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ON THE HALF-HARTLEY TRANSFORM, ITS ITERATION AND COMPOSITIONS WITH FOURIER TRANSFORMS

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ABSTRACT. Employing the generalized Parseval equality for the Mellin transform and elementary trigonometric formulas, the iterated Hartley transform on the nonnegative half-axis (the iterated half-Hartley transform) is investigated in L_2. Mapping and inversion properties are discussed, its relationship with the iterated Stieltjes transform is established. Various compositions with the Fourier cosine and sine transforms are obtained. The results are applied to the uniqueness and universality of the closed form solutions for certain new singular integral and integro-functional equations.

1. Introduction and auxiliary results. The familiar reciprocal pair of the Hartley transforms [**1**]

$$(1.1)\qquad (\mathcal{H}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(xt) + \sin(xt)] f(t)\, dt,\ \ x \in \mathbb{R},$$

$$(1.2)\qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(xt) + \sin(xt)] (\mathcal{H}f)(t)\, dt$$

is well known in connection with various applications in mathematical physics. Mapping and inversion properties of these transforms in L_2, related generalized convolutions as well as their multidimensional analogs were investigated, for instance, in [**6**], [**5**], [**2**], [**8**]. These operators were treated as the so-called bilateral Watson transforms,

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and in some sense they are related to the Fourier cosine and Fourier sine transforms

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xt) f(t)\, dt, \quad x \in \mathbb{R}_+, \tag{1.3}$$

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(xt) f(t)\, dt, \ x \in \mathbb{R}_+. \tag{1.4}$$

Recently **[7]**, the author investigated the Hartley transform (1.1) with the integration over $\mathbb{R}_+$ (the half-Hartley transform)

$$(\mathcal{H}_+ f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\cos(xt) + \sin(xt)] f(t)\, dt, \ x \in \mathbb{R}_+ \tag{1.5}$$

and proved an analog of the Plancherel theorem, establishing its reciprocal inverse operator in $L_2(\mathbb{R}_+)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\sin(xt)\ S(xt) + \cos(xt)\ C(xt)]\, (\mathcal{H}_+ f)(t)\, dt, \tag{1.6}$$

where $S(x),\ C(x)$ are Fresnel sine- and cosine- integrals, respectively,

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \sin(t^2) dt, \quad C(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \cos(t^2)\, dt.$$

Our goal here is to examine the iteration of operator (1.5) $(\mathcal{H}_+^2 f)(x) \equiv (\mathcal{H}_+\mathcal{H}_+ f)(x)$, which will be called the iterated half-Hartley transform and investigate its compositions in L_2 with the Fourier transforms (1.3), (1.4) of the form: $\mathcal{H}_+ F_c$, $\mathcal{H}_+ F_s$, $\mathcal{H}_+ F_c F_s$, $\mathcal{H}_+^2 F_c$, $\mathcal{H}_+^2 F_s$, $\mathcal{H}_+^2 F_c F_s$. The corresponding integral representations of these compositions will be established in L_2 and their boundedness and invertibility will be proved. Moreover, we will apply these results to establish the uniqueness of solutions in the closed form for the corresponding second kind singular integral and integro-functional equations.

Our natural approach is based on the L_2-theory of the Mellin transform **[4]**

$$(\mathcal{M} f)(s) = f^*(s) = \int_0^\infty f(t) t^{s-1} dt, \quad s \in \sigma = \left\{ s \in \mathbb{C},\ s = \frac{1}{2} + i\tau \right\}, \tag{1.7}$$

where the integral is convergent in the mean square sense with respect to the norm in $L_2(\sigma)$. Reciprocally, the inversion formula takes place

$$f(x) = \frac{1}{2\pi i}\int_{\sigma} f^*(s)x^{-s}ds, \quad x > 0 \tag{1.8}$$

with the convergence of the integral in the mean square sense with respect to the norm in $L_2(\mathbb{R}_+)$. Furthermore, for any $f_1, f_2 \in L_2(\mathbb{R}_+)$, the generalized Parseval identity holds

$$\int_0^\infty f_1\,(xt)\,f_2(t)dt = \frac{1}{2\pi i}\int_{\sigma} f_1^*(s)f_2^*(1-s)x^{-s}ds, \quad x > 0 \tag{1.9}$$

with Parseval's equality of squares of L_2- norms

$$\int_0^\infty |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau. \tag{1.10}$$

Finally in this section, we exhibit the known formulas [**4**]

$$\int_0^\infty \frac{\sin t}{t} t^{s-1}dt = \frac{\Gamma(s)}{1-s}\cos\left(\frac{\pi s}{2}\right), \quad s \in \sigma, \tag{1.11}$$

$$\int_0^\infty \frac{1-\cos t}{t} t^{s-1}dt = \frac{\Gamma(s)}{1-s}\sin\left(\frac{\pi s}{2}\right), \quad s \in \sigma, \tag{1.12}$$

where $\Gamma(s)$ is the Euler gamma-function, which will be used in the sequel.

2. Plancherel's type theorems. It is widely known [**4**] via the classical Plancherel theorem in $L_2(\mathbb{R}_+)$ that the Fourier cosine and Fourier sine transforms extend to bounded invertible and isometric mappings, having the properties $F_c^2 = I$, $F_s^2 = I$, where I is the identity operator. We begin, demonstrating our method on the simple composition F_cF_s of operators (1.3), (1.4). Precisely, it drives us to the Plancherel theorem for the Hilbert transform [**4**].

Theorem 1. *The composition $F(x) = (F_cF_sf)(x)$ extends to a bounded invertible and isometric map $F : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ and can be written in the form of the Hilbert transform in L_2*

$$F(x) = \frac{2}{\pi}PV\int_0^\infty \frac{tf(t)}{t^2 - x^2}\,dt, \quad x \in \mathbb{R}_+. \tag{2.1}$$

Reciprocally,

$$f(x) = \frac{2}{\pi} PV \int_0^\infty \frac{x\ F(t)}{x^2 - t^2}\, dt, \quad x \in \mathbb{R}_+ \tag{2.2}$$

and this map is isometric, i.e., $||F|| = ||f||$ *for all* $f \in L_2(\mathbb{R}_+)$.

Proof. Let f belong to the space $C_c^{(2)}(\mathbb{R}_+)$ of continuously differentiable functions of compact support, which is dense in $L_2(\mathbb{R}_+)$. Then integrating by parts in (1.3), (1.4) and (1.7), we find that $(F_c f)(x) = O(x^{-2})$, $(F_s f)(x) = O(x^{-2})$, $x \to \infty$ and $s^2 f^*(s)$ is bounded on σ. Therefore, $(F_c f)(x), (F_s f)(x) \in L_2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, $f^*(s) \in L_2(\sigma) \cap L_1(\sigma)$. Hence, minding equalities (1.11), (1.12), the generalized Parseval equality (1.9) and the supplement formula for the gamma-function, we derive for all $x > 0$ (cf., [**4,** Section 8.4])

$$\begin{aligned}
F(x) &= \frac{1}{2\pi i} \int_\sigma F^*(s) x^{-s} ds \\
&= \frac{2}{\pi} \frac{1}{2\pi i} \int_\sigma \Gamma(s)\Gamma(1-s) \cos^2\left(\frac{\pi s}{2}\right) f^*(s) x^{-s} ds \\
&= \frac{1}{2\pi i} \int_\sigma f^*(s) \cot\left(\frac{\pi s}{2}\right) x^{-s} ds = \frac{2}{\pi} PV \int_0^\infty \frac{t f(t)}{t^2 - x^2}\, dt.
\end{aligned}$$

Hence, reciprocally via (1.8), we obtain

$$\begin{aligned}
f(x) &= \frac{1}{2\pi i} \int_\sigma f^*(s) x^{-s} ds = \frac{1}{2\pi i} \int_\sigma F^*(s) \tan\left(\frac{\pi s}{2}\right) x^{-s} ds \\
&= \frac{1}{2\pi i} \int_\sigma F^*(1-s) \cot\left(\frac{\pi s}{2}\right) x^{s-1} ds = \frac{2}{\pi x} PV \int_0^\infty \frac{F(t)}{1 - (t/x)^2}\, dt \\
&= \frac{2}{\pi} PV \int_0^\infty \frac{x F(t)}{x^2 - t^2}\, dt.
\end{aligned}$$

Thus, we proved (2.1) and (2.2) for any $f \in C_c^{(2)}(\mathbb{R}_+)$. Further, since $C_c^{(2)}(\mathbb{R}_+)$ is dense in $L_2(\mathbb{R}_+)$, there is a unique extension of F as an invertible continuous map $F : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$. Clearly, it is isometric by virtue of the Plancherel theorem for Fourier cosine and Fourier sine transforms. □

Extending this approach, we prove the Plancherel type theorem for the iterated half-Hartley transform $\mathcal{H}_+^2$. Indeed, we have

Theorem 2. *The iterated half-Hartley transform extends to a bounded invertible map* $\mathcal{H}_+^2 : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ *and the following reciprocal formulas hold*

$$(\mathcal{H}_+^2 f)(x) = 2f(x) + \frac{2}{\pi}\int_0^\infty \frac{f(t)}{x+t}dt, \quad x > 0, \tag{2.3}$$

$$f(x) = \frac{1}{2}(\mathcal{H}_+^2 f)(x) - \frac{1}{\pi^2}\int_0^\infty \frac{\sqrt{xt}\,[\log x - \log t]}{x^2 - t^2}(\mathcal{H}_+^2 f)(t)\,dt. \tag{2.4}$$

Moreover, the norm inequalities take place

$$||f|| \le \left|\left|\mathcal{H}_+^2 f\right|\right| \le 8 \;\; ||f||\,. \tag{2.5}$$

Proof. Assuming again $f \in C_c^{(2)}(\mathbb{R}_+)$, and taking into account (1.8), (1.9) and relation (8.4.2.5) in [**3,** Vol. 3], we derive in the same manner the equalities

$$\begin{aligned}(\mathcal{H}_+^2 f)(x) &= \frac{1}{2\pi i}\frac{2}{\pi}\int_\sigma \Gamma(s)\Gamma(1-s)\left(\cos\left(\frac{\pi s}{2}\right) + \sin\left(\frac{\pi s}{2}\right)\right)^2 f^*(s)\; x^{-s}ds \\ &= \frac{1}{\pi i}\int_\sigma \frac{1+\sin(\pi s)}{\sin(\pi s)} f^*(s)\; x^{-s}ds \\ &= 2f(x) + \frac{1}{2\pi i}\frac{2}{\pi}\int_\sigma \Gamma(s)\Gamma(1-s) f^*(s)\; x^{-s}ds \\ &= 2f(x) + \frac{2}{\pi}\int_0^\infty \frac{f(t)}{x+t}dt,\end{aligned} \tag{2.6}$$

which prove representation (2.3), involving the classical Stieltjes transform [**8, 4**]. Conversely, appealing to relation (8.4.6.11) in [**3,** Vol. 3], and elementary properties of the Mellin transform, it gives

$$\begin{aligned}f(x) &= \frac{1}{4\pi i}\int_\sigma \frac{\sin(\pi s)}{1+\sin(\pi s)}(\mathcal{H}_+^2 f)^*(s)\; x^{-s}ds = \frac{1}{2}(\mathcal{H}_+^2 f)(x) \\ &- \frac{1}{8\pi i}\int_\sigma \frac{(\mathcal{H}_+^2 f)^*(s)}{\sin^2(\pi(s+1/2)/2)}\; x^{-s}ds = \frac{1}{2}(\mathcal{H}_+^2 f)^*(x) \\ &- \frac{1}{2\pi i}\frac{1}{4\pi^2}\int_\sigma \left[\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)\right]^2 (\mathcal{H}_+^2 f)^*(s)\, x^{-s}ds \\ &= \frac{1}{2}(\mathcal{H}_+^2 f)(x) - \frac{1}{\pi^2}\int_0^\infty \frac{\sqrt{xt}\,[\log x - \log t]}{x^2 - t^2}(\mathcal{H}_+^2 f)(t)\,dt,\end{aligned} \tag{2.7}$$

which proves (2.4), involving the iterated Stietltjes transform recently treated in **[9]**. In order to establish inequalities (2.5), we call the Parseval equality (1.10) for the Mellin transform, which yields (see (2.6), (2.7))

$$||\mathcal{H}_+^2 f|| = 2\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left(1+\cosh(\pi\tau)\right)^2}{\cosh^2(\pi\tau)}\left|f^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau\right)^{1/2}$$
$$= 4\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cosh^4(\pi\tau/2)}{\cosh^2(\pi\tau)}\left|f^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau\right)^{1/2} \le 8\,||f||,$$

and, on the other hand,

$$||f|| = \frac{1}{4\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} \frac{\cosh^2(\pi\tau)}{\cosh^4(\pi\tau/2)}\left|(\mathcal{H}_+^2 f)^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau\right)^{1/2} \le ||\mathcal{H}_+^2 f||\,.$$

Now the same argument of the denseness of $C_c^{(2)}(\mathbb{R}_+)$ in $L_2(\mathbb{R}_+)$ drives us to a unique extension of $\mathcal{H}_+^2$ as an invertible continuous map $\mathcal{H}_+^2 : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$. $\square$

Remark 1. As we observe via the Schwarz inequality, the convergence of integrals (2.3), (2.4) is pointwise.

Concerning the composition $\mathcal{H}_+ F_c$ it has

Theorem 3. *The composition $F(x) = (\mathcal{H}_+ F_c f)(x)$ extends to a bounded invertible map $F : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ and the following reciprocal formulas hold:*

$$F(x) = f(x) + \frac{2}{\pi} PV \int_0^{\infty} \frac{x f(t)}{x^2 - t^2}\, dt, \quad x > 0, \tag{2.8}$$

$$f(x) = \frac{1}{\pi} PV \int_0^{\infty} \frac{\sqrt{xt}}{t^2 - x^2} F(t)\, dt. \tag{2.9}$$

Moreover, the norm inequalities are valid

$$||f|| \le ||F|| \le 2\sqrt{2}\,||f||\,. \tag{2.10}$$

Proof. For $f \in C_c^{(2)}(\mathbb{R}_+)$, we obtain

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\frac{2}{\pi}\int_\sigma \Gamma(s)\Gamma(1-s) \\ &\quad\times\left(\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)\right)\sin\left(\frac{\pi s}{2}\right) f^*(s)\ x^{-s}ds \\ &= \frac{1}{2\pi i}\int_\sigma\left(1+\tan\left(\frac{\pi s}{2}\right)\right)f^*(s)\ x^{-s}ds \\ &= f(x)+\frac{2}{\pi}PV\int_0^\infty \frac{xf(t)}{x^2-t^2}dt, \end{aligned}$$

which prove representation (2.8), relating again to the classical Hilbert transform in $L_2(\mathbb{R}_+)$. The inverse operator (2.9) can be deduced via the equality

(2.11)

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_\sigma \frac{F^*(s)}{1+\tan(\pi s/2)}\ x^{-s}\,ds \\ &= \frac{1}{2\sqrt{2}\ \pi i}\frac{d}{dx}\int_\sigma F^*(s)\frac{\Gamma((s+1/2)/2)\,\Gamma((3/2-s)/2)}{\Gamma((1+s)/2)\,\Gamma((1-s)/2)}\ \frac{x^{1-s}}{1-s}\,ds, \end{aligned}$$

where the differentiation is allowed under the integral sign via the absolute and uniform convergence. In the meantime, due to the residue theorem

$$\begin{aligned} \frac{1}{2\pi i}\int_\sigma \frac{\Gamma((s+1/2)/2)\,\Gamma((3/2-s)/2)}{\Gamma((1+s)/2)\,\Gamma((1-s)/2)}\ \frac{x^{1-s}}{1-s}\,ds &= \frac{\sqrt{2}}{\pi}\sum_{n=0}^\infty \frac{x^{2n+3/2}}{2n+3/2} \\ &= \frac{\sqrt{2}}{\pi}\int_0^x \frac{\sqrt{y}\,dy}{1-y^2}, \\ & 0<x<1, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\pi i}\int_\sigma \frac{\Gamma((s+1/2)/2)\,\Gamma((3/2-s)/2)}{\Gamma((1+s)/2)\,\Gamma((1-s)/2)}\ \frac{x^{1-s}}{1-s}\,ds &= \frac{\sqrt{2}}{\pi}\sum_{n=0}^\infty \frac{x^{-2n-1/2}}{2n+1/2} \\ &= \frac{\sqrt{2}}{\pi}\int_x^\infty \frac{\sqrt{y}\,dy}{y^2-1}, \\ & x>1. \end{aligned}$$

Therefore, returning to (2.11), differentiating with respect to x under the integral sign and using (1.9), we write it in the form

$$f(x) = \frac{1}{\pi} PV \int_0^\infty \frac{\sqrt{x/t}}{1-(x/t)^2} F(t) \frac{dt}{t}.$$

Thus, we come up with (2.9). Finally, in a similar manner, we derive inequalities (2.10). In fact, we have

$$||f|| \leq ||F|| = 2 \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh^2(\pi\tau/2)}{\cosh(\pi\tau)} \left| f^* \left(\frac{1}{2} + i\tau \right) \right|^2 d\tau \right)^{1/2} \leq 2\sqrt{2} ||f||.$$

Hence, extending F on the whole $L_2(\mathbb{R}_+)$ as an invertible continuous mapping, we complete the proof. □

The Plancherel theorem for the composition $\mathcal{H}_+ F_s$ can be proved analogously, and we leave it without proof.

Theorem 4. *The composition* $F(x) = (\mathcal{H}_+ F_s f)(x)$ *extends to a bounded invertible map* $F : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$, *and the following reciprocal formulas hold*:

$$F(x) = f(x) + \frac{2}{\pi} PV \int_0^\infty \frac{t f(t)}{t^2 - x^2} dt, \quad x > 0, \tag{2.12}$$

$$f(x) = \frac{1}{\pi} PV \int_0^\infty \frac{\sqrt{xt}}{x^2 - t^2} F(t)\, dt. \tag{2.13}$$

Moreover, the norm inequalities (2.10) *are valid.*

The case $\mathcal{H}_+ F_c F_s$ can be treated with the use of Theorem 1. Precisely, we state:

Theorem 5. *The composition* $F(x) = (\mathcal{H}_+ F_c F_s f)(x)$ *extends to a bounded invertible map* $F : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$, *having the integral* *representation*:

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\sin(xt) - \cos(xt) + \frac{2}{\pi}\sqrt{xt}\ S_{-1/2,1/2}(xt)\right] f(t)\, dt, \quad x > 0, \tag{2.14}$$

where $S_{-1/2,1/2}(x)$ *is the Lommel function and the integral converges in the mean square sense. The inverse operator is written in the form of the integral*

$$f(x) = 2\sqrt{\frac{2}{\pi}} \int_0^\infty \ [(1 - S(xt))\sin(xt) - C(xt)\cos(xt)]\, F(t)\, dt, \tag{2.15}$$

which converges in the mean square sense as well. Moreover, the norm inequalities (2.10) *take place.*

Proof. In fact, for $f \in C_c^{(2)}(\mathbb{R}_+)$ we write via (1.9)

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\left(\frac{2}{\pi}\right)^{3/2} \int_\sigma \Gamma^2(s)\Gamma(1-s) \\ &\quad \times \left(\cos\left(\frac{\pi s}{2}\right) + \sin\left(\frac{\pi s}{2}\right)\right) \sin^2\left(\frac{\pi s}{2}\right) f^*(1-s)\ x^{-s}\, ds \\ &= \frac{1}{2\pi i}\sqrt{\frac{2}{\pi}} \int_\sigma \Gamma(s)\sin\left(\frac{\pi s}{2}\right)\left(1 + \tan\left(\frac{\pi s}{2}\right)\right) f^*(1-s)\ x^{-s}\, ds \\ &= (F_s f)(x) - (F_c f)(x) + \frac{1}{2\pi i}\sqrt{\frac{2}{\pi}} \int_\sigma \frac{\Gamma(s)}{\cos(\pi s/2)} f^*(1-s)\ x^{-s} ds. \end{aligned} \tag{2.16}$$

Meanwhile, the latter integral can be calculated, appealing to relations (8.4.2.5) in [**3,** Vol. 3], (8.4.23.1) in [**3,** Vol. 3] and (2.16.3.14) in [**3,** Vol. 2]. But first, employing the supplement and duplication formulas for the gamma-function, we find the following inverse Mellin transform written in terms of the Mellin type convolution with the modified Bessel function, namely,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma}\frac{\Gamma(s)}{\cos(\pi s/2)}\ x^{-s}ds \\ &= \frac{1}{4\pi^{5/2}i}\int_{\sigma}\Gamma^2\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\ (x/2)^{-s}ds \\ &= \frac{2}{\pi}\int_0^{\infty}\frac{\sqrt{t}\ K_0(2\sqrt{t})}{\sqrt{1+(x^2/4t)}}\frac{dt}{t} = \frac{2}{\pi}\int_0^{\infty}\frac{y\ K_0(y)}{\sqrt{y^2+x^2}}dy \\ &= \frac{2\sqrt{x}}{\pi}\quad S_{-1/2,1/2}(x),\ \ x>0,\end{aligned}$$

where $S_{\mu,\nu}(z)$ is the Lommel function [**3,** Vol. 3]. Therefore, returning to (2.16) and recalling the generalized Parseval equality (1.9), we get

$$\begin{aligned}\frac{1}{2\pi i}\sqrt{\frac{2}{\pi}}\int_{\sigma}\frac{\Gamma(s)}{\cos(\pi s/2)}f^*(1-s)\ x^{-s}ds \\ &= \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_0^{\infty}\sqrt{xt}\ S_{-1/2,1/2}(xt)f(t)dt, \quad x>0,\end{aligned}$$

and the latter integral is absolutely convergent for any $f\in L_2(\mathbb{R}_+)$. Combining with (2.16), we come up with representation (2.14) for the dense set $C_c^{(2)}(\mathbb{R}_+)$ of $L_2(\mathbb{R}_+)$. Moreover, the norm inequalities (2.10) follow immediately from (2.16). In fact, we have

$$\begin{aligned}||f||\le||F|| &= \frac{\sqrt{2}}{\pi}\left(\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}+i\tau\right)\right|^2\cosh^2(\pi\tau/2)\left|f^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau\right)^{1/2} \\ &= \sqrt{\frac{2}{\pi}}\left(\int_{-\infty}^{\infty}\frac{\cosh^2(\pi\tau/2)}{\cosh(\pi\tau)}\left|f^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau\right)^{1/2}\le 2\sqrt{2}||f||.\end{aligned}$$

Concerning the inverse operator (2.15), we recall (2.16) and reciprocal formulas (1.7) and (1.8) of the Mellin transform. It yields

$$\begin{aligned}(2.17)\quad f(x) &= \frac{1}{2\pi i}\sqrt{\frac{\pi}{2}}\int_{\sigma}\frac{F^*(1-s)}{\Gamma(1-s)\cos(\pi s/2)\,(1+\cot(\pi s/2))}\ x^{-s}ds \\ &= \frac{1}{4\pi i}\int_{\sigma}\frac{\Gamma((s+1)/2)\,\Gamma((s+1/2)/2)\,\Gamma((3/2-s)/2)}{\Gamma(s/2)\,\Gamma^2(1-s/2)} \\ &\quad\times F^*(1-s)\ (x/2)^{-s}ds.\end{aligned}$$

Meanwhile, the residue theorem and relation (7.14.4.6) in [**3,** Vol. 3] lead us to the value of the integral with the ratio of gamma-functions in terms of Fresnel's integrals (see above)

$$\frac{1}{2\pi i}\int_{\sigma}\frac{\Gamma\left((s+1)/2\right)\Gamma\left((s+1/2)/2\right)\Gamma\left((3/2-s)/2\right)}{\Gamma\left(s/2\right)\Gamma^{2}\left(1-s/2\right)}(x/2)^{-s}ds$$
$$=2\sqrt{\frac{2}{\pi}}\left[(1-S(x))\sin x-C(x)\cos x\right].$$

Hence, returning to (2.17), we easily come up with (2.15), and after extension of F on the whole $L_2(\mathbb{R}_+)$ as an invertible continuous mapping, complete the proof of Theorem 5. □

The Plancherel theorem for compositions $\mathcal{H}_+^2F_c$, $\mathcal{H}_+^2F_s$ is related to Theorem 2 and can be stated as follows:

Theorem 6. *Compositions $F(x)=(\mathcal{H}_+^2F_cf)(x)$, $G(x)=\mathcal{H}_+^2F_s$ extend to bounded invertible mappings $F,G : L_2(\mathbb{R}_+)\to L_2(\mathbb{R}_+)$, having integral representations*

(2.18)
$$F(x)=2\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\cos(xt)+\frac{1}{\pi}\sqrt{xt}\,S_{-3/2,-1/2}(xt)\right]f(t)\,dt,\quad x>0,$$

(2.19)
$$G(x)=2\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\sin(xt)+\frac{1}{\pi}\sqrt{xt}\,S_{-1/2,1/2}(xt)\right]f(t)\,dt,\quad x>0,$$

where both integrals converge in the mean square sense. Inverse operators are written, respectively, in the form

$$f(x)=\int_0^\infty k_c(xt)F(t)\,dt,\quad x>0 \tag{2.20}$$

where

$$k_c(x)=\frac{\sqrt{x}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{2k}}{(3/2)_{2k}}\left[\frac{2}{\pi}\psi(-1/2-2k)-\frac{2}{\pi}\log x+1\right]$$

and $\psi(x)$ is the psi-function,

$$f(x)=\int_0^\infty k_s(xt)G(t)\,dt, \tag{2.21}$$

where

$$k_s(x) = \frac{\sqrt{x}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k}}{(3/2)_{2k}} \left[1 - \frac{2}{\pi}\psi(-1/2 - 2k) + \frac{2}{\pi}\log x\right]$$

and integrals converge in the mean square sense. Moreover, the norm inequalities (2.5) *take place*

$$||f|| \leq \left\{ \begin{array}{c} ||F|| \\ ||G|| \end{array} \right\} \leq 8\,||f||\,. \tag{2.22}$$

Proof. Indeed, for $f \in C_c^{(2)}(\mathbb{R}_+)$ we use (2.6) to obtain

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\left(\frac{2}{\pi}\right)^{3/2} \int_\sigma \Gamma(s)\Gamma^2(1-s) \\ &\quad \times \left(\cos\left(\frac{\pi s}{2}\right) + \sin\left(\frac{\pi s}{2}\right)\right)^2 \sin\left(\frac{\pi s}{2}\right) f^*(s)\ x^{s-1} ds \\ &= \frac{1}{2\pi i}\sqrt{\frac{2}{\pi}} \int_\sigma \frac{1+\sin(\pi s)}{\sin(\pi s/2)} \Gamma(s) f^*(1-s)\ x^{-s} ds \qquad (2.23) \\ &= 2\ (F_c f)(x) + \frac{1}{2\pi i}\sqrt{\frac{2}{\pi}} \int_\sigma \frac{\Gamma(1-s)}{\cos(\pi s/2)} f^*(s)\ x^{s-1} ds. \end{aligned}$$

Hence, making similar calculations, which were done for the latter integral in (2.16), we find

$$\begin{aligned} \frac{1}{2\pi i}\int_\sigma \frac{\Gamma(1-s)}{\cos(\pi s/2)}\ x^{-s} ds &= \frac{2x^{-2}}{\pi}\int_0^\infty \frac{y\ K_0(y)}{\sqrt{y^2+x^{-2}}}\, dy \\ &= \frac{2x^{-3/2}}{\pi}\ S_{-3/2,-1/2}\left(\frac{1}{x}\right), \quad x > 0. \end{aligned}$$

Therefore,

$$F(x) = 2\ (F_c f)(x) + \left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty \sqrt{xt}\ S_{-3/2,-1/2}(xt) f(t)\, dt,$$

which coincides with (2.18). Similarly,

$$\begin{aligned}G(x) &= \frac{1}{2\pi i}\left(\frac{2}{\pi}\right)^{3/2}\int_{\sigma}\Gamma(s)\Gamma^2(1-s)\\&\quad\times\left(\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)\right)^2\cos\left(\frac{\pi s}{2}\right)f^*(s)\,x^{s-1}ds\\&= \frac{1}{2\pi i}\sqrt{\frac{2}{\pi}}\int_{\sigma}\frac{1+\sin(\pi s)}{\sin(\pi s/2)}\Gamma(1-s)f^*(s)\,x^{s-1}ds\\&= 2\left(F_s f\right)(x)+\frac{1}{2\pi i}\sqrt{\frac{2}{\pi}}\int_{\sigma}\frac{\Gamma(s)}{\cos(\pi s/2)}\,f^*(1-s)\;x^{-s}ds,\end{aligned}$$

and we end up with (2.19), appealing again to the latter integral in (2.16). Concerning inverse operator (2.20), we write, recalling (2.23) and formula (1.8) of the inverse Mellin transform

(2.24)
$$\begin{aligned}f(x) &= \frac{1}{2\pi i}\sqrt{\frac{\pi}{2}}\int_{\sigma}\frac{\cos(\pi s/2)}{(1+\sin(\pi s))\Gamma(1-s)}F^*(1-s)\,x^{-s}ds\\&= \frac{1}{2}(F_c F)(x)-\frac{1}{2\pi i}\frac{1}{2\sqrt{2\pi}}\int_{\sigma}\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}F^*(1-s)\,x^{-s}ds,\\&\qquad x>0.\end{aligned}$$

In the meantime, the integral

$$\frac{1}{2\pi i}\int_{\sigma}\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}\,x^{-s}ds$$

can be calculated by the residue theorem. It has

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\sigma}\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}\,x^{-s}ds\\&= \sum_{k=0}^{\infty}\mathrm{Res}_{s=-k}\left[\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}\;x^{-s}\right]\\&\quad+\sum_{k=0}^{\infty}\mathrm{Res}_{s=-1/2-2k}\left[\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}\,x^{-s}\right],\quad x>0.\end{aligned}\tag{2.25}$$

The first sum in the right-hand side of the latter equality contains residues in simple poles $s=-k$, $k\in\mathbb{N}_0$ of the gamma-function and, by straightforward calculations, it gives

$$\sum_{k=0}^{\infty} \mathrm{Res}_{s=-k} \left[\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)} \ x^{-s} \right] = 2\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \cos(\pi k/2) = 2\cos x.$$

The second sum involves double poles $s = -1/2 - 2k$, $k \in \mathbb{N}_0$ of the integrand, and we find

$$\begin{aligned}
&\mathrm{Res}_{s=-1/2-2k} \left[\frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)} \ x^{-s} \right] \\
&= \lim_{s\to -1/2-2k} \frac{d}{ds} \left[(s+1/2+2k)^2 \frac{x^{-s}\ \Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)} \right] \\
&= \frac{\sqrt{2}(-1)^k}{\pi} x^{2k+1/2} \Gamma(-1/2-2k) \\
&\qquad \left[\frac{2}{\pi}\psi(-1/2-2k) - \frac{2}{\pi}\log x + 1 \right],
\end{aligned}$$

where $\psi(x)$ is the psi-function (the logarithmic derivative of the gamma-function). Therefore, substituting these values in (2.25), we obtain

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\sigma} \frac{\Gamma(s)\cos(\pi s/2)}{\sin^2(\pi(s+1/2)/2)} \, x^{-s} ds \\
&= 2\cos x - 2\sqrt{\frac{2x}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k}}{(3/2)_{2k}} \\
&\quad \times \left[\frac{2}{\pi}\psi(-1/2-2k) - \frac{2}{\pi}\log x + 1 \right], \quad x > 0.
\end{aligned}$$

Hence, returning to (2.24) and employing the generalized Parseval equality (1.9), we come up with inversion formula (2.20). Analogously, we establish (2.21). The norm inequalities (2.22) are immediate consequences of (2.5) and isometry properties of the Fourier cosine and sine transforms in L_2. To end the proof, we extend F, G on the whole $L_2(\mathbb{R}_+)$ as invertible continuous mappings. □

Finally in this section we prove the Plancherel theorem for composition $\mathcal{H}_+^2 F_c F_s$. We have:

Theorem 7. *The composition* $F(x) = (\mathcal{H}_+^2 F_c F_s f)(x)$ *extends to a bounded invertible map* $F : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ *and*

$$F(x) = \frac{2^{3/2}}{\sqrt{\pi}} PV \int_0^\infty \left[\frac{1}{\pi}\log\left(\frac{x}{t}\right) - 1\right] \frac{t f(t)}{x^2 - t^2}\, dt, \quad x > 0. \tag{2.26}$$

The inverse operator is written in the form of the integral

$$f(x) = \sqrt{\frac{2}{\pi^3}}\, PV \int_0^\infty \frac{\sqrt{xt}}{x^2 - t^2} F(t)\, dt. \tag{2.27}$$

Moreover, the composition F *satisfies the norm inequalities* (2.5).

Proof. Let $f \in C_c^{(2)}(\mathbb{R}_+)$. Then Theorems 1 and 2 yield

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\left(\frac{2}{\pi}\right)^{3/2} \\ &\quad \times \int_\sigma \left[\Gamma(s)\Gamma(1-s)\left(\cos\left(\frac{\pi s}{2}\right) + \sin\left(\frac{\pi s}{2}\right)\right)\cos\left(\frac{\pi s}{2}\right)\right]^2 f^*(s)\, x^{-s} ds \\ &= \frac{1}{2\pi i}\sqrt{\frac{\pi}{2}} \int_\sigma \frac{1 + \sin(\pi s)}{\sin^2(\pi s/2)} f^*(s)\, x^{-s} ds \\ &= \left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty \frac{\log x - \log t}{x^2 - t^2}\, t f(t)\, dt \\ &\quad - \frac{2\sqrt{2}}{\sqrt{\pi}} PV \int_0^\infty \frac{t f(t)}{x^2 - t^2}\, dt. \end{aligned}$$

Hence we arrive at (2.26). Further, to derive (2.27), we have, reciprocally,

$$f(x) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_\sigma \frac{\sin^2(\pi s/2)\; F^*(s)}{\sin^2(\pi(s+1/2)/2)} \frac{x^{1-s}}{1-s} ds, \tag{2.28}$$

where the differentiation is allowed under the integral sign via the absolute and uniform convergence. Meanwhile, calculating the convergent integral,

$$\frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \int_\sigma \frac{\sin^2(\pi s/2)}{\sin^2(\pi(s+1/2)/2)} \frac{x^{1-s}}{1-s}\, ds$$

with the use of the residue theorem, involving the left-hand double poles of the integrand $s = -2k - 1/2$, $k \in \mathbb{N}_0$, when $0 < x < 1$, we obtain

$$\frac{1}{2\pi i}\frac{1}{\sqrt{2\pi}}\int_\sigma \frac{\sin^2(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}\frac{x^{1-s}}{1-s}\,ds$$
$$= -\frac{\sqrt{2}}{\pi\sqrt{\pi}}\left[1+\frac{\log x}{\pi}\right]\sum_{k=0}^{\infty}\frac{x^{2k+3/2}}{2k+3/2} + \frac{\sqrt{2}}{\pi^2\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{x^{2k+3/2}}{(2k+3/2)^2}$$
$$= -\frac{\sqrt{2}}{\pi\sqrt{\pi}}\left[1+\frac{\log x}{\pi}\right]\int_0^x \frac{y^{1/2}\,dy}{1-y^2} + \frac{\sqrt{2}}{\pi^2\sqrt{\pi}}\int_0^x \frac{y^{1/2}\log(x/y)\,dy}{1-y^2},$$
$$0 < x < 1.$$

When $x > 1$, we should employ the right-hand double poles $s = 2k - 1/2$, $k \in \mathbb{N}$ and the simple pole $s = 1$. This gives the value of the integral

$$\frac{1}{2\pi i}\frac{1}{\sqrt{2\pi}}\int_\sigma \frac{\sin^2(\pi s/2)}{\sin^2(\pi(s+1/2)/2)}\frac{x^{1-s}}{1-s}\,ds$$
$$= -\sqrt{\frac{2}{\pi}} - \frac{\sqrt{2}}{\pi\sqrt{\pi}}\left[1+\frac{\log x}{\pi}\right]\int_x^\infty \frac{y^{1/2}\,dy}{y^2-1}$$
$$+ \frac{\sqrt{2}}{\pi^2\sqrt{\pi}}\int_0^{1/x}\frac{y^{-1/2}\log(xy)}{1-y^2}\,dy, \quad x > 1.$$

Hence, returning to (2.28) and appealing again to the generalized Parseval equality (1.9), we come up with the inversion formula (2.27) after differentiation under the integral sign, which can be motivated similar to formulas of the Hilbert transform in L_2 (see, [**4,** Theorem 90]). The norm inequalities (2.5) follow immediately from the isometry property of the Fourier transform in L_2. Extending the composition on the whole $L_2(\mathbb{R}_+)$ as an invertible continuous mapping, we complete the proof. □

3. Integral and integro-functional equations. In this section we will apply Plancherel theorems for the considered half-Hartley transform (1.5), its iteration (2.3) and compositions with the Fourier transforms to investigate the uniqueness and universality of the closed form solutions of certain singular integral and integro-functional equations. We begin with an immediate corollary of Theorem 2.

Corollary 1. *Let $g \in L_2(\mathbb{R}_+)$ be a given function. The second kind integral equation with the Stieltjes kernel*

$$f(x) + \frac{1}{\pi} \int_0^{\infty} \frac{f(t)}{x+t}\, dt = g(x), \quad x > 0, \tag{3.1}$$

has a unique solution in $L_2(\mathbb{R}_+)$ given by the formula

$$f(x) = g(x) - \frac{2}{\pi^2} \int_0^{\infty} \frac{\sqrt{xt}\,[\log x - \log t]}{x^2 - t^2}\, g(t)\, dt. \tag{3.2}$$

Conversely, for a given $f \in L_2(\mathbb{R}_+)$, integral equation (3.2) *has a unique solution $g \in L_2(\mathbb{R}_+)$ via formula* (3.1).

On the other hand, Theorem 1 leads us to the solvability criterium in $L_2(\mathbb{R}_+)$ of the following integro-functional equations with the Hilbert kernel

$$\frac{1}{x} f\left(\frac{1}{x}\right) = \frac{2}{\pi} \int_0^{\infty} \frac{t f(t)}{t^2 - x^2}\, dt, \quad x \in \mathbb{R}_+, \tag{3.3}$$

$$\frac{1}{x} f\left(\frac{1}{x}\right) = \frac{2}{\pi} \int_0^{\infty} \frac{x f(t)}{x^2 - t^2}\, dt, \quad x \in \mathbb{R}_+. \tag{3.4}$$

In fact, substituting in (3.3) and (3.4) x instead of $1/x$, we arrive at the corresponding second kind homogeneous singular integral equations

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{x t f(t)}{x^2 t^2 - 1}\, dt, \quad x > 0, \tag{3.5}$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{f(t)}{1 - x^2 t^2}\, dt, \quad x > 0. \tag{3.6}$$

Corollary 2. *In order for an arbitrary function $f \in L_2(\mathbb{R}_+)$ to be a solution of either homogeneous integro-functional equation* (3.3) *or second kind integral equation* (3.5)*, it is necessary and sufficient that f have the form of the integral*

$$f(x) = \frac{1}{2\pi i} \int_{\sigma} \frac{\varphi(s)}{\cos(\pi s/2)} x^{-s} ds, \quad x > 0, \tag{3.7}$$

which is convergent in the mean square sense. It is written in terms of some function $\varphi(s)$, satisfying condition $\varphi(s) = \varphi(1-s)$, $s \in \sigma$, i.e., $\varphi(1/2 + i\tau)$ is even with respect to $\tau \in \mathbb{R}$. Analogously, in order for an arbitrary function $f \in L_2(\mathbb{R}_+)$ to be a solution of either homogeneous *integro-functional equation* (3.4) *or second kind integral equation* (3.6), *it is necessary and sufficient that* f *have the form of the integral*

$$f(x) = \frac{1}{2\pi i} \int_{\sigma} \frac{\rho(s)}{\sin(\pi s/2)} x^{-s} ds, \quad x > 0, \tag{3.8}$$

which is convergent in the mean square sense and written in terms of some function $\rho(s)$, *which satisfies condition* $\rho(s) = \rho(1-s),\ s \in \sigma$.

Proof. Necessity. Let $f \in L_2(\mathbb{R}_+)$ be a solution of equation (3.3). In terms of the Mellin transform it can be written as the following functional equation (see the proof of Theorem 1)

$$f^*(s) \cot\left(\frac{\pi s}{2}\right) = f^*(1-s), \quad s \in \sigma.$$

Hence,

$$f^*(s) \cos\left(\frac{\pi s}{2}\right) = f^*(1-s) \sin\left(\frac{\pi s}{2}\right) = \varphi(s), \quad s \in \sigma, \tag{3.9}$$

and we observe that $\varphi(s) = \varphi(1-s)$, $s \in \sigma$. Therefore,

$$f^*(s) = \frac{\varphi(s)}{\cos(\pi s/2)},$$

and, inverting the Mellin transform, we end up with (3.7).

Sufficiency. Conversely, if $\varphi(1/2 + i\tau)$ is an even function, then from (3.7) we get equalities (3.9). Hence, the uniqueness theorem for the Mellin transform in L_2 drives us at (3.3). The same concerns the integral equation (3.5) by virtue of its equivalence to (3.3). In a similar manner, we treat the pair of equations (3.4) and (3.6). □

Theorems 3 and 4 drive us to the following results.

Corollary 3. *Let* $g \in L_2(\mathbb{R}_+)$ *be a given function. The second kind integral equations with the Hilbert kernel*

$$f(x) + \frac{2}{\pi} \int_0^\infty \frac{x f(t)}{x^2 - t^2}\, dt = g(x), \quad x > 0,$$

$$f(x) + \frac{2}{\pi} \int_0^\infty \frac{t f(t)}{t^2 - x^2} dt = g(x), \quad x > 0,$$

have unique solutions in $L_2(\mathbb{R}_+)$ given by formulas, respectively,

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{xt}}{t^2 - x^2} g(t)\, dt,$$
$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{xt}}{x^2 - t^2} g(t)\, dt.$$

Theorem 8. *Let $\lambda \in \mathbb{C}$, $|1-\lambda| \neq 1$. In order for an arbitrary function $f \in L_2(\mathbb{R}_+)$ to be a solution of the homogeneous integro-functional equation*

$$f(x) + \frac{2}{\pi} \int_0^\infty \frac{xf(t)}{x^2 - t^2}\, dt = \frac{\lambda}{x} f\left(\frac{1}{x}\right), \quad x > 0, \tag{3.10}$$

it is necessary that f have the form of the mean square sense convergent integral

$$f(x) = \frac{1}{2\pi i} \int_\sigma \frac{\varphi(s)\, x^{-s}}{\tan(\pi s/2) + 1 - \lambda}\, ds \tag{3.11}$$

of some function $\varphi(s) \in L_2(\sigma)$, which satisfies the condition $\varphi(s) = -\varphi(1-s)$, $s \in \sigma$. This condition and the form of solutions (3.11) *are also sufficient for those φ, whose reciprocal inverse Mellin transform $\mu(x)$ is a solution of the integral equation*

$$(2 - \lambda^2)\mu(x) + \frac{2}{\pi} \int_0^\infty \frac{\mu(t)}{x+t}\, dt = 0, \quad x \in \mathbb{R}_+, \tag{3.12}$$

where integral (3.12) *converges absolutely. Analogously, in order for an arbitrary function $f \in L_2(\mathbb{R}_+)$ to be a solution of the homogeneous integro-functional equation*

$$f(x) + \frac{2}{\pi} \int_0^\infty \frac{tf(t)}{t^2 - x^2}\, dt = \frac{\lambda}{x} f\left(\frac{1}{x}\right), \quad x > 0, \tag{3.13}$$

it is necessary that f have the form of the integral

$$f(x) = \frac{1}{2\pi i} \int_\sigma \frac{\varphi(s)\, x^{-s}}{\cot(\pi s/2) + 1 - \lambda}\, ds, \tag{3.14}$$

which converges in the mean square sense and depends on some function $\varphi(s) \in L_2(\sigma)$, satisfying the condition $\varphi(s) = -\varphi(1-s)$. This condition and the form of solutions (3.14) *are sufficient for those φ,* *whose reciprocal inverse Mellin transform* $\mu(x)$ *is a solution of integral equation* (3.12).

Proof. Let $f \in L_2(\mathbb{R}_+)$ be a solution of equation (3.10). In terms of the Mellin transform it can be written as the following functional equation (see the proof of Theorem 3)

$$f^*(s)\left(1+\tan\left(\frac{\pi s}{2}\right)\right) = \lambda f^*(1-s), \quad s \in \sigma. \tag{3.15}$$

Hence,

$$f^*(s)\tan\left(\frac{\pi s}{2}\right) = \lambda f^*(1-s) - f^*(s)$$

and, changing s on $1-s$ in the previous equality, we get

$$f^*(1-s)\cot\left(\frac{\pi s}{2}\right) = \lambda f^*(s) - f^*(1-s).$$

Thus, adding these two equations, we find

$$f^*(s)\left[\tan\left(\frac{\pi s}{2}\right)+1-\lambda\right] + f^*(1-s)\left[\cot\left(\frac{\pi s}{2}\right)+1-\lambda\right] = 0.$$

Denoting by $\varphi(s) = f^*(s)[\tan(\pi s/2)+1-\lambda]$, we observe that $\varphi(s) = -\varphi(1-s)$, $s \in \sigma$ and $\varphi(s) \in L_2(\sigma)$ if and only if $f^*(s) \in L_2(\sigma)$ because

$$0 < |1-|1-\lambda|| \leq |\tan(\pi s/2)+1-\lambda| \leq 2+|\lambda|, \quad s \in \sigma. \tag{3.16}$$

Hence, $f^*(s) = \varphi(s)[\tan(\pi s/2)+1-\lambda]^{-1}$, and formula (1.8) drives us to solution (3.11).

Assuming now the existence of such a function $\varphi(s) \in L_2(\sigma)$ under condition $\varphi(s) = -\varphi(1-s)$, we substitute the value $f^*(s) = \varphi(s)[\tan(\pi s/2)+1-\lambda]^{-1}$ into equation (3.15). We have

$$\varphi(s)\left[\frac{1+\tan(\pi s/2)}{\tan(\pi s/2)+1-\lambda} + \frac{\lambda}{\cot(\pi s/2)+1-\lambda}\right] = 0,$$

or, via (3.16) and after simple calculations

$$\varphi(s)\left[2-\lambda^2+\frac{2}{\sin(\pi s)}\right] = 0, \quad s \in \sigma. \tag{3.17}$$

Taking the inverse Mellin transform of both sides of the latter equality, we arrive at equation (3.12). Thus, $f(x)$ by formula (3.11) is a solution of integro-functional equation (3.10) for all $\varphi(s)$ under condition $\varphi(s) = -\varphi(1-s)$ such that its inverse Mellin transform is a solution of integral equation (3.12). The absolute convergence of the corresponding integral follows from the Schwarz inequality. In the same manner, integro-functional equation (3.13) and its solution (3.14) can be treated. $\square$

Corollary 4. *Let* $\lambda \in (-\sqrt{2}, \sqrt{2})$. *Then the only trivial solution satisfies integro-functional equations* (3.10) *and* (3.13).

Proof. When $\lambda = 0$, then the condition on λ in Theorem 8 fails. However, the solution of (3.10) is trivial via Corollary 3. Otherwise, since $2 - \lambda^2 + 2/\sin(\pi s) > 0$, $s \in \sigma$, we have from (3.17) $\varphi(s) \equiv 0$ on σ. Therefore, $f^*(s) \equiv 0$ and the inverse Mellin transform implies $f = 0$, i.e., the solution of (3.10) is trivial. The same concerns integro-functional equation (3.13). $\square$

Composition operator (2.14) is involved to investigate the solvability of the corresponding homogeneous second kind integral equation

(3.18)
$$\lambda f(x) + \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\sin(xt) - \cos(xt) + \frac{2}{\pi}\sqrt{xt}\; S_{-1/2,1/2}(xt) \right] f(t)\, dt = 0,$$
$$x > 0, \quad \lambda \in \mathbb{C}.$$

We have:

Theorem 9. *Let* $|\lambda| < 2$. *In order for an arbitrary function* $f \in L_2(\mathbb{R}_+)$ *to be a solution of the homogeneous integro-functional equation* (3.18), *it is necessary that* f *have the representation*

(3.19)
$$f(x) = \frac{1}{2\pi i} \int_\sigma \left[\sqrt{\frac{2}{\pi}} \Gamma(1-s) \cos\left(\frac{\pi s}{2}\right)\left(1 + \cot\left(\frac{\pi s}{2}\right)\right) - \lambda \right]^{-1} \varphi(s) x^{-s} ds,$$
$$x > 0,$$

where the integral is convergent in the mean square sense, depending on some function $\varphi(s) \in L_2(\sigma)$, *which satisfies the condition* $\varphi(s) = \varphi(1-s)$, $s \in \sigma$. *This condition and the form of solutions* (3.19) *are also sufficient for those* φ, *whose reciprocal inverse Mellin transform* $\mu(x)$ *is a solution of integral equation* (3.12).

Proof. Let $f \in L_2(\mathbb{R}_+)$ be a solution of (3.18). Then, in terms of the Mellin transform, it can be written as follows (see the proof of Theorem 5)

(3.20)
$$\sqrt{\frac{2}{\pi}}\Gamma(s)\sin\left(\frac{\pi s}{2}\right)\left(1+\tan\left(\frac{\pi s}{2}\right)\right)f^*(1-s) = -\lambda f^*(s), \quad s \in \sigma.$$

Hence, changing s on $1-s$ in the previous equation, we find

$$\sqrt{\frac{2}{\pi}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\left(1+\cot\left(\frac{\pi s}{2}\right)\right)f^*(s) = -\lambda f^*(1-s).$$

Subtracting one equality from another, after simple manipulations we end up with

$$\left[\sqrt{\frac{2}{\pi}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\left(1+\cot\left(\frac{\pi s}{2}\right)\right) - \lambda\right]f^*(s)$$
$$= \left[\sqrt{\frac{2}{\pi}}\Gamma(s)\sin\left(\frac{\pi s}{2}\right)\left(1+\tan\left(\frac{\pi s}{2}\right)\right) - \lambda\right]f^*(1-s).$$

Denoting the left-hand side of the previous equation by $\varphi(s)$, we easily verify the condition $\varphi(s) = \varphi(1-s)$, $s \in \sigma$. Moreover, via elementary calculus, we derive ($s = 1/2 + i\tau,\ \tau \in \mathbb{R}$)

$$\left|\sqrt{\frac{2}{\pi}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\left(1+\cot\left(\frac{\pi s}{2}\right)\right) - \lambda\right|$$
$$\geq \left|\sqrt{\frac{2}{\pi}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\left(1+\cot\left(\frac{\pi s}{2}\right)\right)\right| - |\lambda|$$
$$= \frac{2\sqrt{2}\cosh(\pi\tau/2)}{\cosh^{1/2}(\pi\tau)} - |\lambda| \geq 2 - |\lambda| > 0$$

and

$$\left|\sqrt{\frac{2}{\pi}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\left(1+\cot\left(\frac{\pi s}{2}\right)\right) - \lambda\right|$$
$$\leq \frac{2\sqrt{2}\cosh(\pi\tau/2)}{\cosh^{1/2}(\pi\tau)} + |\lambda| \leq 2\sqrt{2} + |\lambda|.$$

Therefore, $\varphi(s) \in L_2(\sigma)$. Hence, calling in inversion formula (1.8) of the Mellin transform, we come up with solution (3.19) of equation (3.18).

Conversely, assuming the existence of such a function $\varphi(s) \in L_2(\sigma)$ under condition $\varphi(s) = \varphi(1-s)$, we substitute the value

$$f^*(s) = \left[\sqrt{\frac{2}{\pi}}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\left(1+\cot\left(\frac{\pi s}{2}\right)\right) - \lambda\right]^{-1}\varphi(s)$$

into equation (3.20). However, after straightforward calculations, it becomes equation (3.17). Consequently, under same conclusions as in Theorem 8, we complete the proof. □

Similarly to Corollary 4, we establish

Corollary 5. *Let* $\lambda \in (-\sqrt{2}, \sqrt{2})$. *Then the only trivial solution satisfies integral equation* (3.18).

Further, integral operators (2.18) and (2.19) are employed to investigate the L_2- solvability of the following homogeneous integral equations of the second kind:

$$2\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\cos(xt) + \frac{1}{\pi}\sqrt{xt}\, S_{-3/2,-1/2}(xt)\right] f(t)\,dt = \lambda f(x), \quad x>0, \quad \lambda \in \mathbb{C}, \tag{3.21}$$

$$2\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\sin(xt) + \frac{1}{\pi}\sqrt{xt}\, S_{-1/2,1/2}(xt)\right] f(t)\,dt = \lambda f(x), \quad x>0, \quad \lambda \in \mathbb{C}. \tag{3.22}$$

Precisely, we arrive at

Theorem 10. *Let* $|\lambda| < 2$. *In order for an arbitrary function* $f \in L_2(\mathbb{R}_+)$ *to be a solution of the integral equation* (3.21), *it is necessary that* f *have the representation*

$$f(x) = \frac{1}{2\pi i}\int_\sigma \left[\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s) + \lambda\right]^{-1}\varphi(s)x^{-s}\,ds, \quad x>0, \tag{3.23}$$

where the integral is convergent in the mean square sense and depends on some function $\varphi(s) \in L_2(\sigma)$, *which satisfies the condition* $\varphi(s) = \varphi(1-s)$, $s \in \sigma$. *This condition and the form of solutions* (3.23) *are* *also sufficient for those φ, whose reciprocal inverse Mellin transform $\mu(x)$ is a solution of the integral equation*

(3.24)
$$(4-\lambda^2)\mu(x)+\frac{8}{\pi}\int_0^\infty \frac{\mu(t)}{x+t}\,dt+\frac{4}{\pi^2}\int_0^\infty \frac{\log(x/t)\mu(t)}{x-t}\,dt=0, \quad x\in\mathbb{R}_+,$$

where the integrals converge absolutely. Analogously, in order for an arbitrary function $f\in L_2(\mathbb{R}_+)$ to be a solution of the integral equation (3.22)*, it is necessary that f have the representation*

(3.25)
$$f(x)=\frac{1}{2\pi i}\int_\sigma\left[\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\sin(\pi s/2)}\Gamma(1-s)+\lambda\right]^{-1}\varphi(s)x^{-s}ds, \quad x>0,$$

where the integral is convergent in the mean square sense and depends on some function $\varphi(s)\in L_2(\sigma)$, which satisfies the condition $\varphi(s)=\varphi(1-s)$, $s\in\sigma$. This condition and the form of solutions (3.24) *are also sufficient for those φ, whose reciprocal inverse Mellin transform $\mu(x)$ is a solution of integral equation* (3.24)*.*

Proof. Let $f\in L_2(\mathbb{R}_+)$ be a solution of (3.21). Using the same technique of the Mellin transform and appealing to the proof of Theorem 6, we obtain

$$\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\sin(\pi s/2)}\Gamma(s)f^*(1-s)=\lambda f^*(s), \quad s\in\sigma. \tag{3.26}$$

The change s on $1-s$ gives

$$\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s)f^*(s)=\lambda f^*(1-s).$$

Subtracting one equality from another, we define the function φ as

$$\begin{aligned}\varphi(s)&=\left[\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s)+\lambda\right]f^*(s)\\&=\left[\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\sin(\pi s/2)}\Gamma(s)+\lambda\right]f^*(1-s),\end{aligned}$$

which evidently satisfies the equation $\varphi(s)=\varphi(1-s)$, $s\in\sigma$. Moreover, via elementary calculus we derive ($s = 1/2 + i\tau,\ \tau \in \mathbb{R}$)

$$\left|\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s)+\lambda\right| \geq \left|\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s)\right| - |\lambda|$$
$$= \frac{4\cosh^2(\pi\tau/2)}{\cosh(\pi\tau)} - |\lambda| \geq 2 - |\lambda| > 0$$

and

$$\left|\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s)+\lambda\right| \leq \frac{4\cosh^2(\pi\tau/2)}{\cosh(\pi\tau)} + |\lambda| \leq 4 + |\lambda|.$$

Therefore, $\varphi(s) \in L_2(\sigma)$. Hence, calling in inversion formula (1.8) of the Mellin transform, we come up with solution (3.23) of equation (3.21). Conversely, assuming the existence of such a function $\varphi(s) \in L_2(\sigma)$ under condition $\varphi(s) = \varphi(1-s)$, we substitute the value

$$f^*(s) = \left[\sqrt{\frac{2}{\pi}}\frac{1+\sin(\pi s)}{\cos(\pi s/2)}\Gamma(1-s)+\lambda\right]^{-1}\varphi(s)$$

into equation (3.26). But, after straightforward calculations, it becomes

$$\varphi(s)\left[\lambda^2 - 4 - \frac{4}{\sin^2(\pi s)} - \frac{8}{\sin(\pi s)}\right] = 0, \quad s \in \sigma.$$

Taking the inverse Mellin transform of both sides of the latter equality, we derive integral equation (3.24) (cf., (2.6) and (2.7)). In the same manner, we examine integral equation (3.22) and its solution (3. 25). □

Corollary 6. *Let $\lambda \in (-2, 2)$. Then the only trivial solution satisfies integral equations* (3.21), (3.22).

The final result is the solvability of the integro-functional equation, corresponding to the composition operator (2.26)

(3.27)
$$\frac{2^{3/2}}{\sqrt{\pi}}\int_0^\infty \left[\frac{1}{\pi}\log\left(\frac{x}{t}\right) - 1\right]\frac{tf(t)}{x^2-t^2}dt = \frac{\lambda}{x}f\left(\frac{1}{x}\right), \quad x > 0,\ \lambda \in \mathbb{C}.$$

Theorem 11. *Let $|\lambda| < \sqrt{2\pi}$. In order for an arbitrary function $f \in L_2(\mathbb{R}_+)$ to be a solution of the integro-functional equation* (3.27), *it is necessary that f have the representation*

$$f(x) = \frac{1}{2\pi i}\int_\sigma \left[\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\sin^2(\pi s/2)} + \lambda\right]^{-1} \varphi(s) x^{-s} ds, \quad x > 0, \tag{3.28}$$

where the integral is convergent in the mean square sense and depends on some function $\varphi(s) \in L_2(\sigma)$, which satisfies the condition $\varphi(s) = \varphi(1-s)$, $s \in \sigma$. This condition and the form of solutions (3.28) *are also sufficient for those φ, whose reciprocal inverse Mellin transform $\mu(x)$ is a solution of the integral equation*

$$(2\pi - \lambda^2)\mu(x) + 4\int_0^\infty \frac{\mu(t)}{x+t}\, dt + \frac{2}{\pi}\int_0^\infty \frac{\log(x/t)\mu(t)}{x-t}\, dt = 0, \quad x \in \mathbb{R}_+, \tag{3.29}$$

where the integrals converge absolutely.

Proof. Let $f \in L_2(\mathbb{R}_+)$ be a solution of (3.28). Similarly as above (see the proof of Theorem 7), we find

$$\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\sin^2(\pi s/2)} f^*(s) = \lambda f^*(1-s), \quad s \in \sigma. \tag{3.30}$$

Changing s on $1-s$, we have

$$\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\cos^2(\pi s/2)} f^*(1-s) = \lambda f^*(s).$$

Hence, we define the function φ as

$$\varphi(s) = \left[\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\sin^2(\pi s/2)} + \lambda\right] f^*(s) = \left[\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\cos^2(\pi s/2)} + \lambda\right] f^*(1-s),$$

and clearly, $\varphi(s) = \varphi(1-s)$, $s \in \sigma$. Moreover, $(s = 1/2 + i\tau,\ \tau \in \mathbb{R})$

$$\left|\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\sin^2(\pi s/2)} + \lambda\right| \geq \frac{2\sqrt{2\pi}\cosh^2(\pi\tau/2)}{\cosh(\pi\tau)} - |\lambda| \geq \sqrt{2\pi} - |\lambda| > 0$$

and

$$\left|\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\sin^2(\pi s/2)} + \lambda\right| \leq \frac{2\sqrt{2\pi}\cosh^2(\pi\tau/2)}{\cosh(\pi\tau)} + |\lambda| \leq 2\sqrt{2\pi} + |\lambda| > 0.$$

Therefore, $\varphi(s) \in L_2(\sigma)$. Hence, calling in inversion formula (1.8) of the Mellin transform, we come up with solution (3.28) of integro-functional equation (3.27). Conversely, assuming the existence of such a function $\varphi(s) \in L_2(\sigma)$ under condition $\varphi(s) = \varphi(1-s)$, we substitute the value

$$f^*(s) = \left[\sqrt{\frac{\pi}{2}}\frac{1+\sin(\pi s)}{\sin^2(\pi s/2)} + \lambda\right]^{-1} \varphi(s)$$

into equation (3.30). But, after straightforward calculations, it becomes

$$\varphi(s)\left[\lambda^2 - 2\pi\left[1 + \frac{1}{\sin^2(\pi s)} + \frac{2}{\sin(\pi s)}\right]\right] = 0, \quad s \in \sigma.$$

Taking the inverse Mellin transform of both sides of the latter equality we end up with integral equation (3.29) and complete the proof. $\square$

Corollary 8. *Let* $\lambda \in (-\sqrt{2\pi}, \sqrt{2\pi})$. *Then the only trivial solution satisfies integro-functional equation* (3.27).

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