JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 2, Number 1, Winter 1989

COLLOCATION METHODS FOR SECOND KIND INTEGRAL EQUATIONS WITH NON-COMPACT OPERATORS

S. AMINI AND I.H. SLOAN

ABSTRACT. We study the uniform convergence of collocation methods for integral equations on the half line, where the integral operator is a compact perturbation of a Wiener-Hopf operator. We prove that the collocation and the iterated collocation solutions converge to the exact solution with optimal orders of convergence, provided the meshes are appropriately graded to take account of the asymptotic behavior of the solution. As a consequence of the analysis similar optimal convergence results are proved for the case of boundary integral equations on polygonal domains.

1. Introduction. Initially, consider second-kind integral equations of the form

$$(I-\mathbf{K})x = y, \tag{1.1}$$

where $\mathbf{K}$ is a bounded linear operator on X^+, the Banach space of bounded continuous functions on $\mathbf{R}^+ = [0,\infty)$ with the supremum norm, and is given by

$$(\mathbf{K}x)(s) = \int_0^\infty K(s,t)x(t)\,dt, \quad s\in\mathbf{R}^+,\ x\in X^+. \tag{1.2}$$

Consider the case where the kernel of the half-line operator is of the form

$$K(s,t) = \kappa(s-t) + K_1(s,t), \tag{1.3}$$

where $\kappa\in L_1(\mathbf{R})$ and $K_1(s,t)$ is a "short ranged" kernel satisfying

$$\sup_{s\in\mathbf{R}^+}\int_0^\infty |K_1(s,t)|\,dt < \infty, \tag{1.4a}$$

$$\lim_{s'\to s}\int_0^\infty |K_1(s',t)-K_1(s,t)|\,dt = 0 \quad \text{uniformly for } s\in\mathbf{R}^+, \tag{1.4b}$$

$$\lim_{s\to\infty}\int_0^\infty |K_1(s,t)|\,dt = 0. \tag{1.4c}$$

The function spaces we will be generally working with are subspaces of X^+, namely

$$X_l^+ = \{x \in X^+ : x(\infty) \text{ exists}\}, \tag{1.5a}$$

$$X_0^+ = \{x \in X^+ : x(\infty) = 0\}, \tag{1.5b}$$

(1.5c)
$$C_l^{r,\mu} = \{x \in X_l^+ : \|\bar{x}\|_{r,\mu} < \infty \text{ with } \bar{x}(s) = x(s) - x(\infty),\ s \in \mathbf{R}^+\},$$

where r is a non-negative integer, $\mu \in \mathbf{R}^+$ and

$$\|x\|_{r,\mu} = \sup_{s\in\mathbf{R}^+}\{|e^{\mu s}D^l x(s)| : \quad 0 \le l \le r\}, \tag{1.5d}$$

where Dx is the derivative of x. The conditions (1.4) ensure that the operator $\mathbf{K}_1$ with kernel $K_1(s,t)$ is a compact operator from X^+ to X_0^+ and hence also from X_l^+ to X_0^+, (see [**2**]). The condition $\kappa \in L_1(\mathbf{R})$ together with (1.3) and (1.4a) imply that

$$\|\mathbf{K}\| = \sup_{s\in\mathbf{R}^+}\int_0^\infty |K(s,t)|\,dt < \infty \tag{1.6}$$

and so $\mathbf{K}$ is a bounded operator from X^+ to X^+, and also from X_l^+ to X_l^+ or from X_0^+ to X_0^+; (see [**2,16**]). In general, however, the operator $\mathbf{K}$ is non-compact, as its spectrum $\sigma(\mathbf{K})$ in X^+ can be shown to contain a non-discrete set. For example, the spectrum of $\mathbf{K}$ with $K(s,t) = \kappa(s-t) = e^{-|s-t|}$ with respect to X^+ is [0,2], (see [**1, 2, 13, 18**] for more information). Assume throughout that the problem (1.1) is well-posed in the sense that $(I-\mathbf{K})^{-1}$ is bounded on X^+ and hence on X_l^+.

Initially in this paper assume x, the solution of (1.1), belongs to $C_0^{r,\mu} = C_l^{r,\mu} \cap X_0^+$ for some $\mu > 0$. Later, in §4, we generalize the results to the case $x \in C_l^{r,\mu}$. In the case of the pure Wiener-Hopf equations, $K(s,t) = \kappa(s-t)$, there are many practical examples in which $x \in C_0^{r,\mu}$ or $x \in C_l^{r,\mu}$; for examples, see [**9**] and §6.

To solve (1.1) numerically for the case $x \in C_0^{r,\mu}$ we first approximate it by the finite section equation

$$(1.7) \qquad (I - \mathbf{K}_\beta)x_\beta = y, \quad \beta > 0,$$

with

$$(1.8) \qquad (\mathbf{K}_\beta x)(s) = \int_0^\beta K(s,t)x(t)\,dt, \quad s \in \mathbf{R}^+.$$

The recent work of [**1**] on the finite section approximation was primarily concerned with the convergence of x_β to x on finite intervals. In this work, however, uniform convergence becomes possible, because $x \in C_0^{r,\mu}$ (or, later, $x \in C_l^{r,\mu}$). Throughout this paper we shall be interested in uniform convergence results.

The equation (1.7) is now discretised by a collocation method, which can be written as a projection method of the form

$$(1.9) \qquad (I - P_n\mathbf{K}_\beta)x_{\beta,n} = P_n y.$$

To specify the projection operator P_n let Π_n denote the mesh partitioning of $[0,\infty]$ given by

$$(1.10) \qquad 0 = s_1^{(n)} < s_2^{(n)} < \cdots < s_n^{(n)} = \beta < s_{n+1}^{(n)} = +\infty,$$

and let $S_r(\Pi_n)$ denote the space of polynomial splines of order r (i.e. degree not greater than $r-1$), on each subinterval $I_i^{(n)} = [s_i^{(n)}, s_{i+1}^{(n)})$ for $i = 1,2,\ldots,n$. The splines are required to be bounded on $\mathbf{R}^+$ but are allowed to be discontinuous at the knots; they are well defined even at the knots because of the assumed right continuity at $s_1^{(n)}, s_2^{(n)}, \ldots, s_n^{(n)}$. We now choose r points $\{\xi_j : 1 \le j \le r\}$ with $0 \le \xi_1 < \xi_2 < \cdots < \xi_r < 1$, which we refer to as the "basic quadrature nodes", and define our collocation points by

$$(1.11) \qquad s_{ij}^{(n)} = s_i^{(n)} + \xi_j h_i^{(n)}, \quad i = 1,2,\ldots,n-1;\ j = 1,2,\ldots,r,$$

where $h_i^{(n)} = s_{i+1}^{(n)} - s_i^{(n)}$. For ease of notation, we shall not explicitly show the dependence of certain quantities on n. Now define the interpolatory projection $P_n : S_r(\Pi_n) + X_l^+ \to S_r(\Pi_n)$ as follows:

$$(1.12a) \qquad (P_n v)(s) = \sum_{j=1}^{r} l_{ij}(s)v(s_{ij}), \quad \text{for } s \in I_i;\ i = 1,2,\ldots,n-1,$$

and

$$(1.12b) \qquad (P_n v)(s) \equiv 0, \quad \text{for } s \in I_n,$$

where

$$(1.12c) \qquad l_{ij}(s) = \prod_{\substack{k=1 \\ k \neq j}}^{r} \frac{(s - s_{ik})}{(s_{ij} - s_{ik})},$$

are the Lagrangian basis functions. (To avoid technical difficulties in defining $\mathbf{K}P_n$ later, consider $\mathbf{K} : S_r(\Pi_n) + X_l^+ \to X_l^+$.)

In this setting, we prove that if appropriate knowledge of the asymptotic behavior of x is available, it is possible to "grade" the meshes (1.10) as n increases so as to obtain

$$(1.13) \qquad \|x - x_{\beta,n}\| \leq Cn^{-r},$$

which is the (optimal) result one expects for second-kind integral equations with compact operators [**3**]. (Throughout this paper C is a generic constant, the value of which may be different at different instances.) Clearly, convergence requires $\beta = \beta(n) \to \infty$ as $n \to \infty$.

To study the convergence of the "global error" $\|x - x_{\beta,n}\|$ we shall study the "truncation error" $\|x - x_\beta\|$ and the "discretisation error" $\|x_\beta - x_{\beta,n}\|$ separately in §2 and §3 respectively. We prove in §5 that if the basic quadrature nodes $\{\xi_j : 1 \leq j \leq r\}$ are chosen appropriately, superconvergence may be observed at the collocation points. Some numerical results are provided in §6. Our results for the half-line equations extend those of [**18**], and complement similar results proved for the Nystrom method in [**9**]. In the final section we show how simple modifications of our analysis for the half-line problems can yield uniform convergence results for collocation methods for boundary integral equations on polygonal domains, thereby extending the results of [**8**].

Sometimes it will be convenient in practice to choose a basic quadrature node at each end point; that is to choose $0 = \xi_1 < \xi_2 < \cdots < \xi_r = 1$. All of our analysis remains valid for that case if $S_r(\Pi_n)$ is reinterpreted as the space of continuous polynomial splines of order r.

2. Study of the truncation error. It follows from (1.1) and (1.7) that

$$(I - \mathbf{K}_\beta)(x - x_\beta) = (\mathbf{K} - \mathbf{K}_\beta)x. \tag{2.1}$$

Here we wish to find conditions under which $\|x - x_\beta\| = O(n^{-r})$. It has been proved in [**1**] that provided $\mathbf{K}$ satisfies the conditions in §1 and $(I - \mathbf{K})^{-1}$ exists as a bounded operator on X^+ then $(I - \mathbf{K}_\beta)^{-1}$ exists as a bounded operator on X^+, and furthermore

$$\|(I - \mathbf{K}_\beta)^{-1}\| \leq B_0 < \infty, \tag{2.2}$$

for all β sufficiently large. It is then easy to deduce that (2.2) also holds in the space X_l^+; see [**18**].

It now follows from (2.1) and (2.2) that for β sufficiently large

$$\|x - x_\beta\| \leq B_0 \|(\mathbf{K} - \mathbf{K}_\beta)x\|. \tag{2.3}$$

We now wish to obtain conditions under which $\|(\mathbf{K}-\mathbf{K}_\beta)x\| = O(n^{-r})$. We have

$$|(\mathbf{K} - \mathbf{K}_\beta)x(s)| \leq \int_\beta^\infty |K(s,t)x(t)|\,dt \leq \|\mathbf{K}\| \sup_{t \geq \beta} |x(t)|. \tag{2.4}$$

If $x \in C_0^{r,\mu}$ then it follows from (1.5c), (1.5d) and (2.4) that

$$\|(\mathbf{K} - \mathbf{K}_\beta)x\| \leq \|\mathbf{K}\| \;\; \|x\|_{r,\mu} e^{-\mu\beta}. \tag{2.5}$$

Clearly, therefore, if $\beta = \beta(n)$ is appropriately chosen, namely

$$\beta(n) = (r/\mu) \ln n + O(1), \tag{2.6}$$

then we obtain the required result, $\|(\mathbf{K} - \mathbf{K}_\beta)x\| = O(n^{-r})$. To obtain (2.5) it would have been sufficient to require $x \in C_0^{0,\mu}$, but in the next section, in the study of the discretisation error, we need the stronger condition $x \in C_0^{r,\mu}$.

Let us state precisely the result just proved.

THEOREM 2.1. *Let x, the solution of* (1.1), *belong to $C_0^{r,\mu}$ for some non-negative integer r and real $\mu > 0$ and let $K(s,t)$ satisfy the* *conditions in* §1. *Then if* $\beta(n)$ *satisfies* (2.6), *for* n *sufficiently large we have*

$$\|x - x_{\beta(n)}\| = O(n^{-r}). \tag{2.7}$$

3. Study of the Discretisation Error. It follows from (1.7) and (1.9) that

$$(I - P_n\mathbf{K}_\beta)(x_\beta - x_{\beta,n}) = (I - P_n)x_\beta. \tag{3.1}$$

We wish to obtain conditions, consistent with (2.6), under which $\|x_\beta - x_{\beta,n}\| = O(n^{-r})$. This we will do by establishing the stability condition

$$\|(I - P_n\mathbf{K}_\beta)^{-1}\| \leq B_1 < \infty \tag{3.2}$$

for n sufficiently large, together with the consistency of order r of the discretisation by proving that

$$\|(I - P_n)x_\beta\| = O(n^{-r}). \tag{3.3}$$

Let us initially concentrate on (3.3).

CONSISTENCY 3.1. We have

$$(I - P_n)x_\beta = (I - P_n)x - (I - P_n)(x - x_\beta). \tag{3.4}$$

It follows from (1.12) that

$$\|P_n\| = \max_{1\leq i\leq n-1} \|P_n\|_i = \max_i \sup_{s\in I_i} \sum_{j=1}^{r} |l_{ij}(s)| = C_r. \tag{3.5}$$

From (3.4) and (3.5) we have

$$\|(I - P_n)x_\beta\| \leq \|(I - P_n)x\| + (1 + C_r)\|x - x_\beta\|. \tag{3.6}$$

If $x \in C_0^{r,\mu}$ and $\beta(n)$ satisfies (2.6), it follows from Theorem 2.1 and (3.6) that to establish (3.3) it suffices to show

$$\|(I - P_n)x\| = O(n^{-r}). \tag{3.7}$$

Now (3.7) is relatively easy to establish, as it involves the error in interpolation by r-th order piecewise polynomials, which can be appropriately controlled provided the correct mesh grading is used. We have

$$\|(I-P_n)x\| = \max_{1\le i\le n} \|(I-P_n)x\|_i = \max_i \sup_{s\in I_i} |(I-P_n)x(s)|. \tag{3.8}$$

Let us consider first $\|(I-P_n)x\|_n$, i.e. the error in the infinite interval $I_n = [\beta,\infty)$. By the definition (1.12b), $(P_n v)(s) = 0$ for $s \in I_n$ and it follows from (1.5) and (2.6) that

$$\|(I-P_n)x\|_n = \|x\|_n = \sup_{s\ge\beta} |x(s)| \le e^{-\mu\beta}\|x\|_{r,\mu} = O(n^{-r}). \tag{3.9}$$

For $i = 1,2,\dots,n-1$ we have

$$\begin{aligned} \|(I-P_n)x\|_i &= \sup_{s\in I_i} \left| x(s) - \sum_{j=1}^{r} l_{ij}(s)x(s_{ij}) \right| \\ &\le Ch_i^r \sup_{s\in I_i} |D^r x(s)| \\ &\le Ch_i^r e^{-\mu s_i} \|x\|_{r,\mu}. \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that an appropriate choice of $\{s_i\}$ to ensure (3.7) is given by

$$s_i = (r/\mu)\ln\left(\frac{m}{m+1-i}\right), \quad i = 1,2,\dots,n, \tag{3.11}$$

where $m \ge n$. In fact we shall choose $m = n + l$ where l is some small non-negative number independent of n, the significance of which will be made apparent shortly. Notice that with this choice $s_n = \beta(n)$ satisfies (2.6) and furthermore

$$h_i = (r/\mu)\ln\left(1+\frac{1}{m-i}\right) \le \left(\frac{r}{\mu}\right)\left(\frac{1}{m-i}\right), \quad i = 1,2,\dots,n-1. \tag{3.12}$$

There are many other choices of the knots $\{s_i\}$ which will be satisfactory in yielding the required results in (3.10); any such mesh is referred to as a (r,μ)-graded mesh [**9**, **11**].

Let us now summarize our result on the order r consistency of the approximation scheme, the proof of which follows from (3.6) through (3.12).

THEOREM 3.1. *Let the conditions of Theorem 2.1 be satisfied. If the knots $\{s_i\}$ are given by (3.11) where $m = n + l \geq n$, then*

$$\|(I - P_n)x_\beta\| = O(n^{-r}).$$

STABILITY 3.2. In order to establish (3.2), i.e. the property of uniform boundedness of $(I - P_n\mathbf{K}_\beta)^{-1}$, we use the result of [**1**], quoted in (2.2). In fact it turns out to be easier to first prove the uniform boundedness of $(I - \mathbf{K}_\beta P_n)^{-1}$, (a result which will in any case be required in §5 for the study of superconvergence of the iterated collocation method) and then deduce (3.2) from the identity

$$(I - P_n\mathbf{K}_\beta)^{-1} = I + P_n(I - \mathbf{K}_\beta P_n)^{-1}\mathbf{K}_\beta. \tag{3.13}$$

We can write, at least formally at this stage,

$$(I - \mathbf{K}_\beta P_n)^{-1} = L_{\beta,n}^{-1}(I - \mathbf{K}_\beta + \mathbf{K}_\beta P_n), \tag{3.14}$$

where

$$L_{\beta,n} = (I - \mathbf{K}_\beta) + (\mathbf{K}_\beta - \mathbf{K}_\beta P_n)\mathbf{K}_\beta P_n. \tag{3.15}$$

If we can prove that

$$\alpha_n := \|(\mathbf{K}_\beta - \mathbf{K}_\beta P_n)\mathbf{K}_\beta P_n\| \leq (\delta_0/B_0) \quad \text{for some } 0 < \delta_0 < 1, \tag{3.16}$$

then (from (2.2)) using Banach's Lemma, the existence and uniform boundedness of $L_{\beta,n}^{-1}$ is established. The required stability result (3.2) then follows from (3.13), (3.14), (1.6) and (3.5). Ideally therefore we wish to be able to prove that for any $\epsilon > 0$, (3.16) is satisfied with ϵ replacing δ_0/B_0, provided n is sufficiently large. We shall see that this is the case if we choose $l = m - n$ sufficiently large.

Now study the left hand side of (3.16) by considering $(\mathbf{K}_\beta - \mathbf{K}_\beta P_n)\mathbf{K}_\beta P_n x = \mathbf{K}_\beta(I - P_n)v$, where $v = \mathbf{K}_\beta P_n x$, with $x \in X_l^+ +$

$S_r(\Pi_n)$. The smoothness of $v(s)$ depends on the smoothness properties of $K(s,t)$. At this stage, not wishing to restrict the class of integral equations covered by our analysis, we assume only properties of v that follow from the definition above. We have, using Jackson's Theorem [**6**, p.33] and (3.5)

$$\begin{aligned}
&|\mathbf{K}_\beta(I-P_n)v(s)| \\
&\quad = \Big|\sum_{i=1}^{n-1}\int_{s_i}^{s_{i+1}} K(s,t)\Big[v(t)-\sum_{j=1}^{r} l_{ij}(t)v(s_{ij})\Big]\,dt\Big| \\
&\quad \le C\sum_{i=1}^{n-1}\omega(v,I_i;\delta_i)\int_{s_i}^{s_{i+1}}|K(s,t)|\,dt \\
&\quad \le C_0 W_r(v;\Pi_n),
\end{aligned} \tag{3.17}$$

where $\delta_i = h_i/(2r-2)$, and

$$\omega(v,I;\delta) = \sup\{|v(s')-v(s)| : s',\ s\in I,\ |s'-s|<\delta\} \tag{3.18}$$

is the modulus of continuity of v at δ over I. Further

$$\begin{aligned}
W_r(v;\Pi_n) &= \max\{\omega(v,I_i;\delta_i)) : 1\le i\le n-1\} \\
&\le \omega\Big(v,\mathbf{R}^+;\frac{r}{\mu(m+1-n)(2r-2)}\Big),
\end{aligned} \tag{3.19}$$

where the last inequality follows from the monotonicity property of $\omega(v,I;\delta)$ in δ and the fact that $\{h_i\}$ given by (3.12) form an increasing sequence, with $h_i \le h_{n-1} \le \frac{r}{\mu(m+1-n)}$, for $i=1,2,\dots,n-1$. Now

$$\begin{aligned}
|v(s')-v(s)| &= \Big|\int_0^\beta [K(s',t)-K(s,t)](P_nx)(t)\,dt\Big| \\
&\le \int_0^\infty |K(s',t)-K(s,t)|\,dt \quad \|P_n\|\ \|x\|.
\end{aligned} \tag{3.20}$$

It follows from (1.3), (3.5) and (3.17) through (3.20) that

$$\begin{aligned}
\|\mathbf{K}_\beta(I-P_n)\mathbf{K}_\beta P_n\| \le B_r \sup_{|s'-s|<\delta^*}\Big\{&\int_{-\infty}^{\infty}|\kappa(s'-s+t)-\kappa(t)|\,dt \\
&+\int_0^\infty |K_1(s',t)-K_1(s,t)|\,dt\Big\}
\end{aligned}$$

with

$$\delta^* = \frac{r}{\mu(m+1-n)(2r-2)}, \tag{3.21}$$

which can be made arbitrarily small by taking $m - n = l$ appropriately large. Thus (3.16) is satisfied if l is sufficiently large. Observe that the restriction on the value of $l = m - n$ is independent of n and depends, through (3.16) and (2.2), on the conditioning of the finite section equation as well as on r and μ.

We now state a stability result, the proof of which follows from (3.13) through (3.21).

THEOREM 3.2. *If the kernel $K(s,t)$ of (1.1) satisfies the conditions in §1, and if the knots $\{s_i\}$ are given by (3.11), then for n sufficiently large*

$$\|(I - P_n\mathbf{K}_\beta)^{-1}\| \leq C < \infty \quad \textit{uniformly in } n, \tag{3.22}$$

provided $l = m - n$ is appropriately large.

REMARKS. (a) For practical implementation, the choice $l = m - n = 0$ will often be satisfactory. Choosing larger values for l has the effect of significantly decreasing the maximum step-length h_{n-1}, but at the same time (because β is thereby reduced), it increases the truncation error, (though does not affect its $O(n^{-r})$ convergence to zero). In practice the best procedure may be to use initially $l = 0$ in (3.11), i.e. to use the formula $s_i = (r/\mu)\ln(n/(n+1-i))$. Then larger values of l can be tested without recalculating the matrix: all that is necessary is to define a new value of 'n' by $n' = n - l$, and a new value of 'β' by $\beta' = s_{n'}$, and to omit the last lr rows and columns of the matrix.

(b) In the case $\|\mathbf{K}\| < \delta < 1$, the stability result could be proved easily. We have (see [**8**] for similar discussions)

$$\|P_n\mathbf{K}_\beta x\| = \max_{1\leq i\leq n-1} \|P_n\mathbf{K}_\beta x\|_i = \max_i \sup_{s\in I_i} |(P_n\mathbf{K}_\beta)x(s)|.$$

Now similar to (3.17) we have

$$\begin{aligned} &\|P_n\mathbf{K}_\beta x\|_i \\ (3.23)\qquad &\quad \le \|\mathbf{K}_\beta x\| + \|(I-P_n)\mathbf{K}_\beta x\|_i \\ &\quad \le \|\mathbf{K}\|\ \|x\| + B_r \sup_{|s'-s|\le\delta^*} \int_0^\infty |K(s',t)-K(s,t)|\,dt \quad \|x\|, \end{aligned}$$

with δ^* as in (3.21). It follows from (3.23) and the arguments following (3.20) that, provided $l = m - n$ is large enough, we can ensure that $\|P_n\mathbf{K}_\beta\| \le \delta < 1$, obtaining the required stability result in the form

$$\|(I - P_n\mathbf{K}_\beta)^{-1}\| \le 1/(1-\delta). \tag{3.24}$$

Returning to the general case, we combine Theorems 3.1 and 3.2 to establish the required bound for the discretisation error $\|x_\beta - x_{\beta,n}\|$, and, with the aid of Theorem 2.1, a bound for the total error $\|x - x_{\beta,n}\|$, as follows:

THEOREM 3.3. *Let the conditions of Theorem 3.1 be satisfied. Then for n sufficiently large*

$$\|x_\beta - x_{\beta,n}\| = O(n^{-r})$$

and

$$\|x - x_{\beta,n}\| = O(n^{-r}),$$

provided $l = m - n$ is chosen appropriately large.

4. Extension to $\mathbf{x} \in \mathbf{C}_1^{\mathbf{r},\mu}$. The results so far have relied on the exponential decay to zero of the solution of (1.1) by requiring $x \in C_0^{r,\mu}$. In the X_l^+ setting it is possible to obtain $x(\infty)$ explicitly, as shown in [**18**]. Assuming that $K(s,t)$ satisfies the conditions of §1 and $y \in X_l^+$, then it can be shown that the unique solution of (1.1) satisfies

$$x(\infty) = \frac{1}{1-\chi}y(\infty), \tag{4.1}$$

where

$$\chi = \int_{-\infty}^{\infty} \kappa(u)\,du,$$

and where $\chi \neq 1$, since otherwise 1 would be in the spectrum of $\mathbf{K}$; see [**13, 18**].

If we know that x, the solution of (1.1), belongs to $C_l^{r,\mu}$, then the new unknown $\bar{x}(s) = x(s) - x(\infty) \in C_0^{r,\mu}$ and satisfies the equation

$$(I - \mathbf{K})\bar{x} = \bar{y} \tag{4.2}$$

where

$$\bar{y}(s) = y(s) + \Big(\int_0^\infty K(s,t)\,dt - 1 \Big) x(\infty) = y(s) - \frac{1 - \int_0^\infty K(s,t)\,dt}{1-\chi} y(\infty).$$

Equation (4.2) is of the appropriate form for application of Theorem 3.3. Thus provided $\bar{y}(s)$ can be evaluated analytically (or sufficiently accurately) it will follow, under suitable conditions, that $x_{\beta,n} = \bar{x}_{\beta,n} + x(\infty)$ will satisfy $\|x - x_{\beta,n}\| = \|\bar{x} - \bar{x}_{\beta,n}\| = O(n^{-r})$. See [**9**] for similar results for the Nystrom method.

5. Superconvergence. It is well-known that for second kind equations with smooth kernels over finite intervals, the collocation approximation exhibits superconvergence at the collocation points, provided the collocation points are chosen appropriately; see [**17**] and references therein. We wish to prove similar results for (1.1), even though here $\mathbf{K}$ is non-compact. Recall that the collocation equation for (1.1) is given by

$$(I - P_n \mathbf{K}_\beta) x_{\beta,n} = P_n y. \tag{5.1}$$

Once $x_{\beta,n}$ is obtained we define the iterated collocation approximation by

$$x^*_{\beta,n} = \mathbf{K}_\beta x_{\beta,n} + y. \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$x_{\beta,n} = P_n x^*_{\beta,n}, \tag{5.3}$$

and therefore by definition of P_n

$$x_{\beta,n}(s_{ij}) = x^*_{\beta,n}(s_{ij}), \quad 1 \le i \le n-1,\ 1 \le j \le r. \tag{5.4}$$

Substituting (5.3) into (5.2) the second kind equation is obtained

$$(I - \mathbf{K}_\beta P_n)x^*_{\beta,n} = y. \tag{5.5}$$

Subtracting (5.5) from (1.7), we obtain

$$x_\beta - x^*_{\beta,n} = (I - \mathbf{K}_\beta P_n)^{-1}[\mathbf{K}_\beta(I - P_n)x_\beta]. \tag{5.6}$$

Recall that the uniform boundedness of $(I - \mathbf{K}_\beta P_n)^{-1}$ has already been established in §3.2 under the conditions of Theorem 3.2.

We will now prove that if the basic quadrature nodes $\{\xi_j : 1 \leq j \leq r\}$ with $0 \leq \xi_1 < \xi_2 < \cdots < \xi_r < 1$, are appropriately chosen, then $\|\mathbf{K}_\beta(I - P_n)x_\beta\| = O(n^{-R})$, with $R > r$. From this it will follow that the iterated collocation approximation is globally superconvergent and, from (5.4), that the collocation approximation exhibits superconvergence at the collocation points. Our first step will be to establish the order of convergence for $\|\mathbf{K}_\beta(I - P_n)x\|$, where x is the solution of (1.1).

Let Q_r be the interpolatory projection of $C[0,1]$ into $\mathbf{P}_r$, the space of polynomials of order r (i.e. degree $\leq r-1$), based on the nodes $\{\xi_i : 1 \leq i \leq r\}$. That is, for $x \in C[0,1]$

$$(Q_r x)(s) = \sum_{i=1}^{r} l_{r,i}(s)x(\xi_i), \tag{5.7}$$

where

$$l_{r,i}(s) = \prod_{\substack{j=1\\ j\neq i}}^{r} \frac{(s-\xi_j)}{(\xi_i - \xi_j)} \in \mathbf{P}_r, \quad i = 1, 2, \ldots, r,$$

are the Lagrangian basis functions. Now in general $(I - Q_r)x \equiv 0$ only if $x \in \mathbf{P}_r$. However, it is possible that if $\{\xi_i : 1 \leq i \leq r\}$ are appropriately chosen then

$$\int_0^1 (I - Q_r)x(s)\,ds = 0 \quad \text{for } x \in \mathbf{P}_{r+r'}, \tag{5.8}$$

with $r \geq r' > 0$. In fact if $\{\xi_i\}$ are the Gauss-Legendre nodes on [0,1], then $r = r'$, while if $\xi_i = (i-1)/(r-1)$, $i = 1, 2, \ldots, r$ with r an odd integer then $r' = 1$. For our present analysis it is more convenient to write (5.8) in the equivalent form

$$\int_0^1 \prod_{i=1}^{r} (s - \xi_i)\phi(s)\, ds = 0 \quad \text{for } \phi \in P_{r'}. \tag{5.9}$$

Assuming that (5.9) is satisfied, in other words that the interpolatory quadrature rule on [0,1] based on the nodes $\{\xi_i\}$ has degree of precision $r + r' - 1$, we wish to find conditions under which

$$\|\mathbf{K}_\beta(I - P_n)x\| = O(n^{-(r+r')}). \tag{5.10}$$

Based on our analysis in §2 and §3, we seek to prove (5.10) under the following assumptions:

$$x \in C_0^{r+r',\mu}, \quad \mu > 0 \tag{5.11a}$$

$$s_i^{(n)} = \frac{r + r'}{\mu} \ln\left(\frac{m}{m + 1 - i}\right), \quad i = 1, 2, \ldots, n, \tag{5.11b}$$

with

$$\beta(n) = s_n^{(n)} = \frac{r + r'}{\mu} \ln\left(\frac{m}{m + 1 - n}\right) = \frac{r + r'}{\mu} \ln(n) + O(1), \tag{5.11c}$$

where $l = m - n \geq 0$ is suitably chosen.

Now, similar to [**14**] we have

$$\begin{aligned}
&[\mathbf{K}_\beta(I - P_n)x](s) \\
&= \sum_{i=1}^{n-1} \int_{s_i}^{s_{i+1}} K(s,t)\Big[x(t) - \sum_{j=1}^{r} l_{ij}(t)x(s_{ij})\Big]\, dt \\
&= \sum_{i=1}^{n-1} \int_{s_i}^{s_{i+1}} K(s,t) \prod_{j=1}^{r} (t - s_{ij}) x[s_{i1}, s_{i2}, \ldots, s_{ir}, t]\, dt \\
&= \sum_{i=1}^{n-1} \varepsilon_{i,n}(s),
\end{aligned} \tag{5.12}$$

where $\Pi(t-s_{ij})x[s_{i1}, s_{i2}, \ldots, s_{ir}, t]$ is the polynomial interpolation error for $x(t)$, written in terms of Newton's divided differences $x[s_{i1}, s_{i2}, \ldots, s_{ir}, t]$, see [**4**]. We can write

$$\varepsilon_{i,n}(s) = \int_{s_i}^{s_{i+1}} \prod_{j=1}^{r} (t - s_{ij}) F_s(t)\, dt, \quad i = 1, 2, \ldots, n-1, \tag{5.13}$$

where

$$F_s(t) = K(s,t)x[s_{i1}, s_{i2}, \ldots, s_{ir}, t]. \tag{5.14}$$

To take advantage of (5.9) expand $F_s(t)$ about s_i as a Taylor polynomial of order r', denoted by $\phi_{s,r'}(t)$ together with an integral remainder, to obtain

$$F_s(t) = \phi_{s,r'}(t) + \frac{1}{(r'-1)!} \int_{s_i}^{t} (t-u)^{r'-1} D^{r'} F_s(u)\, du. \tag{5.15}$$

Now, it follows from (5.9), (5.14) and (5.15) that

$$\varepsilon_{i,n}(s) = \frac{1}{(r'-1)!} \int_{s_i}^{s_{i+1}} \prod_{j=1}^{r} (t - s_{ij}) \Big\{ \int_{s_i}^{t} (t-u)^{r'-1} D^{r'} F_s(u)\, du \Big\}\, dt. \tag{5.16}$$

Using the Leibnitz rule for differentiating a product, we obtain

$$D^{r'} F_s(t) = \sum_{a=0}^{r'} \binom{r'}{a} \Big(\Big(\frac{\partial}{\partial t}^{a}\Big) K(s,t) \Big) \Big(\frac{d}{dt}^{r'-a} \Big) x[s_{i1}, s_{i2}, \ldots, s_{ir}, t]. \tag{5.17}$$

Then using the result

$$\Big(\frac{d}{dt}\Big)^{r'-a} x[s_{i1}, s_{i2}, \ldots, s_{ir}, t] = \frac{(r'-a)!}{(r+r'-a)!} D^{r+r'-a} x(\xi_t) \tag{5.18}$$

for some $\xi_t \in (s_i, s_{i+1})$, (proved in [**14**, p.182]), we obtain

$$\begin{aligned}\varepsilon_{i,n}(s) = &\sum_{a=0}^{r'} \frac{r'}{a!(r+r'-a)!} \int_{s_i}^{s_{i+1}} \prod_{j=1}^{r} (t - s_{ij}) \\ &\Big\{ \int_{s_l}^{t} (t-u)^{r'-1} \Big(\Big(\frac{\partial}{\partial u}\Big)^{a} K(s,u) \Big) D^{r+r'-a} x(\xi_u)\, du \Big\}\, dt.\end{aligned}$$

For $u,\ t \in [s_i, s_{i+1}]$, we have $|t-u|,\ |t-s_{ij}| \leq h_i$, and for $x \in C_0^{r+r',\mu}$

$$|D^b x(t)| \leq e^{-\mu s_i}\|x\|_{r+r',\mu} \quad b = 0, 1, \cdots, r+r'.$$

Therefore

$$|\varepsilon_{i,n}(s)| \leq \left\{ \sum_{a=0}^{r'} C_a \int_{s_i}^{s_{i+1}} \left| \left(\frac{\partial}{\partial u}\right)^a K(s,u) \right| du \right\} \cdot e^{-\mu s_i} h_i^{r+r'} \|x\|_{r+r',\mu}, \tag{5.19}$$

where $C_a = \frac{r'}{a!(r+r'-a)!}$. It follows from (5.11b) that $e^{-\mu s_i} h_i^{r+r'} \leq C/n^{r+r'}$, and hence

$$\begin{aligned} \sum_{i=1}^{n-1} |\varepsilon_{i,n}(s)| &\leq \frac{C}{n^{r+r'}} \|x\|_{r+r',\mu} \sum_{a=0}^{r'} C_a \int_0^{\beta(n)} \left| \left(\frac{\partial}{\partial t}\right)^a K(s,t) \right| dt \\ &\leq (C/n^{r+r'}) \|x\|_{r+r',\mu} \end{aligned} \tag{5.20}$$

provided

$$\sup_{s \in \mathbf{R}^+} \int_0^\infty \left| \left(\frac{\partial}{\partial t}\right)^a K(s,t) \right| dt < \infty, \quad a = 0, 1, 2 \ldots, r'. \tag{5.21}$$

We are now in a position to state and prove the main result of this section:

THEOREM 5.1. *Let the kernel $K(s,t)$ of (1.1) satisfy (5.21), in addition to the conditions in §1. If the interpolatory quadrature rule on [0,1] based on the nodes $\{\xi_j : 1 \leq j \leq r\}$ has a degree of precision $r+r'-1$ and if (5.11) is satisfied, then for n sufficiently large*

$$\|x - x^*_{\beta,n}\| = O(n^{-(r+r')}),$$

provided $l = m - n$ is appropriately large.

PROOF. We have

$$\|x - x^*_{\beta,n}\| \leq \|x - x_\beta\| + \|x_\beta - x^*_{\beta,n}\|. \tag{5.22}$$

With $\beta(n)$ given by (5.11c), and x satisfying (5.11a), it follows from Theorem 2.1 that the truncation error satisfies

$$\|x - x_\beta\| = O(n^{-(r+r')}). \tag{5.23}$$

To show that the discretisation error is of the same order, we write, from (5.6),

$$\|x_\beta - x^*_{\beta,n}\| \leq \|(I - \mathbf{K}_\beta P_n)^{-1}\| \;\; \|\mathbf{K}_\beta(I - P_n)x_\beta\|. \tag{5.24}$$

The uniform boundedness of the first factor on the right hand side follows from the argument used to prove Theorem 3.2. (It can easily be seen that the replacement of r by $r + r'$ in the formula for $s_i^{(n)}$ does not affect the stability argument.) Similar to equation (3.6) we have

$$\|\mathbf{K}_\beta(I - P_n)x_\beta\| \leq \|\mathbf{K}_\beta(I - P_n)x\| + \|\mathbf{K}_\beta\|(1 + \|P_n\|)\|x - x_\beta\|. \tag{5.25}$$

From (5.12), (5.20), (5.23) and (5.25) it follows that $\|\mathbf{K}_\beta(I - P_n)x_\beta\| = O(n^{-(r+r'})$, and using (5.22), (5.23) and (5.24) the proof is complete. □

It appears that, provided the conditions of Theorem 5.1 are satisfied, the highest order of convergence $r + r' = 2r$ is obtained if we choose the basic quadrature nodes $\{\xi_i : 1 \leq i \leq r\}$ to be the Gauss-Legendre points shifted to [0,1].

6. Numerical Experiments. In this section we emphasize some of the important features of the results in Theorem 3.3 and Theorem 5.1 by considering the test problem

$$x(s) - (1/2\lambda) \int_0^\infty e^{-|s-t|} x(t)\, dt = (e^{-s}/2\lambda), \quad s \in \mathbf{R}^+, \tag{6.1}$$

the exact solution of which, for $\lambda \notin [0, 1]$, is (see [**2**])

$$x(s) = (\mu - 1)e^{-\mu s},$$

where

$$\mu = (1 - 1/\lambda)^{1/2}.$$

In particular we consider the example with $\lambda = 4/3$, yielding $\mu = 0.5$.

In Tables 1 to 4 we present the results of the collocation methods based on piecewise-constant ($r = 1$), piecewise-linear ($r = 2$) and piecewise-quadratic ($r = 3$) approximations. In all cases the spline knots are given by

$$s_i = (R/\mu) \ln \left(\frac{n}{n+1-i} \right), \quad i = 1, 2, \ldots, n,$$

(i.e. $l = m - n = 0$), where R takes a range of values in each case. In all the tables the quantity presented is the maximum absolute value of the error over the collocation points $\{s_{ij}\}$ with $1 \leq i \leq n-1$ and $1 \leq j \leq r$. The values of n have been chosen so that the results in a given row correspond to approximations with the same number of collocation points in every table, in order to allow easy comparison between methods with splines of differing order. Recall that for a method based on piecewise polynomials of order r, we choose r collocation points within each of the $n-1$ subintervals I_i. The number of collocation points corresponding to rows 1 to 4 of the tables are 24, 48, 96 and 192 respectively.

Results in Table 1 show that if $R = r = 1$ is chosen, the expected order of convergence $O(n^{-1})$ is obtained, while by choosing $R = r + r' = 2$ the predicted superconvergence is observed with errors of the form $O(n^{-2})$. Increasing R to 3 results in reduced accuracy, though still yields the $O(n^{-2})$ convergence.

Results in Table 2 are based on piecewise-linear approximations, using the Gauss-Legendre points as the basic quadrature nodes. For $R = r = 2$ the predicted $O(n^{-2})$ convergence is observed, while for $R = 3$ and 4 superconvergent result are obtained with errors of the form $O(n^{-3})$ and $O(n^{-4})$ respectively. Again, the choice of $R = 5$ cannot improve the optimal convergence rate $O(n^{-4})$.

Results in Table 3 are again for piecewise-linear approximations, but here, unlike Table 2, we do not choose the Gauss-Legendre points as the basic quadrature nodes. For $R = 2, 3, 4$ and 5 the errors are of the form $O(n^{-2})$.

Finally, in Table 4 we present the results for the piecewise-quadratic case ($r = 3$), where the Gauss-Legendre points are chosen as the basic quadrature nodes. As expected, for $R = 3, 4$ and 5 we obtain $O(n^{-R})$

convergence. For $R = r + r' = 6$, however, we do not obtain the $O(n^{-6})$ convergence that at first glance might have been expected from Theorem 5.1. A closer investigation reveals that the smoothness requirement (5.21) with $a = r' = 3$ is not satisfied by the kernel $K(s,t) = \exp(-|s-t|)$, whereas the case $r' = 2$ is covered by an appropriate extension of the theory. Thus, $O(n^{-5})$ is the best result obtainable from Theorem 5.1. In all cases, therefore, the numerical experiments support the theoretical conclusions.

TABLE 1: Errors for the piecewise-constant case

$n\backslash^R$	1	2	3
25	5.02E-3	3.98E-4	6.47E-4
49	2.56E-3	1.04E-4	1.72E-4
97	1.30E-3	2.66E-5	4.44E-5
193	6.51E-3	6.78E-6	1.13E-5

$r = 1$, with $\xi_1 = 0.5$

TABLE 2: Errors for the piecewise-linear case

$n\backslash^R$	2	3	4	5
13	7.11E-4	4.21E-5	1.41E-5	1.78E-5
25	1.92E-4	5.92E-6	1.10E-6	1.31E-6
49	5.01E-5	7.86E-7	7.60E-8	8.86E-8
97	1.28E-5	1.01E-7	5.00E-9	5.74E-9

$r = 2$, with $\xi_1 = 0.5(1 - 1/\sqrt{3})$
$\xi_2 = 0.5(1 + 1/\sqrt{3})$.

TABLE 3: Errors for the piecewise-linear case

$n\backslash^R$	2	3	4	5
13	9.49E-3	8.42E-4	1.21E-3	1.69E-3
25	2.57E-4	2.29E-4	3.31E-4	4.65E-4
49	6.70E-5	5.98E-5	8.65E-5	1.22E-4
97	1.71E-5	1.53E-5	2.21E-5	3.11E-5

$r = 2$, with $\xi_1 = 0.3$, $\xi_2 = 0.7$

TABLE 4: Errors for the piecewise-quadratic case

$n\backslash^R$	3	4	5	6
9	1.85E-4	2.03E-5	3.34E-6	6.54E-6
17	2.74E-5	1.59E-6	1.39E-7	2.73E-7
33	3.75E-6	1.12E-7	5.05E-9	9.79E-9
65	4.90E-7	7.46E-9	1.70E-10	3.28E-10

$r = 3$, with $\xi_1 = 0.5(1 - \sqrt{0.6})$, $\xi_2 = 0.5$, $\xi_3 = 0.5(1 + \sqrt{0.6})$.

7. Application to boundary integral equations on polygonal domains.

INTRODUCTION 7.1. If we recast a boundary value problem as a second-kind boundary integral equation, the resulting integral operator will, in general, be non-compact if the boundary, Γ, of the domain of interest is only piecewise smooth [**5**, **7**, **10**, **12**]. Furthermore, the solution of the integral equation will exhibit singularities (non-smoothness) near the corners of the domain, the asymptotic form of which can be obtained, for example, by using the local Mellin transform [**7**, **10**, **14**]. Insight into the numerical analysis of such integral equations can be gained by considering equations of the form

$$(I \pm \mathbf{K})x = y, \tag{7.1}$$

where

$$(\mathbf{K}x)(s) = \int_0^1 K(s/t)x(t)\,(dt/t), \qquad s \in [0,1], \tag{7.2a}$$

with

$$(\mathbf{K}x)(0) = \lim_{s\to 0}(\mathbf{K}x)(s) = \Big(\int_0^\infty K(u)\,(du/u)\Big)x(0) \tag{7.2b}$$

if $x \in C[0,1]$. In the case of the Dirichlet problem for Laplace's equation in the interior of a polygonal domain, the kernel K in (7.2) is given by [**10**, **14**]

$$K(u) = \Big(\frac{1}{\pi}\Big)\Big(\frac{\sin(\pi-\alpha)}{u + u^{-1} - 2\cos(\pi-\alpha)}\Big), \qquad u \in \mathbf{R}^+, \tag{7.3}$$

where α denotes the interior angle at one of the corners of the polygon, with $0 < \alpha < 2\pi$. It can be shown that the spectrum of $\mathbf{K}$ in this case is given by

$$\text{(7.4)} \qquad \sigma(\mathbf{K}) = \begin{cases} [0, \frac{\pi-\alpha}{\pi}] & 0 < \alpha < \pi \\ [\frac{\pi-\alpha}{\pi}, 0] & \pi < \alpha < 2\pi \end{cases}.$$

Since the spectrum is an interval, $\mathbf{K}$ and all its iterates are non-compact.

The study of (7.1), (considering the '-' sign only), can be reduced to that of a second-kind equation on the half line. The transformation $J : C[0,1] \to X_l^+$ defined by

$$\text{(7.5)} \qquad (Jx)(s) = x(e^{-s}) = \tilde{x}(s), \quad s \in \mathbf{R}^+,$$

is an isometry (with respect to the uniform norm). Applying J to both sides of (7.1) the Wiener-Hopf equation is obtained

$$\text{(7.6}a\text{)} \qquad (Jx)(s) - (J\mathbf{K}J^{-1})(Jx)(s) = (Jy)(s), \quad s \in \mathbf{R}^+,$$

or more explicitly

$$\text{(7.6}b\text{)} \qquad x(e^{-s}) - \int_0^\infty K(e^{-(s-t)})x(e^{-t})\,dt = y(e^{-s}), \quad s \in \mathbf{R}^+,$$

which with the help of notation (7.5) may be written as the convolution equation

$$\text{(7.7)} \qquad \tilde{x}(s) - \int_0^\infty \tilde{K}(s-t)\tilde{x}(t)\,dt = \tilde{y}(s), \quad s \in \mathbf{R}^+.$$

Using the local Mellin transform [**10**,**14**] it can be shown that the solution of (1.1) near a corner of the boundary has the asymptotic form

$$\text{(7.8)} \qquad x(s) = x(0) + as^\mu + Z(s),$$

where s is the arc length measured from the corner and as^μ represents the leading singular term in the expansion. In particular for the case of the kernel (7.3) we have [**7**,**10**,**14**]

$$\text{(7.9)} \qquad \mu = \pi/(\pi + |\pi - \alpha|) > 1/2.$$

A natural setting for the study of (7.1) with solutions of the form (7.8) with $\mu > 0$ is the space $\hat{C}^{r,\mu} \subset C[0,1]$ defined by

$$(7.10a) \quad \hat{C}^{r,\mu} = \{x \in C[0,1] : \|\hat{x}\|_{r,\mu} < \infty \text{ with } \hat{x}(s) = x(s) - x(0)\},$$

where

$$(7.10b) \qquad \|x\|_{r,\mu} = \sup_{s\in[0,1]} \{|s^{l-\mu} D^l x(s)| : 0 \le l \le r\}.$$

We denote by $\hat{C}_0^{r,\mu}$ the subspace of $\hat{C}^{r,\mu}$ which consists of those functions which vanish at the origin.

With $J : C[0,1] \to X_l^+$ defined by (7.5) and $J^{-1} : X_l^+ \to C[0,1]$ defined as

$$(7.11) \qquad (J^{-1}x)(s) = x(-\ln(s)) = \bar{x}(s), \quad s \in (0,1],$$

it is easy to show that $\hat{C}^{r,\mu}$ defined by (7.10) is isomorphic to $C_l^{r,\mu}$ defined by (1.5c). A similar relation holds between $\hat{C}_0^{r,\mu}$ and $C_0^{r,\mu}$.

The question we address is the following: Given the equivalence of (7.1) with the Wiener-Hopf equation expressed in (7.6b), to what extent can the analysis of §2 through §5 be used to establish uniform convergence of the collocation and iterated collocation methods for (7.1) and (7.2)? As the piecewise-polynomial space used for the $[0,\infty)$ case is not mapped to a piecewise-polynomial space for the [0,1] problem, we must expect at least minor modifications to the analysis. An analysis for (7.1) in subsections 7.2 through 7.5 similar to that in §2 through §5 follows.

To solve (7.1) with $\mathbf{K}$ given by (7.2), first approximate the equation by the truncated equation

$$(7.12a) \qquad (I - \mathbf{K}_\beta)x_\beta = y,$$

where

$$(7.12b) \qquad (\mathbf{K}_\beta x)(s) = \int_{\beta(n)}^1 K(s/t)x(t)(dt/t), \quad s \in [0,1],$$

with $\beta(n) \to 0$ as $n \to \infty$. Equation (7.12) is then discretised by the collocation method written as

$$(7.13) \qquad (I - P_n\mathbf{K}_\beta)x_{\beta,n} = P_n y,$$

where, as in §1, we define on a mesh $0 = s_{n+1} < \beta(n) = s_n < s_{n-1} < \cdots < s_1 = 1$ the projection
(7.14a)

$$(P_n v)(s) = \sum_{j=1}^{r} l_{ij}(s) v(s_{ij}), \quad s \in I_i = (s_{i+1}, s_i], \quad i = 1, 2, \ldots, n-1,$$

$$(P_n v)(s) \equiv 0 \quad \text{for } s \in [0, \beta(n)], \tag{7.14b}$$

where $s_{ij} = s_{i+1} + \xi_j h_i$ with $\{\xi_j : 1 \le j \le r\}$ the basic quadrature nodes on $(0,1]$, and $h_i = s_i - s_{i+1}$. The mesh nodes $\{s_i\}$ are numbered in decreasing order so as to correspond directly to those in (3.11) under the transformation (7.5).

The equation (7.6) (or (7.7)) is a special form of (1.1) and it is easy to establish that the smoothness requirements (1.6) and (5.21) are satisfied by the kernel in (7.6) if

$$\int_0^\infty t^a |D^a K(t)| (dt/t) < \infty, \quad a = 0, 1, 2, \ldots, r'. \tag{7.15}$$

The graded meshes (3.11) can be seen to be transformed to

$$s_i = ((m + 1 - i)/m)^{r/\mu}, \quad i = 1, 2, \ldots, n, \tag{7.16}$$

with $\beta(n) = s_n = ((m+1-n)/m)^{r/\mu} \le ((l+1)/n)^{r/\mu}$, where $l = m - n \ge 0$. It is easy to show that

$$(h_i / s_{i+1}) \le (C/n) s_{i+1}^{-\mu/r}, \tag{7.17}$$

which is precisely the definition of (r, μ)-graded meshes [**6**, **8**, **11**, **15**], of which (7.16) is an example. (If $r/\mu < 1$ we replace the exponent in (7.16) by unity.)

In subsections 7.2 through 7.5 we state and give outline proofs of results analogous to those in §2 through §5. Unless stated otherwise, all norms are uniform norms.

TRUNCATION ERROR 7.2.

THEOREM 7.1. *Let x, the solution of (7.1) with $\mathbf{K}$ defined by (7.2), belong to $\hat{C}_0^{r,\mu}$ for some $\mu > 0$, and let $K(t)$ satisfy (7.15). If $\beta(n)$ is given by (7.16) with $i = n$ then for n sufficiently large we have*

$$\|x - x_\beta\| = O(n^{-r}). \tag{7.18}$$

PROOF. It follows from (7.1) and (7.12) that

$$\|x - x_\beta\| \leq \|(I - \mathbf{K}_\beta)^{-1}\| \; \|(\mathbf{K} - \mathbf{K}_\beta)x\|, \tag{7.19}$$

in which the result of [**1**], quoted in equation (2.2), may be applied to (7.6) to establish the uniform boundedness of the first factor on the right hand side of (7.19), for n sufficiently large (i.e. $\beta(n)$ sufficiently close to zero). For the second factor we write

$$\begin{aligned} |(\mathbf{K} - \mathbf{K}_\beta)x(s)| &\leq \int_0^{\beta(n)} |K(s/t)x(t)(1/t)|\,dt \\ &\leq \|\mathbf{K}\| \sup_{t\in(0,\beta(n))} |x(t)| \\ &\leq \|\mathbf{K}\|\beta(n)^\mu \|x\|_{r,\mu} = O(n^{-r}) \end{aligned} \tag{7.20}$$

which now completes the proof. □

DISCRETISATION ERROR 7.3.

THEOREM 7.2. *Let x, the solution of (7.1), belong to $\hat{C}_0^{r,\mu}$ for some $\mu > 0$, and let (7.15) and (7.16) be satisfied with $\beta(n) = s_n$. Then for n sufficiently large*

$$\|x_\beta - x_{\beta,n}\| = O(n^{-r}) \tag{7.21a}$$

and

$$\|x - x_{\beta,n}\| = O(n^{-r}), \tag{7.21b}$$

provided $l = m - n$ is chosen appropriately large.

PROOF. From (7.12) and (7.13) it follows that

$$\|x_\beta - x_{\beta,n}\| \le \|(I - P_n\mathbf{K}_\beta)^{-1}\| \; \|(I - P_n)x_\beta\|. \tag{7.22}$$

As in §3, equation (3.10), for $i = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} \|(I - P_n)x\|_i &= \sup_{s\in I_i} \left| x(s) - \sum_{j=1}^{r} l_{ij}(s)x(s_{ij}) \right| \\ &\le Ch_i^r \sup_{s\in I_i} |D^r x(s)|, \\ &\le Ch_i^r s_{i+1}^{-r+\mu} \|x\|_{r,\mu} \le (C/n^r)\|x\|_{r,\mu}, \end{aligned} \tag{7.23}$$

with the last inequality following from (7.17). For $i = n$ we have, from (7.14),

$$\|(I - P_n)x\|_{[0,\beta(n)]} = \sup_{s\in[0,\beta(n)]} |x(s)| \le \beta^\mu \|x\|_{0,\mu} \le \beta^\mu \|x\|_{r,\mu}. \tag{7.24}$$

As $\beta = \beta(n) = s_n = \left(\frac{l+1}{l+n}\right)^{r/\mu}$, it follows from (3.6), (7.18), (7.23) and (7.24) that

$$\|(I - P_n)x_\beta\| = O(n^{-r}). \tag{7.25}$$

It remains to establish the uniform boundedness of $(I - P_n\mathbf{K}_\beta)^{-1}$. This could be carried out by an analysis similar to that in §3.2, with (7.15) required only for $a = 0$. Here, an outline of a simpler proof is given, in which it is assumed that the kernel satisfies (7.15) for $a = 0, 1, 2, \ldots, r$, as is the case for most such boundary integral equations. Again we show that $\alpha_n := \|(\mathbf{K}_\beta - \mathbf{K}_\beta P_n)\mathbf{K}_\beta P_n\|$ can be made sufficiently small (see (3.16)) to establish the uniform boundedness of $(I - \mathbf{K}_\beta P_n)^{-1}$ and that of $(I - P_n\mathbf{K}_\beta)^{-1}$, using (3.13). As in (3.17) consider

$$\begin{aligned} &|\mathbf{K}_\beta(I - P_n)v(s)| \\ &\quad = \left| \sum_{i=1}^{n-1} \int_{s_{i+1}}^{s_i} k(s,t) \left\{ v(t) - \sum_{j=1}^{r} l_{ij}(t)v(s_{ij}) \right\} dt \right| \\ &\quad \le C \sum_{i=1}^{n-1} h_i^r \|D^r v\|_i \int_{s_{i+1}}^{s_i} |k(s,t)|\, dt, \end{aligned} \tag{7.26}$$

where $k(s,t) = K(s/t)/t$ is the kernel of $\mathbf{K}$. Now setting $v = \mathbf{K}_\beta P_n x$, we obtain

$$\begin{aligned} \|D^r v\|_i &= \sup_{s\in I_i} |D^r v(s)| \\ &\le \sup_{s\in I_i} \left(\int_0^1 \left| \left(\frac{\partial}{\partial s}\right)^r k(s,t) \right| dt \right) \|P_n\| \, \|x\|. \end{aligned} \tag{7.27}$$

But

$$\begin{aligned} (\partial/\partial s)^r k(s,t) &= t^{-r} D^r K(s/t)(1/t) \\ &= s^{-r}(s/t)^r D^r K(s/t)(1/t). \end{aligned} \tag{7.28}$$

From (7.27) and (7.28) it follows that

$$\|D^r v\|_i \le C s_{i+1}^{-r} \|P_n\| \, \|x\|, \tag{7.29}$$

which in turn, using (7.26), yields

$$\begin{aligned} \alpha_n &= \|\mathbf{K}_\beta (I - P_n) \mathbf{K}_\beta P_n\| \\ &\le C \max_i (h_i/s_{i+1})^r = C(h_{n-1}/s_n)^r \le C^*(m+1-n)^{-r}, \end{aligned} \tag{7.30}$$

where the penultimate step follows because $\{(h_i/s_{i+1})\}$ forms a monotone increasing sequence and the last inequality follows from (7.17). Clearly, α_n can be made sufficiently small if $l = m - n$ is a suitably large number. The rest of the analysis now follows similarly to the proof of Theorems 3.2 and 3.3.

EXTENSION 7.4 to $x \in \hat{C}^{r,\mu}$. From (7.1) and (7.2b) it follows that

$$x(0) = y(0)/(1-\chi),$$

where $\chi = \int_0^\infty K(u)\,du/u \neq 1$. If the solution of (7.1) is known to belong to $\hat{C}^{r,\mu}$, then the new unknown $\hat{x}(s) = x(s) - x(0)$ belongs to $\hat{C}_0^{r,\mu}$, and satisfies

$$(I - \mathbf{K})\hat{x} = \hat{y},$$

where

$$\hat{y}(s) = y(s) - \frac{1 - \int_s^\infty K(u)\,du/u}{1-\chi} y(0),$$

to which the previous analysis in this section can be applied, see [**11**].

SUPERCONVERGENCE 7.5. The iterated collocation solution corresponding to (7.1) satisfies

$$(I - \mathbf{K}_\beta P_n)x^*_{\beta,n} = y. \tag{7.31}$$

Here, similar to the results in §5, Theorem 5.1, it can be shown that if the collocation points are appropriately chosen then the iterated collocation solution is globally superconvergent and the collocation method exhibits superconvergence at the collocation points.

Assume the interpolatory quadrature rule based on the nodes $\{\xi_i : 1 \le i \le r\}$ has degree of precision $r + r' - 1$ with $1 \le r' \le r$. Further assume

$$x \in \hat{C}_0^{r+r',\mu} \text{ for some } \mu \geq 0, \tag{7.32a}$$

$$s_i^{(n)} = ((m+1-i)/m)^{(r+r')/\mu}, \quad i = 1, 2, \ldots, n, \tag{7.32b}$$

with

$$\beta(n) = s_n^{(n)} = ((m+1-n)/m)^{(r+r')/\mu} \tag{7.32c}$$

for some $m \geq n$. We can now prove:

THEOREM 7.4. *Let the interpolatory integration rule based on $\{\xi_i : 1 \le i \le r\}$ have degree of precision $r + r' - 1$ and let (7.15) and (7.32) be satisfied. Then for n sufficiently large*

$$\|x - x^*_{\beta,n}\| = O(n^{-(r+r')})$$

if $l = m - n$ is appropriately large.

PROOF. To establish

$$\|\mathbf{K}_\beta(I - P_n)x\| = O(n^{-(r+r')})$$

proceed as in §5 and obtain

$$[\mathbf{K}_\beta(I-P_n)x](s)=\sum_{i=1}^{n-1}\varepsilon_{i,n}(s),$$

where

$$\begin{aligned}\varepsilon_{i,n}(s)=&\sum_{a=0}^{r'}\frac{r'}{a!(r+r'-a)!}\int_{s_{i+1}}^{s_i}\prod_{j=1}^{r}(t-s_{ij})\\ &\cdot\Big\{\int_{s_{i+1}}^{t}(t-u)^{r'-1}(\partial/\partial u)^a(K(s/u)(1/u))D^{r+r'-a}x(\xi_u)du\Big\}dt,\end{aligned}\tag{7.33}$$

with $\xi_u\in(s_{i+1},s_i)$. For $u,t\in[s_{i+1},s_i]$ we have $|t-u|,\ |t-s_{ij}|\le h_i$ and for $x\in\hat{C}_0^{r+r',\mu}$

$$\begin{aligned}|D^a x(t)|\le t^{-a+\mu}\|x\|_{a,\mu}&\le Cs_{i+1}^{-a+\mu}\|x\|_{a,\mu}\\ &\le Cs_{i+1}^{-a+\mu}\|x\|_{r+r',\mu}.\end{aligned}\tag{7.34}$$

It follows from (7.33) and (7.34) that

$$\begin{aligned}|\varepsilon_{i,n}(s)|\le &C\Big\{\sum_{a=0}^{r'}\int_{s_{i+1}}^{s_i}\Big|\Big(\frac{\partial}{\partial t}\Big)^a K\Big(\frac{s}{t}\Big)\Big(\frac{1}{t}\Big)\Big|\,dt\Big\}\\ &\cdot h_i^{r+r'}s_{i+1}^{-(r+r'-a)+\mu}\|x\|_{r+r',\mu}.\end{aligned}\tag{7.35}$$

It can be shown that

$$(\partial/\partial t)^a K(s/t)(1/t)=t^{-a}\sum_{b=0}^{a}C_{a,b}(s/t)^b D^b K(s/t)(1/t)$$

for some constants $C_{a,b}$ so that for $t\in[s_{i+1},s_i]$ we have

$$|(\partial/\partial t)^a K(s/t)(1/t)|\le Cs_{i+1}^{-a}\sum_{b=0}^{a}|(s/t)^b D^b K(s/t)(1/t)|.\tag{7.36}$$

It follows from (7.35), (7.36) and (7.17) with $r + r'$ replacing r, that

$$
\begin{aligned}
&\sum_{i=1}^{n-1} |\varepsilon_{i,n}(s)| \\
(7.37)\qquad &\le \frac{C}{n^{r+r'}} \sum_{i=1}^{n-1} \sum_{a=0}^{r'} \sum_{b=0}^{a} \int_{s_{i+1}}^{s_i} \left| \left(\frac{s}{t}\right)^b D^b K\left(\frac{s}{t}\right)\left(\frac{1}{t}\right)\right| dt \|x\|_{r+r',\mu} \\
&\le \frac{C}{n^{r+r'}} \sum_{a=0}^{r'} \sum_{b=0}^{a} \int_{0}^{1} \left| \left(\frac{s}{t}\right)^b D^b K\left(\frac{s}{t}\right)\left(\frac{1}{t}\right)\right| dt \|x\|_{r+r',\mu}. \\
&\le (C/n^{r+r'}) \|x\|_{r+r',\mu}.
\end{aligned}
$$

The last inequality follows from (7.15). Thus

$$
\|\mathbf{K}_\beta (I - P_n)x\| \le \sup_{s\in[0,1]} \sum_{i=1}^{n-1} |\varepsilon_{i,n}(s)| \; \|x\|_{r+r',\mu} \le (C/n^{r+r'})\|x\|_{r+r',\mu}.
$$

The result now follows as in the proof of Theorem 5.1. □

CONCLUSIONS. We established optimal order convergence for the collocation method in the uniform norm, for a large class of integral equations on the half-line, by appropriately grading the underlying mesh to take account of the asymptotic behavior of the solution. Provided the collocation points are suitably chosen, superconvergence results can be established for the iterated collocation method.

The analysis has been extended to establish similar optimal order convergence for the collocation and iterated collocation methods for equations arising from the boundary integral formulation of boundary value problems on piecewise-smooth domains.

Acknowledgement. S. Amini expresses his appreciation for financial support through the Australian Research Council Program Grant "Numerical analysis for integrals, integral equations and boundary value problems" and for the hospitality of the University of New South Wales during the time this work was carried out. I.H. Sloan gratefully acknowledges the support of the Australian Research Council.

REFERENCES

1. P.M. Anselone and I.H. Sloan, *Integral equations on the half line*, J. Integral Equations, **9** (Suppl.) (1985), 3-23.

2. K.E. Atkinson, *The numerical solution of integral equations on the half-line*, SIAM J. Numer. Anal. **6** (1969), 375-397.

3. ———, *A survey of numerical methods for the solution of Fredholm integral equations of the second kind*, SIAM Philadelphia, (1976).

4. ———, *An introduction to numerical analysis*, John Wiley & Sons, (1978).

5. ——— and F.R. de Hoog, *The numerical solution of Laplace's equation on a wedge,* IMA. J. Numer. Anal. **4** (1984), 19-41.

6. C. de Boor, *A practical guide to splines*, Springer-Verlag, New York, (1978).

7. G.A. Chandler, *Galerkin's method for boundary integral equations on polygonal domains,* J. Austral. Math. Soc. Ser. B **26** (1984), 1-13.

8. ——— and I.G. Graham, *Product integration-collocation methods for noncompact integral operator equations*, Math. Comp. **50** (1988), 125-138.

9. ——— and ———, *The Convergence of Nystrom methods for Wiener-Hopf equations*, Numer. Math. **52** (1988), 345-364.

10. M. Costabel and E. Stephan, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, Banach-Centre Publications, V.15, PWN-Polish Scientific Publishers, Warsaw, (1985), 175-251.

11. I.G. Graham and G.A. Chandler, *High order methods for linear functionals of solutions of second kind integral equations*, SIAM J. Numer. Anal. (To appear).

12. P. Grisvard, *Elliptic problems in non-smooth domains*, Pitman advanced publishing program (1985). Boston-London-Melbourne.

13. M.G. Krein, *Integral equations on a half-line with kernel depending upon the difference of the arguments.* Amer. Math. Soc. Transl. (2) **22**, (1963), 163-288.

14. W. McLean, *Boundary integral methods for Laplace's equation*, (1985) PhD. Thesis, Australian National University.

15. J.R. Rice, *On the degree of convergence of non-linear spline approximation*, I.J. Schoenberg, ed. *Approximation with special emphasis on spline functions*, Academic Press, New York. (1969).

16. I.H. Sloan, *Quadrature methods for integral equations of the second kind over infinite intervals*, Math. Comp. **36** (1981), 511-523.

17. ———, *Superconvergence*, M. Golberg, ed., *Numerical solution of integral equations*, Plenum Press, to appear.

18. ——— and A. Spence, *Projection methods for integral equations on the half-line*, IMA. J. Numer. Anal. **6** (1986), 153-172.

PLYMOUTH POLYTECHNIC, DEPARTMENT OF MATHEMATICS AND STATISTICS, PLYMOUTH, DEVON, P14 8AA, UNITED KINGDOM

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY, NSW, 2033, AUSTRALIA