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CHROMATIC SERIES WITH PROLATE SPHEROIDAL WAVE FUNCTIONS

GILBERT G. WALTER

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ABSTRACT. Prolate spheroidal wave functions (PSWFs) arise as solutions of an integral equation. This makes them bandlimited functions in a Paley-Wiener space, but because they are also solutions to a Sturm-Liouville problem, they behave very much like polynomials locally. Chromatic series are series expansions in which the coefficients are linear combinations of derivatives of a function. They were introduced by Ignjatovic as a replacement for Taylor's series and are based on orthogonal polynomials. Since the PSWFs are close to orthogonal polynomials they can be used to replace them in the Ignjatovic theory. The theory can be extended further to prolate spheroidal wavelet series that then combine chromatic series with sampling series. This leads to an overdetermined system which can use either local or global data to approximate the original function.

1. Introduction. The theory of chromatic derivatives was introduced by Ignjatovic [5] and colleagues in a series of technical reports of the Kromos Corporation. See, e.g., [4,6,8]. In this theory, he proposed an alternative to Taylor's theorem suitable for signal processing applications. The approximations arising from Taylor's theorem are almost useless for band limited signals since the Taylor series do not converge globally. There is another problem as well as we can see if this we write

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the Taylor series of a $\sigma-$band limited function as

$$f(t) = \sum\nolimits_{n=0}^{\infty} f^{(n)}(0)t^n/n! = \sum\nolimits_{n=0}^{\infty} \left[\frac{1}{2\pi}\int_{-\sigma}^{\sigma}(i\omega)^n \widehat{f}(\omega)d\omega\right] t^n/n!,$$

where $\widehat{f}$ is the Fourier transform of f. This series clearly is not globally convergent even though, as a band limited signal, f is an entire function which may be rapidly decreasing. Also the coefficients given by the integral involve primarily the values of $\widehat{f}$ near the endpoints of $[-\sigma, \sigma]$. In the Ignjatovic theory, other differential operators, which avoid this, are used instead. These operators, based on orthogonal polynomials $\{P_n(i\omega)\}$, may be used to get a good approximation to bandlimited functions. Their coefficients involve integrals of the form

$$\frac{1}{2\pi}\int_{-\sigma}^{\sigma} P_n(i\omega)\widehat{f}(\omega)d\omega = \{P_n(D)f\}(0).$$

Since orthogonal polynomials generally have a much more evenly distributed spectrum, these derivative operators $P_n(D)$ are called *chromatic derivatives*. They lead to a series that is globally convergent as shown in [4,6,8].

This theory is also an alternative to the Shannon sampling theorem which is used to approximate bandlimited functions, but involves the values of the function at infinitely many different points. In contrast to the Taylor series, the partial sums of the sampling series converge uniformly, but are not local since they require knowledge of the function at all the integers. The Ignjatovic theory is also a local theory and requires knowledge of the function (signal) only in a neighborhood of the origin.

Another method giving good local approximation is the use of prolate spheroidal wave functions (PSWFs). These do not strictly lead to chromatic derivatives since they are not polynomials. However, they are entire functions whose expansion converges very rapidly and can be approximated by polynomials of low degree. These functions are solutions of an integral equation and solve the problem of finding the bandlimited function of unit total energy whose energy on a finite (concentration) interval is maximized. Their study was pioneered by Slepian and his colleagues at Bell Labs in the 1960's [11,12], and has recently been revived because of applications in image analysis. These functions form an orthogonal basis both of the space of bandlimited functions and of L^2 on the concentration interval. In this work we show how these different approaches may be combined.

1.1 *Quick Chromatic Derivatives* In this section we present in abbreviated form the theory of chromatic derivatives as expounded in [4,5,6,14] and several additional technical reports of Kromos corporation. This chromatic derivative is a generalization of the ordinary derivative and leads to a series referred to as the *chromatic expansion* of an analytic function f.

Let $\{p_n\}$ be a sequence of orthonormal polynomials with respect to the bounded, non-negative weight function $w(\omega)$ on the real line, i.e.,

$$\int_{-\infty}^{\infty} p_n(\omega)p_k(\omega)w(\omega)d\omega = \delta_{nk.}$$

This weight function will usually, but not always, have compact support. We assume furthermore that $\{p_n\}$ is complete in that the expansion in terms of these polynomials of a appropriate function $g(\omega)$ converges;

$$g(\omega) = \sum\nolimits_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} p_n(\xi)g(\xi)d\xi \right\} p_n(\omega)w(\omega). \tag{1}$$

This happens automatically in the sense of L^2 if $g/w^{1/2}$ also belong to L^2. Then Parseval's equality for the orthonormal system $\{p_n w^{1/2}\}$becomes

$$\int |g|^2/w = \sum \left| < \frac{g}{w^{1/2}}, p_n w^{1/2} > \right|^2 = \sum | < g, p_n > |^2.$$

This equality may be expressed as

$$g(\omega)/w^{1/2}(\omega) = \sum\nolimits_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} p_n(\xi)g(\xi)d\xi \right\} p_n(\omega)w^{1/2}(\omega)$$

where the convergence is in the sense of $L^2(\mathbb{R})$ It, in turn, leads immediately to (1) since multiplication by the bounded function $w^{1/2}$

is a continuous operation in $L^2(\mathbb{R})$ It should be observed that if w has compact support on an interval, so does g on the same or smaller subset.

For other functions we can get a formal expansion if the integral in (1) exists, but cannot deduce that the expansion converges without additional conditions. Nonetheless we shall proceed formally and then try to justify the steps later.

Now let $f(t) \in L^2(\mathbb{R})$ with a Fourier transform given by $\widehat{f}(\omega) = g(\omega)$. Suppose furthermore that the inverse Fourier transform of $p_n w$ is

$$\phi_n(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} p_n(\omega) w(\omega) d\omega.$$

Then formally we find

$$\begin{aligned} f(t) &= \sum\nolimits_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} p_n(\xi) g(\xi) d\xi \right\} \phi_n(t) \\ &= \sum\nolimits_{n=0}^{\infty} 2\pi \{p_n(-iD) f\}(0) \phi_n(t) \\ &= \sum\nolimits_{n=0}^{\infty} a_n \phi_n(t) \end{aligned} \tag{2}$$

provided that the inverse Fourier transform of the terms in (1) converges to the inverse Fourier transform of g, namely f.

Again this will happen if $g \in L^2(1/w)$ since, by Schwarz's inequality we have

$$\begin{aligned} &\left| f(t) - \sum\nolimits_{n=0}^{N} a_n \phi_n(t) \right| \\ &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} (g(\omega) - \sum\nolimits_{n=0}^{N} a_n p_n(\omega) w(\omega)) d\omega \right| \\ &\leq \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \left| \frac{g(\omega)}{w^{1/2}(\omega)} - \sum\nolimits_{n=0}^{N} a_n p_n(\omega) w^{1/2}(\omega) \right|^2 d\omega \right\}^{1/2} \\ &\qquad \left\{ \int_{-\infty}^{\infty} w(\omega) d\omega \right\}^{1/2}. \end{aligned}$$

The series expansion of $g/w^{1/2}$ converges to it in the sense of L^2 and hence the last line converges to 0. It is also independent of t and therefore the left hand side converges to 0 uniformly for $t \in \mathbb{R}$ as $N \to \infty$.

The second line in (2) follows from the fact that the Fourier transform changes derivatives into multiplication by $i\omega$, or in terms of the inverse transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\widehat{f}(\omega)(\omega^n)d\omega = \{(-iD_t)^n f\}(t) = (-i)^n f^{(n)}(t).$$

This is evaluated at 0 to deduce that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)(\omega^n)d\omega = \{(-iD_t)^n f\}(0)$$

which then gives us the formula (2) we want because of the linearity of the Fourier transform. The expression in the last line of (2) is the *chromatic expansion*, since it involves the differential operator $P(-iD)$, the *chromatic derivative.*

This procedure works for all the classical orthogonal polynomials, but is most interesting when a closed form can be found for $\phi_n(t)$. This happens for Chebyshev polynomials, in which ϕ_n is a Bessel function, for Legendre polynomials, in which it is spherical Bessel function, and for Hermite polynomials, in which it is the product of a monic polynomial and a Gaussian function [4,7].

2. A generalization We first observe that we do not need the orthogonality, but merely need a *biorthogonal* pair $\{p_n, h_k\}$ for which the integral satisfies

$$\int_{-\infty}^{\infty} p_n(\omega)h_k(\omega)d\omega = \delta_{nk} \tag{3}$$

in order to get the expansion in (2). The completeness of the orthonormal sequence is replaced by the requirement that the $\{h_n\}$ be a Schauder basis. We can also replace the polynomial p_n by an entire function θ_n provided its power series converges sufficiently rapidly. Then the chromatic derivative will involve an infinite series of derivatives rather than a finite sum.

We can even replace the function h_k by a generalized function (distribution) and the integral in (3) by the value of this distribution on the test function θ_n, which we denote by $< h_k, \theta_n >$. We also need to apply Fourier transforms to these distributions. There are several theories of the Fourier transform that we can use based on different types of generalized functions, but we shall initially restrict ourselves to $\mathcal{S}'$, the space of tempered distributions [25].

The series in (1) then has the form

$$g = \sum_{n=0}^{\infty} a_n h_n$$

where g is an element of $\mathcal{S}'$ as well. The convergence of this series is weak* convergence, that is,

$$< g, \theta >= \sum_{n=0}^{\infty} a_n < h_n, \theta >$$

for each test function $\theta \in \mathcal{S}$. In particular, it holds for θ_k and hence $< g, \theta_k >= a_k$. The Fourier transform and its inverse are both continuous operations in $\mathcal{S}'$, and hence the inverse Fourier transform f of g is given by

$$f = \sum_{n=0}^{\infty} a_n \widetilde{h}_n$$

where $\widetilde{h}_n$ is the inverse Fourier transform of h_n. In order for this to be a proper chromatic series, the coefficients must be given by linear combinations of derivatives of f or limits of such linear combinations on the support of g. This will work if each θ_k is analytic on the support of g since then $a_n = [\theta_n(-iD)f](0)$ makes sense.

A more general space of generalized functions is the space $\mathcal{Z}'$, the space of "ultradistributions" whose test function space $\mathcal{Z}$ is composed of entire functions so that the condition that the θ_k be analytic is automatically satisfied. A less general space is the space $\mathcal{E}'$ of "distributions of compact support". Its test function space $\mathcal{E}$ contains all C $^{\infty}$ functions. Both can be used to obtain chromatic series.

Example 1. As an example let $\theta_n = p_n$ be given by $p_n(\omega) = \omega^n/n!$, which belongs to the space $\mathcal{E}$. Then if we take $h_k = \delta^(k)$, we find that

$$< h_k, p_n >=< \delta^{(k)}(\omega), \omega^n/n! >= (-1)^k < \delta(\omega), \omega^{(n-k)}/(n-k)! >$$

by the well known properties of the delta function ([25], p.188). Since $< \delta, \phi >= \phi(0)$ by definition, it follows that

$$< h_k, p_n >=< \delta^{(k)}(\omega), \omega^n/n! >= \delta_{nk}.$$

In this case p_n, although a member of the space $\mathcal{E}$, is unbounded. To find its inverse Fourier transform, we need to consider it as a tempered distribution from which we find it to be exactly $(-i)^n \delta^{(n)}/n!$ [25], p.188. The inverse Fourier transform of h_k in this case is given by $\phi_k(t) = \frac{1}{2\pi}(-it)^k$. The chromatic expansion (2) for then will be

$$\begin{aligned} f(t) &= \sum\nolimits_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} (\xi^n/n!) g(\xi) d\xi \right\} \phi_n(t) \\ &= \sum\nolimits_{n=0}^{\infty} 2\pi \{(iD)^n f\}(0) \phi_n(t) \\ &= \sum\nolimits_{n=0}^{\infty} f^{(n)}(0)(t^n/n!), \end{aligned}$$

a familiar expression. For the convergence implicit in this series to work, we must have $g \in \mathcal{E}'$,i.e., have compact support, and be analytic near 0. But the end result holds for analytic functions, of course, but only locally. It does not converge in the sense of $\mathcal{E}'$.

Example 2. As another example, we replace p_n by the function $e^{in\omega}/2\pi$ with n now ranging from $-\infty$ to ∞. We take h_k to be given by $e^{-ik\omega}\chi_\pi(\omega)$, where χ_π is the indicator function of $[-\pi, \pi]$. The inverse Fourier transform of $e^{in\omega}$ is $\delta(t-n)$, while that of h_k is the familiar sinc function

$$\phi_k(t) = \frac{\sin \pi(t-k)}{\pi(t-k)}.$$

The chromatic expansion for this example will be

$$\begin{aligned} f(t) &= \sum\nolimits_{n=-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{in\xi}) g(\xi) d\xi \right\} \phi_n(t) \\ &= \sum\nolimits_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \end{aligned}$$

another familiar formula. In this case the convergence holds for all $\pi-$ bandlimited functions, and is uniform for all t on the real line.

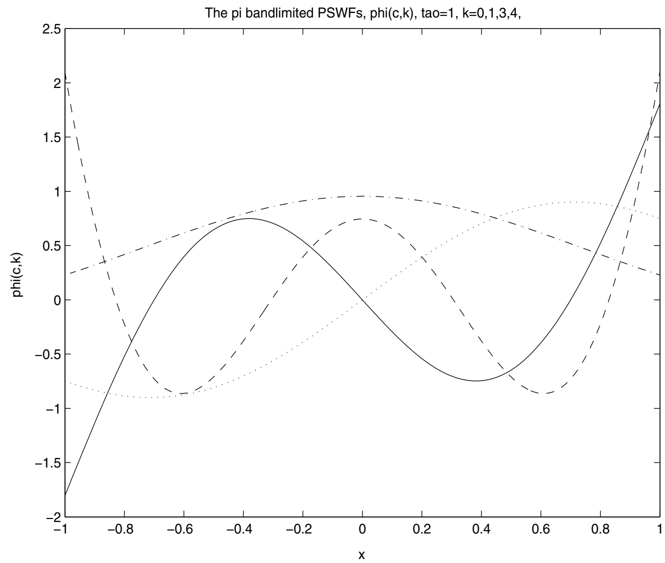

FIGURE 1. The PSWFs $\varphi_0, \varphi_1, \varphi_3, \varphi_4$, with parameter values $\sigma = \pi$ and $\tau = 1$ on the concentration interval $[-1, 1]$.

3. Prolate spheroidal wave functions The prolate spheroidal wave functions (PSWFs) have a combination of local and global properties that might make them a suitable generalization of chromatic expansions. They are obtained by separating the variables in the wave equation

$$\nabla^2 w + \kappa^2 w = 0$$

on a prolate spheroid. This leads to the differential equation eigenvalue problem

$$P_{\sigma,\tau} y := (\tau^2 - t^2)\frac{d^2 y}{dt^2} - 2t\frac{dy}{dt} - \sigma^2 t^2 y = \mu y, \tag{4}$$

which is similar to that of the Legendre polynomials. In Figure 1 we show four of the PSFWs on their concentration interval $[-\tau, \tau]$ where τ is the parameter appearing in (4); their similarity to Legendre polynomials is apparent. But the solutions, i.e. the eigenfunctions $\{\varphi_n\}$, are not polynomials, although they share many of their properties. These solutions are the PSWFs and constitute an orthogonal basis of $L^2[-\tau, \tau]$.

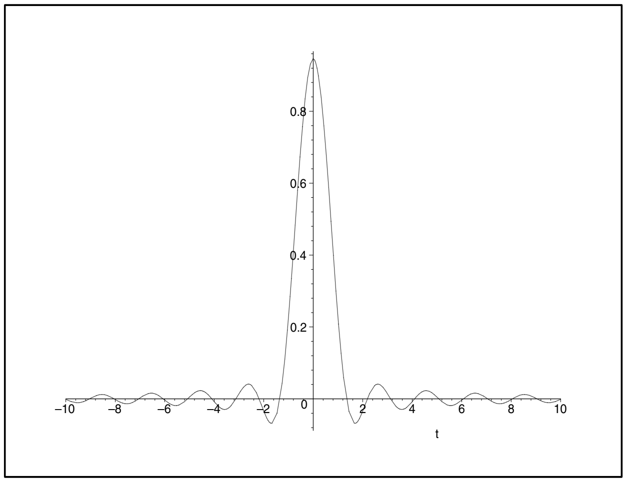

FIGURE 2. The PSWFs φ_0 with parameter values $\sigma = \pi$ and $\tau = 1$ on a larger interval.

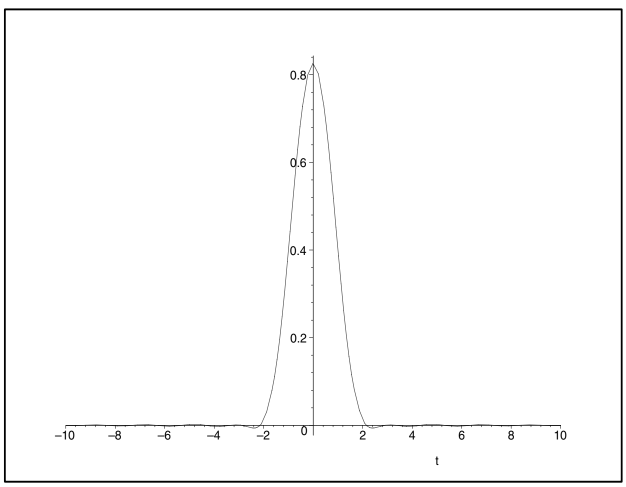

FIGURE 3. The PSWFs φ_0 with parameter values $\sigma = \pi$ and $\tau = 2$ on a larger interval.

In common with polynomials they can be extended to the entire real line and even to the complex plane. However, in contrast to polynomials, they are bounded on the real line and are entire functions of exponential type in the complex plane. Figure 2 shows φ_0, with parameter values $\sigma = \pi$ and $\tau = 1$, on a larger interval; it is positive on $[-1, 1]$ and is smaller outside this interval. If τ is taken to have the larger value 2 as in Figure 3, this concentration is even more apparent.

It appears almost to have compact support, but it cannot since it is an entire function. In fact, as we shall see later, more than 99.9% of the energy of φ_0 is concentrated on $[-2, 2]$.

They are also solutions to another problem, that of maximizing the energy on an interval $[-\tau, \tau]$ of a $\sigma-$bandlimited function. This problem is important in communications theory since all real signals are bandlimited; as such they cannot be time limited as a consequence of the Heisenberg uncertainty principle. The next best thing is to maximize the ratio

$$\rho = \frac{\int_{-\tau}^{\tau} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}, \tag{5}$$

which lead to an integral equation

$$[\mathbb{S}_{\sigma,\tau} f](t) := \int_{-\tau}^{\tau} f(x) S_\sigma(t-x) dx = \lambda f(t), \tag{6}$$

where S_σ is a rescaled sinc function. The function which maximizes the ratio in (5) is the first PSWF φ_0, the function among those orthogonal to φ_0 that maximizes this ratio is the next PSWF φ_1, etc.

The reason the same functions are eigenfunctions of both the differential operator in (4) and the integral operator in (6) is that the two operators commute. This was called a "lucky accident" by Slepian, one of the founders of the theory in the 1960s [11]. These PSWFs have many additional unusual properties [21,23], some of which we list here.

3.1 *Properties of PSWFs.* We now restrict ourselves to $\sigma = \pi$ in order to simplify some of the formulas, but continue to allow τ to be a variable parameter but suppress it in our notation. In addition to the equation (6), the $\{\varphi_n\}$ satisfy an integral equation over $(-\infty, \infty)$ as well since all functions in B_π do:

$$\int_{-\infty}^{\infty} \varphi_n(x) S(t-x) dx = (\varphi_n * S)(t) = \varphi_n(t)$$

This leads to a dual orthogonality [23]

$$\int_{-\tau}^{\tau} \varphi_n(x)\varphi_m(x) dx = \lambda_n \delta_{nm}, \qquad \int_{-\infty}^{\infty} \varphi_n(x)\varphi_m(x) dx = \delta_{nm}.$$

In fact, the $\{\varphi_n\}$ constitute an orthogonal basis of $L^2(-\tau, \tau)$, as well as an orthonormal basis of the subspace B_π of $\pi-$ bandlimited functions in $L^2(-\infty, \infty)$. Because of the isometric property of the Fourier transform, it follows that $\{\widehat{\varphi_n}\}$ is also an orthogonal basis of $L^2[-\pi, \pi]$.

As one might expect, the PSWFs are closely related to the Fourier transforms. Indeed, the Fourier transform is given by

$$\widehat{\varphi}_n(\omega) = (-1)^n \sqrt{\frac{2\tau}{\lambda_n}} \varphi_n\left(\frac{\tau\omega}{\pi}\right) \chi_\pi(\omega)$$

where $\chi_\pi(\omega)$ is the characteristic function of $(-\pi, \pi)$. By a change of scale we see that the PSWFs are also eigenfunctions of still another integral operator [11]. In fact, if $\tau = \pi$, then they are exactly eigenfunctions of the Fourier transform on the interval $[-\pi, \pi]$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (-1)^n \sqrt{\frac{2\pi}{\lambda_n}} \varphi_n(\omega) e^{i\omega t} d\omega = \varphi_n(t).$$

They have some nice discrete properties arising from two types of discrete orthogonality [21],

$$\sum_{k=-\infty}^{\infty} \varphi_n(k)\varphi_m(k) = \delta_{mn}, \qquad \sum_{n=0}^{\infty} \varphi_n(k)\varphi_n(m) = \delta_{mk}. \tag{7}$$

These may be used when one has only the sequence of sampled values $\{f(k)\}$ of the signal. It has the series expansion

$$f(k) = \sum_n a_n \varphi_n(k).$$

The inverse is given by the series

$$a_n = \sum_k f(k)\varphi_n(k) \tag{8}$$

because of the orthogonality given by (7). These are the same coefficients as in the continuous expansion.

4. PSWF series as chromatic expansions The are several ways in which series of these PSWFs can be interpreted as the general chromatic expansions talked about in the section 2. We consider two of them. The first involves using the standard orthogonal system $\{\varphi_n(\omega)\}$ discussed above while the second involves the wavelet system based on them introduced in [21].

4.1 *Orthogonal expansions of PSWFs.* We first take the orthogonal system consisting of their Fourier transforms $\{\widehat{\varphi}_n(\omega)\}$ on $[-\pi, \pi]$. Then, for any $g \in L^2[-\pi, \pi]$, we have

$$g(\omega) = \sum\nolimits_{n=0}^{\infty} \left\{ \int_{-\pi}^{\pi} \widehat{\varphi}_n(\xi) g(\xi) d\xi \right\} \widehat{\varphi}_n(\omega)$$

in place of (1), and the inverse Fourier transform gives

$$f(t) = \sum\nolimits_{n=0}^{\infty} 2\pi \{\varphi_n(-iD)f\}(0)\varphi_n(t). \tag{9}$$

Since the φ_n are entire functions, the chromatic derivative is given by a power series which converges very rapidly but only locally. A better choice would be to approximate the φ_n by Legendre polynomials on the interval $[-\pi, \pi]$, which is the procedure used in most numerical calculations [16].

These approximations will be true chromatic derivatives, and will be quite low order polynomials.

Two such polynomials are shown in Figures 4 and 6 with the errors shown in Figures 5 and 7 respectively. Notice the different vertical scales in the graphs of the errors.

The uniform error is less than 10^{-3} in both cases. The Legendre polynomial approximation improves as n increases and for large n, φ_n

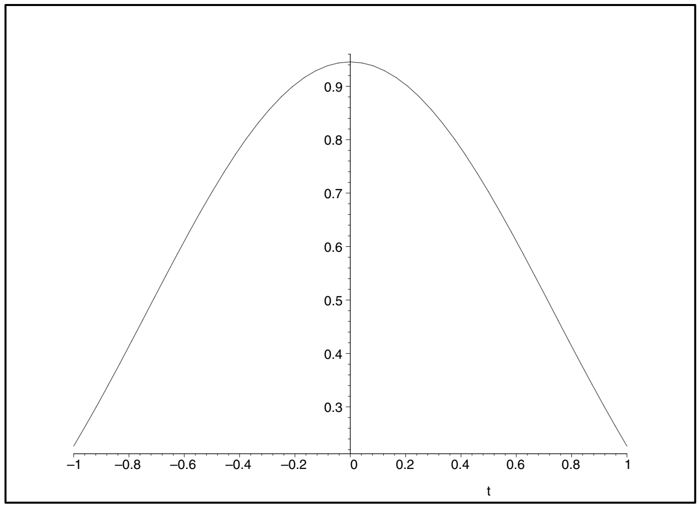

FIGURE 4. The Legendre polynomial P_0 approximation of degree 6 to $\varphi_0, \sigma = \pi, \tau = 1$ on $[-1,1]$.

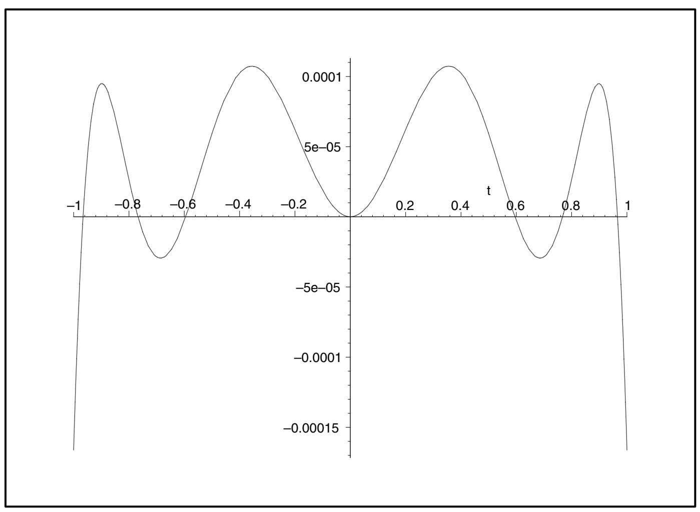

FIGURE 5. The error in the polynomial approximation to φ_0 on the interval $[-1,1]$.

is very close to a polynomial of degree n. The formulae for the two polynomial approximation used in Figures 4 and 6 are given below:

$$P_0(t) := .94558 - 1.07769t^2 + .43213t^4 - .07335t^6$$
$$P_4(t) := .02432 - .30605t^2 + .50760t^4 - .18284t^6 + .02570t^8.$$

The PSWFs in these calculations are normalized such that $\int_{-\infty}^{\infty} |\varphi_n|^2 = 1$. Since as n increases, more of the energy of φ_n will be outside of the interval $[-\tau, \tau]$, both it and the polynomial approximation will be small in this interval and can be ignored eventually. For example if $n = 20$, the maximum on $[-1, 1]$ of $|\varphi_{20}|$ will be about 10^{-20}. Of course, in our theory, these approximations are taken in the Fourier transform domain; the corresponding approximations to the chromatic derivatives are found by taking the inverse Fourier transform, which again will be quite small.

The coefficients in (9) have an alternative expression obtained from (8). It is

$$f(t) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} f(k)\varphi_n(k)\varphi_n(t). \tag{9a}$$

Thus the same series can be interpreted as a chromatic series or as a sampling series. If the data is very localized then the coefficients in (9) are appropriate, whereas if the data is more global, then (9a) is better.

The series in these expressions generally converge very rapidly. This arises because of the properties of the eigenvalues λ_n, which satisfy $1 > \lambda_0 > \lambda_1 > \lambda_3 > ... > 0$. The first $[2\tau]$ are relatively close to 1 while the remaining ones are close to 0 [11]. Thus the series in (9) may be approximated by a small number of terms if the function f has most of its energy concentrated in the interval $[-\tau, \tau]$. In fact, we have the following formulae

$$\sum_{n=0}^{\infty} \lambda_n = 2\tau. \qquad \text{and} \qquad \sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n) \leq A \log^+ \tau + B.$$

These are important because of the version of Parseval's equality for $f \in B_\pi$ when the integrals are taken over the interval $[-\tau, \tau]$:

$$\int_{-\tau}^{\tau} |f(x)|^2 dx = \sum_{n=0}^{\infty} \lambda_n |a_n|^2 \approx \sum_{n=0}^{2\tau} \lambda_n |a_n|^2$$

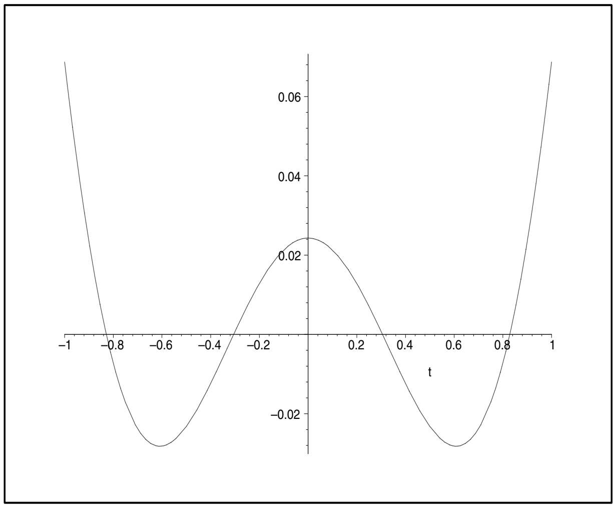

FIGURE 6. The polynomial approximation P_4 to φ_4 on the interval $[-1, 1]$.

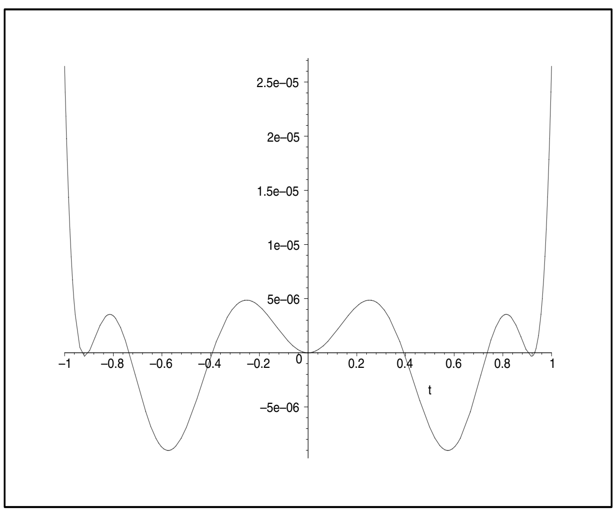

FIGURE 7. The error in the polynomial approximation to φ_4 on the interval $[-1, 1]$.

For example, for $\tau = 1$, the first three eigenvalues are 0.981, 0.749, 0.243, while the sum of all the remaining eigenvalues is 0.027. For $\tau = 2$, the first eigenvalue is 0.9994 which is also the proportion of energy of the first PSWF φ_0 concentrated on $[-2, 2]$. (See [23]). Thus the values of φ_0 outside of this interval will be negligible (as seen in Figure 3).

4.2 *Prolate spheroidal wavelets.* As an alternative, we may use the *prolate spheroidal wavelets* to obtain series representations of functions in B_π.

The first prolate spheroidal wave function, φ_0, the one with maximum concentration on $[-\tau, \tau]$, may be used to generate a basis of B_π by the expedient of taking its integer translates. The set of all such translates $\{\varphi_0(t-n)\}$ has been shown to be a Riesz basis of B_π [21]. In the setting of wavelet theory, φ_0 is a scaling function or father wavelet and the closed linear span of $\{\varphi_0(t-n)\}$ is the space usually denoted as V_0, in this case, our Paley-Wiener space B_π.

A Riesz basis has a dual basis $\{\widetilde{\varphi}_0(t-n)\}$ which together with $\{\varphi_0(t-n)\}$constitute a biorthogonal system. We can get this dual basis by defining the Fourier transform of the dual function $\widetilde{\varphi}_0(t)$ as

$$\widehat{\widetilde{\varphi}}_0(\omega) := \frac{\widehat{\varphi}_0(\omega)}{\sum_k |\widehat{\varphi}_0(\omega - 2\pi k)|^2}. \tag{10}$$

From this it follows that $\sum_k \widehat{\widetilde{\varphi}}_0(\omega - 2\pi k)\overline{\widehat{\varphi}_0(\omega - 2\pi k)} = 1$, which, as shown in wavelet theory, is equivalent to biorthogonality [2]. Since φ_0 has support on $[-\pi, \pi]$, it follows that the sum in the denominator reduces to a single term so that $\widehat{\widetilde{\varphi}}_0(\omega) = (1/\overline{\widehat{\varphi}_0(\omega)})\chi_\pi(\omega)$. Furthermore, since $\widehat{\varphi}_0$ has no zeros in $[-\pi, \pi]$, $\widehat{\widetilde{\varphi}}_0$ will be a bounded function continuous on $[-\pi, \pi]$. Now we can put this into a chromatic series setting by using the generalization to (2). The role of ϕ_n will be played by $\widetilde{\varphi}_0(t-n)$, while the coefficients will be given by

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{\varphi_0(\cdot - n)}(\xi) g(\xi) d\xi &= \int_{-\infty}^{\infty} \widehat{\varphi}_0(\xi) e^{-in\xi} g(\xi) d\xi \\ &= 2\pi\{\varphi_0(-iD)f\}(n).\end{aligned}$$

Hence (2) in this case is

$$f(t) = \sum_n 2\pi\{\varphi_0(-iD)f\}(n)\widetilde{\varphi}_0(t-n). \tag{11}$$

Now $\widehat{\varphi}_0$ is not a polynomial, but again can accurately be approximated by Legendre polynomials since it is just a multiple of φ_0 with a scale change [17].

Thus we have two possible approaches based on PSWFs. The first involves the standard expansion in PSWF given by (9) which uses only local values for the series while the second given by (11) involves global values. If we combine the two, we have an overdetermined system, but can use some terms from (9) to get good local approximation and then some from (11) to get global approximation. If f is highly concentrated on $[-\tau, \tau]$, the former series works better, if not, the latter works better.

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Department of Mathematical Sciences, PO Box 413, University of Wisconsin Milwaukee, Milwaukee, WI 53201, USA
Email address: ggw@uwm.edu