J. Math. Soc. Japan
Vol. 57, No. 4, 2005

Uniqueness of the solution of nonlinear totally characteristic partial differential equations

By Hidetoshi TAHARA

(Received Dec. 7, 2004)

Abstract. Let us consider the following nonlinear singular partial differential equation $(t\partial/\partial t)^m u = F(t, x, \{(t\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j+\alpha \le m, j<m})$ in the complex domain with two independent variables $(t, x) \in \boldsymbol{C}^2$. When the equation is of totally characteristic type, this equation was solved in [**2**] and [**9**] under certain Poincaré condition. In this paper, the author will prove the uniqueness of the solution under the assumption that $u(t, x)$ is holomorphic in $\{(t, x) \in \boldsymbol{C}^2 \,;\, 0 < |t| < r, |\arg t| < \theta, |x| < R\}$ for some $r > 0$, $\theta > 0$, $R > 0$ and that it satisfies $u(t, x) = O(|t|^a)$ (as $t \longrightarrow 0$) uniformly in x for some $a > 0$. The result is applied to the problem of removable singularities of the solution.

1. Introduction.

Notations: $(t, x) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$, $\boldsymbol{N} = \{0, 1, 2, \dots\}$, and $\boldsymbol{N}^* = \{1, 2, \dots\}$. Let $m \in \boldsymbol{N}^*$ be fixed, set $N = \#\{(j, \alpha) \in \boldsymbol{N} \times \boldsymbol{N} \,;\, j + \alpha \le m, j < m\}$ (that is, $N = m(m + 3)/2$), and denote the complex variable $z \in \boldsymbol{C}^N$ by $z = \{z_{j,\alpha}\}_{j+\alpha \le m, j<m}$.

In this paper we will consider the following nonlinear singular partial differential equation:

$$\text{(E)} \qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+\alpha \le m \\ j<m}}\right),$$

where $F(t, x, z)$ is a function of the variables (t, x, z) defined in a neighborhood Δ of the origin of $\boldsymbol{C}_t \times \boldsymbol{C}_x \times \boldsymbol{C}_z^N$, and $u = u(t, x)$ is the unknown function. Set $\Delta_0 = \Delta \cap \{t = 0, z = 0\}$, and set also $I_m = \{(j, \alpha) \in \boldsymbol{N} \times \boldsymbol{N} \,;\, j + \alpha \le m,\ j < m\}$ and $I_m(+) = \{(j, \alpha) \in I_m \,;\, \alpha > 0\}$.

Let us first assume the following conditions:

A$_1$) $F(t, x, z)$ is a holomorphic function on Δ;
A$_2$) $F(0, x, 0) \equiv 0$ on Δ_0.

Then, by expanding $F(t, x, z)$ into Taylor series with respect to (t, z) we have

$$F(t, x, z) = a(x)t + \sum_{\substack{j+\alpha \le m \\ j<m}} b_{j,\alpha}(x) z_{j,\alpha} + \sum_{p+|\nu| \ge 2} g_{p,\nu}(x) t^p z^\nu,$$

2000 *Mathematics Subject Classification.* Primary 35A20; Secondary 35A10, 35G20.

Key Words and Phrases. uniqueness of the solution, nonlinear PDE, totally characteristic PDE.

This research was partially supported by the Grant-in-Aid for Scientific Research (No. 16540169) of Japan Society for the Promotion of Science.

where $a(x)$, $b_{j,\alpha}(x)$ ($j+\alpha \le m, j<m$) and $g_{p,\nu}(x)$ ($p+|\nu| \ge 2$) are all holomorphic functions on Δ_0, $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \boldsymbol{N}^N$, $|\nu| = \sum_{(j,\alpha)\in I_m} \nu_{j,\alpha}$ and $z^\nu = \prod_{(j,\alpha)\in I_m} [z_{j,\alpha}]^{\nu_{j,\alpha}}$. We divide our equation into the following three types:

Type (1) : $b_{j,\alpha}(x) \equiv 0$ for all $(j,\alpha) \in I_m(+)$;
Type (2) : $b_{j,\alpha}(0) \neq 0$ for some $(j,\alpha) \in I_m(+)$;
Type (3) : $b_{j,\alpha}(0) = 0$ for all $(j,\alpha) \in I_m(+)$, but $b_{i,\beta}(x) \not\equiv 0$ for some $(i,\beta) \in I_m(+)$.

Type (1) is called a *Gérard-Tahara type* partial differential equation and it was studied in [**3**], [**4**] and [**10**]; the uniqueness of the solution was studied in [**7**] and [**8**]. Type (2) is called a *spacially nondegenerate type* partial differential equation and it was studied in [**5**]. Type (3) is called a *totally characteristic type* partial differential equation and it was studied in [**2**] and [**9**].

In this paper we will consider the type (3) under the following condition:

A_3) $b_{j,\alpha}(x) = O(x^\alpha)$ (as $x \longrightarrow 0$) for all $(j,\alpha) \in I_m(+)$.

Then, by the condition A_3) we have $b_{j,\alpha}(x) = x^\alpha c_{j,\alpha}(x)$ for some holomorphic functions $c_{j,\alpha}(x)$ ($(j,\alpha) \in I_m$) and therefore our equation (E) is written in the form

$$C\left(x, t\frac{\partial}{\partial t}, x\frac{\partial}{\partial x}\right)u = a(x)\,t + \sum_{p+|\nu|\ge 2} g_{p,\nu}(x)\,t^p \prod_{(j,\alpha)\in I_m} \left[\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right]^{\nu_{j,\alpha}} \tag{1.1}$$

where

$$C(x,\lambda,\rho) = \lambda^m - \sum_{\substack{j+\alpha\le m \\ j<m}} c_{j,\alpha}(x)\lambda^j \rho(\rho-1)\cdots(\rho-\alpha+1).$$

Set

$$L(\lambda,\rho) = \lambda^m - \sum_{\substack{j+\alpha\le m \\ j<m}} c_{j,\alpha}(0)\lambda^j \rho(\rho-1)\cdots(\rho-\alpha+1), \tag{1.2}$$

$$L_m(X) = X^m - \sum_{\substack{j+\alpha= m \\ j<m}} c_{j,\alpha}(0)\,X^j, \tag{1.3}$$

and denote by $c_1,\dots,c_m$ the roots of the equation $L_m(X) = 0$ in X. If we factorize $L(\lambda,l)$ into the form

$$L(\lambda,l) = (\lambda-\lambda_1(l))\cdots(\lambda-\lambda_m(l)), \quad l \in \boldsymbol{N}, \tag{1.4}$$

by renumbering the subscript i of $\lambda_i(l)$ suitably we have

$$\lim_{l\to\infty} \frac{\lambda_i(l)}{l} = c_i \quad \text{for } i = 1,\dots,m.$$

On the existence of a solution of (E) we have the following results.

THEOREM 1. *Assume the conditions* A_1), A_2) *and* A_3). *We have*:

(1)(Chen-Tahara [**2**]) *If* $L(k,l) \neq 0$ *holds for any* $(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}$ *and if* $c_i \in \boldsymbol{C} \setminus [0,\infty)$ *holds for* $i = 1,\dots,m$, *the equation* (E) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ *satisfying* $u(0,x) \equiv 0$.

(2)(Tahara [**9**]) *If* $c_i \in \boldsymbol{C} \setminus [0,\infty)$ *holds for* $i = 1,\dots,m$, *the equation* (E) *has a family of solutions* $u(t,x)$ *of the form*

$$u(t,x) = w\big(t, t\log t, t(\log t)^2, \dots, t(\log t)^{\mu}, x\big),$$

where $w(t,t_1,t_2,\dots,t_\mu,x)$ *is a holomorphic function with* μ *arbitrary constants in a neighborhood of* $(t,t_1,t_2,\dots,t_\mu,x) = (0,0,0,\dots,0,0)$ *satisfying* $w(0,0,0,\dots,0,x) \equiv 0$ *and* μ *is a non-negative integer determined by the equation.*

Chen-Luo [**1**] and Shirai [**6**] have generalized the existence of the unique holomorphic solution in (1) of Theorem 1 to the case of several variables $(t,x) \in \boldsymbol{C} \times \boldsymbol{C}^n$ and $(t,x) \in \boldsymbol{C}^d \times \boldsymbol{C}^n$.

In this paper, we will establish the uniqueness of the solution of the equation (E).

2. Uniqueness of the solution.

We denote:

- $\mathscr{R}(\boldsymbol{C} \setminus \{0\})$ the universal covering space of $\boldsymbol{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathscr{R}(\boldsymbol{C} \setminus \{0\}); |\arg t| < \theta\}$ a sector in $\mathscr{R}(\boldsymbol{C} \setminus \{0\})$,
- $S_\theta(r) = \{t \in S_\theta; 0 < |t| < r\}$,
- $D_R = \{x \in \boldsymbol{C}; |x| \leq R\}$.

Let us define sets of functions $\widetilde{\mathscr{S}_a}$ and $\widetilde{\mathscr{S}_+}$ in which we will prove the uniqueness of the solution of (E).

DEFINITION 1. (1) Let $a > 0$. We denote by $\widetilde{\mathscr{S}_a}$ the set of all $u(t,x)$ satisfying the following i) and ii): i) $u(t,x)$ is a holomorphic function on $S_\theta(r) \times D_R$ for some $\theta > 0$, $r > 0$ and $R > 0$; and ii) $u(t,x)$ satisfies

$$\max_{x \in D_R} \big|u(t,x)\big| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta(r)).$$

(2) We define $\widetilde{\mathscr{S}_+}$ by

$$\widetilde{\mathscr{S}_+} = \bigcup_{a>0} \widetilde{\mathscr{S}_a}.$$

Let $\lambda_1(l),\dots,\lambda_m(l)$ be the ones in (1.4). Our main theorem is as follows:

THEOREM 2. *Assume the conditions* A_1), A_2), A_3) *and*

$$\operatorname{Re} c_i < 0 \quad \textit{for } i = 1,\dots,m. \tag{2.1}$$

Then, if $u_1(t,x)$ *and* $u_2(t,x)$ *are solutions of* (E) *belonging in the class* $\widetilde{\mathscr{S}_+}$ *and if* $u_1 - u_2 \in \widetilde{\mathscr{S}_a}$ *holds for some* $a > 0$ *satisfying*

$$a > \max_{\substack{1 \le i \le m \\ l \ge 0}} \operatorname{Re} \lambda_i(l), \tag{2.2}$$

we have $u_1 = u_2$ *in* $\widetilde{\mathscr{S}_+}$.

Since $\lambda_i(l)/l \longrightarrow c_i$ (as $l \longrightarrow \infty$) for $i = 1, \dots, m$, under the condition (2.1) we have $\operatorname{Re} \lambda_i(l) \longrightarrow -\infty$ (as $l \longrightarrow \infty$) for $i = 1, \dots, m$ and therefore the righthand side of (2.2) is well-defined. We note:

Lemma 1. *Let* $a > 0$. *The following two conditions are equivalent*:
(1) (2.1) *and* (2.2) *hold*;
(2) *there are* $0 \le b < a$ *and* $c > 0$ *such that* $b - \operatorname{Re} \lambda_i(l) \ge c\,l$ *holds for any* $l \in \boldsymbol{N}$ *and* $i = 1, \dots, m$.

Proof. Suppose the condition (1). Set

$$\beta = \max \left[0, \max_{\substack{1 \le i \le m \\ l \ge 0}} \operatorname{Re} \lambda_i(l) \right]$$

and take $b > 0$ so that $\beta < b < a$. Since $\operatorname{Re} \lambda_i(l)/l \longrightarrow \operatorname{Re} c_i$ (as $l \longrightarrow \infty$) for $i = 1, \dots, m$, under the condition (2.1) we can find $\varepsilon > 0$ and $L \in \boldsymbol{N}$ such that $-\operatorname{Re} \lambda_i(l) \ge \varepsilon\, l$ for any $l > L$ and $i = 1, \dots, m$. Then, by taking $c > 0$ so that $0 < c \le \min\{\varepsilon, (b - \beta)/L\}$ we can verify the condition (2) in the following way: if $l > L$ we have $b - \operatorname{Re} \lambda_i(l) \ge -\operatorname{Re} \lambda_i(l) \ge \varepsilon\, l \ge c\,l$, and for $0 \le l \le L$ we have $b - \operatorname{Re} \lambda_i(l) \ge b - \beta \ge cL \ge c\,l$. Thus, we have proved that (1) implies (2).

Conversely, suppose the condition (2). Then we have $(b - \operatorname{Re} \lambda_i(l))/l \ge c$ and so by letting $l \longrightarrow \infty$ we have $-\operatorname{Re} c_i \ge c$ for $i = 1, \dots, m$; this proves (2.1). Since $a > b$ holds, we have $a > b \ge \operatorname{Re} \lambda_i(l) + c\,l \ge \operatorname{Re} \lambda_i(l)$ for any $l \in \boldsymbol{N}$ and $i = 1, \dots, m$; this proves (2.2). Thus, we have proved also that (2) implies (1). □

Thus, Theorem 2 is equivalent to the following

Theorem 2*. *Assume the conditions* A_1), A_2), A_3) *and that there are* $b \ge 0$ *and* $c > 0$ *such that*

$$b - \operatorname{Re} \lambda_i(l) \ge c\,l \quad \text{for any } l \in \boldsymbol{N} \text{ and } i = 1, \dots, m. \tag{2.3}$$

If $u_1(t,x)$ *and* $u_2(t,x)$ *are solutions of* (E) *belonging in the class* $\widetilde{\mathscr{S}_+}$ *and if* $u_1 - u_2 \in \widetilde{\mathscr{S}_a}$ *holds for some* $a > b$, *then we have* $u_1 = u_2$ *in* $\widetilde{\mathscr{S}_+}$.

The rest part of this paper is organized as follows. In the next section 3 we will present basics of the theory of pseudo-differential operators, in section 4 we will prove a uniqueness theorem for some linear pseudo-differential equations, and in section 5 we will prove Theorem 2* by applying the result in section 4 to our nonlinear equations. In the last section 6 we will give an application of Theorem 2 to the problem of removable singularities of the solution of (E).

3. Basics of pseudo-differential operators.

We denote by $\boldsymbol{C}[[x]]$ the ring of formal power series in x with complex coefficients. For a sequence $\lambda(l)$ ($l = 0, 1, 2, \ldots$) of complex numbers, we define the operator $\lambda(\theta)$: $\boldsymbol{C}[[x]] \longrightarrow \boldsymbol{C}[[x]]$ by the following:

$$\boldsymbol{C}[[x]] \ni f = \sum_{l \geq 0} f_l x^l \longmapsto \lambda(\theta) f = \sum_{l \geq 0} f_l \lambda(l) x^l \in \boldsymbol{C}[[x]]. \tag{3.1}$$

If $\lambda(\rho)$ is a mapping from $\boldsymbol{N}$ into $\boldsymbol{C}$, we can define an operator $\lambda(\theta) : \boldsymbol{C}[[x]] \longrightarrow \boldsymbol{C}[[x]]$. In particular, if $\lambda(\rho)$ is a function defined on $\boldsymbol{R}_+ = \{\rho \in \boldsymbol{R}\,;\, \rho \geq 0\}$, we have an operator $\lambda(\theta) : \boldsymbol{C}[[x]] \longrightarrow \boldsymbol{C}[[x]]$. If $\lambda(\rho)$ is a polynomial in ρ, we easily see that $\lambda(\theta) = \lambda(x(d/dx))$ holds as an operator from $\boldsymbol{C}[[x]]$ into $\boldsymbol{C}[[x]]$. Thus, our operator $\lambda(\theta)$ can be regarded as a generalization of a differential operator. From now, we will call this operator $\lambda(\theta)$ as *a pseudo-differential operator*.

If a pseudo-differential operator $\lambda(\theta) : \boldsymbol{C}[[x]] \longrightarrow \boldsymbol{C}[[x]]$ satisfies

$$|\lambda(l)| \leq C(1 + l)^k \quad (l = 0, 1, 2, \ldots)$$

for some $C \geq 0$ and $k \geq 0$, we say that $\lambda(\theta)$ is *a pseudo-differential operator of order* k. We denote by S_k the set of all such pseudo-differential operators of order k as above.

For a formal power series $f(x) = \sum_{l \geq 0} f_l\, x^l \in \boldsymbol{C}[[x]]$, we define

$$|f|(x) = \sum_{l \geq 0} |f_l|\, x^l \quad \text{and} \quad |f|_\rho = |f|(\rho) = \sum_{l \geq 0} |f_l|\, \rho^l. \tag{3.2}$$

Let $R > 0$. Using this norm, we define X_R by

$$X_R = \{f(x) \in \boldsymbol{C}[[x]]\,;\, |f|_R < \infty\}.$$

It is easy to see that X_R is a Banach space with the norm $|\cdot|_R$. We denote by $C^0([0, T], X_R)$ the space of all continuous functions $f(t, x)$ on $[0, T]$ with values in X_R: it is also a Banach space with the norm $\|f\| = \max_{t \in [0,T]} |f(t)|_R$.

The following lemma is an easy consequence of the definition.

LEMMA 2. (1) *Let $\lambda(\theta)$ be a pseudo-differential operator of order* 0. *If $f(t, x) \in C^0([0, T], X_R)$, we have $\lambda(\theta) f(t, x) \in C^0([0, T], X_R)$. Moreover, if $|f(t)|_R = O(t^s)$ (as $t \longrightarrow 0$) for some $s > 0$, we have $|\lambda(\theta) f(t)|_R = O(t^s)$ (as $t \longrightarrow 0$).*

(2) *Let $m \in \boldsymbol{N}^*$ and let $\lambda(\theta)$ be a pseudo-differential operator of order m. If $f(t, x) \in C^0([0, T], X_R)$, we have $\lambda(\theta) f(t, x) \in C^0([0, T], X_{R_1})$ for any $0 < R_1 < R$. Moreover, if $|f(t)|_R = O(t^s)$ (as $t \longrightarrow 0$) for some $s > 0$, we have $|\lambda(\theta) f(t)|_{R_1} = O(t^s)$ (as $t \longrightarrow 0$) for any $0 < R_1 < R$.*

Now, let us consider the following pseudo-differential equation:

$$\Big(t\frac{\partial}{\partial t} - \lambda(\theta)\Big)u = f(t,x). \tag{3.3}$$

For $u(t,x) = \sum_{l\ge 0} u_l(t)x^l$, instead of $|u(t)|_\rho$ we often write

$$|u|(t,\rho) = \sum_{l\ge 0} |u_l(t)|\rho^l.$$

Lemma 3. *Let $R>0$ and $c\ge 0$. Assume that*

$$-\mathrm{Re}\,\lambda(l) \ge c\,l \quad \textit{for any } l=0,1,2,\dots. \tag{3.4}$$

(1)(Integral representation) *If $u(t,x)\in C^1((0,T],X_R)$ satisfies $|u(t)|_R = o(1)$ (as $t\longrightarrow 0$), if $f(t,x)\in C^0([0,T],X_R)$ satisfies $|f(t)|_R = O(t^\varepsilon)$ (as $t\longrightarrow 0$) for some $\varepsilon>0$, and if $u(t,x)$ and $f(t,x)$ satisfy the equation* (3.3) *on $(0,T]\times D_R$, then $u(t,x)$ is expressed in the form*

$$u(t,x) = \int_0^t \sum_{l\ge 0} f_l(\tau)(\tau/t)^{-\lambda(l)}\, x^l\, \frac{d\tau}{\tau} \quad \textit{on } (0,T]\times D_R. \tag{3.5}$$

(2)(A priori estimate) *Under the same conditions as in* (1) *we have*

$$|u|(t,\rho) \le \int_0^t |f|(\tau,(\tau/t)^c\rho)\,\frac{d\tau}{\tau} \quad \textit{on } (0,T]\times[0,R]. \tag{3.6}$$

In the above (3.6) *we can replace "$\le$" by "$\ll$", where $\sum_{l\ge 0} a_l\rho^l \ll \sum_{l\ge 0} b_l\rho^l$ means that $|a_l|\le b_l$ holds for all $l=0,1,2,\dots$.*

(3)(Uniqueness) *If $u_1(t,x)$ and $u_2(t,x)$ are solutions of* (3.3) *belonging in the class $C^1((0,T],X_R)$ and if $|u_1(t)-u_2(t)|_R = o(1)$ (as $t\longrightarrow 0$) holds, we have $u_1(t,x)=u_2(t,x)$ on $[0,T]\times D_R$.*

(4)(Solvability) *If $\lambda(\theta)$ is a pseudo-differential operator of order* 1, *and if $f(t,x)\in C^0([0,T],X_R)$ satisfies $|f(t)|_R = O(t^\varepsilon)$ (as $t\longrightarrow 0$) for some $\varepsilon>0$, the equation* (3.3) *has a unique solution $u(t,x)\in C^0([0,T],X_R)\cap C^1((0,T],X_{R_1})$ for any $0<R_1<R$ such that $|u(t)|_R = O(t^\varepsilon)$ (as $t\longrightarrow 0$) and $|(\partial u/\partial t)(t)|_{R_1} = O(t^{\varepsilon-1})$ (as $t\longrightarrow 0$) for any $0<R_1<R$.*

Proof. Let us prove (1). Set $u(t,x)=\sum_{l\ge 0} u_l(t)x^l$ and $f(t,x)=\sum_{l\ge 0} f_l(t)x^l$. By the equation (3.3) we have

$$\Big(t\frac{\partial}{\partial t} - \lambda(l)\Big)u_l(t) = f_l(t) \quad \text{for any } l=0,1,2,\dots,$$

which is equivalent to $(\partial/\partial t)(t^{-\lambda(l)}u_l(t)) = t^{-\lambda(l)-1}f_l(t)$ for $l = 0, 1, 2, \ldots$. By integrating this from t_0 to t (with $0 < t_0 < t$) we have

$$t^{-\lambda(l)}u_l(t) - {t_0}^{-\lambda(l)}u_l(t_0) = \int_{t_0}^{t} \tau^{-\lambda(l)} f_l(\tau)\,\frac{d\tau}{\tau}. \tag{3.7}$$

Since $-\mathrm{Re}\lambda(l) \geq 0$ is assumed, by the assumption we have

$$\begin{aligned}
&|{t_0}^{-\lambda(l)}u_l(t_0)| \leq |t_0|^{-\mathrm{Re}\lambda(l)}|u_l(t_0)| \leq |u_l(t_0)| = o(1) \ (\text{as } t_0 \longrightarrow 0), \ \text{ and} \\
&|\tau^{-\lambda(l)} f_l(\tau)(1/\tau)| \leq |\tau|^{-\mathrm{Re}\lambda(l)}O(\tau^{\varepsilon-1}) = O(\tau^{\varepsilon-1}) \ (\text{as } \tau \longrightarrow 0) :
\end{aligned}$$

therefore by letting $t_0 \longrightarrow 0$ in (3.7) we have

$$t^{-\lambda(l)}u_l(t) = \int_0^t \tau^{-\lambda(l)} f_l(\tau)\,\frac{d\tau}{\tau} \quad \text{for any } l = 0, 1, 2, \ldots$$

which is equivalent to

$$u_l(t) = \int_0^t (\tau/t)^{-\lambda(l)} f_l(\tau)\,\frac{d\tau}{\tau} \quad \text{for any } l = 0, 1, 2, \ldots.$$

This proves the result (1). By (3.5) we have the result (2) as follows:

$$\begin{aligned}
|u(t,\rho)| = |u(t)|_\rho &\leq \int_0^t \sum_{l \geq 0} |f_l(\tau)|(\tau/t)^{-\mathrm{Re}\lambda(l)}\rho^l\,\frac{d\tau}{\tau} \\
&\leq \int_0^t \sum_{l \geq 0} |f_l(\tau)|(\tau/t)^{cl}\rho^l\,\frac{d\tau}{\tau} = \int_0^t |f|(\tau, (\tau/t)^c\rho)\,\frac{d\tau}{\tau}.
\end{aligned}$$

The result (3) is an easy consequence of the result (1).

Lastly, let us prove (4). By the argument in the proof of (1) it is easy to see that the unique solution $u(t,x)$ is given by

$$u(t,x) = \int_0^t \sum_{l \geq 0} f_l(\tau)(\tau/t)^{-\lambda(l)}\, x^l\,\frac{d\tau}{\tau}.$$

By the assumption on $f(t,x)$ we have

$$|u(t)|_R \leq \int_0^t |f(\tau)|_R\,\frac{d\tau}{\tau} = \int_0^t O(\tau^\varepsilon)\,\frac{d\tau}{\tau} = O(t^\varepsilon) \quad (\text{as } t \longrightarrow 0),$$

and by Lemma 2 we have $|\lambda(\theta)u(t)|_{R_1} = O(t^\varepsilon)$ (as $t \longrightarrow 0$) for any $0 < R_1 < R$;

therefore we obtain $|t(\partial u/\partial t)(t)|_{R_1} \le |f(t)|_{R_1} + |\lambda(\theta)u(t)|_{R_1} = O(t^{\varepsilon})$ (as $t \longrightarrow 0$) for any $0 < R_1 < R$. This proves the result (4). $\square$

Set

$$L = t\frac{\partial}{\partial t} - \lambda(\theta). \tag{3.8}$$

In the proof of Theorem 2* we will use Lemma 3 in the following form:

Proposition 1. *Let $R > 0$ and $c \ge 0$. Assume that $\lambda(\theta)$ is a pseudo-differential operator of order 1 and that*

$$-\mathrm{Re}\,\lambda(l) \ge c\,l \quad \text{for any } l = 0, 1, 2, \ldots. \tag{3.9}$$

Assume also that $u(t,x) \in C^1((0,T], X_R)$ satisfies $|u(t)|_R = O(t^{\varepsilon})$ (as $t \longrightarrow 0$) and $|(\partial u/\partial t)(t)|_R = O(t^{\varepsilon-1})$ (as $t \longrightarrow 0$) for some $\varepsilon > 0$. Set

$$\phi(t,\rho) = \int_0^t |Lu|(\tau, (\tau/t)^c \rho)\,\frac{d\tau}{\tau} \quad \text{on} \quad (0,T] \times [0,R).$$

Then we have $|u|(t,\rho) \le \phi(t,\rho)$ on $(0,T] \times [0,R)$ (or $|u|(t,\rho) \ll \phi(t,\rho)$ as formal power series in ρ) and

$$\left(t\frac{\partial}{\partial t} + c\,\rho\frac{\partial}{\partial \rho}\right)\phi(t,\rho) = |Lu|(t,\rho) \quad \text{on} \quad (0,T] \times [0,R).$$

Proof. Set $f(t,x) = (Lu)(t,x)$; then by the assumption and Lemma 2 we see that $f(t,x) \in C^0([0,T], X_{R_1})$ and $|f(t)|_{R_1} = O(t^{\varepsilon})$ (as $t \longrightarrow 0$) hold for any $0 < R_1 < R$. Since $\phi(t,\rho)$ is nothing but

$$\phi(t,\rho) = \int_0^t |f|(\tau, (\tau/t)^c \rho)\,\frac{d\tau}{\tau} = \int_0^t \sum_{l \ge 0} |f_l(\tau)|(\tau/t)^{cl}\,\rho^l\,\frac{d\tau}{\tau},$$

by (2) of Lemma 3 we have $|u|(t,\rho) \le \phi(t,\rho)$ on $(0,T] \times [0,R_1]$, and by the same argument as in the proof of (4) of Lemma 3 we obtain $(t\partial/\partial t + c\rho\partial/\partial\rho)\phi(t,\rho) = |f|(t,\rho)$ on $(0,T] \times [0,R_1]$. Since $0 < R_1 < R$ is arbitrary, this proves the proposition. $\square$

The following lemma will be used in section 4.

Lemma 4. *Let $p \ge 0$, $k \in \boldsymbol{N}^*$ and let $\lambda(\theta)$ be a pseudo-differential operator of order 1 with the estimates $|\lambda(l)| \le C(1+l)$ $(l = 0, 1, 2, \ldots)$. Then for any $f(x) \in X_R$ and $0 \le \rho \le R$ we have the following results*:

i) $\displaystyle \left|(1+\theta)^p \left(\frac{\partial}{\partial x}\right) f\right|(\rho) \le \left|\left(\frac{\partial}{\partial x}\right)(1+\theta)^p f\right|(\rho) = \frac{\partial}{\partial \rho}|(1+\theta)^p f|(\rho);$

ii) $\left|\lambda(\theta)\left(\dfrac{\partial}{\partial x}\right)f\right|(\rho) \leq C\left|(1+\theta)\left(\dfrac{\partial}{\partial x}\right)f\right|(\rho) \leq C\left|\left(\dfrac{\partial}{\partial x}\right)(1+\theta)f\right|(\rho);$

iii) $\left|\left(\dfrac{\partial}{\partial x}\right)^k \lambda(\theta)f\right|(\rho) \leq C(1+k)\left|(1+\theta)\left(\dfrac{\partial}{\partial x}\right)^k f\right|(\rho);$

iv) $|(1+\theta)f|(\rho) = \left(1+\rho\dfrac{\partial}{\partial\rho}\right)|f|(\rho).$

Proof. i) is verified by the condition: $(1+(l-1))^p \leq (1+l)^p$ $(l=1,2,\dots)$. ii) is verified by the assumption $|\lambda(l)| \leq C(1+l)$ $(l=0,1,\dots)$ and the result i) with $p=1$. iii) is verified by the condition $|\lambda(l)| \leq C(1+k)(1+(l-k))$ for any $l \geq k$. iv) is clear from the definition. □

For a sequence $a(t,x;l) \in C^0([0,T],X_R)$ $(l=0,1,2,\dots)$ we define the operator $a(t,x;\theta)$ by the following:

$$f(t,x) = \sum_{l\geq 0} f_l(t)x^l \longmapsto a(t,x;\theta)f(t,x) = \sum_{l\geq 0} a(t,x;l)f_l(t)x^l. \tag{3.10}$$

We often write $a(t;\theta)f(t)$ instead of $a(t,x;\theta)f(t,x)$. By the definition we have:

Lemma 5. *For any $f(t,x) = \sum_{l\geq 0} f_l(t)x^l \in C^0([0,T],X_R)$ we have*

$$|a(t;\theta)f(t)|_R \leq \sum_{l\geq 0} |a(t;l)|_R\,|f_l(t)|\,R^l \tag{3.11}$$

where $|a(t;l)|_R$ is the norm of $a(t,x;l) \in X_R$ for fixed (t,l).

In view of Lemma 5, we say that $a(t,x;\theta)$ *is a pseudo-differential operator of order k (≥ 0) with symbol in $C^0([0,T],X_R)$*, if it satisfies

$$|a(t;l)|_R \leq C(1+l)^k, \quad 0 \leq t \leq T \text{ and } l=0,1,2,\dots \tag{3.12}$$

for some $C>0$. We denote by $S_k([0,T],X_R)$ the set of all the pseudo-differential operators of order k with symbol in $C^0([0,T],X_R)$.

Proposition 2. (1) *Let $a(t;\theta) = a(t,x;\theta) \in S_0([0,T],X_R)$. Then the mapping $a(t;\theta): C^0([0,T],X_R) \longrightarrow C^0([0,T],X_R)$ is well defined, and we have*

$$|a(t;\theta)f(t)|_R \leq A\,|f(t)|_R \quad \textit{with } A = \sup_{\substack{0\leq t\leq T\\ l\geq 0}} |a(t;l)|_R \tag{3.13}$$

for any $f(t) = f(t,x) \in C^0([0,T],X_R)$.

(2) *Let k be a positive integer and let $a(t;\theta) = a(t,x;\theta) \in S_k([0,T],X_R)$. Then the mapping $a(t;\theta): C^0([0,T],X_R) \longrightarrow C^0([0,T],X_{R_0})$ is well defined for any $0<R_0<R$, and we have*

$$|a(t;\theta)f(t)|_{R_0} \le \frac{A_0}{(1-R_0/R)^{1+k}}\,|f(t)|_R \tag{3.14}$$

with

$$A_0 = \sup_{\substack{0\le t\le T\\ l\ge 0}} \frac{k!\,|a(t;l)|_{R_0}}{(1+l)(2+l)\cdots(k+l)}$$

for any $f(t)=f(t,x)\in C^0([0,T],X_R)$.

Proof. (1) is verified by Lemma 5 and

$$|a(t;\theta)f(t)|_R \le \sum_{l\ge 0} |a(t;l)|_R\,|f_l(t)|\,R^l \le A\sum_{l\ge 0}|f_l(t)|\,R^l = A|f(t)|_R.$$

By Cauchy's inequality we have $|f_l(t)| \le |f(t)|_R/R^l$ for any $l=0,1,2,\dots$: then (2) is verified as follows:

$$\begin{aligned}|a(t;\theta)f(t)|_{R_0} &\le \sum_{l\ge 0}|a(t;l)|_{R_0}\,|f_l(t)|\,{R_0}^l \le \sum_{l\ge 0}\frac{A_0(1+l)(2+l)\cdots(k+l)}{k!}\,\frac{|f(t)|_R}{R^l}\,{R_0}^l\\ &= A_0\,|f(t)|_R\,\frac{1}{(1-R_0/R)^{1+k}}\,. \qquad \square\end{aligned}$$

4. A uniqueness result in some linear equations.

In this section we will prove the uniqueness of the solution of some linear pseudo-differential equations.

Let $T>0$, $R>0$, and let

1) $\lambda_i(\theta)\in S_1 \quad (i=1,\dots,m)$,
2) $a_j(t,x;\theta)\in S_{m-j}([0,T],X_R) \quad (j<m)$,
3) $b_{q,j}(t,x;\theta)\in S_{m-q-j}([0,T],X_R) \quad (q+j\le m, q>0)$,

and set

$$\begin{aligned}\Theta_0 &= 1,\\ \Theta_1 &= \left(t\frac{\partial}{\partial t}-\lambda_1(\theta)\right),\\ \Theta_2 &= \left(t\frac{\partial}{\partial t}-\lambda_2(\theta)\right)\left(t\frac{\partial}{\partial t}-\lambda_1(\theta)\right),\\ &\cdots\cdots\cdots\\ \Theta_m &= \left(t\frac{\partial}{\partial t}-\lambda_m(\theta)\right)\left(t\frac{\partial}{\partial t}-\lambda_{m-1}(\theta)\right)\cdots\left(t\frac{\partial}{\partial t}-\lambda_1(\theta)\right).\end{aligned}$$

Let $\mu \in \boldsymbol{R}$, and let us consider the following linear pseudo-differential equation:

$$\Theta_m u = \sum_{j<m} a_j(t,x;\theta)\Theta_j u + \sum_{\substack{q+j\le m\\ q>0}} b_{q,j}(t,x;\theta)\Big(t^{\mu}\frac{\partial}{\partial x}\Big)^{q}\Theta_j u. \tag{4.1}$$

Main assumptions are:

H_1) there is a $c>0$ such that $-\mathrm{Re}\,\lambda_i(l) \ge c\,l$ holds for any $l \in \boldsymbol{N}$ and $i=1,\dots,m$;
H_2) for $i=0,1,\dots,m-1$ we have

$$\sup_{\substack{0\le t\le T_0\\ l\ge 0}} \frac{|a_j(t;l)|_{R_0}}{(1+l)^{m-j}} = o(1) \quad (\text{as } T_0 \longrightarrow 0 \text{ and } R_0 \longrightarrow 0);$$

H_3) $\mu>0$.

THEOREM 3. *Assume the conditions* H_1), H_2) *and* H_3). *Then, if* $u(t,x)$ *is a solution of* (4.1) *belonging in the class* $C^m((0,T],X_R)$ *and satisfies*

$$\left|\Big(t\frac{\partial}{\partial t}\Big)^{j} u(t)\right|_R = O(t^a) \quad (\textit{as } t \longrightarrow 0) \quad \textit{for } j=0,1,\dots,m-1 \tag{4.2}$$

for some $a>0$, *we have* $u(t,x)\equiv 0$ *on* $(0,\varepsilon]\times D_\delta$ *for some* $\varepsilon>0$ *and* $\delta>0$.

The rest part of this section will be used to prove this theorem. Let $u(t,x) \in C^m((0,T],X_R)$ be a solution of (4.1) satisfying the condition (4.2). First, for $(q,j) \in \boldsymbol{N}\times\boldsymbol{N}$ with $q+j\le m-1$ we set

$$\phi_{q,j}(t,\rho) = \int_0^t |L_{j+1}D_{q,j}u|(\tau,(\tau/t)^c\rho)\,\frac{d\tau}{\tau}, \tag{4.3}$$

where $c>0$ is the constant in H_1),

$$L_{j+1} = \Big(t\frac{\partial}{\partial t} - \lambda_{j+1}(\theta)\Big), \quad j=0,1,\dots,m-1,$$

$$D_{q,j} = (1+\theta)^{m-1-q-j}\Big(t^{\mu}\frac{\partial}{\partial x}\Big)^{q}\Theta_j, \quad q+j\le m-1.$$

By (4.2) we see that $\phi_{q,j}(t,\rho)$ $(q+j\le m-1)$ are well defined on $(0,T]\times[0,R)$. Moreover we have:

LEMMA 6. *In the above context we have*:

c_1) $|D_{q,j}u|(t,\rho) \le \phi_{q,j}(t,\rho)$ *on* $(0,T]\times[0,R)$ (*or* $|D_{q,j}u|(t,\rho) \ll \phi_{q,j}(t,\rho)$ *as formal power series in* ρ).
c_2) $(t\frac{\partial}{\partial t} + c\,\rho\frac{\partial}{\partial\rho})\phi_{q,j}(t,\rho) = |L_{j+1}D_{q,j}u|(t,\rho)$ *on* $(0,T]\times[0,R)$.

c$_3$) $\phi_{q,j}(t,\rho) = O(t^a)$ *(as* $t \longrightarrow 0$*) uniformly in* $\rho \in [0, R_0]$ *(for any* $0 < R_0 < R$*).*

c$_4$) *When* $q \geq 1$ *we have*

$$\begin{aligned} &|L_{j+1} D_{q,j} u|(t,\rho) \\ &\quad \leq t^{\mu} \frac{\partial}{\partial \rho} |D_{q-1,j+1} u|(t,\rho) + (\mu q + C_{j+1}(q+1)) t^{\mu} \frac{\partial}{\partial \rho} |D_{q-1,j} u|(t,\rho) \end{aligned} \tag{4.4}$$

on $(0,T] \times [0,R)$*, where*

$$C_{j+1} = \sup_{l \geq 0} \frac{|\lambda_{j+1}(l)|}{(1+l)}.$$

c$_5$) *When* $q = 0$ *and* $j = 0, 1, \ldots, m-2$ *we have*

$$|L_{j+1} D_{0,j} u|(t,\rho) = \left(1 + \rho \frac{\partial}{\partial \rho}\right) |D_{0,j+1} u|(t,\rho) \quad \text{on } (0,T] \times [0,R). \tag{4.5}$$

c$_6$) *When* $q = 0$ *and* $j = m-1$*, for any* $0 < T_0 \leq T$ *and* $0 \leq R_0 < R$ *we have*

$$\begin{aligned} &|L_m D_{0,m-1} u|(t,\rho) = |\Theta_m u| \\ &\quad \leq \sum_{j<m} A_j \left(1 + \rho \frac{\partial}{\partial \rho}\right) |D_{0,j} u|(t,\rho) + \sum_{\substack{q+j \leq m \\ q>0}} B_{q,j}\, t^{\mu} \frac{\partial}{\partial \rho} |D_{q-1,j} u|(t,\rho) \end{aligned} \tag{4.6}$$

on $(0,T_0] \times [0,R_0]$*, where*

$$A_j = \sup_{\substack{0 \leq t \leq T_0 \\ l \geq 0}} \frac{|a_j(t;l)|_{R_0}}{(1+l)^{m-j}} \quad \text{and} \quad B_{q,j} = \sup_{\substack{0 \leq t \leq T_0 \\ l \geq 0}} \frac{|b_{q,j}(t;l)|_{R_0}}{(1+l)^{m-q-j}}. \tag{4.7}$$

Proof. c$_1$) and c$_2$) are clear from Proposition 1. Let us show c$_3$). Since $u(t,x)$ is a solution of (4.1), by (4.2) and Proposition 2 we have

$$\left| \left(t \frac{\partial}{\partial t}\right)^m u(t) \right|_{R_0} = O(t^a) \quad (\text{as } t \longrightarrow 0)$$

for any $0 < R_0 < R$ and so $|L_{j+1} D_{q,j} u|(t,\rho) = O(t^a)$ (as $t \longrightarrow 0$) uniformly on $[0, R_0]$; therefore c$_3$) is verified by

$$\phi_{q,j}(t,\rho) = \int_0^t O(\tau^a) \frac{d\tau}{\tau} = O(t^a) \ (\text{as } t \longrightarrow 0) \text{ uniformly on } [0, R_0].$$

Let us prove c$_4$). When $q \geq 1$ we have

$$\begin{aligned}L_{j+1}D_{q,j}u &= \Big(t\frac{\partial}{\partial t}-\lambda_{j+1}(\theta)\Big)(1+\theta)^{m-1-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^q\Theta_j u\\ &= (1+\theta)^{m-1-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^q L_{j+1}\Theta_j u+\mu q(1+\theta)^{m-1-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^q\Theta_j u\\ &\quad+(1+\theta)^{m-1-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^q\lambda_{j+1}(\theta)\Theta_j u\\ &\quad-\lambda_{j+1}(\theta)(1+\theta)^{m-1-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^q\Theta_j u.\end{aligned}$$

Therefore, by the conditions $L_{j+1}\Theta_j=\Theta_{j+1}$, $(1+l)^{m-1-q-j}\le(1+l)^{m-q-j}$ ($l=0,1,2,\ldots$), and by Lemma 4 we see:

$$\begin{aligned}|L_{j+1}D_{q,j}u|(t,\rho) &\le t^\mu\frac{\partial}{\partial\rho}\Big|(1+\theta)^{m-1-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^{q-1}\Theta_{j+1}u\Big|(t,\rho)\\ &\quad+\mu q\,t^\mu\frac{\partial}{\partial\rho}\Big|(1+\theta)^{m-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^{q-1}\Theta_j u\Big|(t,\rho)\\ &\quad+C_{j+1}\,q\,t^\mu\frac{\partial}{\partial\rho}\Big|(1+\theta)^{m-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^{q-1}\Theta_j u\Big|(t,\rho)\\ &\quad+C_{j+1}\,t^\mu\frac{\partial}{\partial\rho}\Big|(1+\theta)^{m-q-j}\Big(t^\mu\frac{\partial}{\partial x}\Big)^{q-1}\Theta_j u\Big|(t,\rho)\\ &= t^\mu\frac{\partial}{\partial\rho}|D_{q-1,j+1}u|(t,\rho)+(\mu q+C_{j+1}(q+1))t^\mu\frac{\partial}{\partial\rho}|D_{q-1,j}u|(t,\rho)\end{aligned}$$

which proves the condition c$_4$). When $q=0$ and $j=0,1,\ldots,m-2$, by iv) of Lemma 4 we have

$$\begin{aligned}|L_{j+1}D_{0,j}u|(t,\rho) &= \Big|\Big(t\frac{\partial}{\partial t}-\lambda_{j+1}(\theta)\Big)(1+\theta)^{m-1-j}\Theta_j u\Big|(t,\rho)\\ &= \Big|(1+\theta)(1+\theta)^{m-1-(j+1)}\Theta_{j+1}u\Big|(t,\rho)=\Big(1+\rho\frac{\partial}{\partial\rho}\Big)|D_{0,j+1}u|(t,\rho)\end{aligned}$$

which proves the condition c$_5$). Lastly, let us prove c$_6$). Let $q=0$ and $j=m-1$. Since $u(t,x)$ is a solution of (4.1), we have

$$\begin{aligned}|L_mD_{0,m-1}u|(t,\rho) &= \Big|\Big(t\frac{\partial}{\partial t}-\lambda_m(\theta)\Big)\Theta_{m-1}u\Big|(t,\rho)=|\Theta_m u|(t,\rho)\\ &\le\sum_{j<m}|a_j(t;\theta)\Theta_j u|(t,\rho)+\sum_{\substack{q+j\le m\\ q>0}}\Big|b_{q,j}(t;\theta)\Big(t^\mu\frac{\partial}{\partial x}\Big)^q\Theta_j u\Big|(t,\rho).\end{aligned}\tag{4.8}$$

By the definition of A_j, Proposition 2 and iv) of Lemma 4 we have

$$\begin{aligned}|a_j(t;\theta)\Theta_j u|(t,\rho) &= \left|\frac{a_j(t;\theta)}{(1+\theta)^{m-j}}(1+\theta)^{m-j}\Theta_j u\right|(t,\rho) \\ &\le A_j|(1+\theta)^{m-j}\Theta_j u|(t,\rho) = A_j\left(1+\rho\frac{\partial}{\partial\rho}\right)|D_{0,j}u|(t,\rho)\end{aligned}$$

on $(0,T_0]\times[0,R_0]$. Similarly, by the definition of $B_{q,j}$ we have

$$\begin{aligned}\left|b_{q,j}(t;\theta)\left(t^\mu\frac{\partial}{\partial x}\right)^q\Theta_j u\right|(t,\rho) &= \left|\frac{b_{q,j}(t;\theta)}{(1+\theta)^{m-q-j}}(1+\theta)^{m-q-j}\left(t^\mu\frac{\partial}{\partial x}\right)^q\Theta_j u\right|(t,\rho) \\ &\le B_{q,j}\left|(1+\theta)^{m-q-j}\left(t^\mu\frac{\partial}{\partial x}\right)^q\Theta_j u\right|(t,\rho) \\ &\le B_{q,j}\,t^\mu\frac{\partial}{\partial\rho}\left|(1+\theta)^{m-q-j}\left(t^\mu\frac{\partial}{\partial x}\right)^{q-1}\Theta_j u\right|(t,\rho) \\ &= B_{q,j}\,t^\mu\frac{\partial}{\partial\rho}|D_{q-1,j}u|(t,\rho)\end{aligned}$$

on $(0,T_0]\times[0,R_0]$. Thus, applying these two estimates to (4.8) we obtain the result (4.6). This proves c_6). □

Next, let $\beta_0>0,\beta_1>0,\dots,\beta_{m-1}>0$ and set

$$\Phi(t,\rho)=\sum_{j<m}\beta_j\phi_{0,j}(t,\rho)+\sum_{\substack{q+j\le m-1\\ q>0}}\phi_{q,j}(t,\rho) \tag{4.9}$$

on $(0,T]\times[0,R)$.

Lemma 7. *For any $0<T_0\le T$ and $0\le R_0<R$ we have the following inequality*

$$\begin{aligned}\left(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial\rho}\right)\Phi(t,\rho) \le& M\,t^\mu\frac{\partial}{\partial\rho}\Phi(t,\rho)+\sum_{j\le m-2}\beta_j\left(1+\rho\frac{\partial}{\partial\rho}\right)\phi_{0,j+1}(t,\rho) \\ &+\beta_{m-1}\sum_{j<m}A_j\left(1+\rho\frac{\partial}{\partial\rho}\right)\phi_{0,j}(t,\rho)\end{aligned} \tag{4.10}$$

on $(0,T_0]\times[0,R_0]$, where $c>0$ is the constant in H_1), *$M>0$ is a suitable constant depending on $\beta_0,\beta_1,\dots,\beta_{m-1}$, T_0 and R_0, and A_j is the constant in* (4.7).

Proof. By c_1), c_2) and Lemma 6 we have the following results: when $q\ge1$ by c_4) we have

$$\left(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial\rho}\right)\phi_{q,j}(t,\rho)\le t^\mu\frac{\partial}{\partial\rho}\phi_{q-1,j+1}(t,\rho)+(\mu q+C_{j+1}(q+1))t^\mu\frac{\partial}{\partial\rho}\phi_{q-1,j}(t,\rho)$$

on $(0,T] \times [0,R)$; when $q=0$ and $j=0,1,\dots,m-2$ by c_5) we have

$$\Big(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,j}(t,\rho) \le \Big(1+\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,j+1}(t,\rho)$$

on $(0,T] \times [0,R)$; when $q=0$ and $j=m-1$, for any $0<T_0\le T$ and $0 \le R_0 < R$ we have

$$\Big(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,m-1}(t,\rho) \le \sum_{j<m} A_j\Big(1+\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,j}(t,\rho) + \sum_{\substack{q+j\le m\\ q>0}} B_{q,j}\,t^{\mu}\frac{\partial}{\partial \rho}\phi_{q-1,j}(t,\rho)$$

on $(0,T_0] \times [0,R_0]$. Hence, by applying these inequalities to

$$\Big(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial \rho}\Big)\Phi = \sum_{j<m}\beta_j\Big(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,j} + \sum_{\substack{q+j\le m-1\\ q>0}}\Big(t\frac{\partial}{\partial t}+c\,\rho\frac{\partial}{\partial \rho}\Big)\phi_{q,j}$$

we can obtain the result (4.10). □

Corollary to Lemma 7. *Let $u(t,x)\in C^m((0,T],X_R)$ be a solution of* (4.1) *satisfying the condition* (4.2) *for some $a>0$. Then we can find $\beta_0>0,\beta_1>0,\dots,\beta_{m-1}>0$, $0<b<a$, $M>0$, T_0 and R_0 such that $\Phi(t,\rho)$ defined by* (4.9) *satisfies*

$$t\frac{\partial}{\partial t}\Phi(t,\rho) \le b\,\Phi(t,\rho) + M\,t^{\mu}\frac{\partial}{\partial \rho}\Phi(t,\rho) \quad \text{on } (0,T_0]\times[0,R_0]. \tag{4.11}$$

Proof. We choose $B>0$ so that $B \le \min\{a/3, c/2\}$ and then set $\beta_j = 1/B^j$ for $j=0,1,\dots,m-1$: then we have

$$\begin{aligned}\sum_{j\le m-2}\beta_j\Big(1+\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,j+1}(t,\rho) &= B\sum_{j\le m-2}\beta_{j+1}\Big(1+\rho\frac{\partial}{\partial \rho}\Big)\phi_{0,j+1}(t,\rho)\\ &\le B\Big(1+\rho\frac{\partial}{\partial \rho}\Big)\Phi(t,\rho) \le \frac{a}{3}\,\Phi(t,\rho)+\frac{c}{2}\,\rho\frac{\partial}{\partial \rho}\Phi(t,\rho).\end{aligned} \tag{4.12}$$

By the assumption H_2) we see that the constant A_j satisfies $A_j \longrightarrow 0$ (as $T_0 \longrightarrow 0$ and $R_0 \longrightarrow 0$): therefore, by taking $T_0>0$ and $R_0>0$ sufficiently small we may assume

$$\max_{j<m}\frac{\beta_{m-1}A_j}{\beta_j} \le \min\Big\{\frac{a}{3},\frac{c}{2}\Big\}.$$

Then we have

$$\beta_{m-1}\sum_{j<m} A_j\Big(1+\rho\frac{\partial}{\partial\rho}\Big)\phi_{0,j}(t,\rho)=\beta_{m-1}\sum_{j<m}\frac{A_j}{\beta_j}\Big(1+\rho\frac{\partial}{\partial\rho}\Big)\beta_j\phi_{0,j}(t,\rho)$$
$$\leq\Big(\max_{j<m}\frac{\beta_{m-1}A_j}{\beta_j}\Big)\Big(1+\rho\frac{\partial}{\partial\rho}\Big)\Phi(t,\rho)\leq\frac{a}{3}\Phi(t,\rho)+\frac{c}{2}\rho\frac{\partial}{\partial\rho}\Phi(t,\rho). \tag{4.13}$$

Hence, by applying (4.12), (4.13) to (4.10), and by setting $b=2a/3$ we have the inequality (4.11) on $(0,T_0]\times[0,R_0]$. □

COMPLETION OF THE PROOF OF THEOREM 3. Since

$$|u|(t,\rho)\leq|(1+\theta)^{m-1}u|(t,\rho)=|D_{0,0}u|(t,\rho)\leq\phi_{0,0}(t,\rho)$$

holds, to show Theorem 3 it is sufficient to prove that $\Phi(t,\rho)\equiv 0$ holds on $\{(t,\rho)\,;\,0<t\leq\varepsilon$ and $0\leq\rho\leq\delta\}$ for some $\varepsilon>0$ and $\delta>0$.

Let $b>0$ and $M>0$ be as in Corollary to Lemma 7. Choose $T_1>0$ so that $0<T_1\leq T_0$ and $M{T_1}^{\mu}/\mu\leq R_0$ hold. Define the function $\rho(t)$ by

$$\rho(t)=M\int_t^{T_1}\frac{\tau^{\mu}}{\tau}d\tau=M({T_1}^{\mu}-t^{\mu})/\mu,\quad 0\leq t\leq T_1.$$

Then, $\rho(t)$ is a solution of $t(d\rho/dt)=-Mt^{\mu}$, $0<\rho(0)\leq R_0$, $\rho(T_1)=0$ and $\rho(t)$ is decreasing in t. Set

$$\psi(t)=t^{-b}\Phi(t,\rho(t)),\quad 0<t\leq T_1.$$

Since $\Phi(t,\rho)=O(t^a)$ (as $t\longrightarrow 0$) uniformly on $[0,R_0]$ and since $a>b>0$ holds, we have $\psi(t)=O(t^{a-b})=o(1)$ (as $t\longrightarrow 0$). Moreover, by Corollary to Lemma 7 we have

$$\begin{aligned}
t\frac{d}{dt}\psi(t)&=-b\,t^{-b}\Phi(t,\rho(t))+t^{-b}\,t\frac{\partial\Phi}{\partial t}(t,\rho(t))+t^{-b}\frac{\partial\Phi}{\partial\rho}(t,\rho(t))\,t\frac{d\rho(t)}{dt}\\
&\leq-b\,t^{-b}\Phi(t,\rho(t))+t^{-b}\Big(b\,\Phi(t,\rho(t))+M\,t^{\mu}\frac{\partial}{\partial\rho}\Phi(t,\rho(t))\Big)\\
&\quad+t^{-b}\frac{\partial\Phi}{\partial\rho}(t,\rho(t))\,(-Mt^{\mu})\\
&=0
\end{aligned}$$

and therefore $(d/dt)\psi(t)\leq 0$ for $0<t\leq T_1$. By integrating this from ε to $t(>0)$ we get $\psi(t)\leq\psi(\varepsilon)$ for $0<\varepsilon\leq t\leq T_1$ and by letting $\varepsilon\longrightarrow 0$ we have $\psi(t)\leq 0$ for $0<t\leq T_1$. On the other hand, $\psi(t)\geq 0$ is clear from the definition of $\psi(t)$. Hence, we obtain $\psi(t)=0$ for $0<t\leq T_1$: this implies

$$\Phi(t,\rho)=0\quad\text{on }\{(t,\rho)\,;\,0<t\leq T_1\text{ and }\rho=\rho(t)\}. \tag{4.14}$$

Since $\Phi(t,\rho)$ is increasing in ρ, (4.14) implies

$$\Phi(t,\rho) \equiv 0 \quad \text{on } \{(t,\rho)\,;\, 0 < t \le T_1 \text{ and } 0 \le \rho \le \rho(t)\}.$$

This completes the proof of Theorem 3. □

Let us give a variation. Set

$\mathrm{H}_1)^*$ there are $b \ge 0$ and $c > 0$ such that $b - \operatorname{Re}\lambda_i(l) \ge c\,l$ holds for any $l \in \boldsymbol{N}$ and $i = 1, \dots, m$.

We have

THEOREM 3*. *Assume the conditions* $\mathrm{H}_1)^*$, $\mathrm{H}_2)$ *and* $\mathrm{H}_3)$. *Then, if* $u(t,x)$ *is a solution of* (4.1) *belonging in the class* $C^m((0,T], X_R)$ *and satisfies*

$$\left|\left(t\frac{\partial}{\partial t}\right)^j u(t)\right|_R = O(t^a) \quad (\textit{as } t \longrightarrow 0) \textit{ for } j = 0, 1, \dots, m-1 \tag{4.15}$$

for some $a > b$, *we have* $u(t,x) \equiv 0$ *on* $(0,\varepsilon] \times D_\delta$ *for some* $\varepsilon > 0$ *and* $\delta > 0$.

PROOF. By setting $u^* = t^{-b}u$, $a^* = a - b$ and $\lambda_i^*(\theta) = -b + \lambda_i(\theta)$, we can reduce our problem to the case in Theorem 3*. □

5. Proof of Theorem 2*.

In this section, we will prove Theorem 2* by using Theorem 3*.

Let $\lambda_1(l), \dots, \lambda_m(l)$ ($l \in \boldsymbol{N}$) be the ones in (1.4), and assume that there are $b \ge 0$ and $c > 0$ which satisfy the condition (2.3). Let $u_1(t,x)$ and $u_2(t,x)$ be solutions of (E) belonging in the class $\widetilde{\mathscr{S}_+}$ and assume that $u_1 - u_2 \in \widetilde{\mathscr{S}_a}$ holds for some $a > b$. By the definition of $\widetilde{\mathscr{S}_+}$ we have $u_i(t,x) \in \widetilde{\mathscr{S}_s}$ ($i = 1, 2$) for some $s > 0$.

Set

$$w(t,x) = u_2(t,x) - u_1(t,x) \in \widetilde{\mathscr{S}_a}\,. \tag{5.1}$$

Our aim is to prove that $w(t,x) \equiv 0$ holds on $(0,\varepsilon] \times D_\delta$ for some $\varepsilon > 0$ and $\delta > 0$. Let us show this now.

It is easy to see that $w(t,x) \in C^m((0,T], X_R)$ holds for some $T > 0$ and $R > 0$, and

$$\left|\left(t\frac{\partial}{\partial t}\right)^j w(t)\right|_R = O(t^a) \text{ (as } t \longrightarrow 0) \text{ for } j = 0, 1, \dots, m-1. \tag{5.2}$$

Moreover, since $u_1(t,x)$ and $u_2(t,x)$ are solutions of (1.1) we see that $w(t,x)$ satisfies the following equation

$$C\Big(x, t\frac{\partial}{\partial t}, x\frac{\partial}{\partial x}\Big) w = G\big(t,x,D(u_1+w)\big) - G\big(t,x,Du_1\big) \tag{5.3}$$

where

$$G(t,x,z)=\sum_{p+|\nu|\geq 2,|\nu|\geq 1} g_{p,\nu}(x)\,t^p \prod_{(j,\alpha)\in I_m} [z_{j,\alpha}]^{\nu_{j,\alpha}}, \quad \text{and}$$

$$Du=\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in I_m}$$

with $z=\{z_{j,\alpha}\}_{(j,\alpha)\in I_m}$. Let us write $[\lambda]_0=1$, and $[\lambda]_\alpha=\lambda(\lambda-1)\cdots(\lambda-\alpha+1)$ for $\alpha\in \boldsymbol{N}^*$. Then the eqution (5.3) is written in the form

$$\begin{aligned} L\left(t\frac{\partial}{\partial t},x\frac{\partial}{\partial x}\right)w &= x\sum_{(j,\alpha)\in I_m} S(c_{j,\alpha})(x)\left(t\frac{\partial}{\partial t}\right)^j\left[x\frac{\partial}{\partial x}\right]_\alpha w \\ &\quad + \sum_{(j,\alpha)\in I_m} h_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha w \end{aligned} \tag{5.4}$$

where for $(j,\alpha)\in I_m$

$$\begin{aligned} S(c_{j,\alpha})(x) &= \frac{c_{j,\alpha}(x)-c_{j,\alpha}(0)}{x}, \\ h_{j,\alpha}(t,x) &= \int_0^1 \frac{\partial G}{\partial z_{j,\alpha}}\big(t,x,Du_1(t,x)+\sigma Dw(t,x)\big)d\sigma. \end{aligned}$$

We note the following lemma:

Lemma 8. *Let $\lambda_i(\theta)\in S_1$ be the pseudo-differential operator corresponding to $\lambda_i(l)$ $(l\in \boldsymbol{N})$ for $i=1,\dots,m$, and define the operators Θ_j $(j=0,1,\dots,m)$ as in section 4. We have the following results.*

(1) *We have $L(t\partial/\partial t, x\partial/\partial x)=\Theta_m$ as an operator from $C^m((0,T],X_R)$ into $C^0((0,T],X_{R_1})$ for any $0<R_1<R$.*

(2) *For $j=0,1,\dots,m-1$ the operator $(t\partial/\partial t)^j$ is expressed in the form*

$$\left(t\frac{\partial}{\partial t}\right)^j=\sum_{p=0}^{j}\Lambda_{j,p}(\theta)\Theta_{j-p}$$

for some $\Lambda_{j,p}(\theta)\in S_p$; of course, we have $\Lambda_{j,0}(\theta)=1$.

(3) *$[\theta]_\alpha\in S_\alpha$, that is, $[\theta]_\alpha$ is a pseudo-differential operator of order α.*

(4) *$h_{j,\alpha}(t,x)\in \widetilde{\mathscr{S}}_d$ holds for any $0<d\leq \min\{1,s,a\}$, and therefore, if we take μ satisfying $0<\mu<d/m$ we have $h_{j,\alpha}(t,x)=t^{\mu\alpha}k_{j,\alpha}(t,x)$ for some $k_{j,\alpha}(t,x)\in \widetilde{\mathscr{S}}_+$.*

(5) *We have the relation: $(t^\mu\partial/\partial x)^q\Lambda_{j,p}(\theta)=\Lambda_{j,p}(q+\theta)(t^\mu\partial/\partial x)^q$.*

By using this lemma, we can rewrite our equation (5.4) into

$$\Theta_m w = \sum_{j<m} \sum_{\substack{j\le i<m\\ i+\alpha\le m}} x\, S(c_{i,\alpha})(x)\, \Lambda_{i,i-j}(\theta)\, [\theta]_\alpha\, \Theta_j w$$

$$+ \sum_{\substack{q+j\le m\\ j<m}} \sum_{\substack{j\le i\le m-q\\ i<m}} k_{i,q}(t,x)\, \Lambda_{i,i-j}(q+\theta) \left(t^\mu \frac{\partial}{\partial x}\right)^q \Theta_j w,$$

and hence by setting

$$a_j(t,x;\theta) = \sum_{\substack{j\le i<m\\ i+\alpha\le m}} x\, S(c_{i,\alpha})(x)\, \Lambda_{i,i-j}(\theta)\, [\theta]_\alpha + \sum_{j\le i<m} k_{i,0}(t,x)\, \Lambda_{i,i-j}(\theta),$$

$$b_{q,j}(t,x;\theta) = \sum_{\substack{j\le i\le m-q\\ i<m}} k_{i,q}(t,x)\, \Lambda_{i,i-j}(q+\theta)$$

we obtain the following equation

$$\Theta_m w = \sum_{j<m} a_j(t,x;\theta)\Theta_j w + \sum_{\substack{q+j\le m\\ q>0}} b_{q,j}(t,x;\theta)\left(t^\mu \frac{\partial}{\partial x}\right)^q \Theta_j w \tag{5.5}$$

which is just discussed in section 4.

By the definition we see:

$$a_j(t,x;\theta) \in S_{m-j}([0,T], X_R) \text{ and } b_{q,j}(t,x;\theta) \in S_{m-q-j}([0,T], X_R)$$

for some $T > 0$ and $R > 0$. Moreover, since $|xS(c_{i,\alpha})(x)| = o(1)$ (as $|x| \longrightarrow 0$) and $|k_{i,q}(t,x)| = o(1)$ (as $|t| \longrightarrow 0$) are known, we easily see that

$$\sup_{\substack{0\le t\le T_0\\ l\ge 0}} \frac{|a_j(t;l)|_{R_0}}{(1+l)^{m-j}} = o(1) \quad (\text{as } T_0 \longrightarrow 0 \text{ and } R_0 \longrightarrow 0).$$

Summing up, we have seen that we can apply Theorem 3* to the equation (5.5). Thus, we obtain the conclusion that $w(t,x) \equiv 0$ on $(0,\varepsilon] \times D_\delta$ for some $\varepsilon > 0$ and $\delta > 0$. This completes the proof of Theorem 2*. □

6. Application.

Lastly, let us give an application of Theorem 2 to the problem of removable singularities of solutions of (E). The following theorem asserts the removability of some kind of singularities on $\{t = 0\}$.

THEOREM 4. *Assume* A$_1$), A$_2$), A$_3$), *and the following* i) *and* ii):

i) $\mathrm{Re}\,\lambda_i(l) \le 0$ *for any* $l \in \boldsymbol{N}$ *and* $i = 1, \dots, m$;

ii) $\operatorname{Re} c_i < 0$ *for* $i = 1, \ldots, m$.

Then, if $u(t,x)$ *is a solution of* (E) *belonging in the class* $\widetilde{\mathscr{S}_+}$, $u(t,x)$ *is holomorphic in a full neighborhood of* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$.

Proof. Since the condition i) implies that $L(k,l) \neq 0$ holds for any $(k,l) \in \boldsymbol{N}^* \times \boldsymbol{N}$, by (1) of Theorem 1 we see that the equation (E) has a unique holomorphic solution $u_0(t,x)$ satisfying $u_0(0,x) \equiv 0$.

Let $u(t,x)$ be a solution of (E) in the class $\widetilde{\mathscr{S}_s}$ for some $s > 0$. Then we have $u - u_0 \in \widetilde{\mathscr{S}_a}$ for any $0 < a \leq \min\{1,s\}$, and so by Theorem 2 we have $u = u_0$. This concludes that $u(t,x)$ is holomorphic in a full neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$. □

Conversely, if the condition i) in Theorem 4 is not satisfied we have

Theorem 5. *Assume* A_1), A_2), A_3), *and the following* i) *and* ii):

i) *there is a* (p,l) *such that* $\operatorname{Re}\lambda_p(l) > 0$ *and* $\lambda_p(l) \notin \boldsymbol{N}^*$ *hold*;
ii) $\operatorname{Re} c_i < 0$ *for* $i = 1, \ldots, m$.

Then, the equation (E) *has a solution* $u(t,x)$ *belonging in the class* $\widetilde{\mathscr{S}_+}$ *which has really singularities on* $\{t = 0\}$.

Proof. Set $\beta = \lambda_p(l)$. By the same argument as in [**9**] we can construct an $\widetilde{\mathscr{S}_+}$-solution $u(t,x)$ of the form

$$\begin{aligned} u(t,x) &= w\big(t, t(\log t), \ldots, t(\log t)^{\mu}, t^{\beta}, t^{\beta}(\log t), \ldots, t^{\beta}(\log t)^{\kappa}, x\big) \\ &= \cdots + A t^{\beta} x^{l} + \cdots \end{aligned}$$

where $w(t_0, \ldots, t_\mu, \zeta_0, \ldots, \zeta_\kappa, x)$ is a holomorphic function in a neighborhood of the origin of $\boldsymbol{C}_t^{1+\mu} \times \boldsymbol{C}_\zeta^{1+\mu} \times \boldsymbol{C}_x$ satisfying $w(0, \ldots, 0, x) \equiv 0$, $A \in \boldsymbol{C}$ is an arbitrary constant, $\mu = \#\{(i,l)\,;\, \lambda_i(l) \in \boldsymbol{N}^* \setminus S\}$ with $S = \{p + q\beta\,;\, (p,q) \in \boldsymbol{N} \times \boldsymbol{N}^*\}$, and $1 + \kappa = \#\{(i,l)\,;\, \lambda_i(l) \in S\}$. If we take $A \neq 0$, by looking at the term $At^{\beta}x^{l}$ we can conclude that this solution has really singularities on $\{t = 0\}$. The argument of the construction is almost the same as in [**9**], and so we may omit the details. □

References

[1] H. Chen and Z. Luo, On holomorphic solution of non-linear totally characteristic equations with several space variables, Acta Math. Sci. Ser. B, Engl. Ed., **22** (2002), 393–403.

[2] H. Chen and H. Tahara, On totally characteristic type non-linear partial differential equations in the complex domain, Publ. Res. Inst. Math. Sci., **35** (1999), 621–636.

[3] R. Gérard and H. Tahara, Solutions holomorphes et singulières d'équations aux dérivées partielles singulières non linéaires, Publ. Res. Inst. Math. Sci., **29** (1993), 121–151.

[4] R. Gérard and H. Tahara, Singular nonlinear partial differential equations, Vieweg, 1996.

[5] R. Gérard and H. Tahara, Holomorphic and singular solutions of non-linear singular partial differential equations, II, In: Structure of differential equations, Katata/Kyoto, 1995, (eds. M. Morimoto and T. Kawai), 1996, World Scientific, 135–150.

[6] A. Shirai, Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type, Funkcial. Ekvac., **45** (2002), 187–208.

[7] H. Tahara, Uniqueness of the solution of non-linear singular partial differential equations, J. Math. Soc. Japan, **48** (1996), 729–744.

[8] H. Tahara, On the uniqueness theorem for nonlinear singular partial differential equations, J. Math. Sci. Univ. Tokyo, **5** (1998), 477–506.

[9] H. Tahara, Solvability of partial differential equations of nonlinear totally characteristic type with resonances, J. Math. Soc. Japan, **55** (2003), 1095–1113.

[10] H. Tahara and H. Yamazawa, Structure of solutions of nonlinear partial differential equations of Gérard-Tahara type, Publ. Res. Inst. Math. Sci., **41** (2005), 339–373.

Hidetoshi Tahara

Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku
Tokyo 102-8554
Japan
E-mail: h-tahara@sophia.ac.jp