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Stability of foliations with complex leaves on locally conformal Kähler manifolds

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Abstract. In this paper, we study stability for harmonic foliations on locally conformal Kähler manifolds with complex leaves.

1. Introduction.

The purpose of this paper is to prove a stability theorem for harmonic foliations on compact locally conformal Kähler manifolds. Let (M, J, g_M) be a Hermitian manifold and Ω the fundamental 2-form associated with g_M. Then (M, J, g_M) is a *locally conformal Kähler* manifold if there exists a closed 1-form ω, called the *Lee form*, satisfying $d\Omega = \omega \wedge \Omega$.

Main Theorem. *Let (M, J, g_M) be an n-dimensional compact locally conformal Kähler manifold. If $\mathscr{F}$ is a harmonic foliation on M with bundle-like metric g_M foliated by complex submanifolds, then $\mathscr{F}$ is stable.*

This is an analogue of the theorem "a holomorphic map between two Kähler manifolds is stable as a harmonic map" (see also Corollary 1.2 below), where harmonicity for a foliation $\mathscr{F}$ on a Riemannian manifold (N, g_N) is defined by Kamber and Tondeur in [**6**] as the harmonicity of the canonical projection π from TN onto the normal bundle Q for the foliation $\mathscr{F}$. The key of the proof of Main Theorem is the compatibility of the complex structure with the connection on the normal bundle of the foliation (see Lemma 3.1).

A locally conformal Kähler manifold (M, J, g_M) is called a *Vaisman* manifold if the associated Lee form is non-exact and parallel with respect to the Levi-Civita connection. Although many interesting Vaisman manifolds such as Hopf manifolds are known, some locally conformal Kähler manifolds (e.g. some Inoue surfaces) admits no Vaisman structures (cf. Ornea [**9**], Dragomir and Ornea [**2**], Belgun [**1**]).

For a locally conformal Kähler manifold (M, J, g_M) with the associated Ω and ω, we consider the *Lee vector field* $B = \omega^\sharp$. Here $\sharp$ denotes the raising of indices with respect to g_M. Then on every Vaisman manifold (M, J, g_M), it is known that B and JB generate a complex analytic foliation, called the *canonical foliation*, and g_M is bundle-like (see, e.g., Dragomir and Ornea [**2**, Theorem 5.1]). Now the following is immediate from Main Theorem:

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COROLLARY 1.1. *The canonical foliations on compact Vaisman manifolds are stable.*

The case when the Lee form ω is identically zero, (M, J, g_M) is nothing but a Kähler manifold. Any complex submanifold of a Kähler manifold is also Kähler, and especially, is minimal. Hence, in this case, Main Theorem is written in the following form:

COROLLARY 1.2. *The foliations on compact Kähler manifolds with a bundle-like metric foliated by complex submanifolds are stable.*

This paper is organized as follows. In Section 2, we review the theory of harmonic foliations by Kamber and Tondeur. Then Section 3 is devoted to the proof of Main Theorem above for harmonic foliations. Finally in 3.10, we shall see examples of stable harmonic foliations on Hopf manifolds and Inoue surfaces.

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2. The Jacobi operator and a stability of harmonic foliations.

Let (N, g_N) be an n-dimensional compact Riemannian manifold and let $\mathscr{F}$ be a foliation given by an integrable subbundle $L \subset TN$. We define a torsion free connection ∇ on normal bundle $Q = TN/L$ by

$$\begin{cases} \nabla_X S = \pi[X, Y_S], & \text{for } X \in \Gamma(L),\ S \in \Gamma(Q) \text{ and } Y_S = \sigma(S) \in \Gamma(\sigma(Q)), \\ \nabla_X S = \pi(\nabla^N_X Y_S), & \text{for } X \in \Gamma(\sigma(Q)),\ S \in \Gamma(Q) \text{ and } Y_S = \sigma(S) \in \Gamma(\sigma(Q)), \end{cases} \tag{2.1}$$

where $\sigma : Q \to TN$ is a splitting such that $\sigma(Q)$ coincides with the orthogonal complement $L^{\perp}$ of L in TN with respect to g_N. If the normal bundle Q is equipped with a holonomy invariant fiber metric g_Q, i.e. $X g_Q(S, T) = g_Q(\nabla_X S, T) + g_Q(S, \nabla_X T)$ for all $X \in \Gamma(L)$, the foliation $\mathscr{F}$ is called a *Riemannian foliation* or an *R-foliation*. There is a unique metric g_Q for an R-foliation with a torsion free connection ∇ on the normal bundle Q. A Riemannian metric g_N on N is called a *bundle-like* metric with respect to the foliation $\mathscr{F}$ if the foliation becomes an R-foliation in terms of the fiber metric g_Q induced on Q.

For a foliation $\mathscr{F}$ on a Riemannian manifold (N, g_N), the curvature R^{∇} of the connection ∇ is an $\mathrm{End}(Q)$-valued 2-form on N. Since $i(X)R^{\nabla} = 0$ for $X \in \Gamma(L)$, it follows that the curvature operator $R^{\nabla}(S, T) : Q \to Q$ for $S, T \in \Gamma(Q)$, is well-defined. Define $P^{\nabla}(U, V) : Q \to Q$ by $P^{\nabla}(U, V)S = -R^{\nabla}(U, S)V$ for all $S \in \Gamma(Q)$. The Ricci curvature S^{∇} for $\mathscr{F}$ is then $S^{\nabla}(U, V) = \mathrm{trace} P^{\nabla}(U, V)$ which is a symmetric bilinear form. We define the *Ricci operator* $\rho_{\nabla} : Q \to Q$ as the corresponding self-adjoint operator given by $g_Q(\rho_{\nabla} U, V) = S^{\nabla}(U, V)$, where g_Q denotes the holonomy invariant metric on Q. In terms of an orthonormal basis $e_{p+1}, \ldots, e_n$ of Q_x at some $x \in N$, we have $(\rho_{\nabla} U)_x = \sum_{\alpha=p+1}^{n} R^{\nabla}(U, e_\alpha) e_\alpha$.

Denoting by $\pi \in \Omega^1(N, Q)$ the canonical projection from TN onto Q, we have

$d_\nabla \pi \in \Omega^2(N,Q)$, $d_\nabla^* \pi \in C^\infty(N,Q)$, the Laplacian Δ on $\Omega^1(N,Q)$ and so forth. Then we have the following fact (Kamber and Tondeur [**7**, 3.3]).

FACT. *Let $\mathscr{F}$ be a foliation on a Riemannian manifold (N, g_N). Then the following are equivalent*:
(i) π *is harmonic,*
(ii) *all leaves for the foliation are minimal submanifolds of N.*
If $\mathscr{F}$ is an R-foliation, g_N a bundle-like, and M compact and oriented, then these conditions are equivalent to
(iii) $\Delta\pi = 0$.

A foliation on a Riemannian manifold is said to be *harmonic* if it satisfies (i) or (ii) above.

We next study first and second variations of R-foliation $\mathscr{F}$ on a compact Riemannian manifold (N, g_N) with bundle-like metric g_N. We define the *energy* of the foliation $\mathscr{F}$ by

$$E(\mathscr{F}) = \frac{1}{2}\|\pi\|^2,$$

where π is the canonical projection from TN onto Q and is considered as a Q-valued 1-form on N. Let $\{U_\alpha, f^\alpha, \gamma^{\alpha\beta}\}$ be the Haefliger cocycle representing $\mathscr{F}$. Namely, $\{U_\alpha\}$ is an open cover of N with $f^\alpha : U_\alpha \to \boldsymbol{R}^q$ such that $\gamma^{\alpha\beta}$ are local isometries on $U_\alpha \cap U_\beta (\neq \emptyset)$ satisfying $f^\alpha = \gamma^{\alpha\beta} f^\beta$. Here q denotes the codimension of $\mathscr{F}$. For $\nu \in \Gamma(Q)$, we put

$$\Phi_t^\alpha(x) = \exp_{f^\alpha(x)}(t\nu^\alpha(x)), \qquad x \in U_\alpha,\ t \in (-\varepsilon, \varepsilon),$$

where $\nu^\alpha = \nu|_{U_\alpha}$. We then have a variation Φ_t^α of $f^\alpha = \Phi_0^\alpha$, where ε is sufficiently small. Since $\Phi_t^\alpha(x) = \gamma^{\alpha\beta}\Phi_t^\beta(x)$ on $U_\alpha \cap U_\beta$, the local variations $\{\Phi_t^\alpha\}$ define a variation $\mathscr{F}_t$ of the foliation $\mathscr{F}$. Moreover we have

$$\nabla_{\frac{\partial}{\partial t}|_{t=0}}(\Phi_t^\alpha)_* = \nabla\nu^\alpha \in \Omega^1(U_\alpha, Q). \tag{2.2}$$

To obtain the second variation, we need a 2-parameter variation $\mathscr{F}_{s,t}$ of $\mathscr{F}_{0,0} = \mathscr{F}$ defined locally as $\Phi_{s,t}^\alpha$, where

$$\Phi_{s,t}^\alpha(x) = \exp_{f^\alpha(x)}(s\mu^\alpha(x) + t\nu^\alpha(x)), \qquad x \in U_\alpha,\ s,t \in (-\varepsilon, \varepsilon)$$

for $\nu, \mu \in \Gamma(Q)$. Then by (2.2)

$$\begin{cases} \nabla_{\frac{\partial}{\partial s}|_{s=0,t=0}}(\Phi_{s,t}^\alpha)_* = \nabla\mu^\alpha, \\ \nabla_{\frac{\partial}{\partial t}|_{s=0,t=0}}(\Phi_{s,t}^\alpha)_* = \nabla\nu^\alpha. \end{cases}$$

The second variation formula is now given by

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}\bigg|_{s=0,t=0} E(\mathscr{F}_{s,t}) &= \frac{\partial^2}{\partial s\partial t}\bigg|_{s=0,t=0}\frac{1}{2}\langle \pi_{s,t},\pi_{s,t}\rangle = \frac{\partial}{\partial s}\bigg|_{s=0,t=0}\langle \nabla\nu,\pi_{s,t}\rangle \\ &= \langle \nabla_{\frac{\partial}{\partial s}}\nabla\nu,\pi\rangle + \langle\nabla\nu,\nabla\mu\rangle = \langle R^{\nabla}(\mu,\pi)\nu,\pi\rangle + \langle\nabla\nabla_{\frac{\partial}{\partial s}}\nu,\pi\rangle + \langle d_{\nabla}\nu, d_{\nabla}\mu\rangle \\ &= -\langle R^{\nabla}(\mu,\pi)\pi,\nu\rangle + \langle \nabla_{\frac{\partial}{\partial s}}\nu, d_{\nabla}^{*}\pi\rangle + \langle d_{\nabla}^{*}d_{\nabla}\mu,\nu\rangle = \langle(\Delta-\rho_{\nabla})\nu,\mu\rangle + \langle\nabla_{\frac{\partial}{\partial s}}\nu, d_{\nabla}^{*}\pi\rangle,\end{aligned}$$

where R^{∇} and ρ_{∇} are the curvature and the Ricci operator for Q, respectively. For a harmonic foliation $\mathscr{F}$, we have

$$\frac{\partial^2}{\partial s\partial t}\bigg|_{s=0,t=0} E(\mathscr{F}_{s,t}) = \langle(\Delta-\rho_{\nabla})\mu,\nu\rangle = \langle \mathscr{J}_{\nabla}\mu,\nu\rangle, \tag{2.3}$$

where $\mathscr{J}_{\nabla} = \Delta - \rho_{\nabla}$ is the Jacobi operator of $\mathscr{F}$. Note that the Jacobi operator $\mathscr{J}_{\nabla}$ is a self-adjoint and strongly elliptic with real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to \infty$ for $i \to \infty$. Here the dimension of each eigenspace $V_{\lambda}(\mathscr{F}) = \{\nu \in \Gamma(Q);\ \mathscr{J}_{\nabla}\nu = \lambda\nu\}$ is finite, i.e. $\dim V_{\lambda}(\mathscr{F}) < \infty$.

Definition. The *index* of a harmonic foliation $\mathscr{F}$ is defined by

$$\mathrm{index}(\mathscr{F}) = \sum_{\lambda_i<0} \dim V_{\lambda_i}(\mathscr{F})$$

and a harmonic foliation $\mathscr{F}$ is said to be *stable* if $\mathrm{index}(\mathscr{F}) = 0$, i.e. $\langle \mathscr{J}_{\nabla}\nu,\nu\rangle \geqq 0$ for all $\nu \in \Gamma(Q)$.

Note that this definition makes sense for the case of harmonic foliation $\mathscr{F}$ with bundle-like metric g_N, because if g_N is not bundle-like, then the equality (2.3) does not hold in general.

3. Harmonic foliations on locally conformal Kähler manifolds.

The purpose of this section is to prove Main Theorem in Introduction. The following lemma is crucial in our approach:

Lemma 3.1. *The connection ∇ on Q defined in* (2.1) *satisfies $\nabla_X J_Q S = J_Q \nabla_X S$ for all $X \in \Gamma(TM)$ and $S \in \Gamma(Q)$, where J_Q denotes the almost complex structure on Q induced by J on M.*

Proof. We first note that any complex submanifold N of a locally conformal Kähler manifold M is minimal if and only if the Lee vector field B for M is tangent to N (for instance, see Dragomir and Ornea [**2**, Theorem 12.1]). Let ∇^M be the Levi-Civita connection of (M, g_M). Then for all $X, Y \in \Gamma(TM)$,

$$\nabla^M_X JY = J\nabla^M_X Y + \frac{1}{2}\big\{\theta(Y)X - \omega(Y)JX - g_M(X,Y)A - \Omega(X,Y)B\big\},$$

where $\theta = \omega \circ J$ and $A = -JB$. Then if $X \in \Gamma(\sigma(Q))$ and $Y \in \Gamma(Q)$, we have

$$\begin{aligned}\nabla_X J_Q S - J_Q \nabla_X S &= \pi\big(\nabla^M_X JY_S - J\nabla^M_X Y_S\big) \\ &= \pi\left(\frac{1}{2}\big\{\theta(Y_S)X - \omega(Y_S)JX - g_M(X, Y_S)A - \Omega(X, Y_S)B\big\}\right) = 0\end{aligned}$$

by $\theta(Y_S) = \omega(Y_S) = 0$. On the other hand, if $X \in \Gamma(L)$ and $S \in \Gamma(Q)$, by Proposition 2.2 of Dragomir and Ornea [**2**] (cf. Vaisman [**14**]), we have $[X, JY_S] - J[X, Y_S] \in L$. Then

$$\nabla_X J_Q S - J_Q \nabla_X S = \pi([X, JY_S] - J[X, Y_S]) = 0,$$

and this completes the proof of the lemma. □

We define a linear differential operator $D : \Gamma(Q) \to \Gamma(Q \otimes T^*M)$ of first order by

$$DV(X) = \nabla_{JX} V - J_Q \nabla_X V, \qquad V \in \Gamma(Q) \text{ and } X \in \Gamma(TM).$$

Proof of Main Theorem. It suffices to show

$$\langle \mathscr{J}_\nabla V, V\rangle = \frac{1}{2}\langle DV, DV\rangle \tag{3.2}$$

for all $V \in \Gamma(Q)$. Let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a local orthonormal frame such that $Je_i = f_i$, $Jf_i = -e_i$, $1 \leqq i \leqq n$, and that the frame $\{e_1, \dots, e_p, f_1, \dots, f_p\}$ spans $\mathscr{F}$. Then

$$\begin{aligned}\langle \mathscr{J}_\nabla V, V\rangle &= \langle d^*_\nabla d_\nabla V, V\rangle - \langle \rho_\nabla V, V\rangle = \langle d_\nabla V, d_\nabla V\rangle - \langle R^\nabla(V, \pi)\pi, V\rangle \\ &= \sum_{i=1}^n \left\{\int_M g_Q(\nabla_{e_i} V, \nabla_{e_i} V) v_M + \int_M g_Q(\nabla_{f_i} V, \nabla_{f_i} V) v_M\right\} \\ &\quad - \sum_{i=p+1}^n \left\{\int_M g_Q(R^\nabla(V, e_i)e_i, V) v_M + \int_M g_Q(R^\nabla(V, f_i)f_i, V) v_M\right\}.\end{aligned} \tag{3.3}$$

On the other hand, $\langle DV, DV\rangle$ is written as

$$\begin{aligned}\langle DV, DV\rangle &= \sum_{i=1}^n \left\{\int_M g_Q(DV(e_i), DV(e_i)) v_M + \int_M g_Q(DV(f_i), DV(f_i)) v_M\right\} \\ &= \sum_{i=1}^n \left\{\int_M g_Q(\nabla_{Je_i} V - J\nabla_{e_i} V, \nabla_{Je_i} V - J\nabla_{e_i} V)\right. \\ &\qquad \left. + g_Q(\nabla_{Jf_i} V - J\nabla_{f_i} V, \nabla_{Jf_i} V - J\nabla_{f_i} V) v_M\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_M \big\{ g_Q(\nabla_{Je_i} V, \nabla_{Je_i} V) - 2 g_Q(\nabla_{Je_i} V, J\nabla_{e_i} V) \\
&\qquad + g_Q(J\nabla_{e_i} V, J\nabla_{e_i} V) + g_Q(\nabla_{e_i} V, \nabla_{e_i} V) \\
&\qquad + 2 g_Q(\nabla_{e_i} V, J\nabla_{Je_i} V) + g_Q(J\nabla_{Je_i} V, J\nabla_{Je_i} V) \big\} v_M \\
&= 2 \sum_{i=1}^{n} \int_M \big\{ g_Q(\nabla_{e_i} V, \nabla_{e_i} V) + g_Q(\nabla_{Je_i} V, \nabla_{Je_i} V) \\
&\qquad + g_Q(\nabla_{e_i} V, J\nabla_{Je_i} V) - g_Q(\nabla_{Je_i} V, J\nabla_{e_i} V) \big\} v_M \\
&= 2 \sum_{i=1}^{n} \int_M \big\{ g_Q(\nabla_{e_i} V, \nabla_{e_i} V) + g_Q(\nabla_{Je_i} V, \nabla_{Je_i} V) \\
&\qquad + e_i g_Q(V, J\nabla_{Je_i} V) - g_Q(V, J\nabla_{e_i} \nabla_{Je_i} V) \\
&\qquad - Je_i g_Q(V, J\nabla_{e_i} V) + g_Q(V, J\nabla_{Je_i} \nabla_{e_i} V) \} v_M \\
&= 2 \sum_{i=1}^{n} \int_M \big\{ g_Q(\nabla_{e_i} V, \nabla_{e_i} V) + g_Q(\nabla_{Je_i} V, \nabla_{Je_i} V) \\
&\qquad + e_i g_Q(V, J\nabla_{Je_i} V) - Je_i g_Q(V, J\nabla_{e_i} V) \\
&\qquad - g_Q(V, J R^{\nabla}(e_i, Je_i) V) - g_Q(V, J\nabla_{[e_i, Je_i]} V) \} v_M. \qquad (3.4)
\end{aligned}
$$

We also observe that

$$
\sum_{i=1}^{n} \int_M \big\{ e_i g_Q(V, J\nabla_{Je_i} V) - Je_i g_Q(V, J\nabla_{e_i} V) - g_Q(V, J\nabla_{[e_i, Je_i]} V) \big\} v_M = 0, \qquad (3.5)
$$

because if $X \in \Gamma(TM)$ is defined by $g_M(X, Y) = g_Q(\nabla_{JY} V, JV)$, then the following computation of $\mathrm{div}(X)$ together with $\int_M \mathrm{div}(X) v_M = 0$ allows us to obtain (3.5):

$$
\begin{aligned}
\mathrm{div}(X) &= \sum_{i=1}^{n} \big\{ g_M\big(e_i, \nabla^M_{e_i} X\big) + g_M\big(Je_i, \nabla^M_{Je_i} X\big) \big\} \\
&= \sum_{i=1}^{n} \big\{ e_i g_M(e_i, X) - g_M\big(\nabla^M_{e_i} e_i, X\big) + Je_i g_M(Je_i, X) - g_M\big(\nabla^M_{Je_i} Je_i, X\big) \big\} \\
&= \sum_{i=1}^{n} \big\{ e_i g_Q(\nabla_{Je_i} V, JV) - g_Q\big(\nabla_{J\nabla^M_{e_i} e_i} V, JV\big) \\
&\qquad + Je_i g_Q(\nabla_{JJe_i} V, JV) - g_Q\big(\nabla_{J\nabla^M_{Je_i} Je_i} V, JV\big) \big\} \\
&= \sum_{i=1}^{n} \big\{ e_i g_Q(\nabla_{Je_i} V, JV) - Je_i g_Q(\nabla_{e_i} V, JV) v \\
&\qquad - g_Q\big(\nabla_{\nabla^M_{e_i} Je_i} V, JV\big) + g_Q\big(\nabla_{\nabla^M_{Je_i} e_i} V, JV\big) \big\} \\
&= -\sum_{i=1}^{n} \big\{ e_i g_Q(V, J\nabla_{Je_i} V) - Je_i g_Q(V, J\nabla_{e_i} V) - g_Q\big(V, J\nabla_{[e_i, Je_i]} V\big) \big\}.
\end{aligned}
$$

Now by (3.4) and (3.5), we have

$$\langle DV, DV\rangle = 2\sum_{i=1}^{n}\int_M \{g_Q(\nabla_{e_i}V,\nabla_{e_i}V) + g_Q(\nabla_{Je_i}V,\nabla_{Je_i}V) - g_Q(V, JR^{\nabla}(e_i, Je_i)V)\} v_M. \tag{3.6}$$

Then for $1 \leqq i \leqq p$,

$$\begin{aligned} R^{\nabla}(e_i, Je_i)V &= \nabla_{e_i}\nabla_{Je_i}V - \nabla_{Je_i}\nabla_{e_i}V - \nabla_{[e_i,Je_i]}V \\ &= \pi[e_i, \pi[Je_i, V]] - \pi[Je_i, \pi[e_i, V]] - \pi[[e_i, Je_i], V] \\ &= \pi[e_i, \pi[Je_i, V]] + \pi[Je_i, \pi[V, e_i]] + \pi[V, [e_i, Je_i]] = 0, \end{aligned} \tag{3.7}$$

because the foliation is involutive satisfying

$$\pi[e_i, \pi^{\perp}[Je_i, V]] = 0 = \pi[Je_i, \pi^{\perp}[e_i, V]],$$

where $\pi^{\perp} = \mathrm{id} - \pi$. Furthermore, for $p + 1 \leqq i \leqq n$, the Bianchi identity shows that

$$\begin{aligned} JR^{\nabla}(e_i, Je_i)V &= -JR^{\nabla}(Je_i, V)e_i - JR^{\nabla}(V, e_i)Je_i \\ &= R^{\nabla}(V, Je_i)Je_i + R^{\nabla}(V, e_i)e_i \end{aligned} \tag{3.8}$$

Thus, by (3.3), (3.6), (3.7) and (3.8), we obtain the required identity (3.2) as follows:

$$\begin{aligned} \frac{1}{2}\langle DV, DV\rangle = &\sum_{i=1}^{n}\int_M \{g_Q(\nabla_{e_i}V,\nabla_{e_i}V) + g_Q(\nabla_{Je_i}V,\nabla_{Je_i}V)\} v_m \\ &- \sum_{i=p+1}^{n}\int_M \{g_Q(R^{\nabla}(V, e_i)e_i, V) + g_Q(R^{\nabla}(V, Je_i)Je_i, V)\} v_M \\ = &\langle \mathscr{J}_{\nabla}V, V\rangle. \end{aligned}$$

□

Remark 3.9. (i) More generally, Main Theorem is valid even if M is (not necessarily Kähler and is) just a compact Hermitian manifold, provided that the connection ∇ defined by (2.1) satisfies Lemma 3.1.
(ii) As to stable harmonic foliations, there exists an example foliated by fibers of a Riemannian submersion whose base space is not a complex manifold. A typical example is the twistor space of a quaternionic Kähler manifold.

Example 3.10. We here give examples of stable harmonic foliations on locally conformal Kähler manifolds.

(i) Hopf manifolds: Let λ be a complex number satisfying $|\lambda| \neq 1$. Denote by $\langle\lambda\rangle$ the cyclic group generated by the transformation: $(z_1, \dots, z_n) \mapsto (\lambda z_1, \dots, \lambda z_n)$ of

$\boldsymbol{C}^n - \{0\}$. Since this group acts freely and holomorphically on $\boldsymbol{C}^n - \{0\}$, the quotient space $\boldsymbol{CH}^n := (\boldsymbol{C}^n - \{0\})/\langle\lambda\rangle$ is a complex manifold called a *Hopf manifold*. Consider the Hermitian metric $g_0 = (\Sigma_{k=1}^n dz^k \otimes d\bar{z}^k)/\|z\|^2$ on $\boldsymbol{C}^n - \{0\}$. Then g_0 gives a Vaisman manifold structure on $\boldsymbol{CH}^n$ with Lee form $\omega_0 = -\{\Sigma_{k=1}^n (z^k d\bar{z}^k + \bar{z}^k dz^k)\}/\|z\|^2$. It is well-known that $\boldsymbol{CH}^n$ has a principal $T^1_{\boldsymbol{C}}$-bundle structure over the projective space $\boldsymbol{CP}^{n-1}$. Then the foliation on $\boldsymbol{CH}^n$ defined by the canonical projection $\pi : \boldsymbol{CH}^n \to \boldsymbol{CP}^{n-1}$ is harmonic and is stable by Corollary 1.1, where the metric on $\boldsymbol{CP}^{n-1}$ is the Fubini-Study metric. For examples of canonical foliations on Vaisman compact complex surfaces, see Belgun [**1**].

(ii) Inoue surfaces S_M: Let $\boldsymbol{H} = \{w = w_1 + \sqrt{-1}w_2 \in \boldsymbol{C};\ w_2 > 0\}$ be the upper half-plane and $M = (m_{ij}) \in SL(3, \boldsymbol{Z})$ a unimodular matrix with one real eigenvalue α and two non-real complex eigenvalues $\beta, \bar{\beta}$. Consider the eigenvectors (a_1, a_2, a_3) and (b_1, b_2, b_3) associated to the eigenvalues α and β, respectively. Let G_M be the group of complex automorphisms of $\boldsymbol{H} \times \boldsymbol{C}$ generated by the transformations

$$\begin{aligned}
&(w, z) \mapsto (\alpha w, \beta z), \\
&(w, z) \mapsto (w + a_j, z + b_j), \quad j = 1, 2, 3.
\end{aligned}$$

The quotient space $S_M := \boldsymbol{H} \times \boldsymbol{C}/G_M$ is an Inoue surface. The metric $g_S = w_2^{-2} dw \otimes d\bar{w} + w_2 dz \otimes d\bar{z}$ on $\boldsymbol{H} \times \boldsymbol{C}$ defines a locally conformal Kähler metric, called the *Tricerri* metric, on S_M. We now choose an orthonormal frame for the tangent bundle TS_M as follows:

$$e_1 = w_2 \frac{\partial}{\partial w_1}, \quad f_1 = w_2 \frac{\partial}{\partial w_2}, \quad e_2 = \frac{1}{\sqrt{w_2}} \frac{\partial}{\partial z_1}, \quad f_2 = \frac{1}{\sqrt{w_2}} \frac{\partial}{\partial z_2}.$$

Then the distribution generated by $B = f_1$ and JB defines a harmonic foliation $\mathscr{F}$ with complex leaves by $[e_1, f_1] = -e_1$. We shall now show that $\mathscr{F}$ is stable. Unfortunately, since the Tricerri metric g_S on S_M is not bundle-like, Main Theorem is not applicable. Put $V := ae_2 + bf_2 \in \Gamma(Q)$. To compute the right-hand side of (3.3), we observe that

$$\nabla_{e_2} e_2 = \pi\left(-\frac{1}{2} f_1\right) = 0,\ \nabla_V e_2 = \pi\left(-\frac{1}{2} a f_1\right) = 0,\ \nabla_{[V, e_2]} e_2 = \pi\left(\frac{1}{2}(e_2 a) f_1\right) = 0.$$

Hence $R^\nabla(V, e_2)e_2 = 0$, and by the same computation, we obtain $R^\nabla(V, f_2)f_2 = 0$. It now follows that the second term in the right-hand side of (3.3) vanishes. Thus $\langle \mathscr{J} V, V\rangle \geqq 0$ for all $V \in \Gamma(Q)$.

In conclusion, let us note that a slightly different harmonicity for distributions (not necessarily for foliations) on Riemannian manifolds was studied by Vanhecke et al. [**5**]. If a foliation $\mathscr{F}$ on a Riemannian manifold (M, g_M) is harmonic in the sense of Kamber and Tondeur, the Gauss map from M to the Grasmannian manifold G is then harmonic as a map (cf. Ruh and Vilms [**11**]). Therefore a harmonic foliation $\mathscr{F}$ in Kamber-Tondeur's sense is harmonic in Vanhecke's sense. However, the converse is not true as follows: In view of the beginning of the proof of Lemma 3.1, the orthogonal distributions of canonical foliations on Vaisman manifolds are not harmonic in Kamber-Tondeur's sense, while the orthogonal distributions are harmonic in Vanhecke's sense (cf. [**5**, Proposition 3.4]; see also Ornea and Vanhecke [**10**]).

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