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STABILIZABILITY OF INTEGRODIFFERENTIAL PARABOLIC EQUATIONS

GIUSEPPE DA PRATO AND ALESSANDRA LUNARDI

ABSTRACT. We consider the stabilizability problem for an abstract parabolic integrodifferential equation. Under suitable assumptions, we give a necessary and sufficient condition for stabilizability, generalizing the well known Hautus condition. Then we apply the abstract result to parabolic integrodifferential equations in bounded domains.

Introduction. We consider a parabolic integrodifferential equation in general Banach space X:

$$(0.1)\quad u'(t) = Au(t) + \int_0^t K(t-s)u(s)ds + \Phi f(s), \quad t > 0, u(0) = u_0.$$

Here $A : D(A) \to X$ generates an analytic semigroup, and $K : [0, +\infty[\to \mathrm{L}(D(A), X)$ is a Laplace transformable function. $\Phi \in L(Y, X)$, where Y is a Banach space. Other assumptions are made in order that a spectrum determining condition holds and that the theory developed in [**2**] is applicable.

Roughly speaking, the "resolvent set" in integrodifferential equations of this kind is the set of all $\lambda_0 \in \mathbf{C}$ such that the function $\lambda \to (\lambda - A - \hat{K}(\lambda))^{-1}$ either is well defined or has an analytic extension at λ_0 ($\hat{K}$ is the Laplace transform of the function K). Its complementary set σ is the "spectrum" for problem (0.1).

If $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma\} < 0$, then the free system (with $f \equiv 0$) is exponentially stable: all the solutions decay exponentially to 0 as $t \to +\infty$. If $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma\} \geq 0$, we consider the following problem: find conditions on Φ ensuring that, for each initial value u_0, there exists f such that the solution u of (0.1) converges asymptotically to zero (preferably exponentially, and in the graph norm of A) as $t \to +\infty$. If this happens, system (0.1) is said to be stabilizable. We are also interested in the exponential decay of Cu, where the "observation

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operator" C is a bounded linear operator from $D(A)$ to another Banach space Z.

In [**2**] we studied asymptotic behavior of the solutions of (0.1). Using some results from that paper, we give here a necessary and sufficient condition for system (0.1) to be stabilizable. Our condition reduces to the well known Hautus condition [**4**] in the case where $K \equiv 0$ and X is finite dimensional.

The result is applicable to a large class of parabolic integro-differential equations and systems. In §3 we study in detail a parabolic integrodifferential equation with a completely monotone kernel, and the heat equation in the so called materials of fading memory type.

Although there is a wide literature concerning stabilizability in ordinary and parabolic differential equations (see, e.g., [**10, 7**] and the references quoted there), not much is known in the integrodifferential case, even if X is finite dimensional.

Feedback stabilizability may be recovered (at least, in the case where X is a Hilbert space) by arguments from general control theory, provided one is able to solve suitable integrodifferential Riccati equations. We refer to [**3**] for the relation between feedback stabilizability and solvability of the Riccati equation relevant to hereditary ordinary differential systems.

1. Notation and preliminaries. Let X be a complex Banach space, with norm $\|\cdot\|$, and let $A : D(A) \to X$ generate an analytic semigroup in X. $D(A)$ is endowed with the graph norm. Let $K : [0, +\infty[\to \mathrm{L}(D(A), X)$ be a measurable function such that

(1.1)

(i) there is a $\omega_0 \in \mathbf{R}$ such that $t \to e^{-\omega_0 t} K(t) \in L^1([0, +\infty[\, ; \mathrm{L}(D(A), X))$;

(ii) for each $x \in D(A)$, the Laplace transform $\hat{K}(\cdot)x$ is analytically extendible to a sector $S = \{\lambda \in \mathbf{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$, where $\omega \in \mathbf{R}, \theta \in]\pi/2, \pi[$;

(iii) there are $\beta \in]0, 1], c > 0$, such that $\|\lambda^\beta \hat{K}(\cdot)x\| \leq c\|x\|_{D(A)}$ for each $\lambda \in S$ and $x \in D(A)$.

Under assumption (1.1) it is possible to construct an analytic resolvent operator $R(t) \in \mathrm{L}(X, D(A)), t > 0$, for problem (0.1). In [**1, 6**] we showed that, for every locally α-Hölder continuous function ϕ : $[0, +\infty[\to X$ and every $u_0 \in \overline{D(A)}$, problem (0.1) has a unique classical solution $u \in \mathbf{C}([0, +\infty[\,; X) \cap \mathbf{C}^1(]0, +\infty[\,; X) \cap \mathbf{C}(]0, +\infty[\,; D(A))$, given by the representation formula

$$(1.2) \qquad u(t) = R(t)u_0 + \int_0^t R(t-s)\phi(s)ds, \quad t \geq 0.$$

If ϕ is merely continuous, then u belongs to $\mathbf{C}([0, +\infty[\,; X)$ and is the unique strong solution of problem (0.1).

To treat asymptotic behavior as $t \to +\infty$, we introduce a class of exponentially decaying (or bounded) functions: for $\omega \geq 0, 0 < \alpha < 1$, and any Banach space B, set

(1.3)

$$\mathbf{C}_\omega([0, +\infty[\,; B) = \{u \in \mathbf{C}([0, +\infty[\,; B) : \sup_{t>0} ||u(t)e^{\omega t}||_B < +\infty\},$$

$$||u||_{C_\omega([0,+\infty[,B)} = \sup_{t>0} ||u(t)e^{\omega t}||_B;$$

(1.4)

$$\begin{aligned} \mathbf{C}_\omega^\alpha([0, +\infty[\,; B) = \{u \in \mathbf{C}_\omega([0, +\infty[\,; B) :& \\ \sup_{t>s>0} ||u(t)e^{\omega t} - u(s)e^{\omega s}||_B (t-s)^{-\alpha} &< +\infty\}, \\ ||u||_{C_\omega^\alpha([0,+\infty[,B)} = \sup_{t>0} ||u(t)e^{\omega t}||_B& \\ + \sup_{t>s>0} ||u(t)e^{\omega t} - u(s)e^{\omega s}||_B (t-s)^{-\alpha}&. \end{aligned}$$

We give now some results generalizing the known ones [**4**] concerning asymptotic behavior of the solutions of differential equations in Banach spaces. We fix, once and for all, a maximal analytic extension of $\hat{K}(\cdot)$ (still denoted by $\hat{K}(\cdot)$) and denote by Ω its domain of definition. Set

$$(1.5) \qquad \begin{aligned} \rho_0 &= \{\lambda \in \Omega : \exists(\lambda - A - \hat{K}(\lambda))^{-1}\} \\ F(\lambda) &= (\lambda - A - \hat{K}(\lambda))^{-1} \quad \text{for } \lambda \in \rho_0; \end{aligned}$$

(1.6)

$$\rho = \rho_0 \cup \{\lambda \in \mathbf{C} : \lambda \text{ is an isolated removable singularity of } F(\cdot)\},$$

$$F(\lambda) = \lim_{z \to \lambda} F(z); \lambda \in \rho \backslash \rho_0.$$

The *generalized spectrum* σ is defined by

$$\sigma = \mathbf{C} \backslash \rho. \tag{1.7}$$

In [**2**] we studied the behavior of $F(\cdot)$ near simple poles. Now we consider poles of any order. We recall that, in the case $K \equiv 0$, the poles of $F(\lambda) = R(\lambda, A)$ are the eigenvalues of A having finite ascent.

If λ_0 is a pole of $F(\cdot)$ of order m_0, we set, for λ close to λ_0,

$$F(\lambda) = \sum_{n=0}^{\infty} S_n(\lambda - \lambda_0)^n + \sum_{n=0}^{m_0-1} Q_n(\lambda - \lambda_0)^{-n-1} \tag{1.8}$$

where

$$\begin{aligned} Q_n &= \frac{1}{2\pi i} \int_{\mathbf{C}(\lambda_0, \varepsilon)} F(\lambda)(\lambda - \lambda_0)^n d\lambda, \\ S_n &= \frac{1}{2\pi i} \int_{\mathbf{C}(\lambda_0, \varepsilon)} F(\lambda)(\lambda - \lambda_0)^{-n-1} d\lambda. \end{aligned} \tag{1.9}$$

Here $\mathbf{C}(\lambda_0, \varepsilon)$ is the circle centered at λ_0 with sufficiently small radius $\varepsilon > 0$.

It is convenient to introduce the operators

$$R_{\lambda_0}(t) = \frac{1}{2\pi i} \int_{C(\lambda_0, \varepsilon)} e^{\lambda t} F(\lambda) d\lambda = \sum_{n=0}^{m_0-1} \frac{e^{\lambda_0 t} t^n}{n!} Q_n, \quad t \in \mathbf{R} \tag{1.10}$$

We assume that there is $\omega \geq 0$ such that

$$\sigma \cap \{\lambda \in \mathbf{C} : \operatorname{Re} \lambda = -\omega\} = \emptyset \tag{1.11}$$

and set

$$\sigma_+(\omega) = \sigma \cap \{\lambda \in \mathbf{C} : \operatorname{Re} \lambda > -\omega\}; \sigma_-(\omega) = \sigma \cap \{\lambda \in \mathbf{C} : \operatorname{Re} \lambda < -\omega\} \tag{1.12}$$

The resolvent operator $R(t)$ may be splitted into the sum $R(t) = R_+^\omega(t) + R_-^\omega(t)$, where

$$\begin{aligned} &R_+^\omega(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} F(\lambda) d\lambda, \quad t \in \mathbf{R}; \quad R_-^\omega(t) = R(t) - R_+^\omega(t), \quad t > 0 \\ &R_-^\omega(0) = \mathrm{I} - R_+^\omega(0) \end{aligned} \tag{1.13}$$

and γ is any Jordan curve, oriented counterclockwise, contained in the right half plane $\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > -\omega\}$ surrounding $\sigma_+(\omega)$. It is not difficult to show (see [**2**]) that, for each sufficiently small $\varepsilon > 0$, there is a $c(\varepsilon) > 0$ such that

$$\text{(i) } ||R^\omega_-(t)||_{L(X)} + ||tAR^\omega_-(t)||_{L(X)} \le c(\varepsilon)e^{-(\omega+\varepsilon)t}, t > 0;$$

$$\text{(ii) } ||AR^\omega_-(t) - AR^\omega_-(s)||_{\mathrm{L}(X)} \le c(\varepsilon)e^{-(w+\varepsilon)^t}(1/s - 1/t),$$

$$(1.14) \qquad t > s > 0;$$

$$\text{(iii) } ||R^\omega_+(t)||_{L(X,D(A))} \le c(\varepsilon)e^{(\omega+\varepsilon)t};\ t \le 0.$$

From now on we assume

$$(1.15) \qquad \begin{aligned} &\sigma_+(\omega) = \{\lambda_1, \cdots, \lambda_N\}, \text{ where, for each } j = 1, \cdots, N, \\ &\lambda_j \text{ is a pole of } F(\cdot) \text{ of order } m_j \end{aligned}$$

In this case we have

$$(1.16) \qquad R^\omega_+(t) = \sum_{j=1}^N R_{\lambda_j}(t), \quad t \in \mathbf{R}.$$

PROPOSITION 1.1. *Let* (1.1) *and* (1.15) *hold, and let* $\phi \in \mathbf{C}_\omega([0,+\infty[\,; X)$, $u_0 \in \overline{D(A)}$. *Then the function* u *given by* (1.2) *belongs to* $\mathbf{C}_\omega([0,+\infty[\,; X)$ *if and only if*

$$(1.17) \qquad R^\omega_+(t)u_0 = -\int_0^{+\infty} R^\omega_+(t-s)\phi(s)ds, \quad t \ge 0.$$

If (1.17) *holds, and, in addition,* ϕ *belongs to* $\mathbf{C}^\alpha_\omega([0,+\infty[\,; X)$ *for some* $\alpha \in]0,1[$, *then* u *belongs to* $\mathrm{C}^\alpha_\omega([a,+\infty[\,; D(A))$ *for every* $a > 0$*; in particular,*

$$(1.18) \qquad \sup_{t>a} ||u(t)e^{\omega t}||_{D(A)} < +\infty \text{ for each } a > 0.$$

PROOF. The proposition was shown in [**2**, Theorem 2.10] in the case $K \in \mathrm{L}^1([0,+\infty[\,; \mathrm{L}(D(A),X))$ and $\omega = 0$. We now show the statement in the general case. We set $u(t) = v(t) + z(t)$, with

$$v(t) = R_-^\omega(t)u_0 + \int_0^t R_-^\omega(t-s)\phi(s)ds - \int_t^{+\infty} R_+^\omega(t-s)\phi(s)ds, \quad t \geq 0,$$

$$z(t) = R_+^\omega(t)u_0 + \int_0^{+\infty} R_+^\omega(t-s)\phi(s)ds, \quad t \geq 0.$$

Using estimates (1.14)(i),(iii), it is not difficult to see that v belongs to $C_\omega([0,+\infty[\,;X)$ so that u belongs to $C_\omega([0,+\infty[\,;X)$ if and only if z does. Thanks to (1.10), (1.16) we have

$$z(t) = \sum_{j=1}^{N} \sum_{n=0}^{m_j-1} e^{\lambda_j t} t^n y_{jn}, \quad t \geq 0,$$

where $y_{jn} \in D(A)$. Since $\operatorname{Re}\lambda_j > -\omega$ for each j and the functions $t \to e^{\lambda_j t}t^n$ are linearly independent, then z does not belong to $C_\omega([0,+\infty[\,;X)$ unless it vanishes, i.e., unless (1.17) holds.

If (1.17) holds, and, in addition, ϕ belongs to $C_\omega^\alpha([0,+\infty[\,;X)$ for some $\alpha \in\,]0,1[$, then, using estimates (1.14)(ii),(iii), one can show that $u = v$ belongs to $C_\omega^\alpha([a,+\infty[\,;D(A))$ for every $a > 0$: the proof is similar to the corresponding one in [**2**, Theorem 2.10] and it is omitted. □

2. Stabilizability in abstract integrodifferential equations. We consider here stabilizability for system (0.1). Due to Proposition 1.1, the problem is not trivial if

$$\sigma \cap \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0\} \neq \emptyset. \tag{2.1}$$

We fix ω satisfying (1.11). For the sake of simplicity, we consider first the case where $\sigma_+(\omega)$ consists of simple poles of $F(\cdot)$.

THEOREM 2.1. *Let* (1.1), (1.11) *hold; assume that* $\sigma_+(\omega)$ *consists of* N *simple poles* $\lambda_1, \ldots, \lambda_N$ *of* $F(\cdot)$, *and that the respective residues* Q_j *at* $\lambda = \lambda_j$ *are finite rank operators. Then the following statements are equivalent:*

(i) *For every* $u_0 \in \overline{D(A)}$ *there exists* $f \in C_\omega([0,+\infty[\,;Y)$ *such that the function* u *given by* (1.2) *belongs to* $C_\omega([0,+\infty[\,;X)$;

(ii) *For every* $j = 1, \dots, N$, Range $Q_j^* \cap \ker\Phi^* = \{0\}$.

If either of the above equivalent conditions holds, then, for every $u_0 \in X$, *there exists* $f \in C_\omega^\alpha([0, +\infty[\,; Y)$ *such that the function* u *given by* (1.2) *belongs to* $C_\omega^\alpha([a, +\infty[\,; D(A))$ *for each* $a > 0$.

PROOF. Thanks to Proposition 1.1, condition (i) is equivalent to the following: for each $u_0 \in \overline{D(A)}$ there exists $f \in C_\omega([0, +\infty[\,; Y)$ such that

$$R_+^\omega(t)u_0 = -\int_0^{+\infty} R_+^\omega(t-s)\Phi f(s)ds, \ t \geq 0, \tag{2.2}$$

Due to formulas (1.10) and (1.16), (2.2) holds if and only if

$$Q_j u_0 = \int_0^{+\infty} e^{-\lambda_j s} Q_j \Phi f(s)ds, \quad j = 1, \dots, N, \tag{2.3}$$

i.e., setting

$$\Gamma : C_\omega([0, +\infty[\,; Y) \to X^N, \quad Q : X \to X^n$$

$$\Gamma f = \left(\int_0^{+\infty} e^{-\lambda_1 s} Q_1 \Phi f(s)ds, \dots, \int_0^{+\infty} e^{-\lambda_N s} Q_N \Phi f(s)ds \right) \tag{2.4}$$

$$Qy = (Q_1 y, \dots, Q_N y),$$

we have $\Gamma f = Q u_0$. In other words, (i) holds if and only if

$$\text{Range}\ \Gamma \supset \text{Range}\ Q. \tag{2.5}$$

By assumption, both ranges of Γ and Q are finite dimensional, hence closed. Therefore (2.5) holds if and only if

$$\ker Q^* \supset \ker\Gamma^*. \tag{2.6}$$

Hence, we have to show that (2.6) is equivalent to (ii).

As easily seen, $\ker\Gamma^*$ consists of all N-tuples $(x_1^*, \dots, x_N^*) \in (X^*)^N$ such that $\int_0^{+\infty} \sum_{j=1}^N e^{-\lambda_j s} \langle f(s), \Phi^* Q_j^* x_j^* \rangle ds = 0$ for each $f \in$

$C_\omega([0,+\infty[\,;Y)$. Since the functions $t \to e^{\lambda_j t}$ are linearly independent, this means that $\Phi^* Q_j^* x_j^* = 0$ for every $j = 1,\dots,N$. Moreover, we have $Q^*(x_1^*,\dots,x_N^*) = \sum_{j=1}^N Q_j^* x_j^*$. Therefore, (2.6) is equivalent to

$$\Phi^* Q_j^* x_j^* = 0 \ \forall j = 1,\dots,N \Rightarrow \sum_{j=1}^N Q_j^* x_j^* = 0, \tag{2.7}$$

which, in its turn is obviously equivalent to (ii). Concerning the last statement of the proposition, it is sufficient to remark that, if either (i) or (ii) holds, then, arguing as in the first part of the proof, one can show that the mapping $\Lambda : C_\omega^\alpha([0,+\infty[\,;Y) \to X^N, \Lambda f = \Gamma f$, is such that Range $\Lambda \supset$ Range Q. The statement follows then from Proposition 1.1. □

REMARK 2.2. We showed in [**2**, Proposition 1.6] that, if $\lambda_0 \in \Omega$ is a simple pole of $F(\cdot)$, then Range $Q_0 = \ker(\lambda_0 - A - \hat{K}(\lambda_0))$. Therefore, in this case condition (ii) of Theorem 2.1 is very similar to the Hautus condition [**4**] concerning ordinary differential equations, and reduces to it if $K \equiv 0$ and X is finite dimensional. □

Let us consider now the general case.

THEOREM 2.3. *Let* (1.1), (1.11), (1.15) *hold, and assume that the residues $Q_{j,n}$ of $F(\cdot)$ at $\lambda = \lambda_j$ are finite rank operators. Then the following statements are equivalent:*

(i) *For every $u_0 \in \overline{D(A)}$ there exists $f \in C_\omega([0,+\infty[\,;Y)$ such that the function u given by* (1.2) *belongs to $C_\omega([0,+\infty[\,;X)$;*

(ii) *For every $j = 1,\dots,N, h = 0,\dots,m_j$,* Range $Q_{j,h}^* \cap \ker\Phi^* = \{0\}$.

If either of the above equivalent conditions holds, then for every $u_0 \in X$, there exists $f \in C_\omega^\alpha([0,+\infty[\,;Y)$ such that the function u given by (1.2) *belongs to $C_\omega^\alpha([a,+\infty[\,;D(A))$ for each $a > 0$.*

PROOF. The idea of the proof is the same as the one of Theorem 2.1. In this case, thanks to formulas (1.10), (1.16), condition (2.2) is equivalent to

$$\sum_{j=1}^{N}\sum_{n=0}^{m_j-1} e^{\lambda_j t}\frac{t^n}{n!}Q_{j,n}u_0 \tag{2.8}$$
$$= -\sum_{j=1}^{N}\sum_{n=0}^{m_j-1}\sum_{k=n}^{m_j-1} e^{\lambda_j t}\frac{t^n}{n!}\int_0^{+\infty} e^{-\lambda_j s}(-s)^{k-n}Q_{j,n}\Phi f(s)ds.$$

Since the functions $t \to e^{\lambda_j t}t^n$ are linearly independent, (2.8) is equivalent to $\Gamma f = Qu_0$, where

$$\Gamma : \mathrm{C}_\omega([0,+\infty[\,;Y) \to X^K, \quad Q : X \to X^K, \quad K = \sum_{j=1}^{N} m_j$$

$$\Gamma f = \left\{ \sum_{k=n}^{m_j-1} \int_0^{+\infty} e^{-\lambda_j s}(-s)^{k-n}Q_{j,n}\Phi f(s)ds \right\}_{j=1,\dots,N;\ n=0,\dots,m_j-1}$$

$$Qy = \{Q_{j,n}y\}_{j=1,\dots,N;\ n=0,\dots,m_j-1}.$$

Therefore, (i) holds if and only if Range $\Gamma \supset$ Range Q. By assumption, the ranges of Γ and Q are finite dimensional so that (i) holds if and only if

$$\ker Q^* \supset \ker \Gamma^*. \tag{2.9}$$

For each $(x^*_{jh})_{j=1,\dots,N;L=0,\dots,m_j-1} \in (X^*)^K$, we have

$$\Gamma^*(x^*_{j,h})_{j=1,\dots,N;h=0,\dots,m_j-1}(f)$$
$$= -\sum_{j=1}^{N}\sum_{k=0}^{m_j-1}\sum_{h=0}^{m_j-1-k}\int_0^{+\infty}\frac{e^{-\lambda_j s}(-s)^k}{k!}\langle f(s), \Phi^* Q^*_{j,k+h}x^*_{j,k+h}\rangle ds$$

$$Q^*(x^*_{j,h})_{j=1,\dots,N;h=0,\dots,m_j-1}(y) = \sum_{j=1}^{N}\sum_{h=0}^{m_j-1}\langle y, Q^*_{j,h}x^*_{j,h}\rangle$$

so that

(2.10)

$$\ker\Gamma^* = \left\{ (x^*_{j,h})_{j=1,\dots,N;h=0,\dots,m_j-1} \in (X^*)^K : \Phi^*\Big(\sum_{h=k}^{m_j-1} Q^*_{j,h} x^*_{j,h} \Big) = 0 \right.$$
$$\left. j = 1,\dots,N; k = 0,\dots,m_j-1 \right\}$$
$$= \{ (x^*_{j,h})_{j=1,\dots,N;h=0,\dots,m_j-1} \in (X^*)^K : \Phi^* Q^*_{j,h} x^*_{j,h} = 0,$$
$$j = 1,\dots,N; h = 0,\dots,m_j-1 \}$$

(2.11)

$$\ker Q^* = \left\{ (x^*_{j,h})_{j=1,\dots,N;h=0,\dots,m_j-1} \in (X^*)^K : \sum_{j=1}^{N} \sum_{h=0}^{m_j-1} Q^*_{j,h} x^*_{j,h} = 0 \right\}.$$

We can now show that (2.9) is equivalent to (ii). It is easy to see that (ii) implies (2.9). Conversely, assume that (2.9) holds, and let $y^* \in \ker\Phi^* Q^*_{j_0,h_0}$ for some j_0, h_0. Set $x^*_{j,h} = \delta_{j_0,h_0} y^*$. Then $\Phi^* Q^*_{j,h} x^*_{j,h} = 0$ for each $j = 1,\dots,N, h = 0,\dots,m_j-1$, so that, by assumption (2.9), we have $0 = \sum_{j=1}^{N} \sum_{h=0}^{m_j-1} Q^*_{j,h} x^*_{j,h} = Q^*_{j_0,h_0} y^* = 0$. Therefore, for each j_0, h_0, we have $\ker Q^*_{j_0,h_0} \supset \ker\Phi^* Q^*_{j_0,h_0}$ and (ii) holds.

The last part of the proof follows as in Theorem 2.1. □

In the applications it is sometimes useful to establish exponential decay for $Cu(t)$, where C is a linear bounded operator from $D(A)$ to a Banach space Z. Obviously, if $||u(t)||_{D(A)}$ decays exponentially as $t \to \infty$, then $||Cu(t)||_Z$ does also. On the other hand, conditions (ii) of Theorems 2.1 and 2.3 may be too restrictive if one is interested only in the asymptotic behavior of Cu. However, arguing exactly as in the proof of Theorem 2.2, one can show

THEOREM 2.4. *Let $C \in \mathrm{L}(D(A), Z)$, let* (1.1), (1.11), (1.15) *hold, and assume that the residues $Q_{j,n}$ of $F(\cdot)$ at $\lambda = \lambda_j$ are finite rank operators. Then the following statements are equivalent:*

(i) *For every $u_0 \in X$ there exists $f \in \mathrm{C}^\alpha_\omega([0,+\infty[\,;Y)$ such that the function Cu, where u is given by* (1.2), *belongs to $\mathrm{C}_\omega([a,+\infty[\,;Z)$ for each $a > 0$;*

(ii) *For every* $j = 1, \dots, N, h = 0, \dots, m_j$, $\text{Range} Q^*_{j,h} C^* \cap \ker \Phi^* = \{0\}$.

3. Examples and applications.

3.1. *The case where* $K(t) = k(t)A$. We consider here the special equation

$$(3.1)\quad u'(t) = Au(t) + bu(t) + A\int_0^t k(t-s)u(s)ds + \Phi f(s), \quad u(0) = u_0,$$

where A generates an analytic semigroup, $b \in \mathbf{R}$ and $k : [0, +\infty[\to \mathbf{R}$ is such that

(3.2)
- (i) $k \in L^1([0, +\infty[)$;
- (ii) for each $x \in D(A)$ the Laplace transform $\hat{k}(\lambda)$ is analytically extendible to a sector $S = \{\lambda \in \mathbf{C} : \lambda \neq \omega; |\arg(\lambda - \omega)| < \theta\}$ where $\omega \in \mathbf{R}$ and $\theta \in]\pi/2; \pi[$;
- (iii) there are $0 < \beta < 1$ and $c > 0$ such that $|\lambda^\beta \hat{k}(\lambda)| \leq c$ for each $\lambda \in S$.

In this case

$$(3.3)\qquad \rho_0 = \{\lambda \in \Omega : \hat{k}(\lambda) \neq -1, (\lambda - b)/(\hat{k}(\lambda) + 1) \in \rho(A)\}$$

$$(3.4)\qquad \begin{aligned} F(\lambda) &= (\lambda - A - b - \hat{k}(\lambda)A)^{-1} \\ &= \frac{1}{\hat{k}(\lambda) + 1} R\Big(\frac{\lambda - b}{\hat{k}(\lambda) + 1}, A\Big), \quad \lambda \in \rho_0. \end{aligned}$$

To apply Theorems 2.1 and 2.3 we need to know the poles of $F(\cdot)$ and the respective residues. The following proposition gives a characterization of the poles of $F(\cdot)$ contained in Ω.

PROPOSITION 3.1. $\lambda_0 \in \Omega \backslash \{b\}$ *is a pole of* $F(\cdot)$ *of order* m *if and only if, setting*

$$(3.5)\qquad \phi(\lambda) = (\lambda - b)/(\hat{k}(\lambda) + 1),$$

then

$$(3.6)\quad \begin{array}{l} \text{(i) } \hat{k}(\lambda_0) \neq -1; \\ \text{(ii) } z_0 = \phi(\lambda_0) \textit{ is a pole of } R(\cdot, A) \textit{ of } A \textit{ of order } \\ \mu \geq 1 \textit{ (i.e., an eigenvalue with ascent } \mu); \\ \text{(iii) } \lambda \to \phi(\lambda) - \phi(\lambda_0) \textit{ has a zero of order } h \textit{ at } \\ \lambda = \lambda_0 \textit{ and } \mu h = m. \end{array}$$

If b belongs to Ω and $\hat{k}(b) \neq -1$, then $\lambda_0 = b$ is a pole of $F(\cdot)$ of order m if and only if (3.6)(ii),(iii) *hold. Moreover,*

$$Q_n = \sum_{k=0}^{\mu-1} c_{kn}(A - z_0)^k P_{z_0}, \tag{3.7}$$

where

$$c_{kn} = \frac{1}{2\pi i} \int_{C(\lambda_0,\varepsilon)} (\lambda - \lambda_0)^n (\phi(\lambda) - z_0)^{-k-1} (\hat{k}(\lambda) + 1)^{-1} d\lambda$$

and

$$P_{z_0} = \frac{1}{2\pi i} \int_{C(z_0,\varepsilon)}$$

$R(z, A)dz$.

If b belongs to Ω, and $\hat{k}(b) = -1, \hat{k}'(b) \neq 0$, then $\lambda_0 = b$ is a pole of $F(\cdot)$ of order 1 if and only if $1/\hat{k}'(b)$ belongs to $\rho(A)$. In this case we have

$$Q_0 = \frac{1}{\hat{k}'(b)} R\Big(\frac{1}{\hat{k}'(b)}, A\Big). \tag{3.8}$$

It is a pole of order $m > 1$ if and only if $1/\hat{k}'(b)$ is a pole of $R(\cdot, A)$ of order $\mu \geq 1, \lambda \to \phi(\lambda) - 1/\hat{k}'(b)$ has a zero of order h at $\lambda = b$, and $\mu h = m - 1$.

If b belongs to Ω and $\hat{k}(b) = -1, \hat{k}'(b) = 0$, then $\lambda_0 = b$ is either a pole of $F(\cdot)$ of order 1 *or an essential singularity. In the case that $\lambda_0 = b$ is a pole of $F(\cdot)$ of order* 1 *we have*

$$Q_0 = \mathrm{I}. \tag{3.9}$$

In particular, if $D(A) \neq X$, and $\hat{k}(b) = -1, \hat{k}'(b) = 0$, then $\lambda_0 = b$ is not a pole of $F(\cdot)$.

PROOF. The proposition was proved in [**2**, Proposition 5.3] in the case $m = 1$. We consider here the general case.

Let $\lambda_0 \in \Omega$ be any pole of $F(\cdot)$ of order m. There is a neighborhood U of λ_0 such that $U \backslash \{\lambda_0\}$ is contained in ρ_0 and

$$\begin{aligned} &\text{(i) } (\lambda - \lambda_0)F(\lambda) - (\hat{k}(\lambda) + 1)AF(\lambda) = \mathrm{I} \\ &\text{(ii) } F(\lambda)(\lambda - \lambda_0) - F(\lambda)(\hat{k}(\lambda) + 1)A = \mathrm{I} \ \text{ on } D(A), \end{aligned} \tag{3.10}$$

so that

$$\begin{aligned} (\lambda - \lambda_0)^m(\lambda - b)F(\lambda) &- (\lambda - \lambda_0)^m(\hat{k}(\lambda) + 1)AF(\lambda) \\ &= (\lambda - \lambda_0)^m \mathrm{I}, \ \lambda \in U \backslash \{\lambda_0\}. \end{aligned} \tag{3.11}$$

Letting $\lambda \to \lambda_0$ in (3.11), we find

$$[(\lambda_0 - b) - (\hat{k}(\lambda_0) + 1)A]Q_{m-1} = 0, \tag{3.12}$$

where $Q_{m-1} \neq 0$ by assumption. Now there are two possibilities: either

(a) $\hat{k}(\lambda_0) + 1 \neq 0$, or

(b) $\hat{k}(\lambda_0) + 1 = 0$ and $\lambda_0 = b$.

Moreover, for every $\lambda \in U \backslash \{\lambda_0\}$, (3.4) holds.

Let us consider case (a). Since the functions $\lambda \to 1/(\hat{k}(\lambda) + 1)$ and $\lambda \to \phi(\lambda)$ are holomorphic in U, then $z_0 = \phi(\lambda_0)$ is an isolated element of $\sigma(A)$. Therefore, for every z close to z_0,

$$\begin{aligned} R(z, A) = &\sum_{n=0}^{\infty} (-1)^n S_{z_0}^{n+1} (z - z_0)^n \\ &+ \sum_{n=0}^{\infty} (A - z_0)^n (z - z_0)^{-n-1} P_{z_0}, \end{aligned} \tag{3.13}$$

where

$$P_{z_0} = \frac{1}{2\pi i} \int_{C(z_0, \varepsilon)} R(z, A)dz, \quad S_{z_0} = \lim_{z \to z_0} (1 - P_{z_0})R(z, A). \tag{3.14}$$

Replacing (3.13) in (3.4) we get

$$
\begin{aligned}
F(\lambda) = & \frac{1}{\hat{k}(\lambda)+1}\left\{ \sum_{n=0}^{\infty}(-1)^n S_{z_0}^{n+1}(\phi(\lambda)-z_0)^n \right. \\
& \left. + \sum_{n=0}^{\infty}(A-z_0)^n(\phi(\lambda)-z_0)^{-n-1}P_{z_0} \right\}.
\end{aligned} \tag{3.15}
$$

Since λ_0 is a pole of F of order m, it follows that there exists $\mu \in \mathbf{N}$ such that $(A-z_0)^n P_{z_0} = 0$, for every $n \geq \mu$, and $(A-z_0)^{\mu-1}P_{z_0} \neq 0$. Consequently, λ_0 is a zero of $\phi(\cdot) - z_0$ of order $h = m/\mu$. Formula (3.7) follows now easily, recalling (1.9).

Let us consider case (b). If $\hat{k}'(b) \neq 0$ and $m > 1$, the situation is similar to the previous one: actually, we have $z_0 \doteq \lim_{\lambda\to b}\phi(\lambda) = 1/\hat{k}'(b)$. Then z_0 is an eigenvalue of A, and, multiplying both members of (3.10)(i) by $(\lambda-b)^{m-1}$,we find that b is a zero of $\phi(\cdot)-z_0$ of order $h = (m-1)/\mu$.

If $\hat{k}'(b) \neq 0$ and $m = 1$, from equalities (3.10) with $\lambda_0 = b$, we get, letting $\lambda \to b, (1-\hat{k}'(b)A)Q_0 = \mathrm{I}$, $Q_0(1-\hat{k}'(b)A) = \mathrm{I}$ on $D(A)$. This implies $Q_0 \neq 0$ and that $(1-\hat{k}'(b)A)$ is invertible, with inverse Q_0. In particular, z_0 belongs to the resolvent set of A and

$$
Q_0 = 1/\hat{k}'(b)R(1/\hat{k}'(b), A). \tag{3.16}
$$

In the case where $\hat{k}'(b) = 0$, we get $m = 1$ and $Q_0 = \mathrm{I}$; actually, if m were greater than 1, multiplying both members of (3.10)(i) by $(\lambda-b)^{m-1}$ and letting $\lambda \to b$, we would get $Q_{m-1} = 0$. Hence $m = 1$, and letting $\lambda \to b$ in (3.10)(i), we obtain $Q_0 = \mathrm{I}$. This is clearly impossible if $D(A) \neq X$.

Conversely, if $\lambda_0 \in \Omega$ is such that (3.6) holds, then (3.13) becomes

$$
R(z,A) = \sum_{n=0}^{\infty}(-1)^n S_{z_0}^{n+1}(z-z_0)^n + \sum_{n=0}^{\mu-1}(A-z_0)^n(z-z_0)^{-n-1}P_{z_0} \tag{3.17}
$$

for every z close to z_0. Plugging (3.17) into (3.4),

$$
\begin{aligned}
F(\lambda) = & \frac{1}{\hat{k}(\lambda)+1}\left\{ \sum_{n=0}^{\infty}(-1)^n S_{z_0}^{n+1}(\phi(\lambda)-z_0)^n \right. \\
& \left. + \sum_{n=0}^{\mu-1}(A-z_0)^n(\phi(\lambda)-z_0)^{-n-1}P_{z_0} \right\}
\end{aligned} \tag{3.18}
$$

so that λ_0 is a pole of F of order $m = \mu h$.

Let $\lambda_0 = b$ and $\hat{k}(b) = -1$. Then $(\lambda_0 - b)\mathrm{I} - (1 + \hat{k}(b))A = 0$, so that b does not belong to ρ_0. Since $b \in \Omega$, it is not a removable singularity of $F(\cdot)$ (see [**2**, Lemma 1.3]). Therefore, b does not belong to ρ.

If, in addition, $\hat{k}'(b) \neq 0$ and $1/\hat{k}'(b) \in \rho(A)$, from (3.4) we get easily that b is a simple pole of $F(\cdot)$, and (3.8) holds. On the other hand, if $\hat{k}'(b) \neq 0$ and $1/\hat{k}'(b)$ is a pole of $R(\cdot, A)$ of order $\mu \geq 1, b$ is a zero of order h of the function $\lambda \to \phi(\lambda) - 1/\hat{k}'(b)$ so that, using equality (3.18) (with $z_0 = \lim_{\lambda \to b} \phi(\lambda) = 1/\hat{k}'(b)$), we find that b is a pole of $F(\cdot)$ of order $m = \mu h + 1$.

Finally let $\lambda_0 = b, \hat{k}(b) = -1, \hat{k}'(b) = 0$. From equality (3.10)(i) it follows that either b is a pole of $F(\cdot)$ of order 1, or it is an essential singularity. □

In many applications the function k is a completely monotone kernel, i.e.,

$$k(t) = \int_0^{+\infty} e^{-t\xi} \mu(d\xi), \quad t > 0, \tag{3.19}$$

where μ is a positive Borel measure (see [**10**, Chapter 4] for equivalent definitions and properties). We also assume

$$\text{(i)} \int_0^{+\infty} \frac{\mu(d\xi)}{\xi} < +\infty; \quad \text{(ii)} \int_0^{+\infty} \frac{\mu(d\xi)}{\xi^{1-\beta}} < +\infty \tag{3.20}$$

for some $\beta \in]0,1[$. Condition (3.20)(i) means that k belongs to L^1 $([0,+\infty[)$ and implies that (3.2)(ii) holds; condition (3.20)(ii) implies that (3.2)(iii) holds. Moreover, $\hat{k}(\lambda)$ has a maximal analytic extension in the domain $\Omega = \mathbf{C} \backslash \{-\operatorname{supp} \mu\}$, given by

$$\hat{k}(\lambda) = \int_0^{+\infty} \frac{\mu(d\xi)}{\lambda + \xi}, \quad \lambda \in \Omega. \tag{3.21}$$

(see [**2**, Lemma 5.4]).

PROPOSITION 3.2. *Let k be defined by* (3.19) *and satisfy* (3.20). *Assume, in addition,*

(i) $0 \notin \operatorname{supp} \mu$;

(ii) $z_0 \in \sigma(A)$ *is not a pole of* $R(\cdot, A), \lambda_0 \in \phi^{-1}(z_0) \Rightarrow \operatorname{Re} \lambda_0 < 0$;

(iii) $z_0 \in \sigma(A)$ *is a pole of* $R(\cdot, A), \dim \operatorname{Range} P_{z_0} = +\infty,\ \lambda_0 \in \phi^{-1}(z_0) \Rightarrow \operatorname{Re} \lambda_0 < 0$.

Then either $\{\lambda \in \sigma : \operatorname{Re} \lambda \geq 0\} = \emptyset$ *or* $\{\lambda \in \sigma : \operatorname{Re} \lambda \geq 0\} = \{\lambda_1, \ldots, \lambda_N\}$, *where, for each* $j = 1, \ldots, N, \lambda_j$ *is a pole of* $F(\cdot)$ *such that all the corresponding residues of any order are finite rank operators. Moreover, if* $\phi(\lambda_j)$ *is a simple pole of* $R(\cdot, A)$ *and* $\phi'(\lambda_j) \neq 0$, *then* λ_j *is a simple pole of* $F(\cdot)$.

PROOF. Thanks to Proposition 5.6 of [**2**], we have $\{\lambda \in \sigma : \operatorname{Re} \lambda \geq 0\} = \{\lambda \in \Omega : \operatorname{Re} \lambda \geq 0, \hat{k}(\lambda_0) + 1 \neq 0, \phi(\lambda) \in \sigma(A)\}$. Therefore, by assumptions (ii), (iii), if $\lambda_0 \in \sigma$ and $\operatorname{Re} \lambda_0 \geq 0$, then $z_0 = \phi(\lambda_0)$ is a pole of $R(\cdot, A)$, and dim Range $P_{z_0} < +\infty$. Proposition 3.1 implies now that λ_0 is a pole of $F(\cdot)$, and $\dim \operatorname{Range} Q_n = 0$ for each n. Since $\{\lambda \in \sigma : \operatorname{Re} \lambda \geq 0\}$ is closed, bounded (see [**1, 6**]) and consists of poles, then it is finite. □

REMARK 3.3. Under the assumptions of Proposition 3.2, there exists $\varepsilon > 0$ such that the strip $\{\lambda \in \mathbf{C} : -\varepsilon < \operatorname{Re} \lambda < 0\}$ is contained in ρ. Therefore Theorem 2.2 is applicable with any $\omega \in]0, \varepsilon[$. □

EXAMPLE 3.4. Consider the integrodifferential equation

$$\begin{aligned} u_t(t,x) &= \Delta u(t,x) + bu(t,x) + \int_0^t k(t-s)\Delta u(s,x)ds \\ &\quad + \sum_{k=1}^{K} \langle \phi_k, f_k(t,\cdot)\rangle \phi_k(x), \quad t > 0, x \in \overline{\Omega}, \\ u(0,x) &= u_0(x), \quad x \in \overline{\Omega}, \\ u(t,x) &= 0, \ t > 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.22}$$

where Ω is a bounded open set in $\mathbf{R}^n$ with $\mathbf{C}^2$ boundary $\partial\Omega$, and $\phi_h, h = 1, \ldots, H$ are linearly independent continuous functions. We have set $\langle \phi, \gamma \rangle = \int_\Omega \phi(x)\gamma(x)dx$. The nonzero kernel k satisfies the assumptions of Proposition 3.2.

We choose

$$(3.23)\quad X = C(\overline{\Omega}),\quad D(A) = \{g \in \cap_{p>1} W^{2,p}(\Omega) : \Delta g \in X, g_{|\partial\Omega} = 0\},$$
$$Ag = \Delta g.$$

As it is well known, A generates an analytic semigroup in X (see [**9**]), and its spectrum consists of a sequence of real negative semi-simple eigenvalues $-\zeta_j, j \in \mathbf{N}$, with $\lim_{j\to+\infty} \zeta_j = +\infty$ and $\zeta_j < \zeta_{j+1}$ for each j. In order to apply Theorems 2.1, 2.2, we have to describe the set $\{\lambda \in \sigma : \operatorname{Re}\lambda \geq 0\}$. From Proposition 3.2 we know that $\{\lambda \in \sigma : \operatorname{Re}\lambda \geq 0\} = \cup_{j\in\mathbf{N}}\sigma_j^+$, where $\sigma_j^+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0, (\lambda - b)/(\hat{k}(\lambda)+1) = -\zeta_j\}$ is finite. We set, for each $j \in \mathbf{N}$,

$$(3.24)\qquad m_j = \min\left\{x + \xi_j + \zeta_j \int_0^{+\infty} [x+\xi]^{-1}\mu(d\xi) : x \in [0; b]\right\}$$

$$\Lambda_j = \left\{(x;y) : 0 \leq x \leq b/2; \zeta_j \int_0^{+\infty} \xi/[(x+\xi)^2 + y^2]\mu(d\xi) = 1\right\}$$

$$n_j = \min\left\{2x + \zeta_j + \zeta_j \int_0^{+\infty} \xi/[(x+\xi)^2+y^2]\mu(d\xi) : (x,y) \in \Lambda_j\right\} \text{ if } \Lambda_j \neq \emptyset.$$

PROPOSITION 3.5. *Let k satisfy the assumptions of Proposition 3.2. Then the following statements hold:*

(i) *If $m_j < b$, then $\sigma_j^+ \neq \emptyset$ and consists of simple poles of $F(\cdot)$.*

(ii) *If $m_j = b$ and $\zeta_j + \zeta_j \int_0^{+\infty} \xi^{-1}\mu(d\xi) > b$, then σ_j^+ contains a real not simple pole. If $\zeta_j + \zeta_j \int_0^{+\infty} \xi^{-1}\mu(d\xi) = b$, then σ_j^+ contains a real pole which is simple if and only if $\zeta_j \int_0^{+\infty} \xi^{-2}\mu(d\xi) \neq 1$. In both cases, the other elements of σ_j^+ are simple poles.*

(iii) *If $m_j > b$ and $\Lambda_j \neq \emptyset, n_j \leq b$, then $\sigma_j^+ \neq \emptyset$, and it consists of simple poles.*

(iv) *If $m_j > b$ and either $\Lambda_j = \emptyset$ or $\Lambda_j \neq \emptyset$ and $n_j > b$, then $\sigma_j^+ = \emptyset$.*

PROOF. From Proposition 3.2 we know that σ_j^+ is a finite set for every $j \in \mathbf{N}$ and that it coincides with the set of solutions with nonnegative real part of the equation

$$(3.25)\qquad \lambda - b + \zeta_j(1 + \hat{k}(\lambda)) = 0.$$

Since ζ_j is a simple pole of $R(\cdot, A)$, all the solutions λ of (3.25) are simple poles of $F(\cdot)$ provided

$$\hat{k}'(\lambda) \neq -1/\zeta_j. \tag{3.26}$$

Setting $\lambda = x + iy$, equation (3.25) is equivalent to the system

$$\begin{aligned} &x + \zeta_j + \zeta_j \int_0^{+\infty} \frac{x+\xi}{(x+\xi)^2 + y^2} \mu(d\xi) = b \\ &y\Big(1 - \zeta_j \int_0^{+\infty} \frac{1}{(x+\xi)^2 + y^2} \mu(d\xi)\Big) = 0. \end{aligned} \tag{3.27}$$

Let us consider the real solutions of (3.25) ($y = 0$ in (3.27)). The first equation of (3.27) becomes

$$G(x) \doteq x + \zeta_j + \zeta_j \int_0^{+\infty} [x+\xi]^{-1} \mu(d\xi)) = b. \tag{3.28}$$

Obviously, all the possible nonnegative solutions of (3.28) belong to $[0, b]$.

If $m_j > b$, equation (3.28) has no solutions.

If $m_j < b$, since G is strictly convex, there are precisely two solutions x_1, x_2 provided $G(0) \geq b$, and a unique solution x_3 provided $G(0) < b$. From the strict convexity of G it follows that $G'(x_i) = 1 - \zeta_j \hat{k}'(x_i) \neq 0$, $i = 1, 2, 3$, so (3.26) holds.

If $m_j = b$, there is a unique solution $\overline{x}$. If $G(0) > b$, then $0 < \overline{x} < b$ and $\overline{x}$ is not a simple pole of $F(\cdot)$. If $G(0) = b$, then $\overline{x} = 0$, and it is a simple pole if and only if

$$\zeta_j \int_0^{+\infty} \xi^{-2} \mu(d\xi) \neq 1.$$

Let us consider now the solutions of (3.27) with $y \neq 0$: they have to be found in the set Λ_j. Λ_j is void for every $j \geq j_0$, where j_0 is the smallest positive integer such that

$$\zeta_{j_0} \int_0^{+\infty} \xi^{-2} \mu(d\xi) > 1.$$

If $\Lambda_j \neq \emptyset$ and $n_j < b$, there are at least two solutions $(x, \pm y)$ of (3.27) which correspond to a couple of complex conjugate solutions $\lambda, \overline{\lambda}$ of (3.25). Condition (3.26) holds both at λ and at $\overline{\lambda}$: actually, the imaginary parts of $\hat{k}'(\lambda)$ and of $\hat{k}'(\overline{\lambda})$ vanish if and only if $y = 0$, and our assumption now is $y \neq 0$. □

Proposition 3.5 gives, in particular, a necessary and sufficient condition in order that $\{\lambda \in \sigma : \operatorname{Re}\lambda \geq 0\} = \emptyset$: actually, from Proposition 3.5, it follows that, for every $j \in \mathbf{N}$, we have $\sigma_j^+ = \emptyset$ if and only if $m_j > b$ and either $\Lambda_j = \emptyset$ or $\Lambda_j \neq \emptyset$ and $n_j < b$ (this condition is satisfied, in particular, if $b < \zeta_1$).

If $\{\lambda \in \sigma : \operatorname{Re}\lambda \geq 0\} = \emptyset$, then the free system is exponentially stable, hence it is trivially stabilizable (it is sufficient to choose $f_k = 0$ for each k). The stabilizability problem is significant if the free system is not asymptotically stable, i.e., if $\sigma_j^+ \neq \emptyset$ for some j. In this case we set

$$J = \{j \in \mathbf{N} : \sigma_j^+ \neq \emptyset\}, \tag{3.29}$$

and, for each $j \in J$, we choose any basis $\{\psi_1, \ldots, \psi_{N_j}\}$ of $\ker(\zeta_j - A)$.

PROPOSITION 3.6. *Let k satisfy the assumptions of Proposition 3.2, and let $J \neq \emptyset$, with $m_j \neq b$ for each $j \in J$. Then system (3.2) is stabilizable if and only if, for each $j \in J$, the rank of the matrix*

$$[A_{hk}] = [\langle \psi_h, \phi_k \rangle]_{h=1,\ldots,N_j;k=1,\ldots,K} \tag{3.30}$$

is N_j.

PROOF. From Proposition 3.5 it follows that, for every $j \in J, \sigma_j^+$ consists of simple poles of $F(\cdot)$ different from b so that Theorem 2.1 is applicable. From Proposition 3.1 (with $m = 1$), we get that, for each $\lambda_0 \in \sigma_j^+$, we have $\phi'(\lambda_0) \neq 0$ (the function ϕ is defined in (3.5)), and the residue Q of F at $\lambda = \lambda_0$ is $(\phi'(\lambda_0))^{-1} P_j$, where P_j is a projection on the eigenspace of A corresponding to the eigenvalue ζ_j. Therefore Range Q^* = Range $P_j^* = \ker(\zeta_j - A^*)$. Hence condition (ii) of Theorem 2.1 may be reformulated as

$$\forall j \in J, \ \ker(\zeta_j - A^*) \cap \ker\Phi^* = \{0\}. \tag{3.31}$$

As is easily seen, $\ker(\zeta_j - A^*)$ is spanned by the linear functionals $\psi_h^*, h = 1, \ldots, N_j$, where $\psi_h^*(g) = \langle g, \psi_h \rangle$. Therefore Φ^*: $\ker(\zeta_j - A^*) \to (\mathbf{R}^k)^* = \mathbf{R}^k$ may be represented, with respect to the basis $\{\psi_1^*, \ldots, \psi_{N_j}^*\}$, by the matrix $[A_{hk}]$. Hence, Φ^* is injective on $\ker(\zeta_j - A^*)$ if and only if $N_j \geq k$ and the rank of $[A_{hk}]$ is N_j. □

3.2. *The heat equation with memory.* We consider now the heat equation in materials of "fading memory" type, introduced by Nunziato in [**8**]:

$$(A3.32) \qquad b_0 u_t(t,x) + d/dt \int_0^t \beta(t-s) u(s,x) ds$$

$$= c_0 \Delta u(t,x) - \int_0^t \gamma(t-s) \Delta u(s,x) ds + \phi(t,x), \;\; t > 0, x \in \overline{\Omega},$$

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega},$$

$$\mathcal{B} u(t,x) = 0, \quad t > 0, x \in \partial\Omega,$$

where either $\mathcal{B} u = u$ or $\mathcal{B} u = \partial u / \partial n$. Ω is a bounded open set in $\mathbf{R}^n$ with $\mathbf{C}^2$ boundary $\partial\Omega, u(t,x)$ is the temperature of the point $x \in \overline{\Omega}$ at the time t, the constants b_0, c_0 are positive, and ϕ is the heat supply, which we assume to be of the type

$$(3.33) \qquad \phi(t,x) = (\Phi h(t))(x), \quad t \geq 0, x \in \overline{\Omega},$$

where h is a function (to be determined) from $[0, +\infty[$ to a Banach space Y, Φ belongs to $\mathrm{L}(Y, X)$, and X is either $\mathrm{L}^p(\Omega)$ or $\mathrm{C}(\overline{\Omega})$. The kernels β, γ are measurable positive functions (usually, linear combinations of exponential functions with positive coefficients) satisfying

$$(3.34) \qquad c_0 - \int_0^{+\infty} \gamma(s) ds > 0.$$

We assume here that β and γ are completely monotone kernels, with

$$(3.35) \qquad \gamma(t) = \int_0^{+\infty} e^{-\omega t} \mu(d\omega), \; \beta(t) = \int_0^{+\infty} e^{-\omega t} v(d\omega), \; t \geq 0,$$

where μ, v are positive Borel measures with

$$\text{supp}\,\mu, \text{supp}\,v \subset \,]a, +\infty[\,, \quad a > 0. \tag{3.36}$$

We denote by $\hat{\gamma}(\lambda), \hat{\beta}(\lambda)$ the analytic extensions of the Laplace transforms of γ and β respectively to $\mathbf{C}\backslash(-\text{supp}\,\mu), \mathbf{C}\backslash(-\text{supp}\,v)$.

We showed in [**7**] that equation (3.32) may be rewritten as an evolution equation in the space X in the form (0.1), with

$$A : D(A) = \{\phi \in X : \Delta\phi \in X, \mathcal{B}f = 0\} \to X; \tag{3.37}$$

$$(Af)(x) = b_0^{-1}[c_0\Delta\phi(x) - \beta(0)\phi(x)]$$

$$(K(t)\phi)(x) = b_0^{-1}[-\beta'(t)\phi(x) - \gamma(t)\Delta\phi(x)], \ t > 0,$$

$$f(t) = b_0^{-1}h(t), \ t \geq 0.$$

In fact, we considered in [**7**] only the case $X = \mathrm{C}(\overline{\Omega})$, but the same arguments can be carried out in the case $X = \mathrm{L}^p(\Omega)$. We showed also that

$$\rho_0 = \left\{\lambda \in \mathbf{C} : c_0 - \hat{\gamma}(\lambda) \neq 0, \frac{\lambda(b_0 + \hat{\beta}(\lambda))}{c_0 - \hat{\gamma}(\lambda)} \neq -\lambda_n\right\} \tag{3.38}$$

and

$$F(\lambda) = \frac{b_0}{c_0 - \hat{\gamma}(\lambda)} R\Big(\frac{\lambda(b_0 + \hat{\beta}(\lambda))}{c_0 - \hat{\gamma}(\lambda)}, A\Big), \quad \lambda \in \rho_0, \tag{3.39}$$

where $\{-\lambda_n\}$ is the (decreasing) sequence of the eigenvalues of A. Hence (see [**7**, Proposition 3.1]),

$$\{\lambda \in \sigma : \text{Re}\,\lambda \geq 0\} = \begin{cases} \emptyset & \text{in the case } \mathcal{B}u = u, \\ \{0\} & \text{in the case } \mathcal{B}u = \partial u/\partial n \end{cases} \tag{3.40}$$

and

$$\sup\{\text{Re}\,\lambda : \lambda \in \sigma, \text{Re}\,\lambda < 0\} < 0. \tag{3.41}$$

In particular, system (3.32) is asymptotically stable in the case of the Dirichlet boundary condition. However, one may ask whether it is possible to get $||u(t)||_X \le ce^{-\omega t}$, with ω arbitrary. A partial answer may be found in Proposition 3.7. To state Proposition 3.7, we set

$$\mathcal{U} = \{x \in]-a, 0[\,:\, \hat{\gamma}(x) = -c_0\}. \tag{3.42}$$

The set $\mathcal{U}$ may be possibly void. It is easy to see that all the solutions of the equation $\hat{\gamma}(x) = -c_0$ are real and negative.

PROPOSITION 3.7. *Let $\omega \in]0, a[$ be such that*

$$\frac{\lambda(b_0 + \hat{\beta}(\lambda))}{c_0 - \hat{\gamma}(\lambda)} \neq -\lambda_n \quad \forall n \in \mathbf{N}, \forall \lambda \textit{ with } \operatorname{Re}\lambda = -\omega,$$

and

$$-\omega > \sup\mathcal{U} \quad \textit{if}\, \mathcal{U} \neq \emptyset.$$

Then the following statements are equivalent:

(i) *For every $u_0 \in \overline{D(A)}$ there exists $h \in \mathrm{C}_\omega([0, +\infty[\,; \mathbf{Y})$ such that the solution u of* (3.32) *belongs to* $\mathrm{C}_\omega([0, +\infty[\,; X)$;

(ii) *For every $\lambda \in \mathbf{C}$ with* $\operatorname{Re}\lambda > \omega$ *we have*

$$\ker\Big(\frac{\lambda(b_0 + \hat{\beta}(\lambda))}{c_0 - \hat{\gamma}(\lambda)} - A^*\Big) \cap \ker\Phi^* = \{0\}.$$

If any of the equivalent above conditions holds, then, for every $u_0 \in X$, there exists $h \in \mathrm{C}^\alpha_\omega([0, +\infty[\,; Y)$ such that the solution u of (3.32) *belongs to $\mathrm{C}^\alpha_\omega([a, +\infty[\,; D(A))$ for each $a > 0$.*

PROOF. Let us check the assumptions of Theorem 2.2. Here, A generates an analytic semigroup and K belongs to $\mathrm{L}^1([0, +\infty[\,; \mathrm{L}(D(A), X))$ thanks to assumption (3.36). The set $\sigma_+(\omega)$ consists of all the zeros with real part greater than $-\omega$ of the functions

$$\psi_n(\lambda) = \frac{\lambda(b_0 + \hat{\beta}(\lambda))}{c_0 - \hat{\gamma}(\lambda)} - \lambda_n, \quad n \in \mathbf{N}. \tag{3.43}$$

Since $\lim_{n\to+\infty} -\lambda_n = -\infty$, then $\sigma_+(\omega)$ is finite. Moreover, if it is not empty, it consists of poles of $F(\cdot)$ due to (3.39): actually, for each $n \in \mathbf{N}, 1/\psi_n$ is holomorphic, so that is has only finite order zeros, and the eigenvalues of A are semisimple. If λ_0 is a zero of order m of ψ_n, then we have

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^m F(\lambda) = \frac{m!b_0}{(c_0-\hat{\gamma}(\lambda_0))\psi_n^{(m)}(\lambda_0)}P_n,$$

where P_n is a projection on the eigenspace of A corresponding to the eigenvalue λ_n:

$$P_n = \frac{1}{2\pi i}\int_{C(\lambda_n,\varepsilon)} R(z,A)dz.$$

Therefore λ_0 is a pole of $F(\cdot)$ of order m, with $Q_m = m!b_0P_n/(c_0-\hat{\gamma}(\lambda_0))\psi_n^{(m)}(\lambda_0)$. Consequently, $\mathrm{Range}Q_m^* = \mathrm{Range}P_n^* = \ker(\lambda_n - A^*)$. The statement follows now applying Theorem 2.1. □

In the case of the Neumann boundary condition, the free system is not asymptotically stable because $\lambda_1 = 0$, so that 0 belongs to σ. But 0 is a zero of order 1 of ψ_1 because $\hat{\beta}(0) > 0$, so that it is a simple pole of $F(\cdot)$ (see the proof of Proposition 3.7). Since the eigenspace of A^* with eigenvalue 0 is spanned by the measure

$$\mu^*(\phi) = (\mathrm{meas}\,\Omega)^{-1}\int_\Omega \phi(x)dx,$$

applying Proposition 3.7 we get that system (3.32) with the Neumann boundary condition is stabilizable if and only if $\Phi^*(\mu^*) \neq 0$. In particular, if

$$\Phi : \mathbf{R} \to X, \quad \Phi(y) = y\xi(\cdot),$$

where $\xi \in X$, then system (3.32) is stabilizable if and only if $\mu^*\xi \neq 0$, i.e., the mean value of ξ is not zero.

In the applications, one is often interested in observing the value of the temperature u only at some points $x_1,\ldots,x_H \in \overline{\Omega}$. Then we consider the linear operator

$$C : \mathrm{C}(\overline{\Omega}) \to \mathbf{R}^H, \; C\phi = (\phi(x_1),\ldots,\phi(x_H)).$$

By Theorem 2.4, for each $u_0 \in \mathrm{C}(\overline{\Omega})$, there is f such that $Cu(t,\cdot)$ decays exponentially in the sup norm if and only if $\ker A^*C^* \cap \ker\Phi^* = \{0\}$. In our case, $A^*C^*(x_1^*,\ldots,x_H^*) = \sum_{h=1}^H x_h^*\mu_h^*$, where $\mu_h^*(\phi) =$

$\Delta\phi(x_h)$. Therefore Range $A^*\mathcal{C}^*$ is spanned by the measures $\mu_h^*, h = 1,\ldots,H$, and $\ker A^*\mathcal{C}^* \cap \ker\Phi^* = \{0\}$ means that $\Phi^*\mu_h^* \neq 0$ for each $h = 1,\ldots,H$. In particular, if Φ is as before, then we have $\ker A^*\mathcal{C}^* \cap \ker\Phi^* = \{0\}$ if and only if $\Delta\xi(x_h) \neq 0$ for each $h = 1,\ldots,H$.

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SCUOLA NORMALE SUPERIORE, PIAZZA DEI CAVALIERI 6, 52126 PISA, ITALY

DIPARTIMENTO DI MATEMATICA, VIA OSPEDALE 72, 09124 CAGLIARI, ITALY