frac{(n-\frac{3}{2}-x)(n+\frac{1}{2}+x)}{(n-\frac{1}{2}-x)(n-\frac{1}{2}+x)}\\
&= -\frac{N+\frac{1}{2}+x}{N-\frac{1}{2}-x}.
\end{aligned}$$

We calculate (4) as

$$\begin{aligned}
\frac{\tilde{\mathscr{C}}_{2,N}(x+1)}{\tilde{\mathscr{C}}_{2,N}(x)} &= \prod_{n=1}^{N}\left(n-\frac{1}{2}-x\right)^{-(2n-1)}\prod_{n=1}^{N}\left(n-\frac{1}{2}+x\right)^{2n-1} \\
&\quad \times \prod_{n=1}^{N}\left(n-\frac{1}{2}-(1+x)\right)^{2n-1}\prod_{n=1}^{N}\left(n-\frac{1}{2}+(1+x)\right)^{-(2n-1)} \times e^{4N} \\
&= \prod_{n=1}^{N}\left(n-\frac{1}{2}-x\right)^{2}\prod_{n=1}^{N}\left(n-\frac{1}{2}+x\right)^{2} \\
&\quad \times \left(-\frac{1}{2}-x\right)\left(N-\frac{1}{2}-x\right)^{-2N-1}\left(N+\frac{1}{2}+x\right)^{-2N+1}\left(\frac{1}{2}+x\right)^{-1}\cdot e^{4N} \\
&= -\prod_{n=1}^{N}\left(\left(n-\frac{1}{2}\right)^{2}-x^{2}\right)^{2}\cdot e^{4N}\cdot N^{-4N} \\
&\quad \times \left(1+\frac{x+\frac{1}{2}}{N}\right)^{-2N+1}\left(1+\frac{x-\frac{1}{2}}{N}\right)^{-2N-1}.
\end{aligned}$$

Noting that $\prod_{n=1}^{N}\left(n-\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2N-1}{2}=\frac{(2N)!}{N!2^{2N}}$, we obtain

$$\begin{aligned}
&\frac{\tilde{\mathscr{C}}_{2,N}(x+1)}{\tilde{\mathscr{C}}_{2,N}(x)} \\
&= -\left(\frac{(2N)!(N!)^{-1}2^{-2N}}{N^{N}e^{-N}}\right)^{4}\prod_{n=1}^{N}\left(1-\frac{x^{2}}{(n-\frac{1}{2})^{2}}\right)^{2}\cdot\left(1-\frac{(x+\frac{1}{2})^{2}}{N^{2}}\right)^{-2N}\times\frac{N+\frac{1}{2}+x}{N-\frac{1}{2}-x}.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. (1) Using the respective expressions

$$\mathscr{S}_{1,N}(x)=\left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}}\right)^{2}\cdot x\prod_{n=1}^{N}\left(1-\frac{x^{2}}{n^{2}}\right)=\left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}}\right)^{2}\cdot x\prod_{n=2,n:\mathrm{even}}^{2N}\left(1-\frac{4x^{2}}{n^{2}}\right)$$

and

$$\mathscr{S}_{1,2N}(2x)=\left(\frac{(2N)!}{(2N)^{2N+\frac{1}{2}}e^{-2N}}\right)^{2}\cdot 2x\prod_{n=1}^{2N}\left(1-\frac{4x^{2}}{n^{2}}\right),$$

we see that

$$\begin{aligned}
\frac{\mathscr{S}_{1,2N}(2x)}{\mathscr{S}_{1,N}(x)} &= \left(\frac{(2N)!(N!)^{-1}2^{-2N}}{N^{N}e^{-N}}\right)^{2}\times\prod_{n=1,n:\mathrm{odd}}^{2N-1}\left(1-\frac{4x^{2}}{n^{2}}\right) \\
&= \left(\frac{(2N)!(N!)^{-1}2^{-2N}}{N^{N}e^{-N}}\right)^{2}\times\prod_{n=1}^{N}\left(1-\frac{x^{2}}{(n-\frac{1}{2})^{2}}\right)=\mathscr{C}_{1,N}(x).
\end{aligned}$$

(2) Comparing expressions

$$\begin{aligned}\mathscr{S}_{2,N}(x)^2 &= \left(1+\frac{x}{N}\right)^{2N}\prod_{n=1}^{N}\left\{\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}}\right)^{2n}e^{4x}\right\}\\ &= \left(1+\frac{x}{N}\right)^{2N}\prod_{n=2,n:\mathrm{even}}^{2N}\left\{\left(\frac{1-\frac{2x}{n}}{1+\frac{2x}{n}}\right)^{n}e^{4x}\right\}\end{aligned}$$

with

$$\mathscr{S}_{2,2N}(2x) = \left(1+\frac{x}{N}\right)^{2N}\prod_{n=1}^{2N}\left\{\left(\frac{1-\frac{2x}{n}}{1+\frac{2x}{n}}\right)^{n}e^{4x}\right\},$$

we obtain

$$\frac{\mathscr{S}_{2,2N}(2x)}{\mathscr{S}_{2,N}(x)^2} = \prod_{n=1,n:\mathrm{odd}}^{2N-1}\left\{\left(\frac{1-\frac{2x}{n}}{1+\frac{2x}{n}}\right)^{n}e^{4x}\right\} = \prod_{n=1}^{N}\left\{\left(\frac{1-\frac{x}{n-\frac{1}{2}}}{1+\frac{x}{n-\frac{1}{2}}}\right)^{2n-1}e^{4x}\right\} = \tilde{\mathscr{C}}_{2,N}(x).$$

This proves the duplication formulas for finite ladders. □

5. Divergent factors and higher orders.

In order to arrive the finite ladders corresponding to the multiple sine functions of higher order, we start by the following general problem. Let

$$F(x) = \prod_{n=1}^{\infty}(a_n - x) = \mathrm{Det}(A-x) \quad \text{and} \quad G(x) = \prod_{n=1}^{\infty}(b_n - x) = \mathrm{Det}(B-x)$$

be zeta regularized products giving solutions to the problem (E), where $0 < a_1 \le a_2 \le a_3 \le \cdots \uparrow +\infty$, $0 < b_1 \le b_2 \le b_3 \le \cdots \uparrow +\infty$, $A = \mathrm{diag}(a_1, a_2, a_3, \cdots)$ and $B = \mathrm{diag}(b_1, b_2, b_3, \cdots)$. Assume that $\prod_{n=1}^{\infty} a_n$ takes the form of

$$\prod_{n=1}^{\infty} a_n = \lim_{N\to\infty}\left[\prod_{n=1}^{N} a_n\right]_{\mathrm{reg}} \quad \text{with} \quad \left[\prod_{n=1}^{N} a_n\right]_{\mathrm{reg}} = \frac{\prod_{n=1}^{N} a_n}{\left[\prod_{n=1}^{N} a_n\right]_{\mathrm{div}}}$$

and $\prod_{n=1}^{\infty} b_n$ is so, where reg denotes a regularization and div indicates a divergent factor in an appropriate sense. This type of expression frequently occurs (usually written in an additive way) when we use the Euler-Maclaurin summation formula (see e.g. [**T**]). For instance, if we put $[N!]_{\mathrm{div}} = N^{N+\frac{1}{2}}e^{-N}$, then the expression

$$\prod_{n=1}^{\infty} n = \lim_{N\to\infty}\left[\prod_{n=1}^{N} n\right]_{\mathrm{reg}} = \lim_{N\to\infty}[N!]_{\mathrm{reg}} = \lim_{N\to\infty}\frac{N!}{[N!]_{\mathrm{div}}} = \lim_{N\to\infty}\frac{N!}{N^{N+\frac{1}{2}}e^{-N}} = \sqrt{2\pi}.$$

may explain the situation. Here we used the Stirling formula. Remark that the coefficients of $\mathscr{S}_{1,N}(x)$ in Theorem 3.1 and $\mathscr{C}_{1,N}(x)$ in Theorem 3.2 are expressed in this manner. Then, there arises a natural question as follows: Let I_N be a finite set indexed by an integer N. Then, one can ask

Do the sequences of pair of functions of the form

$$F_N(x) = \Big[\prod_{n\in I_N} |a_n|^{c_n}\Big]_{\text{reg}} \cdot \prod_{n\in I_N} P_r\Big(\frac{x}{a_n}\Big)^{c_n} \quad \text{and}$$

$$G_N(x) = \Big[\prod_{n\in I_N} |b_n|^{d_n}\Big]_{\text{reg}} \cdot \prod_{n\in I_N} P_\ell\Big(\frac{x}{b_n}\Big)^{d_n} \tag{L}$$

give natural candidates of solutions to the problem (E_N) for some appropriate choices of the sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ and integers r, ℓ? Moreover, do the limits $\lim_{N\to\infty} F_N(x)$ and $\lim_{N\to\infty} G_N(x)$ exist and have expressions via certain zeta regularized products?

For instance, the case of the pair $(\mathscr{S}_{2,N}(x), \mathscr{S}_{1,N}(x))$ can be interpreted in this setting. In fact, first taking $I_N = \{-N, \dots, -1, 1, 2, \dots, N\}$, $b_n = n$, $d_n = 1$ and $\ell = 1$, we have

$$\Big[\prod_{n\in I_N} |b_n|^{d_n}\Big]_{\text{reg}} = \Big[\prod_{n\in I_N} |n|\Big]_{\text{reg}} = \big[(N!)^2\big]_{\text{reg}} = \frac{(N!)^2}{\big[(N!)^2\big]_{\text{div}}}$$

with $[(N!)^2]_{\text{div}} = N^{2N+1}e^{-2N}$. It follows that $xG_N(x) = \mathscr{S}_{1,N}(x) \to \mathscr{S}_1(x)$. Moreover, if we take $a_n = n, c_n = n, r = 2$ and I_N as above then, since $\prod_{n\in I_N} a_n^{c_n} = \prod_{n=1}^N |n|^n \cdot \prod_{n=1}^N |n|^{-n} = 1$ we have $\big(1 + \frac{x}{N}\big)^N F_N(x) = \mathscr{S}_{2,N}(x) \to \mathscr{S}_2(x)$. By Theorem 3.1, this observation shows that the pair $(F_N(x), G_N(x))$ gives essentially finite ladders;

$$\frac{F_N(x+1)}{F_N(x)} = -x\Big\{\Big(1-\frac{x}{N}\Big)^{-N}\Big(1+\frac{x+1}{N}\Big)^{-N}\Big(1-\frac{x}{N}\Big)^{-1}\Big\} \times G_N(x).$$

Here, the part $\{\ \}$ clearly becomes 1 when $N \to \infty$.

To be more explicit the situation about a divergent part, we recall the asymptotic formula of $\zeta'(s)$ which is obtained from the Euler-Maclaurin formula (see [**T**], [**H**]). The quantity

$$-\zeta'(s) - \sum_{n=1}^N n^{-s}\log n$$

has an asymptotic expansion

$$-\frac{N^{1-s}\log N}{1-s} + \frac{N^{1-s}}{(1-s)^2} - \frac{1}{2}N^{-s}\log N + \frac{s}{12}N^{-s-1}\log N - \frac{1}{12}N^{-s-1} + \cdots.$$

Let us truncate these series after the last term which doesn't tend to 0 with N^{-1}. Then one obtains, for every complex number $s \neq 1$,

$$-\zeta'(s) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} n^{-s}\log n - R(s,N) \right\},$$

where $R(s,N)$ is the truncated series.

As we mentioned above, it is easy to see that we have $R(0,N) = \left(N+\frac{1}{2}\right)\log N - N$ and

$$-\zeta'(0) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \log n - \left(\left(N+\frac{1}{2}\right)\log N - N \right) \right\} = \lim_{N\to\infty} \log\left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}} \right),$$

which gives the Stirling formula (one knows $\zeta'(0) = -\frac{1}{2}\log 2\pi$ from the functional equation of $\zeta(s)$. See p. 335 in [**H**]). Furthermore, if $s=-2$, for instance, one has

$$R(-2,N) = \left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right)\log N - \frac{N^3}{9} + \frac{N}{12}.$$

It follows that

$$-\zeta'(-2) = \lim_{N\to\infty} \left\{ \log \prod_{n=1}^{N} n^{n^2} - \log e^{R(-2,N)} \right\}.$$

Hence, if we put $\left[\prod_{n=1}^{N} n^{n^2}\right]_{\mathrm{div}} = e^{R(-2,N)} = N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}} e^{-\frac{N^3}{9}+\frac{N}{12}}$, by the definition of our regularized product, we obtain

$$\prod_{n=1}^{\infty} (\!(n)\!)^{n^2} = e^{-\zeta'(-2)} = \lim_{N\to\infty} \frac{\prod_{n=1}^{N} n^{n^2}}{e^{R(-2,N)}} = \lim_{N\to\infty} \frac{\prod_{n=1}^{N} n^{n^2}}{[\prod_{n=1}^{N} n^{n^2}]_{\mathrm{div}}} = \lim_{N\to\infty} \left[\prod_{n=1}^{N} n^{n^2} \right]_{\mathrm{reg}}.$$

This observation would suggest the definition of the finite ladders of the multiple sine functions of order 3 and 4. In fact, we have the

THEOREM 5.1. *Define $\mathscr{S}_{3,N}(x)$ by*

$$\mathscr{S}_{3,N}(x) = \left(\frac{\prod_{n=1}^{N} n^{n^2}}{e^{-\zeta'(-2)} N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}} e^{-\frac{N^3}{9}+\frac{N}{12}}} \right)^2 \cdot e^{\frac{x^2}{2}} \prod_{n=1}^{N} \left\{ \left(1-\frac{x^2}{n^2}\right)^{n^2} e^{x^2} \right\}.$$

Also, define $\mathscr{S}_{4,N}(x)$ by

$$\mathscr{S}_{4,N}(x) = e^{\frac{x^3}{3}} \prod_{n=1}^{N} \left\{ \left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^{n^3} \exp\left(2n^2x + \frac{2}{3}x^3\right) \right\}.$$

Then we have

$$(1)\quad \lim_{N\to\infty}\mathscr{S}_{3,N}(x)=e^{\frac{x^2}{2}}\prod_{n=1}^{\infty}\left\{\left(1-\frac{x^2}{n^2}\right)^{n^2}e^{x^2}\right\}=\mathscr{S}_3(x).$$

$$(2)\quad \lim_{N\to\infty}\mathscr{S}_{4,N}(x)=e^{\frac{x^3}{3}}\prod_{n=1}^{\infty}\left\{\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}}\right)^{n^3}\exp\left(2n^2x+\frac{2}{3}x^3\right)\right\}=\mathscr{S}_4(x).$$

$$(3)\quad \frac{\mathscr{S}_{3,N}(x+1)}{\mathscr{S}_{3,N}(x)}=-\mathscr{S}_{2,N}(x)^2\mathscr{S}_{1,N}(x)$$
$$\times\left\{\left(1-\frac{x^2}{N^2}\right)\left(1-\frac{x}{N}\right)^{-1}\cdot\left(1+\frac{1+x}{N}\right)^{N^2}\left(1-\frac{x}{N}\right)^{-N^2}\right.$$
$$\left.\times\exp\left(-2Nx-N+x+\frac{1}{2}\right)\right\}.$$

Here the term in $\{\ \}$ goes to 1 as $N\to\infty$. In particular,

$$\frac{\mathscr{S}_3(x+1)}{\mathscr{S}_3(x)}=-\mathscr{S}_2(x)^2\mathscr{S}_1(x).$$

$$(4)\quad \frac{\mathscr{S}_{4,N}(x+1)}{\mathscr{S}_{4,N}(x)}=-e^{6\zeta'(-2)}\mathscr{S}_{3,N}(x)^3\mathscr{S}_{2,N}(x)^3\mathscr{S}_{1,N}(x)$$
$$\times\left\{\left(1+\frac{x}{N}\right)^{-3N}\left(1-\frac{x}{N}\right)^{-(N+1)^3}\left(1+\frac{1+x}{N}\right)^{-N^3}\right.$$
$$\left.\times\exp\left(-\left(N+\frac{1}{2}\right)x^2-(4N-1)x+N^2-\frac{N}{2}+\frac{1}{3}\right)\right\}.$$

The factor in $\{\ \}$ goes to 1 as $N\to\infty$. Especially, we have

$$\frac{\mathscr{S}_4(x+1)}{\mathscr{S}_4(x)}=-e^{6\zeta'(-2)}\mathscr{S}_3(x)^3\mathscr{S}_2(x)^3\mathscr{S}_1(x).$$

PROOF. The assertion (1) follows immediately from the limit formula we have already seen; $\lim_{N\to\infty}\prod_{n=1}^{N}n^{n^2}/N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}}e^{-\frac{N^3}{9}+\frac{N}{12}}=e^{-\zeta'(-2)}(=\prod_{n=1}^{\infty}(\!(n)\!)^{n^2})$. The assertion (2) is clear. To show (3) we calculate as

$$\frac{\mathscr{S}_{3,N}(x+1)}{\mathscr{S}_{3,N}(x)}=e^{(N+\frac{1}{2})(2x+1)}\prod_{n=1}^{N}\left(\frac{n^2-(x+1)^2}{n^2-x^2}\right)^{n^2}$$
$$=e^{(N+\frac{1}{2})(2x+1)}\prod_{n=1}^{N}\frac{(n-1-x)^{n^2}(n+1+x)^{n^2}}{(n-x)^{n^2}(n+x)^{n^2}}$$
$$=e^{(N+\frac{1}{2})(2x+1)}\prod_{n=1}^{N}\frac{(n-x)^{(n+1)^2-n^2}}{(n+x)^{n^2-(n-1)^2}}\times(-x)(N-x)^{-(N+1)^2}(N+1+x)^{N^2}$$

$$
\begin{aligned}
&= -e^{(N+\frac{1}{2})(2x+1)} \prod_{n=1}^{N} \left(\frac{n-x}{n+x}\right)^{2n} \\
&\quad \times x \prod_{n=1}^{N} (n-x)(n+x) \times (N-x)^{-(N+1)^2} (N+1+x)^{N^2} \\
&= -e^{(N+\frac{1}{2})(2x+1)} \left(\mathscr{S}_{2,N}(x) \left(1+\frac{x}{N}\right)^{-N} e^{-2Nx}\right)^2 \\
&\quad \times \left(\mathscr{S}_{1,N}(x)(N!)^2 \left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}}\right)^{-2}\right) (N-x)^{-(N+1)^2} (N+1+x)^{N^2} \\
&= -\mathscr{S}_{2,N}(x)^2 \mathscr{S}_{1,N}(x) \left(1-\frac{x^2}{N^2}\right)^{-2N} \left(1-\frac{x}{N}\right)^{-1} \\
&\quad \times \left(1+\frac{1+x}{N}\right)^{N^2} \left(1-\frac{x}{N}\right)^{-N^2} \exp\left(-2Nx-N+x+\frac{1}{2}\right).
\end{aligned}
$$

This proves the former part of the assertion (3). The latter follows from this equation and

$$
\begin{aligned}
&\lim_{N\to\infty} \left(1+\frac{1+x}{N}\right)^{N^2} \left(1-\frac{x}{N}\right)^{-N^2} \exp\left(-2Nx-N+x+\frac{1}{2}\right) \\
&= \lim_{N\to\infty} \exp\left(N^2 \log\left(1+\frac{1+x}{N}\right) - N^2 \log\left(1-\frac{x}{N}\right) - 2Nx - N + x + \frac{1}{2}\right) \\
&= \lim_{N\to\infty} \exp\left(N^2\left(\frac{1+x}{N} - \frac{1}{2}\left(\frac{1+x}{N}\right)^2\right) + N^2\left(\frac{x}{N} - \frac{1}{2}\left(\frac{x}{N}\right)^2\right) - 2Nx - N + x + \frac{1}{2}\right) \\
&= 1.
\end{aligned}
$$

(4) Since

$$
\mathscr{S}_{4,N}(x) = \exp\left(\frac{2N+1}{3}x^3 + 2(1^2+\cdots+N^2)x\right) \prod_{n=1}^{N} \left(\frac{n-x}{n+x}\right)^{n^3},
$$

we see that

$$
\begin{aligned}
\frac{\mathscr{S}_{4,N}(x+1)}{\mathscr{S}_{4,N}(x)} &= \exp\left\{\frac{2N+1}{3}(3x^2+3x+1) + 2(1^2+\cdots+N^2)\right\} \\
&\quad \times \prod_{n=1}^{N} \left\{\frac{(n-1-x)(n+x)}{(n-x)(n+1+x)}\right\}^{n^3}.
\end{aligned}
$$

Here we have

$$\begin{aligned}
&\prod_{n=1}^{N}\left\{\frac{(n-1-x)(n+x)}{(n-x)(n+1+x)}\right\}^{n^3}\\
&=-x\prod_{n=1}^{N}\frac{(n-x)^{(n+1)^3-n^3}}{(n+x)^{n^3-(n-1)^3}}\times(N-x)^{-(N+1)^3}(N+1+x)^{-N^3}\\
&=-x\prod_{n=1}^{N}\left\{(n-x)^{3n^2+3n+1}(n+x)^{3n^3-3n+1}\right\}\times(N-x)^{-(N+1)^3}(N+1+x)^{-N^3}\\
&=-\left(\prod_{n=1}^{N}(n^2-x^2)^{n^2}\right)^3\cdot\left(\prod_{n=1}^{N}\left(\frac{n-x}{n+x}\right)^n\right)^3\cdot x\prod_{n=1}^{N}(n^2-x^2)\\
&\quad\times(N-x)^{-(N+1)^3}(N+1+x)^{-N^3}.
\end{aligned}$$

If we put $A_N=\left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}}\right)^2$ and $B_N=\left(\frac{\prod_{n=1}^{N}n^2}{e^{-\zeta'(-2)}N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}}e^{-\frac{N^3}{9}+\frac{N}{12}}}\right)^2$, then we notice that

$$\begin{aligned}
x\prod_{n=1}^{N}(n^2-x^2)&=A_N^{-1}(N!)^2\mathscr{S}_{1,N}(x),\\
\prod_{n=1}^{N}\left(\frac{n-x}{n+x}\right)&=\left(1+\frac{x}{N}\right)^{-N}e^{-2Nx}\mathscr{S}_{2,N}(x)
\end{aligned}$$

and

$$\prod_{n=1}^{N}(n^2-x^2)^{n^2}=B_N^{-1}\left(\prod_{n=1}^{N}n^{n^2}\right)^2e^{-(N+\frac{1}{2})x^2}\mathscr{S}_{3,N}(x).$$

Thus we obtain

$$\begin{aligned}
\frac{\mathscr{S}_{4,N}(x+1)}{\mathscr{S}_{4,N}(x)}=&-\mathscr{S}_{3,N}(x)^3\mathscr{S}_{2,N}(x)^3\mathscr{S}_{1,N}(x)\\
&\times\left\{\exp\left(\frac{2N+1}{3}(3x^2+3x+1)+2(1^2+\cdots+N^2)\right)\times A_N^{-1}(N!)^2\right.\\
&\times\left(1+\frac{x}{N}\right)^{-3N}e^{-6Nx}\times B_N^{-3}\left(\prod_{n=1}^{N}n^{n^2}\right)^6e^{-(3N+\frac{3}{2})x^2}\\
&\left.\cdot N^{-N^3-(N+1)^3}\left(1-\frac{x}{N}\right)^{-(N+1)^3}\left(1+\frac{1+x}{N}\right)^{-N^3}\right\}.
\end{aligned}$$

It follows from the definition of A_N and B_N that

$$\begin{aligned}\frac{\mathscr{S}_{4,N}(x+1)}{\mathscr{S}_{4,N}(x)} = &-e^{-6\zeta'(-2)}\mathscr{S}_{3,N}(x)^3\mathscr{S}_{2,N}(x)^3\mathscr{S}_{1,N}(x)\\ &\times\left\{\left(1+\frac{x}{N}\right)^{-3N}\left(1-\frac{x}{N}\right)^{-(N+1)^3}\left(1+\frac{1+x}{N}\right)^{-N^3}\right.\\ &\left.\cdot\exp\left(-\left(N+\frac{1}{2}\right)x^2-(4N-1)x+N^2-\frac{N}{2}+\frac{1}{3}\right)\right\}.\end{aligned}$$

Here, the factor $\{\ \}$ goes to 1 as $N\to\infty$. In fact, this can be seen from

$$\begin{aligned}&\lim_{N\to\infty}\left\{\left(1-\frac{x}{N}\right)^{-(N+1)^3}\left(1+\frac{1+x}{N}\right)^{-N^3}\right.\\ &\qquad\left.\cdot\exp\left(-\left(N+\frac{1}{2}\right)x^2-(4N-1)x+N^2-\frac{N}{2}+\frac{1}{3}\right)\right\}\\ &=\lim_{N\to\infty}\exp\left\{-(N+1)^3\log\left(1-\frac{x}{N}\right)\right.\\ &\qquad\left.-N^3\log\left(1+\frac{1+x}{N}\right)-\left(N+\frac{1}{2}\right)x^2-(4N-1)x+N^2-\frac{N}{2}+\frac{1}{3}\right\}\\ &=\lim_{N\to\infty}\exp\left\{(N+1)^3\left(\frac{x}{N}+\frac{1}{2}\left(\frac{x}{N}\right)^2+\frac{1}{3}\left(\frac{x}{N}\right)^3\right)\right.\\ &\qquad-N^3\left(\frac{1+x}{N}-\frac{1}{2}\left(\frac{1+x}{N}\right)^2+\frac{1}{3}\left(\frac{1+x}{N}\right)^3\right)\\ &\qquad\left.-\left(N+\frac{1}{2}\right)x^2-(4N-1)x+N^2-\frac{N}{2}+\frac{1}{3}\right\}\\ &=\lim_{N\to\infty}\exp 3x=e^{3x}.\end{aligned}$$

This completes the proof of the theorem. □

REMARK 1. As in the case of $\mathscr{S}_{2,N}(x)$ we may put polynomials in x instead of the exponential in front of the product for $\mathscr{S}_{3,N}(x)$ and $\mathscr{S}_{4,N}(x)$, but we do not pursue this here.

REMARK 2. Look at the case of $\mathscr{S}_{3,N}(x)$. Let $I_N=\{-N,\dots,-1,1,\dots,N\}, a_n=n, c_n=n^2$ and $r=3$ in the notation we used for describing the question (L) in the beginning of this section. Then $\prod_{n\in I_N}|a_n|^{c_n}=\left(\prod_{n=1}^N n^{n^2}\right)^2$ and $\left[\prod_{n\in I_N}|a_n|^{c_n}\right]_{\mathrm{reg}}=\prod_{n\in I_N}|a_n|^{c_n}/\left[\prod_{n\in I_N}|a_n|^{c_n}\right]_{\mathrm{div}}$ with $\left[\prod_{n\in I_N}|a_n|^{c_n}\right]_{\mathrm{div}}=\left(N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}}e^{-\frac{N^3}{9}+\frac{N}{12}}\right)^2$. Therefore, by Theorem 5.1 and Theorem 2.1 we see that $\lim_{N\to\infty}\left[\prod_{n\in I_N}|a_n|^{c_n}\right]_{\mathrm{reg}}=\left(\prod_{n=1}^\infty(\!(n)\!)^{n^2}\right)^2=e^{-2\zeta'(2)}$ and $\mathscr{S}_{3,N}(x)=e^{2\zeta'(2)}e^{\frac{x^2}{2}}F_N(x)\to\mathscr{S}_3(x)$.

REMARK 3. By Theorem 3.1, the sequence of functions $\mathscr{S}_{1,N}(x)$ can be also regarded as a regularization factors of the products $x\prod_{n=1}^N(n-x)\prod_{n=1}^N(n+x)$ in the following sense.

$$\begin{aligned}\mathscr{S}_1(x) &= \lim_{N\to\infty} \mathscr{S}_{1,N}(x)\\ &= \lim_{N\to\infty}\left[x\prod_{n=1}^{N}(n-x)\prod_{n=1}^{N}(n+x)\right]_{\mathrm{reg}} = \lim_{N\to\infty}\frac{x\prod_{n=1}^{N}(n-x)\prod_{n=1}^{N}(n+x)}{\left[x\prod_{n=1}^{N}(n-x)\prod_{n=1}^{N}(n+x)\right]_{\mathrm{div}}},\end{aligned}$$

with $\left[x\prod_{n=1}^{N}(n-x)\prod_{n=1}^{N}(n+x)\right]_{\mathrm{div}} = (N^{N+\frac{1}{2}}e^{-N})^2$. Furthermore, $\mathscr{S}_{3,N}(x)$ can be also a regularization factor of the product $e^{(N+\frac{1}{2})x^2}\prod_{n=1}^{N}(n-x)^{n^2}\prod_{n=1}^{N}(n+x)^{n^2}$. Actually, if we put $\left[e^{(N+\frac{1}{2})x^2}\prod_{n=1}^{N}(n-x)^{n^2}\prod_{n=1}^{N}(n+x)^{n^2}\right]_{\mathrm{div}} = \left(e^{-\zeta'(2)}N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}}e^{-\frac{N^3}{9}+\frac{N}{12}}\right)^2$ then $\mathscr{S}_{3,N}(x) = \left[e^{(N+\frac{1}{2})x^2}\prod_{n=1}^{N}(n-x)^{n^2}\prod_{n=1}^{N}(n+x)^{n^2}\right]_{\mathrm{reg}} \to \mathscr{S}_3(x)$ when $N\to\infty$ by Theorem 5.1.

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Nobushige Kurokawa
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Tokyo, 152-0033
Japan
E-mail: kurokawa@math.titech.ac.jp

Masato Wakayama
Faculty of Mathematics
Kyushu University
Hakozaki, Fukuoka, 812-8581
Japan
E-mail: wakayama@math.kyushu-u.ac.jp