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UNIFORM CONVERGENCE ESTIMATES FOR A COLLOCATION METHOD FOR THE CAUCHY SINGULAR INTEGRAL EQUATION

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ABSTRACT. The authors study the convergence and the stability of a collocation and a discrete collocation method for Cauchy singular integral equations with weakly singular perturbation kernels in some weighted uniform norms. Uniform error estimates are also given.

1. Introduction. We consider the Cauchy singular integral equation with constant coefficients

$$au(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{u(t)}{t-x}\,dt + \int_{-1}^{1} k(x,t)u(t)\,dt = f(x), \qquad |x| < 1, \tag{1.1}$$

where the first integral in (1.1) is to be understood in the sense of Cauchy principal value. Here u is the unknown solution, a and b are given real constants such that $a^2 + b^2 = 1$, $b \neq 0$, f is a Hölder continuous function, and k is a weakly singular function of the form

$$k(x,t) = \frac{H(x,t)}{|t-x|^{\mu}}, \qquad 0 < \mu < 1, \tag{1.2}$$

with $H(x,t) \in \mathrm{Lip}_{\nu}([-1,1]^2)$, $0 < \nu \leq 1$. Here $\mathrm{Lip}_{\nu}(A)$ is the space defined by

$$\mathrm{Lip}_{\nu}(A) := \left\{ g \in C^0(A) : \sup_{x \neq y \in A} \frac{|g(x) - g(y)|}{|x-y|^{\nu}} := M_g < \infty \right\}$$

and equipped with the norm

$$\|g\|_{\nu} := \|g\| + M_g, \quad \text{where} \quad \|g\| = \max_{x \in A} |g(x)|.$$

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It is well known that (see [**18**]), even if f and k are smooth functions, the solution u can be unbounded at one or both of the points 1 and -1. So we can search for the unknown solution u in the following form,

$$u(x) = w^{(\alpha,\beta)}(x)v(x), \tag{1.3}$$

where v is a smooth function and

$$w^{(\alpha,\beta)}(x) := (1-x)^{\alpha}(1+x)^{\beta}.$$

The exponents are defined by the relations

$$\begin{aligned} a + ib = e^{i\pi\alpha_0}, \qquad & 0 < |\alpha_0| < 1, \\ -1 < \alpha := M - \alpha_0, \qquad & \beta := N + \alpha_0 < 1. \end{aligned} \tag{1.4}$$

The integers N and M are chosen such that the inequalities (1.4) are fulfilled. Then, the index $\chi = -(\alpha+\beta) = -(N+M)$ can only take the values $-1, 0, 1$. In this paper, we consider the cases $\chi = 1$, i.e., $\beta = -1-\alpha$, and $\chi = 0$, i.e., $\beta = -\alpha$.

Let us denote by D and K the dominant and the perturbation operators corresponding to (1.1), i.e.,

$$\begin{aligned} Dv(x) &:= aw^{(\alpha,\beta)}(x)v(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{v(t)}{t-x} w^{(\alpha,\beta)}(t)\, dt, \\ Kv(x) &:= \int_{-1}^{1} \frac{H(x,t)}{|t-x|^{\mu}} v(t) w^{(\alpha,\beta)}(t)\, dt. \end{aligned} \tag{1.5}$$

Then equation (1.1) can be rewritten as

$$Dv + Kv = f. \tag{1.6}$$

Now denote by $L^2_w = L^2_w(-1,1)$ the Hilbert space of all complex-valued functions on $(-1,1)$ with the scalar product

$$(u,v)_w = \frac{1}{\pi}\int_{-1}^{1} u(t)\overline{v(t)}w(t)\, dt$$

and the norm $\|u\|_w = \sqrt{(u,u)_w}$. Then (see, e.g., [**16, 21**]), the operator D is a linear bounded Fredholm operator with index χ acting from L^2_w into $L^2_{1/w}$ and its adjoint operator $\hat{D}$ is defined by

$$\hat{D}v(x) := aw^{(-\alpha,-\beta)}(x)v(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{v(t)}{t-x} w^{(-\alpha,-\beta)}(t)\, dt. \tag{1.7}$$

Moreover, if $\chi = 0$ then D is invertible and $\hat{D}$ is its inverse, i.e.,

$$\hat{D}D = I, \qquad D\hat{D} = I, \tag{1.8}$$

where I is the identity operator. If $\chi = 1$ then $\hat{D}$ is a right inverse operator to D. However (see [**21**]), if we define the Hilbert space $L^2_{w,0} := \{v \in L^2_w : (v, 1)_w = 0\}$, then we have

$$\begin{aligned} \hat{D}Dv &= v \quad \text{for all } v \in L^2_{w(\alpha,\beta),0}, \\ D\hat{D}\overline{v} &= \overline{v} \quad \text{for all } \overline{v} \in L^2_{w(-\alpha,-\beta)}. \end{aligned} \tag{1.9}$$

We agree that in case $\chi = 1$ we look for a solution $v \in L^2_{w(\alpha,\beta),0}$. Then, by multiplying equation (1.6) by $\hat{D}$ we obtain

$$v + \hat{D}Kv = \hat{D}f \tag{1.10}$$

in both cases $\chi = 0$ and $\chi = 1$. Therefore, by recalling that each solution $v \in L^2_{w(\alpha,\beta)}$ of (1.10) automatically belongs to $L^2_{w(\alpha,\beta),0}$ and D is always the left inverse operator to $\hat{D}$, equation (1.10) is equivalent to (1.6).

If the operator $\hat{D}K$ is compact from L^2_w into itself, then (1.10) is called the regularized Fredholm equation of (1.6), and then this equation satisfies the Fredholm alternative. For these reasons, historically some numerical methods, the so called "indirect methods", set out to solve (1.10) instead of equation (1.6). However it can be very difficult to know the kernel of the operator $\hat{D}K$ analytically; and so the most efficient numerical methods are those solving (1.6) directly (the so called "direct methods").

Nevertheless equation (1.10) is very important to establish if and in what space equation (1.6) has solutions.

In fact if we can prove that the linear, bounded operator $\hat{D}K$ is defined and compact from a named space into itself, then we automatically know that in this space we can search for solutions of equation (1.6).

Several authors have considered the integral equation (1.1) and have chosen an approximation to the unknown solution by using projection methods like collocation, Galerkin, or quadrature procedures (cf., [**7,**

8, 9, 10, 11, 12, 13, 15, 16, 17, 20, 21, 22] and the references given by these authors). Several convergence results and error estimates are obtained in L^2_w.

Nevertheless, if we deduce uniform error estimates from those obtained in weighted L^2-norm, we obtain pessimistic estimates about the rate of convergence. For this reason, in this paper, we prove the convergence and the stability of some collocation methods directly in a weighted space $C_{w^{(\rho,\tau)}}$ of continuous functions defined by

$$C_{w^{(\rho,\tau)}}[-1,1] := \{v(x) : w^{(\rho,\tau)}(x)v(x) \in C^0([-1,1])\},$$

$$\rho = \max\{0,\alpha\}, \qquad \tau = \max\{0,\beta\}, \qquad -(\alpha+\beta) = \chi \in \{0,1\}, \tag{1.11}$$

and equipped with the norm

$$||v||_{w^{(\rho,\tau)}} := \max_{x\in[-1,1]} |v(x)w^{(\rho,\tau)}(x)|.$$

We point out that if $\chi = 1$, i.e. $\rho = \tau = 0$, then $C_{w^{(\rho,\tau)}}[-1,1] = C^0([-1,1])$ and

$$\|u\|_{w^{(0,0)}} := \|u\| = \max_{x\in[-1,1]} |u(x)|$$

is the usual uniform norm.

Therefore, a crucial point for our purposes is to prove that operator $\hat{D}K$ is compact in $C_{w^{(\rho,\tau)}}[-1,1]$. We establish this result in Lemma 4.3. Subsequently, in the present paper, after having proved the convergence and the stability of the collocation methods described below, we give also the rate of convergence in both $C_{w^{(\rho,\tau)}}[-1,1]$ and $C^0[-1,1]$.

2. The collocation method. Let $\{p_n(w^{(\alpha,\beta)})\}$ be the sequence of the orthonormal polynomials in $[-1,1]$ with positive leading coefficient corresponding to the weight function $w^{(\alpha,\beta)}$, where the exponents α,β are defined by (1.4).

Let $T = \{x_{m,k}, k = 1,\dots,m, m = 1,2,\dots\}$ be a matrix of collocation points. The collocation method consists in approximating the unknown solution v by a polynomial of the kind

$$v_n(x) = \sum_{j=0}^{n+\chi} a_j p_j(w^{(\alpha,\beta)}; x), \tag{2.1}$$

where a_j, $j = 0, 1, \dots, n + \chi$, are unknown constants.

In order to evaluate the coefficients a_j, $j = 0, 1, \dots, n+\chi$, we require that v_n is the solution of equation (1.6) on the points of T, i.e.,

$$Dv_n(x_{m,k}) + Kv_n(x_{m,k}) = f(x_{m,k}), \quad k = 1, \dots, m = n + 1. \tag{2.2}$$

In this way, the problem of finding a solution of (1.6) is reduced to the solution of a linear system.

Equations (2.2) represent a linear system of $m = n + 1$ equations in the $n + \chi + 1$ unknowns a_j, $j = 0, \dots, n + \chi$.

For $\chi = 0$ we have $n + 1$ equations in $n + 1$ unknowns.

If $\chi = 1$ then we have $n+1$ equations in $n+2$ unknowns and therefore we need another equation. This is usually obtained by requiring that the approximate solution v_n belongs to $L^2_{w^{(\alpha,\beta)},0}$, that means,

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x) v_n(x) dx = 0, \quad \text{i.e. } a_0 = 0. \tag{2.3}$$

Thus, the integral equation (1.6) is replaced by the linear system (2.2) if $\chi = 0$ and by the system (2.2), (2.3) if $\chi = 1$.

Now let $L_m(T)$ be the interpolating Lagrange operator related to the collocation matrix T. Let $\{||L_m(T)||\}_m$ be the sequence of the Lebesgue constants defined as follows

$$\|L_m(T)\| := \max_{|x|\le 1} \sum_{k=1}^{m} |l_{m,k}(x)|, \tag{2.4}$$

where

$$l_{m,k}(x) = \prod_{\substack{i=1 \\ i\neq k}}^{m} \frac{x - x_{m,i}}{x_{m,k} - x_{m,i}}.$$

We will choose T such that $||L_m(T)|| = O(\log m)$. Matrices that satisfy this condition are well known: classical examples are the zeros of Jacobi polynomials with exponents $\alpha, \beta \le -1/2$ and the practical abscissae $\{-\cos k\pi/(m + 1), k = 0, \dots, m+1\}$. Moreover, in [**6, 14**], the authors showed that, beginning from the zeros of the Jacobi polynomials, it is possible to obtain wide classes of knots satisfying the condition

$$||L_m(T)|| = O(\log m).$$

Now, we recall a fundamental property of the operator D. Let $\{p_n(w^{(-\alpha,-\beta)})\}$ be the sequence of the orthonormal polynomials in $[-1,1]$ with positive leading coefficient corresponding to the weight function $w^{(-\alpha,-\beta)}$. It is well known that

$$\begin{aligned} D(p_n(w^{(\alpha,\beta)})) &= (-1)^M p_{n-\chi}(w^{(-\alpha,-\beta)}), \\ \hat{D}(p_n(w^{(-\alpha,-\beta)})) &= (-1)^M p_{n+\chi}(w^{(\alpha,\beta)}), \end{aligned} \tag{2.5}$$

where $p_{-1}(w^{(-\alpha,-\beta)}) = p_{-1}(w^{(\alpha,\beta)}) = 0$, (cf. [**12**]). We note that in the righthand sides of relations (2.5) the factors $2^{-\chi}$ and 2^{χ}, respectively, appear if orthogonal polynomials instead of orthonormal ones are used.

In view of relation (2.5), the system (2.2) can be rewritten in the form

$$\begin{aligned} \sum_{j=0}^{n+\chi}(-1)^M a_j p_{j-\chi}(w^{(-\alpha,-\beta)}; x_{m,k}) + Kv_n(x_{m,k}) \\ = f(x_{m,k}), \qquad k = 1, \dots, m. \end{aligned} \tag{2.6}$$

2.1. *The discrete collocation method.* If we look at the system (2.6) we note that "a priori" we don't know whether the integral $Kv_n(x)$ can be exactly evaluated or not. To avoid this problem, we choose a suitable quadrature formula for calculating $Kv_n(x)$. Let N be the number of knots of the quadrature formula, we call K_N the discrete operator obtained from K applying the chosen quadrature formula.

In this way the system (2.6) becomes

$$\begin{aligned} Dv_n(x_{m,k}) + K_N v_n(x_{m,k}) = f(x_{m,k}), \\ k = 1, \dots, m. \end{aligned} \tag{2.7}$$

In this paper we will choose a product quadrature formula. More precisely, we approximate $Kv_n(x)$ by

$$K_N v_n(x) := \sum_{i=1}^{N} w_{N,i}(x) H(x, t_{N,i}) v_n(t_{N,i}), \tag{2.8}$$

where H is defined by (1.2), $t_{N,i}$ are the knots of a Jacobi matrix $\overline{T}$,

$$\begin{aligned} \overline{T} &:= \{t_{N,i}(w^{(\overline{\alpha},\overline{\beta})}), w^{(\overline{\alpha},\overline{\beta})}(x) \\ &= (1-x)^{\overline{\alpha}}(1+x)^{\overline{\beta}}, -1 < \overline{\alpha}, \overline{\beta} \leq -1/2\}, \end{aligned} \tag{2.9}$$

and

$$w_{N,i}(x) := \int_{-1}^{1} \frac{l_{N,i}(t) w^{(\alpha,\beta)}(t)}{|t-x|^{\mu}} \, dt.$$

Here $l_{N,i}$ are the usual fundamental Lagrange polynomials associated to the matrix $\overline{T}$; N is an integer such that $N \geq 2n + \chi + 1$. We recall that in this case $||L_N(\overline{T})|| = O(\log N)$.

3. The stability and the convergence of the methods. The main results of the present paper are stated in the following three theorems.

Theorem 3.1. *Let* $f \in \mathrm{Lip}_{\theta}([-1,1])$, $0 < \theta \leq 1$, $k(x,t) = H(x,t)/|t-x|^{\mu}$, $0 < \mu < 1 - \max\{|\alpha|, |\beta|\}$, $H(x,t) \in \mathrm{Lip}_{\nu}([-1,1]^2)$, $0 < \nu \leq 1$. *Assume that the problem* (1.6) *has a unique solution* $v \in C_{w^{(\rho,\tau)}}([-1,1])$ *for* $\chi = 0$ *or* $v \in C_{w^{(\rho,\tau)}}([-1,1]) \cap L^2_{w^{(\alpha,\beta)},0}$ *for* $\chi = 1$, *and the collocation points are the knots of a given matrix* T, *such that the Lebesgue constants satisfy the condition* $\|L_m(T)\| \leq C \log m$. *Then the system of equations* (2.2) *and* (2.3) *for* $\chi = 1$, *or the system* (2.2) *for* $\chi = 0$, *is uniquely solvable for all sufficiently large* n *and*

$$\|v - v_n\|_{w^{(\rho,\tau)}} = O\left(\frac{\log^2 n}{n^r}\right), \tag{3.1}$$

where v_n *is defined by* (2.1) *and* $r = \min\{\nu, \theta, 1 - \mu - \max\{|\alpha|, |\beta|\}\}$.

Theorem 3.2. *If the points* $t_{N,i}$ *are the knots of the matrix* $\overline{T}$ *given by* (2.9), *with* $N \geq 2n + \chi + 1$ *and if the assumptions of Theorem* 3.1 *are satisfied, then the system of equations* (2.7), (2.3) *for* $\chi = 1$, *or the system* (2.7) *for* $\chi = 0$, *is uniquely solvable for all sufficiently large* n *and*

$$\|v - v_n\|_{w^{(\rho,\tau)}} = O\left(\frac{\log^2 n}{n^r}\right), \tag{3.2}$$

where v_n is defined by (2.1) *and* $r = \min\{\nu, \theta, 1-\mu-\max\{|\alpha|, |\beta|\}\}$.

We observe that if $\chi = 1$ the norm in (3.1) and in (3.2) is the uniform norm; in the case $\chi = 0$, we obtain the convergence results in a weighted norm. However, we are able to prove the uniform convergence of the method by means of

Theorem 3.3. *Let $\chi = 0$, and let the assumptions of Theorem* 3.1 *or of Theorem* 3.2 *be satisfied. Then the system* (2.2) *or the system* (2.7) *is uniquely solvable for all sufficiently large n and*

$$\|v - v_n\| = O\left(\frac{\log^2 n}{n^{r-2|\alpha|}}\right), \tag{3.3}$$

where v_n is defined by (2.1), $r = \min(\nu, \theta, 1-\mu-|\alpha|)$ *and* $r - 2|\alpha| > 0$.

4. The proofs of the main results. In order to prove the main results of this paper presented in the previous section, some notations and preliminary results are needed. In the following, the symbol "C" stands for a generic constant taking different values at different places.

Lemma 4.1. *Let* $0 < \mu < 1 - \max\{0, -\gamma, -\delta\}$

$$\varphi(x) = \int_{-1}^{1} \frac{(1-t)^{\gamma}(1+t)^{\delta}}{|t-x|^{\mu}}\,dt, \qquad |x| \le 1, \quad \gamma, \delta > -1.$$

Then $\varphi(x) \in \mathrm{Lip}_{\lambda}[-1,1]$ *with* $\lambda = 1 - \mu - \max\{0, -\gamma, -\delta\}$. *Moreover, if* $-1 < \gamma$, $\delta < 0$, *then* $\varphi(x) \in \mathrm{Lip}_{1-\mu}[a,b]$, $0 < \mu < 1$, *for every closed subset* $[a,b] \subset (-1,1)$.

Proof. In order to prove the assertion, it is sufficient to consider $-1 < \gamma$, $\delta < 0$, $x_1 < x_2 \in [-1,0]$ such that $x_2 - x_1 = h < 1/4$ and the two cases

i) $-1 \le x_1 \le -1 + h$, for all $x_2 \in [-1,0]$;

ii) $-1 + h \le x_1 < x_2 \le 0$.

Firstly we consider the case i). We can write

(4.1)
$$
\begin{aligned}
|\varphi(x_2)-\varphi(x_1)| \le \Big\{ \int_{-1}^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{x_2+h} + \int_{x_2+h}^{1/2} + \int_{1/2}^{1} \Big\} \\
\times \left| \frac{1}{|t-x_2|^\mu} - \frac{1}{|t-x_1|^\mu} \right| (1-t)^\gamma (1+t)^\delta \, dt \\
:= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

We have

(4.2)
$$
\begin{aligned}
I_1 &= \int_{-1}^{x_1} \left[\frac{1}{(x_1-t)^\mu} - \frac{1}{(x_1+h-t)^\mu} \right] (1-t)^\gamma (1+t)^\delta \, dt \\
&\le \int_{-1}^{x_1} \frac{(1+t)^\delta}{(x_1-t)^\mu} \, dt \\
&= (1+x_1)^{1-\mu+\delta} \int_0^1 (1-y)^{-\mu} y^\delta \, dy \\
&\le h^{1-\mu+\delta} \int_0^1 (1-y)^{-\mu} y^\delta \, dy.
\end{aligned}
$$

(4.3)
$$
\begin{aligned}
I_2 &= \int_{x_1}^{x_1+h} \left| \frac{1}{(x_1+h-t)^\mu} - \frac{1}{(t-x_1)^\mu} \right| (1-t)^\gamma (1+t)^\delta \, dt \\
&\le h^{1-\mu} \left[\int_0^1 \frac{(1+x_1+hy)^\delta}{(1-y)^\mu} \, dy + \int_0^1 \frac{(1+x_1+hy)^\delta}{y^\mu} \, dy \right] \\
&\le h^{1-\mu+\delta} \left[\int_0^1 (1-y)^{-\mu} y^\delta \, dy + \int_0^1 y^{\delta-\mu} \, dy \right] \\
&\le h^{1-\mu+\delta} \left[\int_0^1 (1-y)^{-\mu} y^\delta \, dy + 1/(1-\mu+\delta) \right].
\end{aligned}
$$

(4.4)
$$
\begin{aligned}
I_3 &\le (1+x_2)^\delta \int_{x_2}^{x_2+h} \left| \frac{1}{(t-x_1)^\mu} - \frac{1}{(t-x_2)^\mu} \right| dt \\
&\le h^\delta \int_{x_2}^{x_2+h} \frac{1}{(t-x_2)^\mu} \, dt \\
&\le \frac{h^{1-\mu+\delta}}{1-\mu}.
\end{aligned}
$$

Since, for $x_2+h \le t \le 1$ and $x_{12} \in [x_1,x_2]$, we have

$$\left|\frac{1}{|t-x_2|^\mu}-\frac{1}{|t-x_1|^\mu}\right| = \frac{1}{(t-x_2)^\mu}-\frac{1}{(t-x_1)^\mu} = \frac{\mu h}{(t-x_{12})^{\mu+1}} \le \frac{\mu h}{(t-x_2)^{\mu+1}},$$

$$\begin{aligned}
I_4 &\le (1+x_2+h)^\delta \int_{x_2+h}^{1/2}\left[\frac{1}{(t-x_2)^\mu}-\frac{1}{(t-x_1)^\mu}\right]dt \\
&\le \mu h^{1+\delta}\int_{x_2+h}^{1/2}\frac{1}{(t-x_2)^{1+\mu}}\,dt \\
&\le \mu h^{1+\delta}\int_{x_2+h}^{\infty}\frac{1}{(t-x_2)^{1+\mu}}\,dt \\
&= h^{1-\mu+\delta};
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
I_5 &\le \int_{1/2}^{1}\left[\frac{1}{(t-x_2)^\mu}-\frac{1}{(t-x_1)^\mu}\right](1-t)^\gamma\,dt \\
&\le \mu h\int_{1/2}^{1}\frac{(1-t)^\gamma}{(t-x_2)^{1+\mu}}\,dt \\
&\le \frac{2^{\mu-\gamma}\mu h}{\gamma+1}.
\end{aligned} \tag{4.6}$$

Therefore, substituting (4.2)–(4.6) into (4.1), we obtain for case i)

$$|\varphi(x_2)-\varphi(x_1)| \le Ch^{1-\mu+\delta},$$

where C is a positive constant only depending on μ,γ,δ.

Secondly, we consider case ii) and write

$$\begin{aligned}
&|\varphi(x_2)-\varphi(x_1)| \\
&\quad \le \left\{\int_{-1}^{-1+h}+\int_{-1+h}^{x_1}+\int_{x_1}^{x_2}+\int_{x_2}^{x_2+h}+\int_{x_2+h}^{1/2}+\int_{1/2}^{1}\right\} \\
&\left|\frac{1}{|t-x_2|^\mu}-\frac{1}{|t-x_1|^\mu}\right|(1-t)^\gamma(1+t)^\delta\,dt \\
&\qquad := J_1+J_2+J_3+I_3+I_4+I_5.
\end{aligned} \tag{4.7}$$

Since, for $-1 \le t \le -1+h$, it holds

$$\left| \frac{1}{|t-x_2|^\mu} - \frac{1}{|t-x_1|^\mu} \right| \le \frac{1}{(x_1-t)^\mu} \le \frac{1}{(h-1+t)^\mu},$$

we get

(4.8)

$$\begin{aligned} J_1 &\le \int_{-1}^{-1+h} \frac{(1+t)^\delta}{(h-1+t)^\mu}\, dt \\ &= h^{1-\mu+\delta} \int_0^1 (1-y)^{-\mu} y^\delta \, dy. \end{aligned}$$

(4.9)

$$\begin{aligned} J_2 &\le h^\delta \int_{-1+h}^{x_1} \left[\frac{1}{(x_1-t)^\mu} - \frac{1}{(x_2-t)^\mu} \right] dt \\ &= h^\delta \int_0^{x_1-h+1} \left[\frac{1}{y^\mu} - \frac{1}{(y+h)^\mu} \right] dy \\ &\le \frac{h^\delta}{1-\mu} |(x_1+1-h)^{1-\mu} - (x_1+1)^{1-\mu}| \\ &\le \frac{h^{1-\mu+\delta}}{1-\mu}. \end{aligned}$$

(4.10)

$$\begin{aligned} J_3 &\le (1+x_1)^\delta \left[\int_{x_1}^{x_1+h} \frac{dt}{(t-x_1)^\mu} + \int_{x_1}^{x_1+h} \frac{dt}{(x_1+h-t)^\mu} \right] \\ &= \frac{2(1+x_1)^\delta h^{1-\mu}}{1-\mu} \le \frac{2}{1-\mu} h^{1-\mu+\delta}. \end{aligned}$$

Therefore, substituting (4.4)–(4.6) and (4.8)–(4.10) in (4.7), also in case ii) we obtain

$$|\varphi(x_2) - \varphi(x_1)| \le C h^{1-\mu+\delta},$$

where C, as before, is a positive constant only depending on μ, γ, δ.

For all $x_1 \ne x_2 \in [-1, 0]$ the latter leads to

$$\left| \frac{\varphi(x_2) - \varphi(x_1)}{|x_2-x_1|^{1-\mu+\delta}} \right| \le \begin{cases} 8||\varphi||_\infty & \text{for } |x_2-x_1| \ge 1/4, \\ C(\gamma,\delta,\mu) & \text{for } |x_2-x_1| < 1/4, \end{cases}$$

and the lemma is completely proved. □

Lemma 4.2. *The operator* $\hat{D}$ *defined by* (1.7) *maps the space* $\mathrm{Lip}_\sigma[-1,1]$, $0<\sigma\le 1$ *into the space* $C_{w^{(\rho,\tau)}}$ *and is bounded.*

Proof. In order to prove the assertion, it is sufficient to prove that

$$\|\hat{D}g\|_{w^{(\rho,\tau)}} \le C\|g\|_\sigma \quad \forall\, g \in \mathrm{Lip}_\sigma[-1,1].$$

Firstly we consider the case $\chi = 1$, that means $-\alpha, \alpha+1 > 0$ and $C_{w^{(\rho,\tau)}} = C^0$. Then we obtain

$$\begin{aligned}|\hat{D}g(x)| &= \bigg| a(1-x)^{-\alpha}(1+x)^{1+\alpha}g(x) \\ &\qquad - \frac{b}{\pi}\int_{-1}^{1}\frac{(1-t)^{-\alpha}(1+t)^{1+\alpha}g(t)}{t-x}\,dt\bigg| \\ &\le 2a|g(x)| + \frac{b}{\pi}|g(x)|\bigg|\int_{-1}^{1}\frac{(1-t)^{-\alpha}(1+t)^{1+\alpha}}{t-x}dt\bigg| \\ &\quad + \frac{b}{\pi}\int_{-1}^{1}\frac{(1-t)^{-\alpha}(1+t)^{1+\alpha}|g(t)-g(x)|}{|t-x|}\,dt.\end{aligned}$$

By recalling that for every weight function $w^{(\gamma,\delta)}(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma,\delta > -1$, we have (see [**5**, Theorem 2.1])

$$\left|\int_{-1}^{1}\frac{w^{(\gamma,\delta)}(t)}{t-x}\,dt\right| \le C w^{(\min\{0,\gamma\},\min\{0,\delta\})}(x), \tag{4.11}$$

and setting $M_g(\sigma) = \sup_{x\ne t}|g(t)-g(x)|/|t-x|^\sigma$, we obtain

$$\begin{aligned}|\hat{D}g(x)| &\le C|g(x)| + \frac{2b}{\pi}M_g(\sigma)\int_{-1}^{1}\frac{dt}{|t-x|^{1-\sigma}} \\ &\le C[\|g\|_\infty + M_g(\sigma)] \\ &= C\|g\|_\sigma.\end{aligned}$$

Consider now the case $\chi = 0$. Without loss of generality, we can assume $\alpha > 0$; the case $\alpha < 0$ can be treated similarly. Then in this case we have $C_{w^{(\rho,\tau)}} = C_{w^{(\alpha,0)}}$ and

$$\begin{aligned}\hat{D}g(x) &= a(1-x)^{-\alpha}(1+x)^{\alpha}g(x) \\ &\qquad - \frac{b}{\pi}\int_{-1}^{1}\frac{g(t)(1-t)^{-\alpha}(1+t)^{\alpha}}{t-x}\,dt.\end{aligned}$$

Thus, using relation (4.11) and recalling that $g \in \mathrm{Lip}_{\sigma}[-1,1]$, this leads to

$$\begin{aligned}
|(1-x)^{\alpha}\hat{D}g(x)| &\le 2^{\alpha}a|g(x)| + \frac{b(1-x)^{\alpha}}{\pi} \\
&\quad\times \left| \int_{-1}^{1} \frac{g(t)(1-t)^{-\alpha}(1+t)^{\alpha}}{t-x}\,dt \right| \\
&\le 2^{\alpha}a|g(x)| + \frac{b(1-x)^{\alpha}|g(x)|}{\pi} \\
&\quad\times \left| \int_{-1}^{1} \frac{(1-t)^{-\alpha}(1+t)^{\alpha}}{t-x} dt \right| + \frac{b(1-x)^{\alpha}}{\pi} \\
&\quad\times \int_{-1}^{1} \frac{|g(t)-g(x)|(1-t)^{-\alpha}(1+t)^{\alpha}}{|t-x|}\,dt \\
&\le C|g(x)| + \frac{b(1-x)^{\alpha}M_g(\sigma)}{\pi} \\
&\quad\times \int_{-1}^{1} \frac{(1-t)^{-\alpha}(1+t)^{\alpha}}{|t-x|^{1-\sigma}}\,dt \\
&:= C|g(x)| + \frac{b(1-x)^{\alpha}M_g(\sigma)}{\pi}\mathcal{I}.
\end{aligned}$$

Now, we write

$$\mathcal{I} = \left\{ \int_{-1}^{x} + \int_{x}^{1} \right\} \frac{(1-t)^{-\alpha}(1+t)^{\alpha}}{|t-x|^{1-\sigma}}\,dt := \mathcal{I}_1 + \mathcal{I}_2.$$

Let us start by considering the case $0 \le x \le 1$. In this case, it follows that $1-t > 1-x$. Therefore

$$\mathcal{I}_1 \le (1-x)^{-\alpha} \int_{-1}^{x} \frac{(1+t)^{\alpha}}{|t-x|^{1-\sigma}}\,dt \le C(1-x)^{-\alpha}.$$

On the other hand, using the linear transformation $1-t = (1-x)y$, we get

$$\begin{aligned}
\mathcal{I}_2 &\le C \int_{x}^{1} \frac{(1-t)^{-\alpha}}{(t-x)^{1-\sigma}}\,dt \\
&= C(1-x)^{-\alpha+\sigma} \int_{0}^{1} \frac{y^{-\alpha}}{(1-y)^{1-\sigma}}\,dy \\
&\le C(1-x)^{-\alpha}.
\end{aligned}$$

Thus,

$$\mathcal{I} \leq C(1-x)^{-\alpha}$$

and, for $0 \leq x \leq 1$, it holds

$$|(1-x)^{\alpha}\hat{D}g(x)| \leq C[\|g\|_{\infty} + M_g(\sigma)] = C\|g\|_{\sigma}. \tag{4.12}$$

Now consider the case $-1 \leq x \leq 0$. By this condition we deduce $1-t \geq 1$ and

$$\mathcal{I}_1 \leq C\int_{-1}^{x} \frac{dt}{(x-t)^{1-\sigma}} \leq C.$$

Using the same linear transformation as before we obtain

$$\mathcal{I}_2 \leq C\int_{x}^{1} \frac{(1-t)^{-\alpha}}{(t-x)^{1-\sigma}}dt \leq C(1-x)^{-\alpha}.$$

Therefore, relation (4.12) holds also in the case $-1 \leq x \leq 0$, and the lemma is completely proved. □

Lemma 4.3. *Let $w(x) = w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $-1 < \alpha, \beta < 1$, $-(\alpha+\beta) \in \{0,1\}$, and consider the kernel $k(x,t) = H(x,t)/|t-x|^{\mu}$, $H(x,t) \in \mathrm{Lip}_{\nu}([-1,1]^2)$. Then the operator $K : C_{w^{(\rho,\tau)}} \to \mathrm{Lip}_{\sigma}([-1,1])$ is continuous, where $\sigma = \min\{\nu, 1-\mu-\max\{|\alpha|,|\beta|\}\}$ if $0 < \mu < 1-\max\{|\alpha|,|\beta|\}$. Moreover, if the last condition on μ is fulfilled, the linear integral operator $\hat{D}K$ defined by (1.5) and (1.7) maps the space $C_{w^{(\rho,\tau)}}[-1,1]$ into itself and is compact.*

Proof. Let us recall that $-(\alpha+\beta) \in \{0,1\}$, i.e., $\beta = -1-\alpha$, $-1 < \alpha, \beta < 0$ for $\chi = 1$ and $\beta = -\alpha$ for $\chi = 0$. Set $\varepsilon = \min\{0, \alpha\}$, $\xi = \min\{0, \beta\}$. Then, summing and subtracting $H(y,t)/|t-x|^{\mu}$ leads to

$$\begin{aligned}|Kg(x) - Kg(y)| &\\ &\leq C||g||_{w^{(\rho,\tau)}}\bigg\{\int_{-1}^{1}\bigg[\frac{|H(x,t)-H(y,t)|}{|t-x|^{\mu}} \\ &\qquad + |H(y,t)|\bigg|\frac{1}{|t-x|^{\mu}} - \frac{1}{|t-y|^{\mu}}\bigg|\bigg]w^{(\varepsilon,\xi)}(t)\,dt\bigg\}.\end{aligned}$$

Now, recalling that $H(x,t) \in \mathrm{Lip}_\nu([-1,1]^2)$ and taking Lemma 4.1 into account, we have
(4.13)
$$|Kg(x)-Kg(y)| \le C||g||_{w^{(\rho,\tau)}}\{|x-y|^\nu + |x-y|^{1-\mu-\max\{0,-\varepsilon,-\xi\}}\}$$
$$\le C||g||_{w^{(\rho,\tau)}}|x-y|^\sigma,$$

where $\sigma = \min\{\nu, 1-\mu-\max\{-\varepsilon,-\xi\}\}$, if $0<\mu<1-\max\{-\varepsilon,-\xi\}$. Recalling that $|Kg(x)| \le C||g||_{w^{\rho,\tau}}$, hence the operator K maps the space $C_{w^{(\rho,\tau)}}$ into the space Lip_σ, and it is bounded. Since the space Lip_σ is compactly embedded in $\mathrm{Lip}_{\tilde\sigma}$ for $0<\tilde\sigma<\sigma$, the operator $K: C_{w^{(\rho,\tau)}} \to \mathrm{Lip}_{\tilde\sigma}$ is compact. Moreover, by Lemma 4.2, we deduce that the operator $\hat D$ maps the space $\mathrm{Lip}_{\tilde\sigma}$ into $C_{w^{(\rho,\tau)}}$ and is bounded.

Thus, the operator $\hat D K$ maps the space $C_{w^{(\rho,\tau)}}$ into itself and is compact as the product of a bounded operator and a compact operator. The lemma is completely proved. □

Let $L_m(f)$ be the Lagrange interpolating polynomial of a bounded function f with respect to the knots $x_{m,k}$ of a given matrix $T = \{x_{m,k}, k=1,\dots,m, m=1,2\dots\}$ i.e.

$$L_m(f;x) = \sum_{k=1}^{m} l_{m,k}(x) f(x_{m,k}),$$

where
$$l_{m,k}(x) = \prod_{\substack{i=1\\ i\neq k}}^{m} \frac{x-x_{m,i}}{x_{m,k}-x_{m,i}}, \qquad k=1,\dots,m.$$

Let us introduce the remainder term of Lagrange interpolation

$$R_m(f;x) = f(x) - L_m(f;x). \tag{4.14}$$

The following result can be found in [**2**].

Lemma 4.4. *Let $T = \{x_{m,k}, k=1,\dots,m, m=1,2\dots\}$ be a given matrix of knots. Then for every function $f \in C^{p+\sigma}([-1,1]), p\ge 0, 0<\sigma\le 1$, we have*

$$\|\hat D(R_m(f))\|_{w^{(\rho,\tau)}} \le C\ \|f^{(p)}\|_\sigma \frac{\log m}{m^{p+\sigma}} \|L_m(T)\|, \tag{4.15}$$

where $\hat{D}$ is the operator defined by (1.7), *ρ and τ are defined by* (1.11), *$\|L_m(T)\|$ is defined by* (2.4) *and $R_m(f)$ denotes the Lagrange interpolation error* (4.14).

Proof of Theorem 3.1. Start with the observation that Dv_n, according to (2.5), is a polynomial of degree at most n. Thus, since degree$(L_m(f)) = m-1 = n$, the discrete system (2.1) is equivalent to the operator equation

$$Dv_n + K_m v_n = f_m, \qquad m = n+1, \tag{4.16}$$

where

$$K_m v(x) := L_m K v(x) := \int_{-1}^{1} w^{(\alpha,\beta)}(t) k_m(x,t) v(t)\, dt,$$
$$k_m(x,t) := \sum_{k=1}^{m} l_{m,k}(x) k(x_{m,k}, t),$$
$$f_m(x) := L_m f(x) := \sum_{k=1}^{m} l_{m,k}(x) f(x_{m,k}).$$

Applying $\hat{D}$ to (4.16) and remembering (1.8) and (1.9), we write

$$(I + \hat{D}K_m) v_n = \hat{D} f_m. \tag{4.17}$$

Now we can show that the inverse operators $(I + \hat{D}K_m)^{-1}$ exist and are uniformly bounded in $C_{w^{(\rho,\tau)}}[-1,1]$. Indeed, we note that we can write

$$I + \hat{D}K_m = I + \hat{D}K + \hat{D}(K_m - K).$$

Lemma 4.3 and the assumptions of Theorem 3.1 show that the operator $I + \hat{D}K$ has a bounded inverse in the space $C_{w^{(\rho,\tau)}}[-1,1]$. Thus, it remains to prove that the operator $\hat{D}(K_m - K)$, acting from $C_{w^{(\rho,\tau)}}[-1,1]$ into itself, satisfies the condition

$$\|\hat{D}(K_m - K)\|_{w^{(\rho,\tau)}} = o(1), \qquad m \to \infty.$$

By (4.15) we have, for σ sufficiently small,

$$\begin{aligned} \|(\hat{D}(K_m - K))v\|_{w^{(\rho,\tau)}} &= \|\hat{D}(R_m(Kv))\|_{w^{(\rho,\tau)}} \\ &\le C\ \|Kv\|_\sigma \frac{\log m}{m^\sigma} \|L_m(T)\| \\ &\le C\ \|v\|_{w^{(\rho,\tau)}} \frac{\log m\ \|L_m(T)\|}{m^\sigma} \\ &\le C\ \|v\|_{w^{(\rho,\tau)}} \frac{\log^2 m}{m^\sigma}, \quad m \to \infty. \end{aligned} \tag{4.18}$$

Hence, (see, for example, [**1**, Theorem 5]) the operator $(I + \hat{D}K_m)^{-1}$ exists and is uniformly bounded. More precisely,

$$||(I + \hat{D}K_m)^{-1}||_{w^{(\rho,\tau)}} \le \frac{||(I+\hat{D}K)^{-1}||_{w^{(\rho,\tau)}}}{1 - ||(I+\hat{D}K)^{-1}||_{w^{(\rho,\tau)}} ||\hat{D}(K_m - K)||_{w^{(\rho,\tau)}}}. \tag{4.19}$$

As a consequence of (4.19) we conclude that (4.17) has a unique solution v_n and

$$||v_n||_{w^{(\rho,\tau)}} \le ||(I + \hat{D}K_m)^{-1}||_{w^{(\rho,\tau)}} ||\hat{D}f_m||_{w^{(\rho,\tau)}} \le C,$$

i.e., the stability of the method. Subtracting (4.17) from (1.10), we find that

$$(I + \hat{D}K)(v - v_n) = \hat{D}(f - f_m) + \hat{D}(K_m - K)v_n.$$

Hence

$$||v - v_n||_{w^{(\rho,\tau)}} \le C[||\widehat{D}(f - f_m)||_{w^{(\rho,\tau)}} + ||\hat{D}(K_m - K)v_n||_{w^{(\rho,\tau)}}].$$

Finally, estimate (3.1) follows again by (4.15) and (4.18) for $\sigma = \min\{\theta, \nu, 1 - \mu - \max\{-\varepsilon, -\xi\}\}$ (cf. Lemma 4.3). Hence the theorem is completely proved. □

Lemma 4.5. *Let K be the operator defined by* (1.5), *where $H(x,t) \in \mathrm{Lip}_\nu([-1,1]^2)$ and K_N is the operator defined by* (2.8) *with the matrix of the knots $\overline{T}$ given by* (2.9). *Then for every $v_n \in \mathbf{P}_{n+\chi}$, we have*

$$||(K - K_N)v_n|| \le \frac{C}{n^\nu} ||v_n||_{w^{(\rho,\tau)}}, \tag{4.20}$$

where $N \geq 2n + \chi + 1$.

Proof. Firstly, we consider the case $\chi = 1$. We have to prove that

$$||(K - K_N)v_n||_\infty \leq \frac{C}{n^\nu}||v_n||_\infty.$$

Now, we recall (see, for example, [**4**]) that for any function $H(x,t) \in \mathrm{Lip}_\nu([-1,1]^2)$ and for any positive integer M, there exists an algebraic polynomial $Q_M(x,t)$ of degree M in x and t separately, such that

$$|H(x,t) - Q_M(x,t)| \leq \frac{C}{M^\nu}, \qquad -1 \leq x, t \leq 1.$$

Thus, if we fix $M = N - n - \chi - 1$, then $Q_M(x,t)v_n(t) \in \mathbf{P}_{N-1}$ with respect to the variable t and so, since the degree of exactness of the chosen quadrature formula is at least $N-1$, we have

(4.21)

$$\begin{aligned}
|(K - K_N)v_n(x)| &\leq \int_{-1}^{1} \frac{|H(x,t) - Q_M(x,t)|}{|t-x|^\mu}|v_n(t)|w^{(\alpha,\beta)}(t)\,dt \\
&\quad + \left|\sum_{i=1}^{N} w_{N,i}(x)[H(x,t_{N,i}) - Q_M(x,t_{N,i})]v_n(t_{N,i})\right| \\
&\leq ||H - Q_M||_\infty ||v_n||_\infty \int_{-1}^{1} \frac{w^{(\alpha,\beta)}(t)}{|t-x|^\mu}\,dt \\
&\quad + \left|\int_{-1}^{1} L_N[(H(x,.) - Q_M(x,.))v_n](t)\frac{w^{(\alpha,\beta)}(t)}{|t-x|^\mu}\,dt\right| \\
&\leq C\frac{||v_n||_\infty}{M^\nu} \\
&\quad + \left|\int_{-1}^{1} L_N[(H(x,.) - Q_M(x,.))v_n](t)\frac{w^{(\alpha,\beta)}(t)}{|t-x|^\mu}\,dt\right| \\
&:= C\frac{||v_n||_\infty}{M^\nu} + I.
\end{aligned}$$

Since we consider $0 < \mu < 1 - \max\{|\alpha|, |\beta|\}$, it is always possible to choose a $p > 1$ such that $p(\mu + \max\{|\alpha|, |\beta|\}) < 1$. Then, if q is the conjugate of p, i.e., $q = p/p-1$, by applying the Hölder inequality we obtain

$$\begin{aligned} I &\le \left[\int_{-1}^{1} \left| L_N[H(x,.) - Q_M(x,.)v_n](t) \right|^q dt \right]^{1/q} \\ &\quad \times \left[\int_{-1}^{1} \frac{w^{(\alpha,\beta)^p}(t)}{|t-x|^{\mu p}} \, dt \right]^{1/p} \\ &\le C||L_N[(H-Q_M)v_n]||_q. \end{aligned}$$

Now, the necessary and sufficient condition for the uniform boundedness of $L_N : C^0 \to L_q$ is [cf., for example, **19**] $1/\sqrt{w^{(\bar\alpha,\bar\beta)}(x)\sqrt{1-x^2}} \in L_q$, and this condition is surely satisfied under the assumptions made on $\bar\alpha, \bar\beta, \mu$ and p. Consequently, we obtain

$$I \le C||H - Q_M||_\infty ||v_n||_\infty \le C\frac{||v_n||_\infty}{M^\nu} \tag{4.22}$$

and inequality (4.20) follows for $\chi = 1$ by relations (4.21) and (4.22), recalling that $M = N - n - \chi - 1$ with $N \ge 2n + \chi + 1$. The case $\chi = 0$ can be treated in a similar way, and Lemma 4.5 is completely proved. □

Proof of Theorem 3.2. Recalling the definition of K_N in (2.8), we define

$$\overline{K}_m = L_m K_N.$$

So, looking at the proof of Theorem 3.1, we observe that the crucial point to prove the stability and the convergence of the discrete collocation method is, like in the continuous case, the evaluation of the quantity

$$||\hat{D}(K - \overline{K}_m)||_{w^{(\rho,\tau)}}.$$

First of all, we can write

$$\begin{aligned} ||\hat{D}(K-\overline{K}_m)||_{w^{(\rho,\tau)}} &\le ||\hat{D}(K-K_m)||_{w^{(\rho,\tau)}} + ||\hat{D}(K_m-\overline{K}_m)||_{w^{(\rho,\tau)}} \\ &= ||\hat{D}(K-K_m)||_{w^{(\rho,\tau)}} + ||\hat{D}L_m(K-K_N)||_{w^{(\rho,\tau)}}. \end{aligned}$$

Hence, since the first term at the righthand side has been investigated in the proof of Theorem 3.1, it remains to estimate the quantity

$$||\hat{D}L_m(K-K_N)||_{w^{(\rho,\tau)}} = \sup_{v_n \in \mathbf{P}_{n+\chi}} \frac{||\hat{D}L_m(K-K_N)v_n||_{w^{(\rho,\tau)}}}{||v_n||_{w^{(\rho,\tau)}}}.$$

Now, by Corollary 2.7 in [**3**], we obtain

$$\begin{aligned} ||\hat{D}L_m(K-K_N)||_{w^{(\rho,\tau)}} &\le C\log m||L_m(K-K_N)|| \\ &\le C\log m||L_m(T)||\ ||K-K_N|| \\ &\le C\log^2 m||K-K_N||. \end{aligned}$$

Therefore, applying Lemma 4.5, we have

$$||\hat{D}L_m(K-K_N)v_n||_{w^{(\rho,\tau)}} \le C\frac{\log^2 n}{n^\nu}||v_n||_{w^{(\rho,\tau)}} \tag{4.23}$$

and, consequently, by (4.18) and (4.23),

$$||\hat{D}(K-\overline{K}_m)v_n||_{w^{(\rho,\tau)}} \le C\frac{\log^2 n}{n^r}||v_n||_{w^{(\rho,\tau)}}, \quad \forall\, v_n \in \mathbf{P}_{n+\chi} \tag{4.24}$$

where $r=\min\{\nu,\theta,1-\mu-\max\{|\alpha|,|\beta|\}\}$. Now, with the help of (4.24) we can show that the operators $I+\hat{D}\overline{K}_m$ are invertible in $\mathbf{P}_{n+\chi}$ and that

$$||(I+\hat{D}\overline{K}_m)v_n||_{w^{(\rho,\tau)}} \ge C_1||v_n||_{w^{(\rho,\tau)}} \tag{4.25}$$

for all $v_n \in \mathbf{P}_{n+\chi}$ and for all sufficiently large n, where C_1 is a positive constant, i.e. the stability of the method.

Moreover, by inequalities (4.15), (4.24) and (4.25), we obtain

$$\begin{aligned} ||v-v_n||_{w^{(\rho,\tau)}} &\le C\big[||\widehat{D}(f-f_m)||_{w^{(\rho,\tau)}} \\ &\qquad + ||\hat{D}(\overline{K}_m-K)v_n||_{w^{(\rho,\tau)}}\big] \\ &\le C\left(\frac{\log^2 n}{n^\theta}+\frac{\log^2 n}{n^r}||v_n||_{w^{(\rho,\tau)}}\right) \\ &\le C\frac{\log^2 n}{n^r}\left(1+\frac{1}{C_1}||\widehat{D}f_m||_{w^{(\rho,\tau)}}\right) \\ &\le C\frac{\log^2 n}{n^r}. \end{aligned}$$

Thus, Theorem 3.2 is completely proved. □

Finally, using a diatic decomposition and the Remez inequality, we can prove the result of Theorem 3.3 (see [**2**, Theorem 3.2] for more details).

5. Numerical examples. In this section we apply the method described in Section 2 to some test equations of the type (1.6). In all of the examples the chosen collocation matrix T is

$$T = \{x_{m,j}(\bar{w}),\ j = 1, \dots , m,\ \bar{w}(x) = (1 - x)^{-.6}(1 + x)^{-.8}\}.$$

Moreover, $\overline{T}$ is the matrix of the Chebyshev knots of the first kind.

In all of the treated examples we don't know the analytical solution of (1.6). The tables show the approximate solution $v_n(x)$ for increasing values of $n \in \mathbf{N}$, evaluated on the equispaced points $x_j = j/5$, $j = 0, \dots , 4$.

Example 1.

$$\frac{1}{\pi}\int_{-1}^{1} \frac{v(t)}{\sqrt{1-t^2}}\frac{1}{t-x}\,dt + \int_{-1}^{1} \frac{v(t)}{\sqrt{1-t^2}}\frac{|x+t|}{|t-x|^{0.4}}\,dt = |x|.$$

Here we have the index $\chi = 1$, $\alpha = \beta = -1/2$, $f(x) \in \mathrm{Lip}_1[-1, 1]$, $H(x, t) \in \mathrm{Lip}_1([-1, 1]^2)$ and $\mu = 0.4$. Therefore, Theorem 3.1 shows a theoretical uniform convergence rate $O(\log^2 n/n^{0.1})$.

TABLE 1.

n	.0	.2	.4	.6	.8
8	.222416	.125122	.100299	.182530	.306456
16	.217407	7.78D-02	.108325	.175603	.295979
32	.215288	8.03D-02	.102773	.175817	.295163
64	.214315	7.83D-02	.101328	.174817	.295044
128	.213851	7.78D-02	.100957	.174743	.295017
256	.213625	7.77D-02	.100888	.174680	.294992

Example 2.

$$
\begin{aligned}
&-\frac{\sqrt{2}}{2}v(x)(1-x)^{-1/4}(1+x)^{-3/4}\\
&-\frac{\sqrt{2}}{2\pi}\int_{-1}^{1}\frac{v(t)}{t-x}(1-t)^{-1/4}(1+t)^{-3/4}\,dt\\
&+\int_{-1}^{1}\frac{v(t)\sqrt{|t+x|}}{|t-x|^{0.1}}(1-t)^{-1/4}(1+t)^{-3/4}\,dt=|x|.
\end{aligned}
$$

Here we have the index $\chi = 1$, $\alpha = -1/4$, $\beta = -3/4$, $f(x) \in \mathrm{Lip}_1[-1,1]$, $H(x,t) \in \mathrm{Lip}_{1/2}([-1,1]^2)$ and $\mu = 0.1$. Therefore, Theorem 3.1 shows a theoretical uniform convergence rate $O((\log^2 n)/n^{3/20})$.

TABLE 2.

n	.0	.2	.4	.6	.8
8	.156210	.129685	-8.92D-02	-.480442	-.959195
16	.187110	.154458	-.108509	-.464751	-.945530
32	.205937	.144728	-.103742	-.464301	-.945431
64	.216437	.146925	-.102718	-.464098	-.944134
128	.221979	.147423	-.102532	-.464102	-.944034
256	.224820	.147540	-.102531	-.464048	-.943899

Example 3. We have also tested the proposed method choosing the known functions $H(x,t)$ and $f(x)$ more than Lipschitz continuous. In such cases we observed that the rate of convergence increases a little bit in comparison with the previous examples.

However, if we choose $H(x,t) = H(x-t) \in C^{p+\nu}([-1,1]^2)$ with $p \geq 1$ and such that $H^{(i)}(0) = 0$, $i = 0,\dots,p-1$ and $f(x) \in C^{p+\nu}([-1,1])$, then, obviously, the rate of convergence of the proposed method grows up as p increases.

For example, if we consider the following integral equation

$$
\frac{1}{\pi}\int_{-1}^{1}\frac{v(t)}{\sqrt{1-t^2}}\frac{1}{t-x}\,dt+\int_{-1}^{1}\frac{v(t)}{\sqrt{1-t^2}}\frac{\sin^5(x-t)}{|t-x|^{0.4}}\,dt=\sin^5(x),
$$

we obtain the results given in Table 3.

TABLE 3.

n	.2	.6
8	-1.D-02	-8.D-02
16	-1.797641D-02	-8.063693D-02
32	-1.79764112139D-02	-8.06369367789D-02
64	-1.797641121391844D-02	-8.06369367789800D-02
128	-1.7976411213918447D-02	-8.0636936778980053D-02

Conclusion. At first we note that, if k is defined as in (1.2), the representation

$$k(x,t) = \frac{h(x,t) - h(x,x)}{t-x} \tag{6.1}$$

holds, where $h(x,t) = (t-x)|t-x|^{-\mu}H(x,t) \in \mathrm{Lip}_{\lambda}([-1,1]^2)$ with $\lambda = \min(1-\mu,\nu)$ (see, e.g., [**18**]). Moreover, we recall that in [**15**], the authors consider a quadrature method for the integral equation (1.1) and obtain the following result.

Theorem 6.1. *Assume* $\chi = 0$ *and* $|\alpha| \le 1/2$ *or* $\chi = 1$ *and* $-1 < \alpha < 0$, $h \in C^{p+\lambda}([-1,1]^2)$ *(recall* (6.1)*), and* $f \in C^{p+\lambda}([-1,1])$, *where* $p \ge 0$ *and* $0 < \lambda \le 1$. *If the problem* (1.1) *has a unique solution* $u = wv$, $v \in L^2_w$ *for* $\chi = 0$ *or* $v \in L^2_{w^{(\alpha,\beta)},0}$ *for* $\chi = 1$, *then the system of equations, relative to the quadrature method, is uniquely solvable for all sufficiently large* m *and*

$$\|v - v_m\|_{L^2_w} = O\left(\frac{\log m}{m^{p+\lambda}}\right),$$

where

$$v_m(x) = \sum_{k=1}^{m} \frac{p_m(w;x)}{(x-x_k)p_m(w;x_k)}\xi_k$$

is the Lagrange interpolating polynomial corresponding to the solution of the system of equations relative to the quadrature method.

Moreover, if $p+\lambda > \gamma := \max\{\alpha,\beta\}+1$ *then* $v \in C[-1,1]$ *and*

$$\max_{-1\le x\le 1} |v(x)-v_m(x)| = O\bigg(\frac{\log m}{m^{p+\lambda-\gamma}}\bigg). \tag{6.2}$$

Further, if $p+\lambda > 1/2$*, then* v *is continuous on any closed subset* $\Delta \subset (-1,1)$ *and*

$$\max_{\Delta} |v(x)-v_m(x)| = O\bigg(\frac{\log m}{m^{p+\lambda-1/2}}\bigg).$$

If we look at estimate (6.2) we recognize that for $p=0$ and $\chi=0$, this relation cannot give uniform convergence for the quadrature method proposed in [**15**].

On the contrary, estimate (3.3) for the collocation methods gives us uniform convergence for $p=0$ if $r-2|\alpha|>0$.

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