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GLOBAL STABILITY CONDITION FOR COLLOCATION METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. The solution of the Volterra integral equation with degenerate kernel

$$y(t) = g(t) + \int_0^t \sum_{i=1}^n a_i(t)b_i(s)y(s)\,ds, \qquad t \geq 0,$$

is bounded provided that g and $\sum_{i=1}^n |a_i(t)|$ are bounded, and $b_j, j = 1, 2, \ldots, n$ are absolutely integrable.

It is shown that under the same hypotheses this property is inherited by the numerical solution resulting from applying exact collocation methods to this equation.

1. Introduction. The purpose of this paper is to investigate stability properties of exact collocation methods for Volterra integral equations (VIEs) of the second kind

$$(1) \qquad y(t) = g(t) + \int_0^t k(t, s, y(s))\,ds, \qquad t \in [0, T],$$

where the functions g and k are continuous. We denote by Y the unique solution of this equation.

Consider the partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the interval $[0, T]$, and put $h_i = t_{i+1} - t_i$, $\sigma_0 = [t_0, t_1], \sigma_i = (t_i, t_{i+1}], i = 1, 2, \ldots, N-1$, $Z_N = \{t_i : i = 1, 2, \ldots, N\}$. Define also the set X of collocation points by

$$X = \cup_{i=0}^{N-1} X_i,$$

where $X_i = \{t_{i,j} := t_i + c_j h_i,\ 0 \leq c_1 < c_2 < \cdots < c_m \leq 1\}$. Here, $c_j, j = 1, 2, \ldots, m$, are given collocation parameters.

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Finally, for given integers $m \geq 0$ and $d \geq -1$ define the space of polynomial splines of degree m and continuity class d by

$$S_m^{(d)} := \{u : u_i := u|_{\sigma i} \in \pi_m, \quad i = 0, 1, \dots, N-1,$$

$$u_{i-1}^{(j)}(t_i) = u_i^{(j)}(t_i), \quad j = 0, 1, \dots, d\}.$$

Here, π_m denotes the space of polynomials of degree less than or equal to m and $u|_{\sigma i}$ stands for the restriction of the function u to the interval σ_i.

The exact collocation method for VIEs approximates the solution Y of (1) by a function $u \in S_{m-1}^{(-1)}$ defined on the interval σ_i by

$$\begin{aligned} u(t_{i,j}) &= g(t_{i,j}) + \int_0^{t_i} k(t_{i,j}, s, u(s))\, ds \\ &\quad + \int_{t_i}^{t_{i,j}} k(t_{i,j}, s, u(s))\, ds, \quad j = 1, 2, \dots, m, \\ u(t) &= \sum_{k=1}^{m} u(t_{i,k}) L_k^i(t), \qquad t \in \sigma_i, \end{aligned} \tag{2}$$

$i = 0, 1, \dots, N-1$, where $L_k^i(t)$ are Lagrange fundamental polynomials for the collocation points $t_{i,j}$

$$L_k^i(t) = \prod_{\substack{j=1 \\ j \neq k}}^{m} \frac{t - t_{i,j}}{t_{i,k} - t_{i,j}}, \qquad k = 1, 2, \dots, m.$$

Observe that if $c_1 = 0$ and $c_m = 1$, then $u \in S_{m-1}^{(0)}$. It is known that if g and k are of class C^p then the method (2) is convergent to Y and the order of convergence is:

$$\|Y - u\|_\infty = \begin{cases} O(h^p) & \text{if } 1 \leq p < m \\ O(h^m) & \text{if } p \geq m \end{cases}$$

where $h := \max_i h_i$, and $\|Y - u\|_\infty := \sup\{|Y(t) - u(t)| : t \in [0, T]\}$.

A number of superconvergence results have also been obtained for (2); see [**3**] or [**4**] for a survey of results in this area. For example, if the collocation parameters $\{c_j\}$ are the Radau II points for $(0, 1]$, then in a sufficiently smooth situation,

$$\max\{|Y(t_i) - u(t_i)| : t_i \in Z_N\} = 0(h^{2m-1}), \quad h \to 0;$$

and if $\{c_j\}$ are the Lobatto points for $[0, 1]$, then

$$\max\{|Y(t_i) - u(t_i)| : t_i \in Z_N\} = 0(h^{2m-2}), \quad h \to 0.$$

However, contrary to the case of ordinary differential equations, there is no superconvergence if $\{c_j\}$ are the Gauss points for $(0, 1)$. Refer to [**3**] for an explanation of this phenomenon.

The purpose of this paper is to investigate stability properties of (2) with respect to the test equation with degenerate kernel

$$y(t) = g(t) + \int_0^t \sum_{i=1}^n a_i(t) b_i(s) y(s)\, ds, \qquad t \geq 0, \tag{3}$$

where g, a_i and b_i are always assumed to be continuous.

The importance of this equation in testing stability properties of numerical methods for VIEs follows from the fact that degenerate kernels are dense in the space of all continuous kernels $k(t, s)$, see [**4**] for the discussion of this topic.

Application of numerical method for VIEs to the equation (3) lead to recurrence relations with variable coefficients which are, in general, difficult to investigate. Therefore, it is not surprising that stability results with respect to (3) obtained up to date are of local nature. They are based, in principle, on "freezing" the variable coefficients in these recursions. For example, van der Houwen and Wolkenfelt [**7**] have studied local stability properties of Volterra linear multistep methods for (1). Similar results have been obtained by Brunner and van der Houwen [**4**] for indirect linear multistep methods. Crisci et. al. [**5**] have formulated local stability conditions for exact collocation methods (3). Refer also to [**1**] and [**6**] for related results.

In this paper, using a completely different approach from that given in the above papers, we have arrived at global stability conditions for the method (2) with respect to the test equation (3).

The essence of the result is the following: assuming that g and $\sum_{i=1}^{n} |a_i(t)|$ are bounded and $b_i, i = 1, 2, \ldots, n$, are absolutely integrable, the solution Y of (3) is bounded (see §2), and it has been proved that every exact collocation method is stable, in the sense that, under the same hypotheses on the equation (3), the numerical solution inherits the property of boundedness (see §3).

2. Boundedness of solutions of the integral equation with degenerate kernel. In this section we will prove the following result.

THEOREM 1. *Assume that g and $\sum_{i=1}^{n} |a_i(t)|$ are bounded in $[0, \infty)$ and that $b_i \in L^1[0, \infty)$*, i.e. *$\int_0^\infty |b_i(s)|\, ds < \infty, i = 1, 2, \ldots, n$. Then the solution Y of* (3) *is also bounded.*

PROOF. It could be proved that the above hypotheses imply those of the theorem 2.1 of [**2**], concerning the uniform stability of (3), choosing as logarithmic norm μ_∞. The following direct proof is more suitable for our purposes, as it gives hints for the analogous proof in the discrete case.

Define $b(t) := \max_j |b_j(t)|$ and denote by A, B, and G constants such that

$$||g||_\infty \leq G, \qquad \sum_{i=1}^{n} |a_i(t)| \leq A, \qquad t \geq 0, \qquad \int_0^\infty b(s)\, ds \leq B.$$

Putting

$$\xi_i(t) = \int_0^t b_i(s) y(s)\, ds, \quad i = 1, 2, \ldots n,$$

the equation (3) can be written in the form

$$y(t) = g(t) + \sum_{i=1}^{n} a_i(t) \xi_i(t),$$

where the functions $\xi_i(t)$ are solutions of the system of differential equations

$$\begin{aligned} \xi_i'(t) &= \sum_{k=1}^{n} a_k(t) b_i(t) \xi_k(t) + b_i(t) g(t), \\ \xi_i(0) &= 0. \end{aligned}$$

Define $\xi(t) := \max_j |\xi_j(t)|$. Then, integrating the above system of differential equations, it follows that

$$|\xi_i(t)| \leq \int_0^t \sum_{k=1}^n |a_k(s)| b(s) \xi(s)\, ds + BG,$$

and since the right hand side is independent of i then

$$\xi(t) \leq A \int_0^t b(s)\xi(s)\, ds + BG.$$

The application of the Gronwall's inequality to this relation yields

$$\xi(t) \leq BG \exp\Big(A \int_0^t b(s)\, ds\Big), \qquad t \geq 0.$$

But $b \in L^1[0,\infty)$, therefore $||\xi||_\infty \leq BG\exp(AB)$ and $||Y||_\infty \leq G(1 + AB\exp(AB))$ which concludes the proof. □

3. Stability analysis of the exact collocation methods. The application of the method (2) to the test equation (3) leads to:

$$\begin{aligned} u_i(t_{i,j}) &= g(t_{i,j}) + \sum_{l=1}^n a_l(t_{i,j}) \int_0^{t_i} b_l(s) u(s)\, ds \\ &\quad + \sum_{l=1}^n a_l(t_{i,j}) \int_{t_i}^{t_{i,j}} b_l(s) u_i(s)\, ds \qquad j = 1, 2, \ldots, m \\ u_i(t) &= \sum_{k=1}^m u_i(t_{i,k}) L_k^i(t) \qquad t \in \sigma_i, \qquad i = 0, 1, \ldots \end{aligned}$$

Put

$$z_j(t) = \int_0^t b_j(s) u(s)\, ds, \qquad j = 1, 2, \ldots, n.$$

It follows that

$$\begin{aligned} u_i(t_{i,j}) &= g(t_{i,j}) + \sum_{l=1}^n a_l(t_{i,j}) z_l(t_i) \\ &\quad + \sum_{k=1}^m \Big(\sum_{l=1}^n a_l(t_{i,j}) \int_{t_i}^{t_{i,j}} b_l(s) L_k^i(s)\, ds \Big) u_i(t_{i,k}) \end{aligned} \tag{4}$$

$$z_j(t_{i+1}) = z_j(t_i) + \sum_{k=1}^{m} \Big(\int_{t_i}^{t_{i+1}} b_j(s) L_k^i(s)\, ds \Big) u_i(t_{i,k}) \tag{5}$$

Define the matrices

$$\mathbf{A}^i = \left[\alpha_{j,k}^i\right]_{j=1,\ k=1}^{m \quad n} \qquad \alpha_{j,k}^i = a_k(t_{i,j})$$

$$\mathbf{B}^i = \left[\beta_{j,k}^i\right]_{j=1,\ k=1}^{n \quad m} \qquad \beta_{j,k}^i = \int_{t_i}^{t_{i+1}} b_j(s) L_k^i(s)\, ds$$

$$\mathbf{S}^i = \left[s_{j,k}^i\right]_{j,k=1}^{n} \qquad s_{j,k}^i = \sum_{l=1}^{n} a_l(t_{i,j}) \int_{t_i}^{t_{i,j}} b_l(s) L_k^i(s)\, ds,$$

and put

$$\begin{aligned} u_{i+1} &= [u_i(t_{i,1}), u_i(t_{i,2}), \dots, u(t_{i,m})]^T \\ z_{i+1} &= [z_1(t_{i+1}), z_2(t_{i+1}), \dots, z_n(t_{i+1})]^T, \\ g_i &= [g(t_{i,1}), g(t_{i,2}), \dots, g(t_{i,m})]^T. \end{aligned}$$

Then the relations (4) and (5) can be written in the following vector form:

$$u_{i+1} = \mathbf{A}^i z_i + \mathbf{S}^i u_{i+1} + g_i, \tag{6}$$

$$z_{i+1} = z_i + \mathbf{B}^i u_{i+1} \tag{7}$$

$i = 0, 1, \dots$. Deriving u_{i+1} from the first relation and substituting in the second one, with easy algebraic manipulations the above relation can be rewritten in the form:

$$u_{i+1} = (\mathbf{I}_m - \mathbf{S}^i)^{-1} \mathbf{A}^i z_i + (\mathbf{I}_m - \mathbf{S}^i)^{-1} g_i, \tag{8}$$

$$z_{i+1} = (\mathbf{I}_n + \mathbf{B}^i (\mathbf{I}_m - \mathbf{S}^i)^{-1} \mathbf{A}^i) z_i + \mathbf{B}^i (\mathbf{I}_m - \mathbf{S}^i)^{-1} g_i, \tag{9}$$

$i = 0, 1, \dots,$ where $\mathbf{I}_n$ stands for n-dimensional identity matrix.

This recurrence equation was obtained before in [**5**]. We implicitly assumed that the collocation equations (6), (7) have a solution, that is $\det(\mathbf{I}_m - \mathbf{S}^i) \neq 0$. This is surely true for sufficiently small $h > 0$, since from the definition of the elements of $\mathbf{S}^i$ it follows:

$$s_{jk}^i = O(h).$$

Now consider the collocation approximation on the knots of the positive real line $[0,\infty)$ $t_i = ih_i$, where $t_i \to \infty$ as $i \to \infty$ (e.g. $h_i = h$ for every i).

We have the following discrete analogue of Theorem 1.

THEOREM 2. *Assume that g and $\sum_{k=1}^{n} |a_k(t)|$ are bounded in $[0,\infty)$; $b_k \in L^1[0,\infty), k = 1,\ldots,n$. Then every exact collocation method is stable.*

PROOF. As stated in the introduction, in order to prove the stability of the methods, we show that every solution of the recurrence relations (8) and (9) is bounded.

First investigate the relation (8). We have

$$z_{i+1} = \mathbf{M}_i z_i + \omega_i, \tag{10}$$

where

$$\mathbf{M}_i := \mathbf{I}_n + \mathbf{B}^i(\mathbf{I}_m - \mathbf{S}^i)^{-1}\mathbf{A}^i,$$
$$\omega_i := \mathbf{B}^i(\mathbf{I}_m - \mathbf{S}^i)^{-1} g_i.$$

The solution of (10) is given by

$$z_i = \prod_{\nu=0}^{i-1} \mathbf{M}_{i-1-\nu} z_0 + \sum_{\mu=0}^{i-1} \prod_{\nu=\mu+1}^{i-1} \mathbf{M}_{i+\mu-\nu}\omega_\mu,$$

$i = 0,1,\ldots, \prod_{\nu=1}^{0} := 1,\ \sum_{\nu=1}^{0} := 0$, and it follows that

$$||z_i||_\infty \le \prod_{\nu=0}^{i-1} ||\mathbf{M}_\nu||_\infty ||z_0||_\infty + \sum_{\mu=0}^{i-1} \prod_{\nu=\mu+1}^{i-1} ||\mathbf{M}_\nu||_\infty ||\omega_\mu||_\infty. \tag{11}$$

To estimate $\prod_{\nu=0}^{i-1} ||\mathbf{M}_\nu||_\infty$ observe that

$$||\mathbf{M}_i||_\infty \le 1 + ||\mathbf{A}^i||_\infty ||\mathbf{B}^i||_\infty ||(\mathbf{I}_m - \mathbf{S}^i)^{-1}||_\infty,$$

and

$$\|\mathbf{A}^i\|_\infty = \max\Big\{\sum_{\mu=1}^{n} |a_\mu(t_{i,\nu})| : \nu = 1,2,\ldots,m\Big\} \le \mathrm{A},$$

$$\begin{aligned}\|\mathbf{B}^i\|_\infty &= \max\Big\{\sum_{\mu=1}^{m}\int_{t_i}^{t_{i+1}} |b_\nu(s)||L_k^i(s)|\,ds : \nu = 1,2,\ldots,n\Big\}\\ &\le Q_m \max\Big\{\int_{t_i}^{t_{i+1}} |b_\nu(s)|\,ds : \nu = 1,2,\ldots,n\Big\}\\ &\le Q_m \int_{t_i}^{t_{i+1}} b(s)\,ds.\end{aligned}$$

Here, A and $b(s)$ are defined as in Section 2 and

$$Q_m := \sup\Big\{\sum_{\mu=1}^{m} |l_\mu(s)| : s \in [0,1]\Big\},$$

where $l_\mu(s)$ are Lagrange fundamental polynomials with respect to $\{c_j\}$. We have also

$$\begin{aligned}\|\mathbf{S}^i\|_\infty &= \max\Big\{\sum_{j=1}^{n} |a_j(t_{i,\nu})| \int_{t_i}^{t_{i,\nu}} |b_j(s)| \sum_{\mu=1}^{m} |L_\mu^i(s)|\,ds : \nu = 1,2,\ldots m\Big\}\\ &\le AQ_m \int_{t_i}^{t_{i+1}} b(s)\,ds \le AQ_m \int_{t_i}^{\infty} b(s)\,ds.\end{aligned}$$

Since $b_\mu \in L^1[0,\infty)$ for $\mu = 1,2\ldots,n$, it follows that $b \in L^1[0,\infty)$ and $\|\mathbf{S}^i\|_\infty \to 0$ as $i \to \infty$.

Therefore, there exists N such that $\|\mathbf{S}^i\|_\infty < 1$ for $i \ge N$ and

$$\|(\mathbf{I}_m - \mathbf{S}^i)^{-1}\|_\infty \le \frac{1}{1 - \|\mathbf{S}^i\|_\infty} \le P, \qquad i \ge N,$$

for some constant P independent of i. Since, if the method is applicable, $\det(\mathbf{I}_m - \mathbf{S}^i) \ne 0, i \ge 0$, we can assume without loss of generality that $\|(\mathbf{I}_m - \mathbf{S}^i)^{-1}\|_\infty \le P$ for any $i \ge 0$. Consequently,

$$\prod_{\nu=0}^{i-1} \|\mathbf{M}_\nu\|_\infty \le \prod_{\nu=0}^{i-1} (1 + AP\|\mathbf{B}^\nu\|_\infty), \qquad i \ge 0.$$

But
$$\sum_{\nu=0}^{\infty} ||\mathbf{B}^{\nu}||_{\infty} \leq Q_m \sum_{\nu=0}^{\infty} \int_{t_\nu}^{t_{\nu+1}} b(s)\, ds \leq BQ_m,$$

hence the infinite product $\prod_{\nu=0}^{\infty}(1+AP||\mathbf{B}^{\nu}||_{\infty})$ is convergent. Therefore, there exists a constant C independent of i such that

$$\prod_{\nu=0}^{i-1} ||\mathbf{M}_{\nu}||_{\infty} \leq C, \qquad i=0,1,\ldots$$

In view of (11) we obtain

$$\begin{aligned} ||z_i||_{\infty} &\leq C\Big(||z_0||_{\infty} + \sum_{\mu=0}^{i-1} ||\omega_{\mu}||_{\infty}\Big) \\ &\leq C\Big(||z_0||_{\infty} + \sum_{\mu=0}^{\infty} \frac{||\mathbf{B}^{\mu}||_{\infty}||g_{\mu}||_{\infty}}{1-||\mathbf{S}^{\mu}||_{\infty}}\Big) \\ &\leq C\Big(||z_0||_{\infty} + GPBQ_m\Big) := D, \end{aligned}$$

which proves that the sequence $\{z_i\}_{i=0}^{\infty}$ is bounded. Taking into account the relation (8) we have

$$||u_i||_{\infty} \leq P(AD+G),$$

and in view of the definition of $u(t)$ for $t \in \sigma_i$ we obtain $||u||_{\infty} \leq Q_m P(AD+G)$. This completes the proof. □

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