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SOLUTION OF THE GENERALIZED ABEL INTEGRAL EQUATION

ALOKNATH CHAKRABARTI

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ABSTRACT. A direct function theoretic method is employed to determine the closed form solution of the generalized Abel integral equation. The present form of the solution involves only weakly singular integrals to be evaluated finally as opposed to the known form that requires evaluation of strongly singular integrals of the Cauchy type.

1. Introduction. The generalized Abel integral equation

$$(1)\quad a(x)\int_\alpha^x \frac{\phi(t)\,dt}{(x-t)^u} + b(x)\int_x^\beta \frac{\phi(t)\,dt}{(t-x)^u} = f(x), \qquad (0<\mu<1)\quad (\alpha\le x\le\beta)$$

where the coefficients $a(x)$ and $b(x)$ do not vanish simultaneously, is solved in closed form, under the specific assumptions on the functions a, b, f and ϕ, though not stated explicitly, which will be clear from the form of the solutions derived later on.

The generalized Abel equation (1) was examined in Gakhovs book [1], under the special assumptions that the coefficients $a(x)$ and $b(x)$ satisfy Hölders conditon in $[\alpha,\beta]$, whereas the forcing term $f(x)$ and the unknown function $\phi(x)$ belong to those classes of functions which admit representations of the form

$$(2)\quad \left.\begin{aligned} f(x) &= [(x-\alpha)(\beta-x)]^\epsilon\, f^*(x),\\ \text{and}\quad \phi(x) &= \frac{\phi^*(x)}{[(x-\alpha)(\beta-x)]^{1-\mu-\epsilon}}\end{aligned}\right\}\quad (\epsilon>0)$$

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where $f^*(x)$ possesses a Hölder continuous derivative in $[\alpha, \beta]$ and $\phi^*(x)$ satisfies Hölders condition in $[\alpha, \beta]$.

The method of solution, as explained in Gakhov [1], requires the solution of a Riemann - Hilbert problem for the determination of the sectionally analytic function of the complex variable z ($z = x+iy$, $i^2 = -1$), belonging to the complex z-plane, cut along the segment $[\alpha, \beta]$ of the real axis, as defined by

$$\Phi(z) = \frac{1}{R(z)} \int_{\alpha}^{\beta} \frac{\phi(t)\, dt}{(t-z)^{\mu}}, \tag{3}$$

with

$$R(z) = [(z-\alpha)(\beta - z)]^{\frac{1}{2}(1-\mu)}, \tag{4}$$

so that

$$\Phi(z) = 0\left(\frac{1}{z}\right), \quad \text{as} \quad |z| \to \infty, \tag{5}$$

and the associated Riemann - Hilbert problem is finally solved by utilizing the Plemelj -Sokhotski formulae involving Cauchy-type singular integrals.

As a particular example of the equation (1) the method of Gakhov [1] gives rise to the solution of the integral equation

$$\int_{\alpha}^{\beta} \frac{\phi(t)\, dt}{|x-t|^{\mu}} = f(x), \quad (\alpha \le x \le \beta) \tag{6}$$

as given by

$$\phi(x) = \frac{\sin(\mu\pi)}{\pi} \frac{d}{dx} \left[\int_{\alpha}^{x} \frac{g(t)\, dt}{(x-t)^{1-\mu}} \right], \tag{7}$$

where

$$g(x) = \frac{1}{2} f(x) - \frac{\cot(\mu\pi/2)}{2\pi} R(x) \int_{\alpha}^{\beta} f(t) \frac{dt}{R(t)(t-x)}. \tag{8}$$

We observe that the form of the solution $\phi(x)$ of equation (6), as given by the expressions (7) and (8), needs the evaluation of a singular integral of Cauchy type that involves stronger singularity than what Abel's integral equation actually requires. Thus, the method of Gakhov has a particular disadvantage in the sense that while solving a singular equation that involves integrals only with weak singularity of the type $(t-x)^{-\mu}(0<\mu<1)$, occurrence of strongly singular integrals involving Cauchy type singularities of the type $(t-x)^{-1}$ has to be permitted.

In the present paper we have followed a straightforward and direct method to solve the original integral equation (1). The final form of the presently determined solution involves only weakly singular integrals of the Abel type and thus Cauchy type singular integrals are avoided altogether.

2 The detailed method. We set

$$\Phi(z)=\int_\alpha^\beta \frac{\phi(t)dt}{(t-z)^\mu} \quad (0<\mu<1) \qquad \left[\equiv \int_\alpha^x \frac{\phi(t)dt}{(t-z)^\mu}+\int_x^\beta \frac{\phi(t)dt}{(t-z)^\mu}\right], \tag{9}$$

and find that as z tends to a point $x\in[\alpha,\beta]$, from above ($z=x+iy, y\longrightarrow o^+$) and below ($z=x+iy, y\longrightarrow o^-)(i^2=-1)$, respectively, the sectionally analytic function $\Phi(z)$ (see Gakhov [1]) as given by (9), tends to the following limiting values:

$$\Phi^\pm(x)=e^{\pm\mu\pi i}(A_1\phi)(x)+(A_2\phi)(x) \tag{10}$$

where

$$\left.\begin{aligned} (A_1\phi)(x)&=\int_\alpha^x \frac{\phi(t)\,dt}{(x-t)^\mu},\\ \text{and}\quad (A_2\phi)(x)&=\int_x^\beta \frac{\phi(t)\,dt}{(x-t)^\mu}, \end{aligned}\right\} \tag{11}$$

The relation (10) can also be expressed as

$$\begin{aligned} (A_1\phi)(x)&=\frac{1}{2i\sin(\mu\pi)}[\Phi^+(x)-\Phi^-(x)],\\ \text{and}\quad (A_2\phi)(x)&=\frac{-1}{2i\sin(\mu\pi)}[e^{-\mu\pi i}\Phi^+(x)-e^{\mu\pi i}\Phi^-(x)], \end{aligned} \tag{12}$$

By using the relations (12) in the given integral equation (1), we obtain:

$$[a(x) - e^{-\mu\pi i}b(x)]\Phi^{+}(x) - [a(x) - e^{+\mu\pi i}b(x)]\Phi^{-}(x) = 2i\sin(\mu\pi)f(x), \qquad (\alpha \le x \le \beta) \tag{13}$$

The above relation (13) represents the special Riemann-Hilbert type problem as given by the relation

$$\Phi^{+}(x) + G(x)\Phi^{-}(x) = g(x), \qquad (\alpha \le x \le \beta) \tag{14}$$

with

$$\begin{aligned} G(x) &= -\left[\frac{a(x) - e^{\mu\pi i}\,b(x)}{a(x) - e^{-\mu\pi i}\,b(x)}\right] \\ &= -\exp\left[-2i\arctan\ \left\{\frac{b(x)\sin(\mu\pi)}{a(x) - b(x)\cos(\mu\pi)}\right\}\right] \end{aligned} \tag{15}$$

and

$$g(x) = \frac{2\,i\sin(\mu\pi)\,f(x)}{a(x) - e^{-\mu\pi i}\,b(x)}. \tag{16}$$

We shall next explain a method of solution of the new Riemann-Hilbert type problem (14), for which the unknown sectionally analytic function $\Phi(z)$, given by equation (9), satisfies the following condition at infinity:

$$\Phi(z) = O\left(\frac{1}{z^{\mu}}\right), \quad \text{as } |z| \to \infty \tag{17}$$

We first solve the homogeneous problem (14), satisfying the relation

$$\Phi_0^{+}(x) + G(x)\Phi_0^{-}(x) = 0, \tag{18}$$

giving

$$\Psi_0^{+}(x) - \Psi_0^{-}(x) = G_0(x), \tag{19}$$

where

$$\Phi_0(z) = \exp\left[\Psi_0(z)\right]$$

and

$$-G(x) = \exp[G_0(x)\,] \tag{20}$$

Now, by utilizing the first of the two relations in (12), we find that we can express the function $\Psi_0(z)$, satisfying (19), as :

$$\Psi_0(z) = \int_\alpha^\beta \frac{\psi_0(t)dt}{(t-z)^\mu} \tag{21}$$

where

$$\psi_0(x) = [2i\sin(\mu\pi)]^{-1}(A_1^{-1}G_0)(x), \tag{22}$$

with

$$(A_1^{-1}G_0)(x) = \left(\frac{\sin\mu\pi}{\pi}\right)\frac{d}{dx}\int_\alpha^x \frac{G_0(t)\,dt}{(x-t)^{1-\mu}}. \tag{23}$$

Next, by utilizing (19) in (14), we obtain

$$\frac{\Phi^+(x)}{\Phi_0^+(x)} - \frac{\Phi^-(x)}{\Phi_0^-(x)} = \frac{g(x)}{\Phi_0^+(x)}, \tag{24}$$

where

$$\Phi_0^\pm(x) = \exp\left[\,\Psi_0^\pm(x)\,\right] \tag{25}$$

with $\Psi_0^\pm(x)$ being obtainable by using the relations (21)- (23), giving rise to results of the type (10).

Then, by utilizing the first of the formulae (12), we can determine the solution of the Riemann-Hilbert type problem (24), as given by :

$$\frac{\Phi(z)}{\Phi_0(z)} = \int_\alpha^\beta \frac{\lambda(t)\,dt}{(t-z)^\mu}, \tag{26}$$

where

$$\lambda(x) = \frac{1}{2\pi i}\cdot\frac{d}{dx}\left[\int_\alpha^x \frac{g(t)\,dt}{\Phi_0^+(t)(x-t)^{1-\mu}}\right]. \tag{27}$$

The relation (27) takes the equivalent form, obtainable by integrating by parts, given by

$$\lambda(x) = \frac{1}{2\pi i}\left[\frac{p(\alpha)}{(\alpha - x)^{1-\mu}} + \int_\alpha^x \frac{p'(t)\,dt}{(t-x)^{1-\mu}}\right], \tag{28}$$

with

$$p(t) = \frac{g(t)}{\Phi_0^+(t)},\ p'(t) = \frac{dp}{dt}, \tag{29}$$

under the special assumptions on the behaviour of the functions a, b and f, which admit the existence of the derivative of the function $p(x)(\alpha < x < \beta)$.

Next, we obtain the following limiting values of the function $\Phi(z)$, as z approaches the point $x \in [\alpha, \beta]$: [see (10)]:

$$\Phi^{\pm}(x) = \Phi_0^{\pm}(x)\ \left[\ e^{\pm\mu\pi i}(A_1\lambda)(x) + (A_2\lambda)(x))\ \right] \tag{30}$$

giving

$$\Phi^+(x) - \Phi^-(x) = h(x)(\text{say}), \tag{31}$$

where

$$\begin{aligned} h(x) = \left[\ e^{+\mu\pi i}\Phi_0^+(x) - e^{-\mu\pi i}\Phi_0^-(x)\right]\ (A_1\lambda)(x) \\ + \left[\Phi_0^+(x) - \Phi_0^-(x)\right]\ (A_2\lambda)(x), \end{aligned} \tag{32}$$

Finally, by utilizing the first formula in (12), once again, we obtain the required solution of the given integral equation (1) in the form:

$$\phi(x) = \frac{1}{2\pi i}\frac{d}{dx}\left[\int_\alpha^x \frac{h(t)\,dt}{(x-t)^{1-\mu}}\right]. \tag{33}$$

The result (33) can also be expressed in the equivalent form [see (27) and (28)]:

$$\phi(x) = \frac{1}{2\pi i}\left[\frac{h(\alpha)}{(x-\alpha)^{1-\mu}} + \int_\alpha^x \frac{h'(t)\,dt}{(x-t)^{1-\mu}}\right], \tag{34}$$

under the special assumptions on the functions a, b and f, which admit the existence of the derivatives of the function $h(x)$.

An alternative form of the solution $\phi(x)$ can be derived as explained below :

We find that if we solve the Riemann-Hilbert problem (14), by first solving a different homogeneous Riemann-Hilbert problem, as given by the relation

$$\Phi_0^+(x) + e^{-2\mu\pi i} G(x) \widehat{\Phi}_0^-(x) = 0, \tag{35}$$

instead of the homogeneous problem (18), we obtain the following alternative representation of the sectionally analytic function $\Phi(z)$:

$$\Phi(z) = \widehat{\Phi}_0(z) \int_\alpha^\beta \frac{\widehat{\lambda}(t)\, dt}{(t-z)^\mu}. \tag{36}$$

with

$$\widehat{\lambda}(x) = \frac{e^{-\mu\pi i}}{2\pi i} \frac{d}{dx} \int_x^\beta \frac{g(t)\, dt}{\widehat{\Phi}_0^+(t)(t-x)^{1-\mu}}, \tag{37}$$

or,

$$\widehat{\lambda}(x) = \frac{e^{-\mu\pi i}}{2\pi i} \left[\frac{\widehat{p}(\beta)}{(\beta-x)^{1-\mu}} - \int_x^\beta \frac{\widehat{p}'(t)\, dt}{(t-x)^{1-\mu}} \right], \tag{38}$$

with

$$\widehat{p}(t) = \frac{g(t)}{\widehat{\Phi}_0^+(t)}, \tag{39}$$

whenever $\widehat{p}(x)$ is differentiable.

Then, utilizing the limiting values $\Phi^\pm(x)$ of the function $\Phi(z)$, as given by the formula (36), along with the second formula in (12), we obtain an alternative representation of the unknown function $\phi(x)$ [the solution of the integral equation (1)] as given by

$$\phi(x) = \frac{1}{2\pi i} \frac{d}{dx} \left[\int_x^\beta \frac{\widehat{h}(t)\, dt}{(t-x)^{1-\mu}} \right], \tag{40}$$

where

$$\widehat{h}(x) = e^{-\mu\pi i}\Phi^{+}(x) - e^{-\mu\pi i}\Phi^{-}(x) \\ = \left[\widehat{\Phi}_0^{+}(x) - \widehat{\Phi}_0^{-}(x)\right](A_1\widehat{\lambda})(x) \\ + \left[e^{-\mu\pi i}\widehat{\Phi}_0^{+}(x) - e^{\mu\pi i}\widehat{\Phi}_0^{-}(x)\right](A_2\widehat{\lambda})(x). \tag{41}$$

We note that we have used above, the well-known formula

$$(A_2^{-1}f)(x) = -\frac{\sin(\mu\pi)}{\pi}\frac{d}{dx}\left[\int_x^{\beta}\frac{f(t)\,dt}{(t-x)^{1-\mu}}\right], \tag{42}$$

The result (40) can also be expressed in the equivalent form

$$\phi(x) = -\frac{1}{2\pi i}\left[\frac{\widehat{h}(\beta)}{(\beta-x)^{1-\mu}} - \int_x^{\beta}\frac{\widehat{h}'(t)\,dt}{(t-x)^{1-\mu}}\right] \tag{43}$$

whenever the function $\widehat{h}(x)$ is differentiable.

We emphasize that though the exact assumptions on the class of functions a, b and f, for which the solution formulae (36) and (37) hold good are not stated explicitly, it is clear that these formulae are valid for a wide range of choices of the functions involved.

In particular, when either $a = 0, b = 1$ or $a = 1, b = 0$, we get back the known solutions of Abel's integral equations, by utilizing the solution formula (33) or (40).

We find that no Cauchy type singular integrals occur in the above analysis.

In the particular case, when $a = b = 1$, we obtain the integral equation (6), and many results derived above simplify a lot giving

$$\Phi_0^{+}(x) = \exp(-i\pi/4)\left[\frac{1-(1-x)^{1/2}}{1+(1-x)^{1/2}}\right]^{1/4}, \\ \text{if } \alpha = 0,\ \beta = 1 \text{ and } \mu = 1/2. \tag{44}$$

We obtain the solution of (6), as given by either of the two formulae (35) and (36), which is different from the known (see Gakhov [1]) result

(8), where a Cauchy-type singular integral is needed to be evaluated, which is in sharp contrast with the formula (31). A similar conclusion holds for the particular case, when $a = b = -1$, for which we obtain

$$\Phi_0^+(x) = \exp(+i\pi/4) \left[\frac{1+(1-x)^{1/2}}{1-(1-x)^{1/2}}\right]^{1/4}, \tag{45}$$
$$\text{if } \alpha = 0,\ \beta = 1 \text{ and } \mu = 1/2.$$

We observe that the final results obtained here are all in computable forms and dealing with specific examples will be the subject matter of future work.

The major findings of the present work can be expressed in the form of a theorem as stated below :

Theorem : The generalized Abel integral equation

$$a(x)(A_1\phi)(x) + b(x)(A_2\phi)(x) = f(x), \quad (\alpha \le x \le \beta) \tag{1*}$$

where the two Abel operators A_1 and A_2 are given by the relations :

$$(A_1\phi)(x) = \int_\alpha^\beta \frac{\phi(t)\,dt}{(x-t)^\mu}, \tag{2*}$$
$$(A_2\phi)(x) = \int_x^\beta \frac{\phi(t)\,dt}{(x-t)^\mu} \quad (0 < \mu < 1)$$

with $a(x), b(x)$ and $f(x)$ representing known functions of sufficiently general class as dictated by the various formulae occurring below, can be solved in closed form which requires evaluation of only weakly singular Abel type integrals as given by the following formula:

either

$$\phi(x) = \frac{1}{2\,i\sin(\mu\pi)}\,(A_1^{-1}h)(x), \quad (\alpha \le x \le \beta), \tag{3*}$$

or

$$\phi(x) = \frac{-1}{2\,i\sin(\mu\pi)}\,(A_2^{-1}\hat{h})(x), \quad (\alpha \le x \le \beta), \tag{4*}$$

with A_1^{-1} and A_2^{-1} representing the well-known Abel's inverse operators, where the functions $h(x)$ and $\hat{h}(x)$ are given by the following relations.

$$\text{(5*)}\qquad \begin{aligned} h(x) &= [e^{+\mu\pi i}\Phi_0^+(x) - e^{-\mu\pi i}\Phi_0^-(x)](A_1\lambda)(x) \\ &\quad + [\Phi_0^+(x) - \Phi_0^-(x)](A_2\lambda)(x), \end{aligned}$$

and

$$\text{(6*)}\qquad \begin{aligned} \hat{h}(x) &= [\hat{\Phi}_0^+(x) - \hat{\Phi}_0^-(x)](A_1\hat{\lambda})(x) \\ &\quad + [e^{-\mu\pi i}\hat{\Phi}_0^+(x) - e^{+\mu\pi i}\hat{\Phi}_0^-(x)](A_2\hat{\lambda})(x) \end{aligned}$$

with

$$\text{(7*)}\qquad \left.\begin{aligned} \lambda(x) &= \frac{1}{2i}\big((A_1^{-1}(\frac{g}{\Phi_0^+})\big)(x), \\ \hat{\lambda}(x) &= \frac{-e^{\mu\pi i}}{2i}\big((A_2^{-1}(\frac{g}{\hat{\Phi}_0^+})\big)(x) \end{aligned}\right\} \quad (\alpha \le x \le \beta)$$

where the functions $\Phi_0^+(x)$ and $\hat{\Phi}_0^+(x)$ are the limiting values of the functions $\Phi_0(z)$ and $\hat{\Phi}_0(z), (z = x + iy, y \to 0^+)$, as given by the formulae :

$$\text{(8*)}\qquad \left.\begin{aligned} \Phi_0(z) &= \exp\left[\int_\alpha^\beta \frac{\psi_0(t)\,dt}{(t-z)^\mu}\right], \\ \hat{\Phi}_0(z) &= \exp\left[\int_\alpha^\beta \frac{\hat{\psi}_0(t)\,dt}{(t-z)^\mu}\right] \end{aligned}\right\}$$

with

$$\text{(9*)}\qquad \left.\begin{aligned} \psi_0(x) &= \frac{1}{2i\sin(\mu\pi)}\,(A_1^{-1}G_0)(x)\,, \\ \widehat{\psi}_0(x) &= \frac{1}{2i\sin(\mu\pi)}\,(A_1^{-1}(G_0 - 2\mu\pi i))(x)\,, \end{aligned}\right\}$$

the functions $g(x)$ and $G_0(x)$ being given by the relations :

$$\text{(10*)}\qquad g(x) = \frac{2\,i\sin(\mu\pi)f(x)}{a(x) - e^{-\mu\pi i}b(x)}$$

and

$$G_0(x) = -2\,i\arctan\left[\frac{b(x)\sin(\mu\pi)}{a(x)-b(x)\cos(\mu\pi)}\right] \tag{11*}$$

The related references to the present work are the papers of Gakhov [2], Lundgren and Chiang [3] and Sakalyuk [4].

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Department of Mathematics, Indian Institute of Science Bangalore - 560 012, India.
Email address: alok@math.iisc.ernet.in