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WEAK SOLUTIONS FOR PARTIAL PETTIS HADAMARD FRACTIONAL INTEGRAL EQUATIONS WITH RANDOM EFFECTS

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ABSTRACT. In this article, we apply Mönch and Engl's fixed point theorems associated with the technique of measure of weak noncompactness to investigate the existence of random solutions for a class of partial random integral equations via Hadamard's fractional integral, under the Pettis integrability assumption.

1. Introduction. Fractional differential and integral equations arise in a variety of areas of biological, physical and engineering applications, see [**22, 32**]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [**1, 2**], Kilbas et al. [**24**], Miller and Ross [**25**] and Zhou [**37, 38**] and the papers of Abbas et al. [**4**], Benchohra et al. [**11**], Wang et al. [**33, 34, 35, 36**], Zhou et al. [**39**], and the references therein. In [**14**], Butzer et al. investigated properties of the Hadamard fractional integral and derivative. In [**15**], they obtained the Mellin transform of the Hadamard fractional integral and differential operators, and in [**30**], Pooseh et al. obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [**31**], and the references therein.

The measure of weak noncompactness is introduced by De Blasi [**16**]. The strong measure of noncompactness was developed first by

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Banaŝ and Goebel **[7]** and subsequently developed and used in many papers, see for example, Akhmerov et al. **[5]**, Alvàrez **[6]**, Benchohra et al. **[12]**, Guo et al. **[19]**, and the references therein. In **[12, 28]**, the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see **[2, 9, 10]**. Existence of random solutions for functional differential and integral equations has extensively been studied in various papers, see **[3, 8, 17]**, and the references therein.

This article deals with the existence of solutions to the following random Hadamard partial fractional integral equation of the form:

$$\begin{aligned} u(x,y,w) = \mu(x,y,w) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \\ \times \frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\, dt\, ds, \end{aligned} \tag{1.1}$$

if $(x,y) \in J$, $w \in \Omega$, where $J := [1,a] \times [1,b]$, $a,b > 1$, $r_1, r_2 > 0$,

$$\mu : J \times \Omega \longrightarrow E, \qquad f : J \times E \times \Omega \longrightarrow E$$

are given continuous functions, $(\Omega, \mathcal{A}, \nu)$ is a measurable space, $\Gamma(\cdot)$ is the Euler gamma function and E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^* such that E is the dual of a weakly compactly generated Banach space X.

2. Preliminaries. In this section, we introduce notation, definitions and preliminary facts which are used throughout this paper.

Let $C := C(J,E)$ be the Banach space of continuous functions $u : J \to E$ with the norm

$$\|u\|_C = \sup_{(x,y)\in J} \|u(x,y)\|_E.$$

Denote by $L^\infty(\Omega,\nu)$ the Banach space of measurable functions

$$u : \Omega \to C,$$

which are essentially bounded and equipped with the norm

$$\|u\|_{L^\infty} := \inf\{c > 0 : \|u(w)\|_C \le c, \text{ a.e. for } \omega \in \Omega\}.$$

Let $(E, w) = (E, \sigma(E, E^*))$ denotes the Banach space E with its weak topology.

Denote by $(E, w) = (E, \sigma(E, E^*))$ the Banach space E with its weak topology.

Definition 2.1. A Banach space X is called *weakly compactly generated* (WCG) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2. A function $h : E \to E$ is said to be *weakly sequentially continuous* if h takes each weakly convergent sequence in E to a weakly convergent sequence in E, i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w).

Definition 2.3 ([**29**])**.** The function $u : J \to E$ is said to be *Pettis integrable* on J if and only if there is an element $u_j \in E$ corresponding to each $j \subset J$ such that

$$\phi(u_j) = \iint_j \phi(u(s,t))\, dt\, ds$$

for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_j = \iint_j u(s,t)\, dt\, ds$).

Let $P(J, E)$ be the space of all E-valued Pettis integrable functions on J, and let $L^1(J, \mathbb{R})$ be the Banach space of Lebesgue measurable functions

$$u : J \longrightarrow \mathbb{R}.$$

Define the class $P_1(J, E)$ by

$$P_1(J, E) = \{u \in P(J, E) : \varphi(u) \in L^1(J, \mathbb{R}), \text{ for every } \varphi \in E^*\}.$$

The space $P_1(J, E)$ is normed by

$$\|u\|_{P_1} = \sup_{\substack{\varphi \in E^* \\ \|\varphi\| \le 1}} \int_1^a \int_1^b |\varphi(u(x,y))|\, d\lambda(x,y),$$

where λ stands for a Lebesgue measure on J.

The next result is due to Pettis, see [**29,** Theorem 3.4, Corollary 3.41].

Proposition 2.4 (**[29]**). *If $u \in P_1(J,E)$ and h is a measurable and essentially bounded $E-$valued function, then $uh \in P_1([0,a],E)$.*

In what follows, the sign "$\int$" denotes the Pettis integral. We recall the definitions of Pettis integral and Hadamard integral of fractional order.

Definition 2.5 (**[20, 24]**). The left sided mixed Pettis Hadamard integral of order $q > 0$ for a function $g \in P_1([1,a],E)$ is defined as

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s}\, ds.$$

Remark 2.6. Let $g \in P_1([1,a],E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^H I_1^r g)(x) = ({}^H I_1^r \varphi g)(x), \quad \text{for almost every } x \in [1,a].$$

Definition 2.7. Let $r_1, r_2 \geq 0$, $\sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in P_1(J,E)$, define the left sided mixed Pettis Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st}\, dt\, ds.$$

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if, for any $B \in \beta_E$, we have

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

In order to define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.8. A mapping

$$T : \Omega \times E \longrightarrow E$$

is called *jointly measurable* if, for any $B \in \beta_E$, we have

$$T^{-1}(B) = \{(w,v) \in \Omega \times E : T(w,v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E, respectively.

Lemma 2.9 ([**17**]). *Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \in E$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then, the map*

$$(w, v) \longmapsto T(w, v)$$

is jointly measurable.

Definition 2.10 ([**21**]). A function

$$f : J \times E \times \Omega \longrightarrow E$$

is called *random Carathéodory* if the following conditions are satisfied:

(i) the map

$$(x, y, w) \longrightarrow f(x, y, u, w)$$

is jointly measurable for all $u \in E$, and

(ii) the map

$$u \longrightarrow f(x, y, u, w)$$

is continuous for almost all $(x, y) \in J$ and $w \in \Omega$.

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a *random operator* if $T(w, u)$ is measurable in w for all $u \in E$, and it is expressed as $T(w)u = T(w, u)$. In this case, we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called *continuous*, respectively, *compact, totally bounded* and *completely continuous*, if $T(w, u)$ is continuous, respectively, compact, totally bounded and completely continuous, in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties may be found in [**23**].

Definition 2.11 ([**18**]). Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and F a mapping from Ω into $\mathcal{P}(Y)$. A mapping

$$T : \{(w, y) : w \in \Omega,\ y \in F(w)\} \longrightarrow Y$$

is called a *random operator with stochastic domain* F if F is measurable, i.e., for all closed $A \subset Y$, $\{w \in \Omega, F(w) \cap A \neq \emptyset\}$ is measurable, and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in F(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called a random (stochastic) fixed point of T if, for P-almost all $w \in \Omega$, $y(w) \in F(w)$ and $T(w)y(w) = y(w)$ and, for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Definition 2.12 ([**16**])**.** Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map

$$\beta : \Omega_E \longrightarrow [0, \infty)$$

defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$;
(b) $\beta(A) = 0 \Leftrightarrow$ A is weakly relatively compact'
(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(d) $\beta(\overline{A}^{\omega}) = \beta(A)$; ($\overline{A}^{\omega}$ denotes the weak closure of A);
(e) $\beta(A + B) \leq \beta(A) + \beta(B)$;
(f) $\beta(\lambda A) = |\lambda|\beta(A)$;
(g) $\beta(\text{conv}(A)) = \beta(A)$;
(h) $\beta(\cup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The next result directly follows from the Hahn-Banach theorem.

Proposition 2.13. *Let E be a normed space and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : J \to E$, we denote by

$$V(x, y) = \{v(x, y) : v \in V\}, \quad (x, y) \in J,$$

and

$$V(J) = \{v(x, y) : v \in V, \quad (x, y) \in J\}.$$

Lemma 2.14 ([**19**]). *Let $H \subset C$ be bounded and equicontinuous. Then, the function*

$$(x,y) \longrightarrow \beta(H(x,y))$$

is continuous on J, and

$$\beta_C(H) = \max_{(x,y)\in J}\beta(H(x,y)),$$

and

$$\beta\bigg(\int\int_J u(s,t)\,dt\,ds\bigg) \leq \int\int_J \beta(H(s,t))\,dt\,ds,$$

where $H(x,y) = \{u(x,y) : u \in H\}$, $(x,y) \in J$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

We need the following fixed point theorems.

Theorem 2.15 ([**27**]). *Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable, locally convex vector space $C(J,E)$ such that $0 \in Q$. Suppose that*

$$T : Q \longrightarrow Q$$

is weakly-sequentially continuous. If the implication

$$\overline{V} = \overline{\mathrm{conv}}(\{0\} \cup T(V)) \Longrightarrow V \text{ is relatively weakly compact,} \tag{2.1}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

Theorem 2.16 ([**18**]). *Let*

$$F : \Omega \longrightarrow 2^Y$$

be measurable with $F(w)$ closed, convex and solid, i.e., $\mathrm{int}F(w) \neq \emptyset$, for all $w \in \Omega$. We assume that there exists measurable

$$y_0 : \Omega \longrightarrow Y$$

with $y_0 \in \mathrm{int}F(w)$ for all $w \in \Omega$. Let T be a continuous, random operator with stochastic domain F such that, for every $w \in \Omega$,

$$\{y \in F(\omega) : T(w)y = y\} \neq \emptyset.$$

Then, T has a stochastic fixed point.

3. Existence results. We begin by defining what we mean by a random solution of the integral equation (1.1).

Definition 3.1. A function $u \in C$ is said to be a random solution of (1.1) if u satisfies equation (1.1) on J.

Next, we state the main result.

Theorem 3.2. *Assume that the following hypotheses hold*:

- (H_1) *the function* $w \mapsto \mu(x,y,w)$ *is measurable and bounded for almost every* $(x,y) \in J$;
- (H_2) *the function* f *is random Carathéodory on* $J \times E \times \Omega$;
- (H_3) *for almost every* $(x,y) \in J$, *and all* $w \in \Omega$, *the function* $u \to f(x,y,u,w)$ *is weakly sequentially continuous*;
- (H_4) *for almost every* $u \in E$, *and all* $w \in \Omega$, *the function* $(x,y) \to f(x,y,u,w)$ *is Pettis integrable almost everywhere on* J;
- (H_5) *there exists a* $p \in C(J,[0,\infty))$ *such that*:
  there exists a function

$$p : J \times \Omega \longrightarrow [0,\infty)$$

with $p(w) \in L^\infty(J,[0,\infty))$ *for each* $w \in \Omega$ *such that, for all* $\varphi \in E*$, *we have*

$$|\varphi(f(x,y,u,w))| \leq \frac{p(x,y,w)\|\varphi\|}{1+\|\varphi\|+\|u\|_E},$$

for almost every $(x,y) \in J$, *and each* $u \in E$, *with*

- (H_6) *for each bounded set* $B \subset E$ *and, for each* $(x,y) \in J$ *and* $w \in \Omega$, *we have*

$$\beta(f(x,y,B,w) \leq p(x,y,w)\beta(B);$$

- (H_7) *there exists a random function* $R : \Omega \to (0,\infty)$ *such that*

$$R(w) > \mu^*(w) + \frac{p^*(w)(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)};$$

where

$$\mu^*(w) = \sup_{(x,y)\in J}|\mu(x,y,w)|, \;\; p^*(w) = \sup_{(x,y)\in J}p(x,y,w).$$

If

$$\ell := \frac{p^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.1}$$

where $p^* = \sup \mathrm{ess}_{w\in\Omega}\, p^*(w)$, *then the integral equation* (1.1) *has at least one random solution defined on* J.

Proof. Transform the integral equation (1.1) into a fixed point equation. Consider the operator

$$N : \Omega \times C \longrightarrow C$$

defined by:

(3.2)

$$(N(w)u)(x,y) = \mu(x,y,w) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1} \times \frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds.$$

From hypotheses (H_2)–(H_4), for each $w \in \Omega$ and almost all $(x,y) \in J$, the function $f(\cdot,\cdot,u(\cdot,\cdot,w),w)$ is Pettis integral on J. From (H_5), we have that, for all $(x,y) \in J$,

$$\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t,w),w)}{st}$$

is Pettis integrable for all $w \in \Omega$. Again, as the map μ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on J, then $N(w)$ defines a mapping

$$N : \Omega \times C \longrightarrow C.$$

Hence, u is a solution for the integral equation (1.1) if and only if $u = (N(w))u$. We shall show that the operator N satisfies all of the assumptions of Theorem 2.16. The proof will be given in several steps.

Step 1. *$N(w)$ is a random operator with stochastic domain on C.* Since $f(x,y,u,w)$ is random Carathéodory, the map

$$w \longrightarrow f(x,y,u,w)$$

is measurable in view of Definition 2.8. Similarly, the product

$$\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st}$$

of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions' therefore, the map

$$w\longmapsto\mu(x,y,w)+\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dt\,ds$$

is measurable. As a result, N is a random operator on $\Omega\times C$ into C.

Let $W:\Omega\to\mathcal{P}(C)$ be defined by

$$\begin{aligned}W(w)=\Big\{u\in C:\|u\|_C\le R(w)\text{ and }&\|u(x_1,y_1,w)-u(x_2,y_2,w)\|_E\\ \le\|\mu(x_1,y_1,w)-\mu(x_2,y_2,w)\|_E&\\ +\frac{p^*(w)}{\Gamma(1+r_1)\Gamma(1+r_2)}&[2(\log y_2)^{r_2}(\log x_2-\log x_1)^{r_1}\\ &+2(\log x_2)^{r_1}(\log y_2-\log y_1)^{r_2}\\ &+(\log x_1)^{r_1}(\log y_1)^{r_2}-(\log x_2)^{r_1}(\log y_2)^{r_2}\\ &-2(\log x_2-\log x_1)^{r_1}(\log y_2-\log y_1)^{r_2}]\Big\}.\end{aligned}$$

Clearly, the subset $W(w)$ is closed, convex and equicontinuous for all $w\in\Omega$. Then, W is measurable by [**18,** Lemma 17]. Therefore, N is a random operator with stochastic domain W.

Step 2. *$N(w)$ is continuous.* Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $\mathcal{C}$. Then, there exists a $\phi\in E^*$ such that

$$\|(N(w)u)(x,y)\|_E=\phi((N(w)u)(x,y)).$$

For each $(x,y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned}
&\|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \\
&\quad = \phi((N(w)u_n)(x,y) - (N(w)u)(x,y)) \\
&\quad \le \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\
&\qquad \times \frac{|\phi(f(s,t,u_n(s,t,w),w) - f(s,t,u(s,t,w),w))|}{\Gamma(r_1)\Gamma(r_2)}\, dt\, ds.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we obtain

$$\|N(w)u_n - N(w)u\|_C \longrightarrow 0 \quad \text{as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that

$$N(w) : W(w) \longrightarrow N(w)$$

is a continuous, random operator with stochastic domain W.

Step 3. *For every* $w \in \Omega$,

$$\{u \in W(w) : N(w)u = u\} \ne \emptyset.$$

For this, we apply Theorem 2.15. The proof will be given in several claims.

Claim 1. *$N(w)$ maps $W(w)$ into itself.* Let $w \in \Omega$ be fixed, and let $u \in W(w)$, $(x,y) \in J$. Assume that $(N(w)u)(x,y) \ne 0$. Then, there exists a $\phi \in E^*$ such that

$$\|(N(w)u)(x,y)\|_E = \phi((N(w)u)(x,y)).$$

Hence, we obtain:

$$\begin{aligned}
&\|(N(w)u)(x,y)\|_E \\
&\quad = \phi\bigg(\mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \\
&\qquad\qquad \times \frac{f(s,t,u(s,t,w),w)}{st}\, dt\, ds\bigg) \\
&\quad = \phi(\mu(x,y,w)) \\
&\qquad + \phi\bigg(\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1}
\end{aligned}$$

$$\times \frac{f(s,t,u(s,t,w),w)}{st}\,dt\,ds\Bigg)$$
$$\leq \mu^*(w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}$$
$$\times \frac{|\phi(f(s,t,u(s,t,w),w))|}{st}\,dt\,ds$$
$$\leq \mu^*(w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y!\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{p(s,t,w)}{st}\,dt\,ds$$
$$\leq \mu^*(w) + \frac{p^*(w)(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$\leq R(w).$$

Next, for any fixed $w\in\Omega$, let $(x_1,y_1),(x_2,y_2)\in J$ be such that $x_1<x_2$ and $y_1<y_2$, and let $u\in W(w)$, with

$$(N(w)u)(x_1,y_1)-(N(w)u)(x_2,y_2)\neq 0.$$

Then, there exists a $\phi\in E^*$ such that

$$\|(N(w)u)(x_1,y_1)-(N(w)u)(x_2,y_2)\|_E$$
$$= \phi((N(w)u)(x_1,y_1)-(N(w)u)(x_2,y_2))$$

and

$$\|\varphi\| = 1.$$

Thus, we have

$$\|(N(w)u)(x_2,y_2)-(N(w)u)(x_1,y_1)\|_E$$
$$= \phi((N(w)u)(x_2,y_2)-(N(w)u)(x_1,y_1))$$
$$\leq \|\mu(x_1,y_1,w)-\mu(x_2,y_2,w)\|_E$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_1^{y_1}\left[\left|\log\frac{x_2}{s}\right|^{r_1-1}\times\left|\log\frac{y_2}{t}\right|^{r_2-1}\right.$$
$$\left.-\left|\log\frac{x_1}{s}\right|^{r_1-1}\left|\log\frac{y_1}{t}\right|^{r_2-1}\right]\frac{\|f(s,t,u(s,t,w),w)\|_E}{st}\,dt\,ds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}$$
$$\times\frac{|\phi(f(s,t,u(s,t,w),w))|}{st}\,dt\,ds$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1}\int_{y_1}^{y_2} \left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1} \\
&\times \frac{|\phi(f(s,t,u(s,t,w),w))|}{st}\,dt\,ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{1}^{y_1} \left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1} \\
&\times \frac{|\phi(f(s,t,u(s,t,w),w))|}{st}\,dt\,ds.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\|(N(w)u)(x_2,y_2) - (N(w)u)(x_1,y_1)\|_E \\
&\quad\le \|\mu(x_1,y_1,w) - \mu(x_2,y_2,w)\|_E \\
&\qquad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1}\int_1^{y_1} \bigg[\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1} \\
&\qquad - \left|\log\frac{x_1}{s}\right|^{r_1-1}\left|\log\frac{y_1}{t}\right|^{r_2-1}\bigg] \frac{p^*(w)}{st}\,dt\,ds \\
&\qquad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} \left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1} \frac{p^*(w)}{st}\,dt\,ds \\
&\qquad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{1}^{x_1}\int_{y_1}^{y_2} \left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1} \frac{p^*(w)}{st}\,dt\,ds \\
&\qquad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{1}^{y_1} \left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1} \frac{p^*(w)}{st}\,dt\,ds \\
&\quad\le \|\mu(x_1,y_1,w) - \mu(x_2,y_2,w)\|_E + \frac{p^*(w)}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&\qquad \times [2(\log y_2)^{r_2}(\log x_2 - \log x_1)^{r_1} + 2(\log x_2)^{r_1}(\log y_2 - \log y_1)^{r_2} \\
&\qquad\quad + (\log x_1)^{r_1}(\log y_1)^{r_2} - (\log x_2)^{r_1}(\log y_2)^{r_2} \\
&\qquad\qquad + (\log x_2 - \log x_1)^{r_1}(\log y_2 - \log y_1)^{r_2}].
\end{aligned}
$$

Hence, $N(W(w)) \subset W(w)$. Therefore,

$$N(w) : W(w) \longrightarrow N(w)$$

maps $W(w)$ into itself.

Claim 2. *$N(w)$ is weakly-sequentially continuous.* Let (u_n) be a sequence in $W(w)$, and let

$$(u_n(x,y,w)) \longrightarrow u(x,y,w)$$

in (E,ω) for any $w\in\Omega$ and each $(x,y)\in J$. Fix $(x,y)\in J$. Since f satisfies assumption (H_3), we have $f(x,y,u_n(x,y,w),w)$ converges weakly uniformly to $f(x,y,u(x,y,w),w)$. Hence, the Lebesgue dominated convergence theorem for the Pettis integral implies $(Nu_n)(x,y,w)$ converges weakly uniformly to $(N(w)u)(x,y)$ in (E,ω). This may be performed for any $w\in\Omega$ and each $(x,y)\in J$; thus, $N(w)(u_n)\to N(w)(u)$. Then,

$$N : W(w) \longrightarrow W(w)$$

is weakly-sequentially continuous.

Claim 3. *The implication* (2.1) *holds.* Let V be a subset of $W(w)$ such that

$$\overline{V} = \overline{\mathrm{conv}}(N(w)(V)\cup\{0\}).$$

Obviously,

$$V(x,y,w) \subset \overline{\mathrm{conv}}(N(w)V)(x,y))\cup\{0\}).$$

Further, as V is bounded and equicontinuous, by [**13,** Lemma 3] the function

$$(x,y,w) \longrightarrow u(x,y,w) = \beta(V(x,y,w))$$

is continuous on $J\times\Omega$. Since the function μ is continuous on $J\times\Omega$, the set

$$\{\mu(x,y,w),\ (x,y)\in J,\ w\in\Omega\}\subset E$$

is compact. From (H_5), Lemma 2.14 and the properties of the measure β, for any $w\in\Omega$ and each $(x,y)\in J$, we have

$$\begin{aligned}
v(x,y,w) &\le \beta((N(w)V)(x,y)\cup\{0\}) \le \beta((N(w)V)(x,y))\\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{p(s,t,w)\beta(V(s,t,w))}{st}\,dt\,ds\\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{p(s,t,w)v(s,t,w)}{st}\,dt\,ds
\end{aligned}$$

$$\leq \frac{\|v\|_C}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \frac{p(s,t,w)}{st}\, dt\, ds$$
$$\leq \frac{p^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|v\|_C.$$

Thus,

$$\|v\|_C \leq \ell \|v\|_C.$$

From (3.1), we obtain $\|v\|_C = 0$, that is,

$$v(x,y,w) = \beta(V(x,y,w)) = 0,$$

for any $w \in \Omega$ and each $(x,y) \in J$. Hence, **[26,** Theorem 2] shows that V is weakly relatively compact in C.

As a consequence of Claims 1–3, and from Theorem 2.15, it follows that, for every $w \in \Omega$,

$$\{u \in W(w) : N(w)u = u\} \neq \emptyset.$$

Now apply Theorem 2.16. Steps 1–3 show that, for each $w \in \Omega$, N has at least one fixed point in W. Since

$$\bigcap_{w \in \Omega} \mathrm{int} W(w) \neq \emptyset$$

and the hypothesis that a measurable selector of $\mathrm{int}W$ exists holds, then N has a stochastic fixed point, i.e., the integral equation (1.1) has at least one random solution on C. □

4. An example. Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$, be equipped with the usual σ-algebra consisting of Bochner measurable subsets of $(-\infty, 0)$. Given a measurable function

$$u : \Omega \longrightarrow C([1,e] \times [1,e]),$$

consider the following partial random Hadamard integral equation of the form

$$\begin{aligned} u(x,y,w) = \mu(x,y,w) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \\ \times \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\, dt\, ds, \end{aligned} \tag{4.1}$$

for $(x, y) \in [1, e] \times [1, e]$, $w \in \Omega$, where

$$r_1, r_2 > 0, \quad \mu(x, y, w) = x \sin w + y^2 \cos w, \quad (x, y) \in [1, e] \times [1, e],$$

and

$$f(x, y, u(x, y)) = \frac{xy^2}{(1 + w^2 + |u(x, y, w)|)e^{x+y+5}}, \\ (x, y) \in [1, e] \times [1, e], \quad w \in \Omega.$$

The function

$$w \longmapsto \mu(x, y, w) = x \sin w + y^2 \cos w$$

is measurable and bounded with

$$|\mu(x, y, w)| \leq e + e^2;$$

hence, the condition (H_1) is satisfied.

The map

$$(x, y, w) \longmapsto f(x, y, u, w)$$

is jointly continuous for all $u \in \mathbb{R}$, and hence, jointly measurable, for all $u \in \mathbb{R}$. In addition, the map

$$u \longmapsto f(x, y, u, w)$$

is continuous for all $(x, y) \in [1, e] \times [1, e]$ and $w \in \Omega$. Therefore, the function f is Carathéodory on

$$[1, e] \times [1, e] \times \mathbb{R} \times \Omega.$$

For each $u \in \mathbb{R}$, $(x, y) \in [1, e] \times [1, e]$ and $w \in \Omega$, we have

$$|f(x, y, u, w)| \leq \frac{xy^2}{e^5(1 + |u|)}.$$

Hence, condition (H_5) is satisfied with $p^* = e^{-2}$.

We show that condition $\ell < 1$ holds with $a = b = e$. Indeed, for each $r_1, r_2 > 0$, we obtain

$$\ell = \frac{p^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \leq \frac{1}{e^2\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1.$$

A simple computation shows that all conditions of Theorem 3.2 are satisfied. It follows that the random integral equation (4.1) has at least one random solution on $[1, e] \times [1, e]$.

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