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CAUCHY PROBLEMS ASSOCIATED WITH CERTAIN INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We develop analytic solutions of a class of linear time evolution type problems for a class of equations in which an integral operator acts on certain of the underlying space variables and a partial differential operator acts on the remaining ones. Two approaches are employed for the constructions: the method of transmutations and the method of quasi inner products. The first of these involves the use of an integral transform while the second makes use of function theoretic arguments. Applications are made to a variety of well-posed and ill-posed generalizations of the wave equation. Clean cut and diffuse effects show up in a number of these "generalized" wave formulas.

1. Introduction. In this paper we employ transmutations and the quasi inner product to develop solution representations and their properties for a class of initial value problems having the forms

$$\begin{aligned} w_{tt}(x,y,t) &= \{I_x + P(D_y)\}w(x,y,t) \\ w(x,y,0) &= 0, \qquad w_t(x,y,0) = \phi(x,y). \end{aligned} \tag{1.1}$$

In this, I_x denotes a linear integral operator acting on the x component of the function w while $P(D_y)$ denotes a partial differential operator acting on the y component of w. A specific example of a type of equation included in (1.1) is

(1.2)
$$w_{tt}(x,y,t) = (\Gamma(p))^{-1}\int_0^x (x-\sigma)^{p-1} w(\sigma,y,t)\,d\sigma + \partial^2 w(x,y,t)/\partial y^2, \quad p > 0.$$

The types of equations considered here are not typical of the usual integro-partial differential equations associated with evolution problems in which the integration involved in the integral operator is carried

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out on the time variable. Rather, we can regard the equation in (1.1) as occurring in some process in which there is smoothing on some of the non-time variables and differentiation on the others. One can also think of obtaining such an equation by perturbing a partial differential equation by adding in an integral term. The solution of the problem in (1.1) corresponding to the conditions

$$w^*(x,y,0) = \phi(x,y), \qquad w_t^*(x,y,0) = 0$$

is given by the relation $w^*(x,y,t) = \partial w(x,y,t)/\partial t$.

The method of transmutations was developed to connect the solutions of pairs of initial or boundary value problems in partial differential equations by means of various integral transforms. By relating the solution of a "higher level" problem to the solution of a simpler and solvable problem, the connecting integral transform then maps the solution of the simpler problem into the solution of the higher level one (see [**1, 4**–**7**]). On the other hand, the method of quasi inner products (qips) is a function theoretic approach whose primary aim is to reduce the solving of initial value problems in partial differential equations to the carrying out of real or complex translations followed by appropriate integrations [**2, 3**]. When reductions of this sort are not feasible, the attendant formulas provide other means for obtaining solutions. These alternative approaches lead to different forms for solution representations, and they emphasize different aspects about a problem. Moreover, some problems can be treated conveniently by one of the methods but not by the other. As an instance of one of the differences, one can conveniently treat the problem (1.1) with the operator I_x replaced by I_x^2 when employing the method of quasi inner products.

Underlying both of these approaches is the interpretation of the exponential of an operator A acting on the appropriate data. This reduces to the problem of assigning a solution to the associated "heat problem"

$$u_t = Au, \quad t > 0; \qquad u(0) = \phi \tag{1.3}$$

in a Banach or other function space X. Symbolically, we write $u(t) = e^{tA}\phi$. If A is the generator of a continuous group $G_A(t)$ in X, then this solution can be expressed as $G_A(t)\phi$ where ϕ is taken to lie in some dense subspace of X. Similarly, if A generates a semigroup of operators $T_A(t)$ in X, then we write $u(t) = T_A(t)\phi$. In the cases in which $A = D_x = \partial/\partial x$ or $A = D_x^2$, the solution $e^{tA}\phi(x)$ defines, respectively, a translation of $\phi(x)(e^{tD_x}\phi(x) = \phi(x+t))$ if $\phi(x) \in C^1$ or a solution $H(x,t)$ of the classical heat problem $H_t(x,t) = H_{xx}(x,t)$, $t > 0$; $H(x,0) = \phi(x)$. If t is negative or complex in these last two cases, it is possible to assign a meaning to $e^{tD_x^2}\phi(x)$ if $\phi(x)$ is entire in x. To treat problems of the form (1.1), to be described in greater detail below, we will need to develop analytic formulas for the groups e^{tl_x} and the "semigroups" $e^{tl_x^2}$ for various choices of the integral operator I_x.

To make the meaning of (1.1) more precise, let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$, and let $D_y = (D_1, \dots, D_m)$ where $D_j\phi(x,y) = \partial\phi(x,y)/\partial y_j$, $j = 1, 2, \dots, m$. Next, let I_x be defined by an integral relation of the form $I_x\phi(x) = \int_{E_x} K(x,\sigma)\phi(\sigma)\,d\sigma$ in which $\sigma = (\sigma_1, \dots, \sigma_n)$, $d\sigma = d\sigma_1 \delta\sigma_2 \cdots d\sigma_n$ and in which $K(x,\sigma)$ is a known kernel function depending on the vectors x and σ. The region of integration E_x may depend upon the vector x as would be the case for convolution or Volterra type integrals. Finally, we let $P(D_y)$ denote a linear partial differential operator in the D_j, $j = 1, 2, \dots, m$ case. We evidently have the commutativity relation $I_x P(D_y)\phi(x,y) = P(D_y) I_x \phi(x,y)$ for suitable $\phi(x,y)$ and the problem (1.1) is well defined. In the examples we give that make use of transmutations, we take $P(D_y)$ to be some variation of the Laplacian operator $\Delta_m = \sum_{j=1}^m D_j^2$. In a number of these cases, we take $m \le 4$ to cut down on the technical details. For the method of qips, we also make restrictions on m depending upon the operator $P(D_y)$ appearing in the equation. As for the choices for the operator I_x, we consider the above mentioned convolution and Volterra integral operators along with Fredholm operators.

Section 2, which focuses on the construction of integral representations for the groups and semigroups generated by the various integral operators I_x, lies at the heart of the development. For one particular class of these operators which depend upon a real parameter p (such as the one appearing in the equation in (1.2)), these integral formulas have kernels that reduce to hypergeometric functions for special values of p. Some specific ones of these will be noted. With this background, we then employ transmutations in Section 3 and the method of quasi inner products in Section 4 to solve a variety of cases of problem (1.1). In the examples considered in Section 3, we shall see how the "clean cut and diffused" effects for the classical wave equations transform into corresponding properties for solutions of closely associated integro-partial differential equations. For the convenience of the reader, the formulas essential to these two methods will be summarized in those sections. It will be seen that the method of transmutations offers considerably more versatility in treating these problems not only from the standpoint of the number of variables permissible but also from the standpoint of the variety of types of integral operators that one can consider in such equations. On the other hand, the operator factorization possibilities that go with the qip approach often permit reducing a solution operator into a much simpler form.

Let us note that one can replace the left hand member of the equation (1.1) by other combinations of t derivatives (e.g., the Euler-Poisson-Darboux case [**6**], etc.) and the right member by a variety of other types of forms involving I_x and $P(D_y)$. We leave it to the reader to consider other such generalizations by referring to a variety of the types of problems considered for partial differential equations in [**3**].

2. Exponentials of some integral operators. We now develop general integral formulas involving kernels for the exponential operators e^{tI_x} and $e^{tI_x^2}$ where I_x is an integral operator of the type mentioned above. This involves terminology and notation that is familiar for standard integral equations. For a particular class of these integral operators which includes the convolution operator in (1.1), the kernels in those integral formulas are often expressible in terms of hypergeometric functions.

In the discussion that follows, we select I_x to be a Fredholm type operator. Let $K(x,\sigma)$ be a bounded and integrable function of the vectors x and σ over a closed bounded set $E \subseteq R^n$ (note: $K(x,\sigma)$ may vanish in various portions of the set E as in the Volterra cases). Let V_E denote the content of this set, and let $\phi(x)$ be continuous. Then we have $I_x\phi(x) = \int_E K(x,\sigma)\phi(\sigma)\,d\sigma$, $I_x^2\phi(x) = \int_E K_1(x,\sigma)\phi(\sigma)\,d\sigma$ where $K_1(x,\sigma) = \int_E K(x,\xi)K(\xi,\sigma)\,d\xi$ and, in general, $I_x^{j+1}\phi(x) = \int_E K_j(x,\sigma)\phi(\sigma)\,d\sigma$ with $K_j(x,\sigma) = \int_E K_{j-1}(x,\xi)K(\xi,\sigma)\,d\sigma$ for $j = 1, 2, \ldots$ and where $K_0(x,\sigma) = K(x,\sigma)$ (these $K_j(x,\sigma)$ functions arise, of course, in the construction of a solution of the Fredholm integral equation $f(x) = \phi(x) + \lambda\int_E K(x,\sigma)f(\sigma)\,d\sigma$ [**12**]). With these and the expansions $\sum_{j=0}^{\infty} t^j I_x^j \phi(x)/j!$ for e^{tI_x} and $\sum_{j=0}^{\infty} t^j I_x^{2j} \phi(x)/j!$ for $e^{tI_x^2}$, we can write

$$
\begin{aligned}
e^{tI_x}\phi(x) &= \phi(x) + \int_E G(x,\sigma,t)\phi(\sigma)\,d\sigma \\
e^{tI_x^2}\phi(x) &= \phi(x) + \int_E S(x,\sigma,t)\phi(\sigma)\,d\sigma
\end{aligned} \tag{2.1}
$$

where

$$
\begin{aligned}
G(x,\sigma,t) &= \sum_{j=1}^{\infty} t^j K_{j-1}(x,\sigma,t)/j! \quad \text{and} \\
S(x,\sigma,t) &= \sum_{j=1}^{\infty} t^j K_{2j-1}(x,\sigma,t)/j!.
\end{aligned} \tag{2.2}
$$

While these functions appear to be well defined, it is nonetheless useful to check out the uniform convergence of these series and determine their regions of validity. This will be of importance in validating term-by-term inversions of Laplace transforms in Section 3 below.

Let $T > 0$, and restrict t so that $|t| \le T$. Next, let M and N be positive constants such that $|K(x,\sigma)| \le M$ and $|\phi(x)| \le N$ in E. It is an easy task to show inductively that $|K_j(x,\sigma)| \le M^{j+1}V_E^j$ and $|\int_E K_{j-1}(x,\sigma)\phi(\sigma)\,d\sigma| \le NM^jV_E^j$. From these, it follows that

$$
\begin{aligned}
|e^{tI_x}\phi(x)| &= \left|\phi(x) + \sum_{j=1}^{\infty} t^j I_x^j \phi(x)/j!\right| \\
&\le |\phi(x)| + \sum_{j=1}^{\infty} \frac{|t|^j}{j!} \cdot \left|\int_E K_{j-1}(x,\sigma)\phi(\sigma)\,d\sigma\right| \\
&\le N \sum_{j=0}^{\infty} (MTV_E)^j/j!
\end{aligned}
$$

and, hence, the exponential operator e^{tI_x} is well defined. A similar argument leads to the same conclusion for the definition of $e^{tI_x^2}$ given in (2.1) and (2.2). Thus, we see that there are no restrictions on t when computing $e^{tI_x}\phi(x)$ or $e^{tI_x^2}\phi(x)$.

When $n = 1$ and I_x is a Volterra operator of the form $I_x\phi(x) = \int_a^x K(x,\sigma)\phi(\sigma)\,d\sigma$, the formulas for the $K_j(x,\sigma)$ become $K_j(x,\sigma) = \int_\sigma^x K_{j-1}(x,\xi)K(\xi,\sigma)\,d\sigma$, and the imposition of the same types of bounds as above on $K(x,\sigma)$ and $\phi(x)$ leads to an even stronger convergence of the sums defining $G(x,\sigma,t)$ and $S(x,\sigma,t)$ in (2.2).

It is useful to consider some cases of the integral operator I_x for which the functions $G(x,\sigma,t)$ and $S(x,\sigma,t)$ can be expressed in terms of classical special functions. For x a single variable, one important class of such operators is defined by the relation

$$I_{x,p}\phi(x) = \frac{1}{\Gamma(p)}\int_a^x \nu(\sigma)[V(x)-V(\sigma)]^{p-1}\phi(\sigma)\,d\sigma \tag{2.3}$$

in which $\nu(x) > 0$ and continuous, $V'(x) = \nu(x)$, and $p > 0$. The choices $\nu(x) = 1$ and $a = 0$ in this yield the operator in the equation (1.1). Another special example is

$$I_x\phi(x) = \int_0^x (x-\sigma)^{-\alpha}e^{x-\sigma}\phi(\sigma)\,d\sigma, \qquad 0 \le \alpha < 1. \tag{2.4}$$

Once we have developed the groups and the semigroups generated by the $I_{x,p}$ operator in (2.3), we can consider problems in (1.1) in which the integral operator I_x^2 is replaced by the product of two integral operators $I_{x_1,p}I_{x_2,q}$ to operate on a function of two space variables x_1 and x_2. This can be extended to handle cases with several space variables. After developing the general formulas for $G_p(x,\sigma,t)$ and $S_p(x,\sigma,t)$ for the operator $I_{x,p}$ in (2.3), we then specialize the choices for p to obtain formulas for these kernels in terms of the modified Bessel functions and the hypergeometric functions. We then note the general formula for G in the two variable case but leave it to the reader to specialize this for particular choices of p and q. Finally, we write out the G function for the operator I_x defined by (2.4) but leave the details of the computation to the reader.

From (2.3) and the definition of the $K_j(x,\sigma)$, we have

$$\begin{aligned} K_1(x,\sigma) &= \int_\sigma^x K(x,\xi)K(\xi,\sigma)\,d\xi \\ &= \frac{\nu(\sigma)}{\Gamma^2(p)}\int_\sigma^x \nu(\xi)[V(x)-V(\xi)]^{p-1}[V(\xi)-V(\sigma)]^{p-1}\,d\xi. \end{aligned}$$

With the change of variables $V(\xi) = \lambda V(x) + (1-\lambda)V(\sigma)$ in this last integral, we find that

(2.5)
$$\begin{aligned} K_1(x,\sigma) &= \frac{1}{[\Gamma(p)]^2}\nu(\sigma)[V(x)-V(\sigma)]^{2p-1}\cdot\left\{\int_0^1[\lambda(1-\lambda)]^{p-1}\,d\lambda\right\} \\ &= \frac{1}{\Gamma(2p)}\int_a^x \nu(\sigma)[V(x)-V(\xi)]^{2p-1}\,d\sigma \end{aligned}$$

where we have evaluated the beta integral in the second member of this in terms of gamma functions to obtain the last member. Repeating these same types of calculations, one can show that

$$K_{n-1}(s,\sigma) = \frac{1}{\Gamma(np)}\nu(\sigma)[V(x)-V(\sigma)]^{np-1}. \tag{2.6}$$

From these formulas and (2.2), we deduce, after a shift of summation indices, the general formulas

$$\text{(2.7)}\quad \begin{aligned} &\text{(a)}\quad G_p(x,\sigma,t) = \nu(\sigma)\sum_{j=0}^{\infty}\frac{t^{j+1}}{(j+1)!}\cdot\frac{\omega^{jp+p-1}}{\Gamma(jp+p)} \\ &\text{(b)}\quad S_p(x,\sigma,t) = \nu(\sigma)\sum_{j=0}^{\infty}\frac{t^{j+1}}{(j+1)!}\cdot\frac{\omega^{2jp+2p-1}}{\Gamma(2jp+2p)} \end{aligned}$$

where $\omega = V(x) - V(\sigma)$. Clearly, the second of these shows that

$$S_p(x,\sigma,t) = G_{2p}(x,\sigma,t) \tag{2.8}$$

and we thus need only compute the $G_p(x,\sigma,t)$ kernels.

If p is any positive integer, we can split $\Gamma(pj+p) = (jp+p-1)!$ into the sets of factors $(p-1)!$ and $(p+l)(2p+l)\cdots(jp+l) = p^j\prod_{k=1}^{j}(k+l/p) = p^j(1+l/p)_j$ for $l = 0,1,\ldots,j$. Hence, $\Gamma(pj+p) = (p-1)!p^{pj}j!(1+1/p)_j(1+2/p)_j\cdots(1+(p-1)/p)_j$. Inserting this last expression into (2.7a) and noting that $(j+1)! = (2)_j$, we get

$$\begin{aligned} G_p(x,\sigma,t) = \frac{t\nu(\sigma)}{(p-1)!}\cdot\omega^{p-1}\cdot{}_0F_p(\_;2,1+1/p,1+2/p,\ldots, \\ 1+(p-1)/p;t\omega^p/p^p). \end{aligned} \tag{2.9}$$

When $p = 1$, this can be expressed in terms of a modified Bessel function $I_1(z)$.

Similarly, if $p = 1/2$, we can split up the sum in (2.7) into one sum taken over the even indices and a second sum taken over the odd indices. By doing this and writing the denominators in these sums as a product of powers of 2 by appropriate Pochhammer symbols, we can show that

$$\begin{aligned} G_{1/2}(x,\sigma,t) = \nu(\sigma)[t(\omega\pi)^{-1/2} \cdot {}_0F_2(\_; 1/2, 3/2; t^2\omega/4) + t^2/2 \\ \cdot {}_0F_2(\_; 3/2, 2; t^2\omega/4]. \end{aligned} \tag{2.10}$$

Formulas for $G_{1/p}$ for p a positive integer can similarly be obtained as a sum of terms involving the hypergeometric functions ${}_0F_p$.

Next we consider a product integral operator $I = I_{x_1,p} \cdot I_{x_2,q}$ in which the first integral operator factor is the operator in (2.3) with x replaced by x_1 and in which the second operator factor is given by $I_{x_2,q}\phi(x_2) = \int_b^{x_2} \mu(\sigma_2)[U(x_2) - U(\sigma_2)]^{q-1}\phi(\sigma_2)\,d\sigma_2$ where $q > 0$ and $U'(x_2) = \mu(x_2)$. Then we can show that this product operator I generates a group given analytically by the formula

$$e^{tI}\phi(x_1,x_2) = \phi(x_1,x_2) + \int_a^{x_1}\int_b^{x_2} G(x_1,x_2,\sigma_1,\sigma_2,t)\phi(\sigma_1,\sigma_2)\,d\sigma_1\,d\sigma_2 \tag{2.11}$$

where

$$\begin{aligned} G(x_1,x_2,\sigma_1,\sigma_2,t) &= \nu(\sigma_1)\mu(\sigma_2) \\ &\cdot \sum_{n=1}^{\infty} \frac{t^n}{n!\Gamma(np)\Gamma(nq)}[V(x_1)-V(\sigma_1)]^{np-1} \\ &\cdot [U(x_2) - U(\sigma_2)]^{nq-1}. \end{aligned} \tag{2.12}$$

As for the integral operator I_x defined by (2.4), it is relatively easy to show that

$$G(x,\sigma,t) = (x-\sigma)^{-1}e^{x-\sigma}\sum_{n=1}^{\infty}\frac{\{t\Gamma(1-\alpha)(x-\sigma)^{1-\alpha}\}^n}{\Gamma(n-n\alpha)\cdot n!}. \tag{2.13}$$

This function can also be expressed in terms of hypergeometric functions for various special choices of α.

3. Transmutation methods. In the following, we construct solution functions for a number of related problems of the form (1.1) in which I_x may be any one of the types of integral operators considered in Section 2 and in which $P(D_y)$ involves a Laplacian operator. Of central importance for these considerations is the following result on transmutations:

Let $u(t)$ denote a solution of the abstract heat problem (1.3). Then a solution of the abstract wave problem

$$w_{tt}(t) = Aw(t), \quad t > 0; \qquad w(0) = 0, \qquad w_t(0) = \phi \tag{3.1}$$

is given by the formula

$$w(t) = \Gamma(3/2)\mathcal{L}_s^{-1}\{s^{-3/2}u(1/(4s))\}_{s\to t^2} \tag{3.2}$$

where $\mathcal{L}_s^{-1}\{\cdots\}_{s\to t^2}$ denotes the inverse Laplace transform in which s is the variable of the transform and t^2 is the variable of inversion [**4**].

In order to illustrate the difference between the solution characteristics for variations in the underlying equations, we first consider an equation in which the right hand member involves only an integral operator. This is followed by an equation that involves an integral operator plus the Laplacian operator. Finally, we modify the second example by adding in a negative constant multiple of the function $w(x, y, t)$. It will be seen that each change in the problem leads to a direct modification of the function $G(x, \sigma, t)$ associated with the integral operator appearing in the problem and its related inverse transform. It will be seen that we are basically constructing fundamental solutions for these integro-partial differential equation problems.

(A) *A strict integral operator.* We now consider the following problem involving a multi-dimensional integral operator:

$$w_{tt}(x, t) = \int_{E_x} K(x, \sigma)w(\sigma, t)\, d\sigma, \qquad w(x, 0) = 0, \qquad w_t(x, 0) = \phi(x) \tag{3.3}$$

with x and σ both n-vectors, $d\sigma = d\sigma_1 d\sigma_2 \cdots d\sigma_n$, and in which the function $\phi(x)$ is continuous. The version of (1.3) associated with this has the solution function

$$u(x,t) = \phi(x) + \int_{E_x} G(x,\sigma,t)\phi(\sigma)\,d\sigma \tag{3.4}$$

in which

$$G(x,\sigma,t) = \sum_{j=1}^{\infty} \frac{t^j}{j!} K_{j-1}(x,\sigma) \tag{3.5}$$

with the K_j as defined in Section 2. Upon applying the transmutation formula (3.2) to this, we arrive at the following result:

Theorem 3.1. *A solution of the problem* (3.3) *is given by the formula*

$$w(x,t) = \phi(x)t + \int_{E_x} \hat{G}(x,\sigma,t)\phi(\sigma)\,d\sigma \tag{3.6}$$

in which

$$\begin{aligned} \hat{G}(x,\sigma,t) &= \Gamma(3/2)\mathcal{L}_s^{-1}\{s^{-3/2}G(x,\sigma,1/4s)\}_{s\to t^2} \\ &= \Gamma(3/2)\sum_{j=1}^{\infty} \frac{t^{2j+1}}{4^j j!\Gamma(j+1/2)} K_{j-1}(x,\sigma) \end{aligned} \tag{3.7}$$

Whether $\hat{G}$ can be expressed in a closed form depends upon the choice of $K(x,\sigma)$. In particular, if $n = 1$ and the lower and upper limits of integration are, respectively, a and x and $K(x,\sigma) = \nu(\sigma)[V(x) - V(\sigma)]^{p-1}$, we have

$$\hat{G}(x,\sigma,t) = \Gamma(3/2)\nu(\sigma)\sum_{j-1}^{\infty} \frac{t^{2j+1}\omega^{jp-1}}{4^j j!\Gamma(j+1/2)\Gamma(jp)} \tag{3.8}$$

with ω as in Section 2. This reduces to $\nu(\sigma)t^3 \cdot {}_0F_2(\_;3/2,2;t^2\omega/2)/4$ when $p = 1$. For a general positive integer p in (3.8), the methods of Section 2 can be employed to express $\hat{G}$ in terms of hypergeometric functions (see [**9**] for more about these functions).

One could, of course, consider the two space variable problem

$$\begin{aligned}(3.9)\qquad w_{tt}(x_1,x_2,t) = I_{x_1,p}I_{x_2,q}w(x_1,x_2,t); \qquad w(x_1,x_2,0)=0,\\ w_t(x_1,x_2,0)=\phi(x_1,x_2).\end{aligned}$$

The problem (1.3) corresponding to this has the solution

(3.10)
$$u(x_1,x_2,t) = \phi(x_1,x_2) + \int_a^{x_1}\int_b^{x_2} G(x_1,x_2,\sigma_1,\sigma_2,t)\phi(\sigma_1,\sigma_2)\,d\sigma$$

with $G(x_1,x_2,\sigma_1,\sigma_2,t)$ given by (2.12). Then, if we apply (3.2) to (3.10), we find

(3.11)
$$w(x_1,x_2,t) = \phi(x_1,x_2)t + \int_a^{x_1}\int_b^{x_2} \hat{G}(x_1,x_2,\sigma_1,\sigma_2,t)\phi(\sigma_1,\sigma_2)\,d\sigma$$

where

$$(3.12)\qquad \hat{G} = \Gamma(3/2)\nu(\sigma_1)\mu(\sigma_2)\sum_{n=1}^{\infty}\frac{t^{2n+1}\omega^{np-1}\psi^{nq-1}}{4^n n!\Gamma(pn)\Gamma(qn)\Gamma(n+1/2)}$$

and $\psi = U(x_2) - U(\sigma_2)$. Again, the reduction of this $\hat{G}$ function in terms of hypergeometric functions depends upon the choices of p and q.

(B) *Adding in a Laplacian operator.* Let us now modify (3.3) to read

$$\begin{aligned}(3.13)\qquad w_{tt}(x,t) = \int_{E_x} K(x,\sigma)w(\sigma,y,t)\,d\sigma + \Delta_m w(x,y,t), \quad t>0\\ w(x,y,0)=0, \qquad w_t(x,y,0)=\phi(x,y)\end{aligned}$$

where $y=(y_1,y_2,\dots,y_m)$ and $\Delta_m = \sum_{k=1}^m \partial^2/\partial y_k^2$, and in which $\phi(x,y)$ is bounded and continuous. Once again, the solution of the problem (1.3) corresponding to this can be expressed symbolically in the form

$$(3.14)\qquad u(x,y,t) = e^{t\cdot(I_x+\Delta_m)}\phi(x,y) = e^{tI_x}H(x,y,t)$$

in which
(3.15)
$$H(x,y,t) = e^{t\Delta_m}\phi(x,y) = (4\pi t)^{-m/2}\int_{E_m} e^{-||y-\xi||^2/4t}\phi(x,\xi)\,d\xi$$

with $\xi = (\xi_1, \xi_2, \dots, \xi_m)$. Here $||y-\xi||^2 = \sum_{j=1}^m (y_j - \xi_j)^2$. Finally, using (3.4), (3.14) becomes

$$u(x,y,t) = H(x,y,t) + \int_{E_x} G(x,\sigma,t)H(\sigma,y,t)\,d\sigma \tag{3.16}$$

with $G(x,\sigma,t)$ as given in (3.5). We now wish to recover the solution $W(x,y,t)$ of (3.13) from this by applying formula (3.2).

Now $\Gamma(3/2)_{\mathcal{L}_s^{-1}}\{s^{-3/2}H(x,y,1/4s)\}_{s\to t^2} = W(x,y,t)$ where $W(x,y,t)$ satisfies the initial wave problem

$$W_{tt}(x,y,t) = \Delta_m W(x,y,t), \qquad W(x,y,0) = 0, \qquad W_t(x,y,0) = \phi(x,y).$$

See [**4**] or [**8**] for the form of this solution. Next, we apply the transmutation formula to the second term in the right member of (3.16). Let

$$\mathcal{L}_s^{-1}\left\{\frac{1}{S^{(3-m)/2}}\,G(x,\sigma,1/4s)\right\}_{s\to\tau} = G^*(x,\sigma,\tau). \tag{3.17}$$

If $m \le 4$, it is easy to see by writing out the expression for G and inverting term-by-term (which was shown to be valid in Section 2), that

$$G^*(x,\sigma,\tau) = \sum_{j=1}^{\infty} \frac{\tau^{j+(1-m)/2}}{j!\Gamma(j+(3-m)/2)} \cdot K_{j-1}(x,\sigma). \tag{3.18}$$

Then by the translation property for Laplace transforms, it follows that
(3.19)
$$\mathcal{L}_s^{-1}\left\{\frac{1}{s^{(3-m)/2}}\,G(x,\sigma,1/4s)e^{-||y-\xi||^2 s}\right\}_{s\to\tau} = G^*(x,\sigma,[\tau - ||y-\xi||^2||]_+)$$

in which $[\tau - ||y-\xi||^2]_+ = \tau - ||y-\xi||^2$ if $\tau \ge ||y-\xi||^2$ and 0 otherwise. This establishes the following conclusion:

Theorem 3.2. *If $m \leq 4$, the solution of the problem* (3.13) *is given by the formula*

$$(3.20)\quad w(x,y,t) = W(x,y,t) + \frac{1}{2\pi^{(m-1)/2}} \cdot \int_{E_x} \int_{B(y,t)} G^*(x,\sigma,t^2 - ||y-\xi||^2)\phi(\sigma,\xi)\,d\xi\,d\sigma$$

in which $B(y,t)$ is a ball centered at y with radius t.

Thus, we see that the region of integration on ξ is restricted to this ball and this shows up directly in the "kernel" function G^* (or fundamental solution) appearing in this solution formula. If we choose $n = 1$ in (3.13) and take the integral operator there to be defined by (2.3) with p a positive integer, then it can be readily shown that, for $m \leq 4$, the function $G^*(x,\sigma,\tau)$ appearing in (3.20) is given by the formula

$$G^*(x,\sigma,\tau) = \frac{\nu(\sigma)[V(x)-V(\sigma)]^{p-1}\tau^{(3-m)/2}}{(p-1)!\Gamma((5-m)/2)} \cdot {}_0F_{p+1}(\_;2,1+1/p,\dots,1+(p-1)/p, (5-m)/2; \tau[(V(x)-V(\sigma))/p]^p).$$

The region of integration E_x in (3.20) reduces to the interval $[a,x]$. The reader can compute the G^* for other choices of p by using the formulas from Section 2.

The restriction $m \leq 4$ permitted us to avoid obtaining distributions in the kernel G^* above. If we remove this restriction, then we can split the inversion in (3.17) into two groupings, one of which involves inverting nonnegative powers of s and the other which involves inverting negative powers of s. The cases m even and m odd lead to different solution forms as we shall see in the following discussion.

Case I. $m = 2r+1$, $r \geq 2$. In this situation, the function $s^{-(3-m)/2}G(x,\sigma,1/4s)$ can be split into the sums $\Sigma_1(x,\sigma,s)$ and

$\Sigma_2(x,\sigma,s)$ where

$$(3.21)\qquad \begin{aligned}\Sigma_1(x,\sigma,s) &= \sum_{j=1}^{r-1} \frac{s^{r-1-j}}{4^j j!} K_{j-1}(x,\sigma) \quad \text{and}\\ \Sigma_2(x,\sigma,s) &= \sum_{j=r}^{\infty} \frac{1}{4^j j! s^{j+1-r}} K_{j-1}(x,\sigma).\end{aligned}$$

Now $\mathcal{L}_s^{-1}\{s^l\}_{s\to\tau} = \delta^{(l)}(\tau)$ for l a nonnegative integer where $\delta(\tau)$ denotes the Dirac delta function. From this, (3.21) and (3.17), we find that

$$(3.22)\qquad G^*(x,\sigma,[t^2-||y-\xi||^2]_+) = \sum_{l=1}^{2} G_l^*(x,\sigma,[t^2-||y-\xi||^2]_+)$$

with

$$(3.23)\qquad \begin{aligned}G_1^* &= \sum_{j=1}^{r-1} \frac{1}{4^j j!} K_{j-1}(x,\sigma)\cdot\left(\frac{d}{dt^2}\right)^{r-1-j} \delta(t^2-||y-\xi||^2)\\ G_2^* &= \sum_{j=r}^{\infty} \frac{1}{4^j j!(j-r)!} K_{j-1}(x,\sigma)\cdot([t^2-||y-\xi||^2]_+)^{j-r}.\end{aligned}$$

The first of these is a generalized function with support on the sphere $\mathcal{S}(y,t)$ of center y and radius t while the second has support in the ball $B(y,t)$. Moreover, $\delta(t^2-||y-\xi||^2) = \delta(t-||y-\xi||)/2t$ [**4**]. Using this fact and letting $d\mathcal{S}_{2n+2}$ denote the surface element on the above sphere, it follows that the formula (3.20) in Theorem 3.2 is then altered, and this leads to

Theorem 3.3. *If $m = 2r+1$ with $r \geq 2$, then a solution of* (3.13) *is given by*

(3.24)

$$\begin{aligned}w(x,y,t) = W(x,y,t) &+ \frac{1}{2\pi^r}\sum_{j=1}^{r-1}\frac{1}{4^j j!}\left(\frac{d}{dt^2}\right)^{r-1-j}\\ &\cdot \int_{E_x}\int_{S(y,t)} (K_{j-1}(x,\sigma)/2t)\phi(\sigma,\xi)\, dS_{2n+1}\, d\sigma\\ &+ \frac{1}{2\pi^r}\int_{E_x}\int_{B(y,t)} G_2^*(x,\sigma,t^2-||y-\xi||^2)\phi(\sigma,\xi)\, d\xi\, d\sigma.\end{aligned}$$

Case II. $m = 2r$, $r \geq 3$. In this case one immediately deduces that the term $s^{-(3-m)/2}S(x, \sigma, 1/4s)$ can again be split into sums $\Sigma_1(x, \sigma, s)$ and $\Sigma_2(x, \sigma, s)$ with

$$\begin{aligned} \Sigma_1(x,\sigma,s) &= \sum_{j=1}^{r-2} \frac{s^{r-1-j}}{4^j j! \sqrt{s}} K_{j-1}(x,\sigma) \\ \Sigma_2(x,\sigma,s) &= \sum_{j=r-1}^{\infty} \frac{1}{4^j j! s^{j+(3-2r)/2}} K_{j-1}(x,\sigma). \end{aligned} \tag{3.25}$$

Using these along with (3.17), we can then deduce a formula analogous to (3.22) with

(3.26)

$$\begin{aligned} G_1^* &= \frac{1}{\sqrt{\pi}} \sum_{j=1}^{r-2} \frac{1}{4^j j!} K_{j-1}(x,\sigma) \cdot \left(\frac{d}{dt^2}\right)^{r-1-j} \{(t^2 - ||y-\xi||^2)\}^{-1/2} \\ G_2^* &= \sum_{j=r-1}^{\infty} \frac{1}{4^j j!(j-r)!\,\Gamma(j+(3-2r)/2)} K_{j-1}(x,\sigma) \\ &\qquad \cdot ([t^2 - ||y-\xi||^2]_+)^{j-r+1/2}. \end{aligned}$$

With these, we can finally establish:

Theorem 3.4. *The solution of the problem* (3.13) *with* $m = 2r$ *and* $r \geq 3$ *given by*

(3.27)

$$\begin{aligned} w(x,y,t) &= W(x,y,t) \\ &\quad + \frac{1}{2\pi^r} \sum_{j=1}^{r-2} \frac{1}{4^j j!} \left(\frac{d}{dt^2}\right)^{r-1-j} \\ &\quad \cdot \int_{E_x} \int_{B(y,t)} K_{j-1}(x,\sigma) \{(t^2 - ||y-\xi||^2\}^{-1/2} \phi(\sigma,\xi)\, d\xi\, d\sigma \\ &\quad + \frac{1}{2\pi^r} \int_{E_x} \int_{B(y,t)} G_2^*(x,\sigma,t^2 - ||y-\xi||^2) \phi(\sigma,\xi)\, d\xi\, d\sigma. \end{aligned}$$

(C) *Adding in a negative multiplier.* As the final example in this section, we choose $m = 3$ and consider the problem

(3.28)
$$w_{tt}(x,t) = \int_{E_x} K(x,\sigma) w(\sigma,y,t)\,d\sigma + (\Delta_3 - a^2) w(x,y,t), \quad t > 0$$
$$w(x,y,0) = 0, \qquad w_t(x,y,0) = \phi(x,y).$$

In this, a is taken to be a positive constant. Corresponding to this, the solution of the associated problem (1.3) is defined by

$$\begin{aligned} u(x,y,t) &= e^{t[I_x + \Delta_3 - a^2]}\phi(x,y) = e^{-a^2 t}\{e^{tI_x} H(x,y,t)\} \\ &= e^{-a^2 t} H(x,y,t) + e^{-a^2 t}\int_{E_x} G(x,\sigma,t) H(\sigma,y,t)\,d\sigma \end{aligned} \tag{3.29}$$

with $H(x,y,t)$ as given by (3.14). We now wish to apply the transmutation formula to this. From the results in [**5**], we have

(3.30)
$$\Gamma(3/2)\mathcal{L}_s^{-1}\Big\{e^{-a^2/4s} H(x,y,1/4s)\Big\}_{s\to t^2}$$
$$= \frac{1}{4\pi}\int_{S(y,t)} \frac{\phi(x,\xi)}{t}\,d\mathcal{S}_3 - \frac{a}{4\pi}\int_{B(y,t)} \frac{J_1(a\sqrt{t^2 - ||y-\xi||^2})}{\sqrt{t^2 - ||y-\xi||^2}}\ \phi(x,\xi)\,d\xi$$

in which $J_1(z)$ denotes a standard Bessel function of index 1. Also, we can rewrite the integral term in the last member of (3.29) in the form

$$\int_{E_x}\int_{E_3} \Big\{(4\pi t)^{-3/2} G(x,\sigma,t) e^{-a^2 t} e^{-||y-\xi||^2/4t}\Big\}\phi(\sigma,\xi)\,d\sigma\,d\xi.$$

Denote the result of applying (3.2) to the term in braces in this integral by $G^*(x,\sigma,(t^2 - ||y-\xi||^2))$; thus,

$$G^*(x,\sigma,(t^2 - ||y-\xi||^2))$$
$$= \frac{1}{2\pi}\sum_{j=1}^{\infty} \frac{1}{4^j j!} K_{j-1}(x,\sigma)\mathcal{L}_s^{-1}\Big\{\frac{1}{s^j} e^{-a^2/4s} e^{-||y-\xi||^2 s}\Big\}_{s\to t^2}.$$

By a standard Laplace inversion result (see [**10**, p. 246]), we find that this becomes

$$\begin{aligned} &G^*(x,\sigma,(t^2 - ||y-\xi||^2)) \\ &= \frac{1}{4\pi}\sum_{j=1}^{\infty} \frac{1}{2^j a^{j-1} j!} K_{j-1}(x,\sigma)\{(t^2 - ||y-\xi||^2)_+\}^{(j-1)/2} \\ &\qquad J_{j-1}(a\sqrt{(t^2 - ||y-\xi||^2)_+}). \end{aligned} \tag{3.31}$$

Since $|J_{j-1}(x)| \leq 1$ for $j = 1, 2, \dots$, it follows that G^* is well defined. Thus, the integral term in the last member of (3.29) inverts into

$$\int_{E_x} \int_{B(y,t)} G^*(x, \sigma, t^2 - ||y - \xi||^2)\phi(\sigma, \xi)\, d\xi\, d\sigma. \tag{3.32}$$

Upon adding this last term to the term in the right member of (3.30), we finally obtain the solution of the problem (3.28).

4. The quasi inner product approach. The development of quasi inner product methods for solving initial value problems in partial differential equations was carried out in [**2**] and [**3**]. In order to describe the results of [**3**] which are pertinent to solving problems analogous to (1.1), let y denote a vector with m components as earlier, and let $Q(D)$ be a linear partial differential operator in $D = (D_1, D_2, \dots, D_m)$ where $D_j = \partial/\partial y_j$. In Section 3 of that paper, it was shown that a solution of the problem

$$\nu_{tt}(y,t) = Q(D)\nu(y,t), \qquad \nu(y,0) = 0, \qquad \nu_t(y,0) = \varphi(y) \tag{4.1}$$

has the integral representation

$$\nu(y,t) = \frac{1}{2} t \int_0^1 \eta^{-1/2} A(y,t,\eta;\varphi)\, d\eta \tag{4.2}$$

in which

$$A(y,t,\eta;\varphi) = (2\pi)^{-1} \int_0^{2\pi} e^{t(1-\eta)e^{i\theta}} \left\{ e^{te^{-i\theta}Q(D)/4} \varphi(y) \right\} d\theta \tag{4.3}$$

where $\varphi(y)$ is entire of growth (ρ, τ) with $\rho \leq 2$. The evaluation of expression $e^{\lambda Q(D)}\varphi(y)$, $\lambda = te^{-i\theta}/4$, can be carried out by employing the reduction methods of [**2**]. If $Q(D)$ can be factored into a product of commutative operators $Q_1(D)$ and $Q_2(D)$, then the properties of quasi inner products permit rewriting (4.3) in the form

$$A(y,t,\eta;\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ e^{t\sqrt{1-\eta}e^{i\theta}Q_1(D)/2} e^{t\sqrt{1-\eta}e^{-i\theta}Q_2(D)/2} \varphi(y) \right\} d\theta. \tag{4.4}$$

In particular, if $Q_1(D) = Q_2(D)$, the integrand in this becomes $e^{t\cos\theta\sqrt{1-\eta}Q_1(D)}\varphi(y)$ which can be evaluated by the techniques of [**2**] or explicitly if $e^{\lambda Q_1(D)}$ defines a translation operator. The development of the formulas (4.2)–(4.4) is not specifically dependent upon $Q(D)$ being a differential operator. We can, in fact, replace this $Q(D)$ by one of the types of integral operators considered in Section 2. For integro-partial differential equation problems of the form

$$w_{tt}(x,y,t) = (I_x^2 + P^2(D))w(x,y,t), \qquad w(x,y,0) = 0, \quad w_t(x,y,0) = \phi(x,y) \tag{4.5}$$

in which I_x and $P(D)$ commute where $P(D)$ is a partial differential operator in the D with D as described above and with $x = (x_1, x_2, \dots, x_n)$, formulas of the form (4.2)–(4.4) continue to hold for appropriate choices of $\phi(x,y)$. With this background, we now focus on the construction technique for a solution of (4.5). This will be followed up by considering some special choices of I_x and $P(D)$ for this problem.

Using the factorization $I_x^2 + P^2(D) = (I_x + iP(D)) \cdot (I_x - iP(D))$, the extended version of (4.4) with $Q_1(D)$ replaced by $I_x + iP(D)$ and $Q_2(D)$ replaced by $I_x - iP(D)$ permits us to write

$$\begin{aligned} &A(x,y,t,\eta;\phi) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Big\{ e^{te^{i\theta}\sqrt{1-\eta}(I_x+iP(D))/2} e^{te^{-i\theta}\sqrt{1-\eta}(I_x-iP(D))/2}\phi(x,y)\Big\}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{(t\cos\theta\sqrt{1-\eta})I_x}\Big\{ e^{-(t\sin\theta\sqrt{1-\eta})P(D)}\phi(x,y)\Big\}\, d\theta. \end{aligned} \tag{4.6}$$

Now the term in braces in the last member of this can be evaluated by the methods of [**2**] if $\phi(x,y)$ is entire in y or more simply if $e^{\lambda P(D)}$ is a translation operator. If we denote this term by $\psi(x,y,t,\eta,\theta;\phi)$, then (4.6) becomes

$$A(x,y,t,\eta;\phi) = \frac{1}{2\pi}\int_0^{2\pi} e^{(t\cos\theta\sqrt{1-\eta})I_x}\psi(x,y,t,\eta,\theta;\phi)\, d\theta. \tag{4.7}$$

The integrand of this can be evaluated by employing the formula (2.1a) and its expression in the form of a function given by

$$\psi(x,y,t,\eta,\theta;\phi) + \int_{E_x} G\Big(x,\xi,t\sqrt{1-\eta}\cos\theta\Big)\psi(\xi,y,t,\eta,\theta;\phi)\, d\xi \tag{4.8}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ and $d\xi = d\xi_1 d\xi_2 \cdots d\xi_n$. With this, the function $A(x, y, t, \eta; \phi)$ is defined in (4.7) and we finally obtain, via (4.2), the general solution formula

$$w(x, y, t) = \frac{1}{2} t \int_0^1 \eta^{-1/2} A(x, y, t, \eta; \phi)\, d\eta. \tag{4.9}$$

Let us now examine a few special cases of (4.5) in which $n = 1$. Since the equation in that problem has a quite special form involving the square of an integral operator I_x, we wish to consider some standard examples of integral operators that can be conveniently expressed as the square of another such operator. From Section 2, we see that the operator defined in equation (2.3) satisfies the condition $I_{x,p}\phi(x) = I^2_{x,p/2}\phi(x)$ for $p > 0$. For the examples to be considered, we employ these operators with $\nu(x) = 1$ and $a = 0$.

Example 1. Consider the problem

(4.10)
$$w_{tt}(x, y, t) = \frac{1}{\Gamma(p)} \int_0^x (x - \sigma)^{p-1} w(\sigma, y, t)\, d\sigma + \partial^2 w(x, y, t)/\partial y^2,$$
$$w(x, y, 0) = 0, \qquad w_t(x, y, 0) = \phi(x, y)$$

with $p > 0$ and $m = 1$. Then, from what was noted in Section 2, it follows that the equation in this can be rewritten in the form

$$w_{tt}(x, y, t) = \{I^2_{x,p/2} + D^2\} w(x, y, t).$$

In this case, $P(D) = D$ and the term in braces in the last member of (4.6) defines a translation on y. Hence, (4.6) becomes

(4.11)
$$A(x, y, t, \eta; \phi) = \frac{1}{2\pi} \int_0^{2\pi} e^{(t\sqrt{1-\eta}\cos\theta) I_{x,p/2}} \phi\Big(x, y - t \cdot \sin\theta \sqrt{1 - \eta}\Big)\, d\theta.$$

Upon using (2.7a) with $\nu(\sigma) = 1$ in the first formula in (2.1), it follows that the integrand in this becomes

(4.12)
$$\phi\Big(x, y - t \cdot \sin\theta \sqrt{1-\eta}\Big)$$
$$+ \int_0^x G_{p/2}\Big(x, \sigma, t \cdot \cos\theta \sqrt{1-\eta}\Big) \cdot \phi\Big(\sigma, y - t \cdot \sin\theta \sqrt{1-\eta}\Big)\, d\theta.$$

Inserting this back into (4.11), we obtain the A function and the solution function $w(x,y,t)$ follows by introducing this back into (4.9). As usual, one can replace the $G_{p/2}$ function in this in terms of appropriate hypergeometric functions for p a positive integer. Observe that it was necessary for the data function $\phi(x,y)$ to be continuous in x and have a continuous derivative with respect to y.

Example 2. Suppose we consider the problem (4.10) with the derivative operator D^2 replaced in the basic equation by Δ_m^2 with Δ_m the Laplacian operator in the m variables y_j as in problem (3.13). In this problem, we take the data function $\phi(x,y)$ to be entire of growth (ρ,τ) in each of the variables y_j. Then the formula (4.6) becomes

(4.13)
$$\begin{aligned} A(x,y,t,\eta;\phi) &= \frac{1}{2\pi}\int_0^{2\pi} e^{(t\cos\theta\sqrt{1-\eta})I_x}\Big\{e^{-(t\sin\theta\sqrt{1-\eta}\Delta_m)}\phi(x,y)\Big\}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{(t\cos\theta\sqrt{1-\eta})I_x} H\Big(x,y,-t\sin\theta\sqrt{1-\eta}\Big)\,d\theta \end{aligned}$$

in which $H(x,y,t)$ is a solution of the heat problem $u_t(x,y,t) = \Delta_m u(x,y,t)$, $u(x,y,0) = \phi(x,y)$ which is valid for $|t| < 1/(4\tau)$ (see [**11**]). One can now replace the exponential of the integral operator acting on H in the last member of (4.3) by

$$\begin{aligned} &H\Big(x,y,-t\cdot\sin\theta\sqrt{1-\eta}\Big) \\ &\quad + \int_0^x G_{p/2}\Big(x,\sigma,t\cdot\cos\theta\sqrt{1-\eta}\Big)\cdot H\Big(\sigma,y,-t\cdot\sin\theta\sqrt{1-\eta}\Big)\,d\theta. \end{aligned}$$

With this evaluation for the integrand in (4.13), we can now obtain the function A. Upon inserting this into (4.9), we obtain the solution function $w(x,y,t)$ for $|t| < 1/(4\tau)$.

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