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A SINGULAR NONLINEAR VOLTERRA INTEGRAL EQUATION

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ABSTRACT. This paper concerns the integral equation

$$x(t) = f(t) + \int_0^t g(s)/x(s)\,ds$$

in which the functions and variables are real-valued and x is the unknown. The interest is in nonnegative continuous solutions of this equation for $t \geq 0$ when $f \in C([0,\infty))$, $f(0) \geq 0$ and $g \in L^1(0,\tau)$ for all $\tau \in (0,\infty)$. Of particular interest is the singular case $f(0) = 0$. This equation arises in the study of travelling waves in nonlinear reaction-convection-diffusion processes. It is shown that the integral equation has none, one or an uncountable number of solutions. Subsequently, it is shown that, even if there is an infinite number of solutions, there is one which is maximal. Moreover, a method for constructing this particular solution is provided. This permits the establishment of necessary and sufficient conditions for the existence of a solution. Comparison principles for solutions of the equation with different sets of coefficients are then presented. Rather detailed analyses follow for the case that $f(0) = 0$ and $g(s) \leq 0$ for almost all s in a right neighborhood of zero and for the case that $f(0) = 0$ and the inequality for g is reversed. These analyses demonstrate that the equation may indeed have none, one or an uncountable number of solutions, among other phenomena.

1. Introduction. This paper concerns the integral equation

$$x(t) = f(t) + \int_0^t g(s)/x(s)\,ds \tag{1}$$

in which the functions and variables are real-valued and x is the unknown. We shall be interested in nonnegative continuous solutions of this equation for $t \geq 0$ when

$$f \in C([0,\infty)) \quad \text{with} \quad f(0) \geq 0 \tag{2}$$

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and

$$g \in L^1(0,\tau) \quad \text{for all} \quad \tau \in (0,\infty). \tag{3}$$

This equation may be classified as a nonlinear Volterra integral equation of the second kind with an integrand which is singular (when $x = 0$). In this respect, we shall be especially interested in the case $f(0) = 0$.

The motivation for studying equation (1) stems from the field of nonlinear reaction-convection-diffusion processes. Many such processes can be modelled by the nonlinear partial differential equation

$$u_t = (a(u))_{xx} + (b(u))_x + c(u) \tag{4}$$

in which subscripts denote partial differentiation. Areas in which an equation of this type arises are nonlinear heat transfer, combustion, reaction chemistry, hydrodynamics, soil-moisture physics, thin viscous fluid flow, and biological population dynamics, to name but a few. In these settings, the unknown u corresponds to a temperature, concentration, density or similar nonnegative variable, and the coefficients have the properties

$$a, b \in C([0,\infty)), \qquad c \in C(0,\infty),$$
$$a \text{ is strictly increasing on } [0,\infty)$$

and

$$a(0) = b(0) = c(0) = 0.$$

Specific prototypes for equation (4) are the Burgers equation, the porous media equation, the Richards equation, the Fishers equation and the KPP equation [**2, 3**]. The search for a travelling-wave solution of equation (4) of the form

$$u(x,t) = U(\xi) \quad \text{with} \quad \xi = x - \lambda t, \tag{5}$$

with U monotonic decreasing for $\xi \in (-\infty,\infty)$ and

$$U(\xi),\ (a(U))'(\xi) \to 0 \quad \text{as} \quad \xi \to \infty$$

formally leads to consideration of the ordinary differential equation

$$-\lambda U' = (a(U))'' + (b(U))' + c(U)$$

upon substitution of (5) into (4). Whence integrating

$$\lambda U(\xi) = -(a(U))'(\xi) - b(U(\xi)) + \int_{\xi}^{\infty} c(U(\eta))\, d\eta$$

for all $\xi \in (-\infty, \infty)$. Subsequently defining the function θ in some right neighborhood of zero by

$$\theta(U(\xi)) = -(a(U))'(\xi)$$

gives rise to the equation

$$\theta(U) = \lambda U + b(U) - \int_0^U c(s)/\theta(s)\, da(s).$$

Changing the dependent variable to $z := a(U)$ finally yields an equation of the form (1) with $f(0) = 0$.

To commence the study of (1), we need to clarify what we mean by a solution of this equation. Ambiguousness in the definition of the integrand in (1) will be avoided by interpreting this as

$$I(s,x) := \begin{cases} g(s)/x & \text{if } x > 0 \\ \infty & \text{if } g(s) > 0 \text{ and } x = 0 \\ 0 & \text{if } g(s) = 0 \text{ and } x = 0 \\ -\infty & \text{if } g(s) < 0 \text{ and } x = 0, \end{cases} \tag{6}$$

and generality will be supported by considering the integral as an improper Lebesgue integral.

Definition 1. *A function x is a* solution *of equation* (1) *if it is defined, real, nonnegative and continuous in a right neighborhood of zero* $[0, \tau)$ *with* $0 < \tau \le \infty$, $I(s, x(s)) \in L^1_{\mathrm{loc}}(0, \tau)$,

$$\int_0^t I(s, x(s))\, ds := \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^t I(s, x(s))\, ds \quad \textit{exists}$$

and satisfies

$$x(t) = f(t) + \int_0^t I(s, x(s))\, ds \quad \text{for all} \quad t \in (0, \tau).$$

Note that if $g \geq 0$ almost everywhere or if $g \leq 0$ almost everywhere in a right neighborhood of zero, the definition infers $I(s, x(s)) \in L^1(0, t)$ for all $t \in (0, \tau)$ although, in general, the latter need not be the case. Either way a solution possesses the attributes

$$\int_0^t I(s, x(s))\, ds \to 0 \quad \text{as} \quad t \downarrow 0$$

and

$$x(0) = f(0).$$

The remainder of this paper is arranged as follows. In the next section we prove some useful preliminary results. Thereafter, in Section 3 we show that equation (1) either has no solution, a unique solution or an uncountable number of solutions. Subsequently, in Section 4 we show that even if (1) does have an infinite number of solutions there is one which is maximal. Moreover, we provide a method for constructing this particular solution. This permits us to establish necessary and sufficient conditions for the existence of a solution of equation (1) in Section 5. Following this, in Section 6, we state and prove comparison principles for solutions of equation (1) with different sets of coefficients. In Section 7 we then present a rather detailed analysis of (1) in the case that $f(0) = 0$ and $g(s) \leq 0$ for almost all s in a right neighborhood of zero. In Section 8 this exercise is repeated with the last-mentioned inequality reversed. These analyses signal examples of equation (1) which may indeed have none, one or an uncountable number of solutions. Thus, in general, our results on the uniqueness of solutions of (1) cannot be improved upon, and the concept of a maximal solution is not superfluous. In the final section we discuss a particular equation of the form (1) which may be solved explicitly.

We refer the interested reader to [**1–3**] for a discussion of consequences of the results in this paper for the study of travelling-wave solutions of equation (4).

2. Preliminaries. Throughout the remainder of this paper, the letters f and g, with or without subscripts or superscripts, will be assumed to denote functions of the types (2) and (3), respectively. Furthermore, any expression of the form g/x will be interpreted in the sense of the right-hand side of (6).

Lemma 1. *If $f(0) > 0$, then (1) has a unique positive solution $x \in C([0,\delta))$ for some $\delta \in (0,\infty)$.*

Proof. Since if $x > 0$ the integrand $I(s,x)$ defined by (6) is locally Lipschitz continuous with respect to x, the lemma is a straightforward consequence of a standard existence theorem for nonlinear Volterra integral equations based on a contraction-mapping principle [**4**, Theorem 12.2.6; **5**, Theorem II.1.2]. $\square$

Lemma 2. *Let x denote a solution of (1) on some interval $[0,T) \subsetneqq [0,\infty)$. Then*

$$x(T) := \lim_{t\uparrow T} x(t) \quad \textit{exists and is finite,} \tag{7}$$

$$\int_0^T g(s)/x(s)\,ds := \lim_{t\uparrow T}\int_0^t g(s)/x(s)\,ds \quad \textit{exists} \tag{8}$$

and is finite, and

$$x(T) = f(T) + \int_0^T g(s)/x(s)\,ds. \tag{9}$$

Moreover, either $x(T) = 0$ or x is continuously extendible as a solution of (1) onto an interval $[0,T')$ with $[0,T) \subsetneqq [0,T') \subseteqq [0,\infty)$.

Proof. Suppose to begin with that

$$\liminf_{t\uparrow T} x(t) < \rho < \limsup_{t\uparrow T} x(t) \tag{10}$$

for some $\rho \in (0,\infty)$. Consider the equation

$$x(t) = \rho + f(t) - f(T) + \int_T^t g(s)/x(s)\,ds. \tag{11}$$

By Lemma 1, this equation has a continuous positive solution x^* on an interval $(T-\delta, T]$ for some $\delta \in (0,T)$. Moreover, in view of (10) there must be a point $t^* \in (T-\delta, T)$ such that $x(t^*) = x^*(t^*) > 0$. However, since both (1) and (11) can be rewritten as

$$x(t) = x(t^*) + f(t) - f(t^*) + \int_{t^*}^{t} g(s)/x(s)\,ds,$$

by Lemma 1 this can only be the case if $x \equiv x^*$ on $[t^*, T)$. This contradicts (10). We therefore deduce that $x(T)$ exists with $0 \leq x(T) \leq \infty$.

Suppose next that $x(T) = \infty$. Then $x_0 := \inf\{x(s) : T-\delta < s < T\} > 0$ for some $\delta \in (0, T)$. So that, by (1),

$$x(t) \leq f(t) + \int_0^{T-\delta} g(s)/x(s)\,ds + \int_{T-\delta}^{t} |g(s)|\,ds/x_0$$

for any $t \in (T-\delta, T)$. However, in the limit $t \uparrow T$ this contradicts the datum that $g \in L^1(0, T)$. We must conclude that $x(T) < \infty$. This proves (7). Furthermore, letting $t \uparrow T$ in (1), it also proves (8) and (9).

Lastly, we note that if $x(T) > 0$, then $g/x \in L^1(T-\delta, T)$ for any $\delta \in (0, T)$ and using Lemma 1 we can subsequently continuously extend x as a solution of (1) beyond $[0, T]$. $\square$

Lemma 3. *For any $\rho > 0$ and $t^* \in (0, \infty)$, the equation*

$$x^*(t) = \rho + f(t) - f(t^*) + \int_{t^*}^{t} g(s)/x^*(s)\,ds \tag{12}$$

has a unique positive solution x^* on a maximal interval of existence (t^-, t^+) with

$$0 \leq t^- < t^* < t^+ \leq \infty \tag{13}$$

such that

$$x^* \in C([t^-, t^+)), \tag{14}$$

and

$$t^- = 0 \quad or \quad x^*(t^-) = 0. \tag{15}$$

Furthermore, x^* solves

$$x^*(t) = x^*(t^-) + f(t) - f(t^-) + \int_{t^-}^{t} g(s)/x^*(s)\,ds \tag{16}$$

on $[t^-, t^+)$ in the sense of Definition 1.

Proof. Apply Lemmata 1 and 2 to equation (12) for $t \geq t^*$ and for $t \leq t^*$. $\square$

Lemma 4. *Let x_1 denote a solution of the equation*

$$x_1(t) = f_1(t) + \int_0^t g_1(s)/x_1(s)\,ds \tag{17}$$

on some interval $[0, \delta)$ with $0 < \delta < \infty$. Suppose that

$$f(0) > f_1(0),$$

$$(f - f_1) \quad \textit{is nondecreasing on} \quad [0, \delta) \tag{18}$$

and

$$g(t) \geq g_1(t) \quad \textit{for almost all} \quad t \in (0, \delta). \tag{19}$$

Then if (1) has a solution x on $[0, \delta)$ such that $x(t) > 0$ for all $t \in [0, \delta)$ there holds $x(t) > x_1(t)$ for all $t \in [0, \delta)$.

Proof. Let us hypothesize that the lemma is false. Then there exists a point $t^* \in (0, \delta)$ such that

$$x(t) > x_1(t) \quad \text{for all} \quad t \in [0, t^*) \tag{20}$$

and

$$x(t^*) = x_1(t^*) > 0. \tag{21}$$

Let $t_0 \in [0, t^*)$ be such that $x_1(t) > 0$ for all $t \in [t_0, t^*]$ and

$$\int_{t_0}^{t^*} \frac{|g(s)|}{x_1(s)x(s)}\, ds \leq \frac{1}{2}.$$

Using (1), (17) and (21),

$$x(t) - x_1(t) = (f - f_1)(t) - (f - f_1)(t^*) + \int_t^{t^*} \left\{ \frac{g_1(s)}{x_1(s)} - \frac{g(s)}{x(s)} \right\} ds$$

for any $t \in [t_0, t^*]$. Whence

$$\begin{aligned} x(t) - x_1(t) &\leq \int_t^{t^*} g(s)\left\{ \frac{1}{x_1(s)} - \frac{1}{x(s)} \right\} ds \\ &\leq \int_t^{t^*} |g(s)| \frac{\{x(s) - x_1(s)\}}{x_1(s)x(s)}\, ds \\ &\leq \frac{1}{2} ||x - x_1||_{L^\infty(t_0, t^*)} \end{aligned}$$

for all $t \in [t_0, t^*]$. However, this is only possible if

$$||x - x_1||_{L^\infty(t_0, t^*)} = 0$$

which contradicts (20). Thus, the lemma cannot be false. $\square$

Combination of Lemmata 1 and 2 yields our first major result.

Theorem 1. *If $f(0) > 0$, then equation* (1) *has a unique positive solution x on an interval $[0, \tau)$ such that $\tau = \infty$ or $x(t) \to 0$ as $t \uparrow \tau$.*

3. Uniqueness. The principal result of this section is the following.

Theorem 2. *Equation* (1) *has none, one or an uncountable number of solutions.*

Proof. To prove this theorem, it is enough to show that if (1) has two distinct solutions, x_1 and x_2 in an interval $[0, \tau) \subseteq [0, \infty)$, then we can actually construct a one-parameter family of such solutions.

Let $t^* \in (0,\tau)$ be such that $0 \le x_1(t^*) < x_2(t^*)$. Next, for fixed $\rho \in (x_1(t^*), x_2(t^*))$ consider the integral equation (12). Lemma 3 states that this equation has a unique positive continuous solution x^* on some interval (t^-, t^+), such that (13)–(15) hold and x^* solves (16) in the sense of Definition 1. Furthermore, by Lemma 4,

$$x_1(t) < x^*(t) < x_2(t) \quad \text{for all} \quad t \in (t^-, t^*]. \tag{22}$$

If now $t^- = 0$, by (14), (16) and (22), x^* must be a solution of (1) on $[0, t^+)$. On the other hand, if $t^- > 0$, then by (14), (15) and (22), necessarily $x^*(t^-) = x_1(t^-) = 0$; in which case, extending x^* by defining $x^* \equiv x_1$ on $[0, t^-)$, it can still be shown that the former function generates a solution of (1) on $[0, t^+)$. We conclude that for any $\rho \in (x_1(t^*), x_2(t^*))$ we can construct a solution of (1) which takes the value ρ at t^*. Since ρ was arbitrary, this confirms the theorem. □

Under the particular constraint that g is nonnegative almost everywhere in a right neighborhood of zero, we can improve on Theorem 2.

Theorem 3. *Suppose that* $\operatorname{ess\,inf}\{g(t) : 0 < t < \tau\} \ge 0$ *for some* $0 < \tau \le \infty$. *Then equation* (1) *has at most one solution in* $[0, \tau)$.

Proof. Suppose, contrary to the assertion of the theorem, that (1) has two distinct solutions on $[0,\tau)$, x_1 and x_2 say. Then there exist a point $t_0 \in [0,\tau)$ and a point $t_1 \in (t_0, \tau)$ such that

$$x_1(t_0) = x_2(t_0)$$

and $x_1(t) \ne x_2(t)$ for all $t \in (t_0, t_1]$. Without loss of generality, we may assume that

$$x_1(t) < x_2(t) \quad \text{for} \quad t \in (t_0, t_1]. \tag{23}$$

Using (1), this infers

$$x_1(t_1) - x_2(t_1) = x_1(t_0) - x_2(t_0) + \int_{t_0}^{t_1} \left\{ \frac{g(s)}{x_1(s)} - \frac{g(s)}{x_2(s)} \right\} ds \ge 0$$

which contradicts (23). ◻

4. The maximal solution. As we have seen with Theorem 1, when $f(0) > 0$ equation (1) is amenable to treatment with the standard theory for nonlinear Volterra integral equations. The real difficulty with the study of (1) is the singularity of its integrand which clearly manifests itself in the event that $f(0) = 0$. To circumvent this difficulty, in this and the following section we shall study equation (1) as the limit as $\mu \downarrow 0$ of the regularized equation

$$x(t) = \mu + f(t) + \int_0^t g(s)/x(s)\, ds \tag{24}$$

where μ is a positive real parameter.

From Theorem 1 we know that (24) has a unique positive solution in a right neighborhood of zero for any $\mu > -f(0)$. We shall denote this solution by $x(t;\mu)$ and its maximal interval of existence by $[0, T(\mu))$, where recalling Theorem 1, either $T(\mu) = \infty$ or $x(t;\mu) \to 0$ as $t \uparrow T(\mu)$. Moreover, by Lemma 4 there holds $T(\mu_1) \leq T(\mu_2)$ and

(25)

$$x(t;\mu_1) < x(t;\mu_2) \quad \text{for all } t \in [0, T(\mu_1)) \quad \text{if } -f(0) < \mu_1 < \mu_2 < \infty.$$

In fact, we can state more about $T(\mu)$.

Lemma 5. *The function T is nondecreasing and continuous from the left on $(-f(0), \infty)$. Moreover,*

$$T(\mu) \uparrow \infty \quad \textit{as} \quad \mu \uparrow \infty. \tag{26}$$

Proof. The monotonicity of T was already established in Lemma 4. Furthermore, in the light of the remarks made in the proof of Lemma 1, we can deduce that T is lower semi-continuous from the standard theory of Volterra integral equations [**4**, Theorem 13.2.3; **5**, Theorem II.4.2]. Together these observations yield the stated monotonicity and continuity. To prove the lemma, it therefore remains to confirm (26).

Suppose that (26) is false. Then

$$t^* := \sup\{T(\mu) : -f(0) < \mu < \infty\} \in (0, \infty).$$

Let $\rho \in (0, \infty)$ and consider (12). Lemma 3 states that this equation has a positive continuous solution x^* on an interval (t^-, t^+) such that (13)–(15) hold and (16) holds in the sense of Definition 1. If, though, $x^*(t^-) > 0$, then $t^- = 0$ by (15). Consequently, $x^*(t)$ can be identified as $x(t; \mu)$ with $\mu = x^*(0) - f(0)$. Whence for this value we have $T(\mu) = t^+ > t^*$, which clearly provides a contradiction of the definition of t^*. On the other hand, if $x^*(t^-) = 0$, choosing μ so large that $T(\mu) > t^-$ there holds $x(t^-; \mu) > 0 = x^*(t^-)$. Whence, by Lemma 4, $T(\mu) \geq t^+$ for any such μ. So, either way, we arrive at a contradiction of the definition of t^*. We can only conclude that $t^* = \infty$ as it were. $\square$

Note that, in general, we are unable to say that the monotonicity of T is strict. By way of illustration consider equation (24) with $f(t) := -4t$ and $g(t) := 1 - t$. It can be verified that $x(t; \mu) = \mu(1 - t)$ is a solution of this equation when $\mu = 2 - \sqrt{3}$ or $\mu = 2 + \sqrt{3}$. Thus, for this combination of coefficients, we have $T(\mu) = 1$ for every $\mu \in [2 - \sqrt{3}, 2 + \sqrt{3}]$.

Lemma 5 and the inequality (25) justify the definition of the function $\tilde{x}(t; 0)$ on $[0, \infty)$ by

$$\tilde{x}(t; 0) = \inf\{x(t; \mu) : \mu \in (0, \infty) \text{ such that } T(\mu) > t\}.$$

Our major assertion is that if equation (1) has a solution, then $\tilde{x}(t; 0)$ constitutes its maximal solution.

Theorem 4. *If equation* (1) *has a solution x on an interval* $[0, \delta)$, *then $\tilde{x}(t; 0)$ is a solution of* (1) *on an interval* $[0, \tau) \supseteq [0, \delta)$ *and $\tilde{x}(t; 0) \geq x(t)$ for all $t \in [0, \delta)$. Furthermore, $\tau = \infty$ or $\tilde{x}(t; 0) \to 0$ as $t \uparrow \tau$.*

To prove this theorem, we introduce the following additional notation. For fixed $\mu \geq 0$, we let

$$S(\mu) := \lim_{\mu' \downarrow \mu} T(\mu').$$

Subsequently, we define

$$T(0) := 0$$

and

$$\Omega := \{t \in (0, \infty) : T(\mu) < t < S(\mu) \text{ for some } \mu \in [0, \infty)\}.$$

These definitions are sensible in view of Lemma 5. Furthermore, since by Lemma 5, T is a monotonic function and a monotonic function has at most a countable number of discontinuities,

$$\Omega = \bigcup_{k=1}^{\infty} (T(\mu_k), S(\mu_k)) \tag{27}$$

for some sequence of values $\{\mu_k\}_{k=1}^{\infty} \subseteq [0, \infty)$.

Using this notation we state and prove five lemmata which culminate in the verification of Theorem 4.

Lemma 6. *The function $\tilde{x}(t; 0)$ is continuous on $[0, \infty)$. Moreover,*

$$\tilde{x}(0; 0) = f(0) \tag{28}$$

and

$$\tilde{x}(t; 0) = 0 \quad \textit{for all} \quad t \in (0, \infty)\backslash\Omega. \tag{29}$$

Proof. Let $t^* \in (0, \infty)$ and recalling Lemma 5 choose $\mu^* \in [0, \infty)$ such that

$$T(\mu^*) \le t^* < T(\mu) \quad \text{for all} \quad \mu > \mu^*.$$

Next, for $\rho > 0$ let x^* denote the positive continuous solution of (12) on its maximal interval of existence $(t^-, t^+) \subseteq (0, \infty)$. This function exists and (13) holds by Lemma 3.

Suppose now that $\rho > \tilde{x}(t^*; 0)$. Then there exists a $\mu > \mu^*$ such that $0 < x(t^*; \mu) < \rho$. Lemma 4 subsequently implies that $t^* \in (0, T(\mu)) \subseteq (t^-, t^+)$ and $x(t; \mu) < x^*(t)$ for all $t \in (0, T(\mu))$. Hence, $\tilde{x}(t; 0) < x^*(t)$ for all $t \in (0, T(\mu))$ and $t^* \in (0, T(\mu))$. This gives

$$\limsup_{t \to t^*} \tilde{x}(t; 0) \le x^*(t^*) = \rho.$$

Noting though that ρ was arbitrary, this establishes the upper semi-continuity of $\tilde{x}$ at t^*.

Suppose next that $\rho < \tilde{x}(t^*;0)$. Then $\rho < x(t^*;\mu)$ for all $\mu > \mu^*$. Whence, by Lemma 4, $(t^-,t^+) \subseteq (0,T(\mu))$ and $x^*(t) < x(t;\mu)$ for any $t \in (t^-,t^+)$ and $\mu > \mu^*$. In the limit $\mu \downarrow \mu^*$ this means that $(t^-,t^+) \subseteq (T(\mu^*),S(\mu^*))$ and that $x^*(t) \le \tilde{x}(t;0)$ for any $t \in (t^-,t^+)$. So $t^* \in (T(\mu^*),S(\mu^*))$ and

$$\liminf_{t\to t^*} \tilde{x}(t;0) \ge x^*(t^*) = \rho.$$

Subsequently, if $\tilde{x}(t^*;0) > 0$ then necessarily $t^* \in \Omega$ and $\tilde{x}$ is lower semi-continuous at t^*.

For $t^* = 0$ we may likewise show that $\tilde{x}(t;0)$ is upper semi-continuous in t^* and that if $\tilde{x}(0;0) > 0$ then $\tilde{x}(t;0)$ is lower semi-continuous in t^*. The only adaptation we have to make to the above argument is to replace the function x^* by $x(t;\rho - f(0))$.

Finally, noting the definition of $\tilde{x}(0;0)$ and that $\tilde{x}(t;0)$ is nonnegative and therefore trivially lower semi-continuous at any point $t^* \in [0,\infty)$ for which $\tilde{x}(t^*;0) = 0$ the above yields the lemma. $\square$

Lemma 7. *Suppose that*

$$g(t)/\tilde{x}(t;0) \in L^1_{\mathrm{loc}}(0,\delta) \tag{30}$$

for some $\delta \in (0,\infty)$. *Then*

$$\begin{aligned}\tilde{x}(t^+;0) \ge \tilde{x}(t^-;0) &+ f(t^+) - f(t^-)\\ &+ \int_{t^-}^{t^+} \min\{0,g(s)\}/\tilde{x}(s;0)\,ds\end{aligned}$$

for any $0 < t^- < t^+ < \delta$. *Moreover, if* $T(\mu_k) \le t^- < t^+ \le S(\mu_k)$ *for some* $\mu_k \in [0,\infty)$, *then*

$$\begin{aligned}\tilde{x}(t^+;0) = \tilde{x}(t^-;0) &+ f(t^+) - f(t^-)\\ &+ \int_{t^-}^{t^+} g(s)/\tilde{x}(s;0)\,ds.\end{aligned} \tag{31}$$

Proof. Fix $0 < t^- < t^+ < \delta$. Define μ^* by $T(\mu^*) \le t^+ < T(\mu)$ for all $\mu > \mu^*$, and set

$$\tilde{x}(t;\mu^*) := \lim_{\mu \downarrow \mu^*} x(t;\mu)$$

for every $t \in [0, t^+]$. Observe that

$$\tilde{x}(t;\mu^*) \ge \tilde{x}(t;0) \quad \text{for} \quad t \in [0, T(\mu^*)) \tag{32}$$

and

$$\tilde{x}(t;\mu^*) = \tilde{x}(t;0) \quad \text{for} \quad t \in [T(\mu^*), t^+]. \tag{33}$$

Now, for any $\mu > \mu^*$, (31) holds with $x(t;\mu)$ in the place of $\tilde{x}(t;0)$ since the former is an appropriate solution of (24). Subsequently, considering (25), (30), (32) and (33) and applying the dominated convergence theorem, (31) also holds with $\tilde{x}(t;\mu^*)$ in lieu of $\tilde{x}(t;0)$. The conclusions of the lemma are now immediate from (32) and (33). $\square$

Lemma 8. *Under the assumptions of Lemma* 7,

$$\tilde{x}(t^+;0) \ge \tilde{x}(t^-;0) + f(t^+) - f(t^-) + \int_{t^-}^{t^+} g(s)/\tilde{x}(s;0)\, ds$$

for any $0 < t^- < t^+ < \delta$.

Proof. Let $\{\mu_k\}_{k=1}^{\infty}$ denote the sequence defined implicitly by (27). Set

$$g_1(t) = \min\{0, g(t)\} \quad \text{for all} \quad t \in (0, \infty)$$

and using induction define

$$g_{k+1}(t) = \begin{cases} g(t) & \text{for } t \in (T(\mu_k), S(\mu_k)) \\ g_k(t) & \text{otherwise} \end{cases} \tag{34}$$

for every $k \ge 1$. Plainly

$$\min\{0, g(t)\} \le g_k(t) \le g_{k+1}(t) \le g(t) \tag{35}$$

for all $t \in (0, \infty)$. Furthermore, $g_k(t) \uparrow g(t)$ as $k \uparrow \infty$ for every $t \in S_1 := \{s \in (0,\infty) : g(s) \le 0 \text{ or } s \in \Omega\}$. However, since by (30) the set $S_2 := \{s \in (0,\delta) : g(s) \neq 0 \text{ and } \tilde{x}(s;0) = 0\}$ must have Lebesgue measure zero, and by (29) the interval $(0,\delta) \subseteq S_1 \cup S_2$ this means that

$$g_k(t) \uparrow g(t) \quad \text{as } k \uparrow \infty \quad \text{for almost all} \quad t \in (0,\delta). \tag{36}$$

We next observe that, as a consequence of (30) and (35),

$$\int_{t^-}^{t^+} |g_k(s)/\tilde{x}(s;0)|\, ds \leq \int_{t^-}^{t^+} |g(s)/\tilde{x}(s;0)|\, ds < \infty \tag{37}$$

for any $0 < t^- < t^+ < \delta$ and $k \geq 1$.

For each $k \geq 1$ we assert that

$$\tilde{x}(t^+;0) \geq \tilde{x}(t^-;0) + f(t^+) - f(t^-) + \int_{t^-}^{t^+} g_k(s)/\tilde{x}(s;0)\, ds \tag{38}$$

for every $0 < t^- < t^+ < \delta$. This assertion is certainly true when $k = 1$ by Lemma 7. Suppose now that it is true for an arbitrary $k \geq 1$. In this event, with $0 < t^- < t^+ < \delta$ fixed, (38) holds with t^+ replaced by $\min\{t^+, \max\{t^-, T(\mu_k)\}\}$. Likewise (38) holds with t^- replaced by $\max\{t^-, \min\{t^+, S(\mu_k)\}\}$. Moreover, (31) holds with $\max\{t^-, \min\{t^+, T(\mu_k)\}\}$ in lieu of t^- and with $\min\{t^+, \max\{t^-, S(\mu_k)\}\}$ in lieu of t^+, by Lemma 7. Adding these three inequalities yields (38) with g_k succeeded by g_{k+1}. By induction the assertion is subsequently true for all $k \geq 1$.

The lemma finally results upon letting $k \uparrow \infty$ in (38). In the light of (36) and (37), the dominated convergence theorem may be invoked to substantiate the desired conclusion. $\square$

Lemma 9. *Let x_1 denote a solution of* (17) *on some interval* $[0,\delta)$ *with* $0 < \delta < \infty$. *Suppose that* $f(0) \geq f_1(0)$ *and that* (18) *and* (19) *hold. Then*

$$\tilde{x}(t;0) \geq x_1(t) \quad \textit{for all} \quad t \in [0,\delta). \tag{39}$$

Proof. For any $\mu > 0$ we have $x(t;\mu) > x_1(t)$ for all $t \in [0, \min\{T(\mu),\delta\})$ by Lemma 4. Whence considering the definition of $\tilde{x}(t;0)$, (39) follows. $\square$

Lemma 10. *Suppose that the hypotheses of Lemma 9 hold and* (30) *holds. Then* $\tilde{x}(t;0)$ *is a solution of* (1) *on* $[0,\delta)$.

Proof. First we show that

$$\tilde{x}(t^+;0) \le \tilde{x}(t^-;0) + f(t^+) - f(t^-) + \int_{t^-}^{t^+} g(s)/\tilde{x}(s;0)\,ds \tag{40}$$

for any $0 < t^- < t^+ < \delta$. The demonstration of this has much in common with the proof of Lemma 8.

Using the sequence $\{\mu_k\}_{k=1}^{\infty}$ defined by (27) we construct a sequence of functions $\{x_k\}_{k=1}^{\infty}$ by

$$x_{k+1}(t) = \begin{cases} \tilde{x}(t;0) & \text{if } t \in [T(\mu_k), S(\mu_k)) \\ x_k(t) & \text{otherwise} \end{cases}$$

and a sequence of functions $\{g_k\}_{k=1}^{\infty}$ by (34). By (29) and (39)

$$x_k(t) \uparrow \tilde{x}(t;0) \quad \text{as} \quad k \uparrow \infty$$

for every $t \in [0,\delta)$. Furthermore,

$$|g_k(t)/x_k(t)| \le |g_1(t)/x_1(t)| + |g(t)/\tilde{x}(t;0)| \tag{41}$$

for any $t \in (0,\delta)$ and $k \ge 1$. So $g_k/x_k \in L^1_{\mathrm{loc}}(0,\delta)$ for every $k \ge 1$. When $k = 1$:

$$x_k(t^+) \le x_k(t^-) + f(t^+) - f(t^-) + \int_{t^-}^{t^+} g_k(s)/x_k(s)\,ds \tag{42}$$

for any $0 < t^- < t^+ < \delta$, since x_1 solves (17) on $(0,\delta)$ and (18) holds. However, recalling (29) and (39), and applying an induction argument similar to that used in the proof of Lemma 8, this infers that (42) actually holds for any $0 < t^- < t^+ < \delta$ and $k \ge 1$. Whence, letting $k \uparrow \infty$ and invoking (41) to justify application of the dominated convergence theorem, we obtain (40) with $g(s)$ replaced by $g_\infty(s) := \sup\{g_k(s) : k \ge 1\}$. Observing though that $g_\infty(s) \le g(s)$ for all $s \in (0,\delta)$, this yields (40) as it stands.

Combining (40) with Lemma 8 yields (40) with equality for any $0 < t^- < t^+ < \delta$. Subsequently, letting $t^- \downarrow 0$ and using the continuity of $\tilde{x}(t;0)$ and (28), we derive that $\tilde{x}(t;0)$ is indeed a solution of (1). $\square$

Theorem 4 follows from Lemmata 9, 10 and 2.

5. Necessary and sufficient conditions for existence. The analysis developed in the previous section can be used to extend our initial knowledge of positive solutions of (24) when $\mu > 0$ to nonnegative ones for all $\mu \geq 0$.

From the previous analysis, we know that (24) has a unique positive solution $x(t;\mu)$ on a maximal interval of existence $[0, T(\mu))$ for any $\mu > 0$. Moreover, we know that

$$\tilde{x}(t;\mu) = \inf\{x(t;\mu') : \mu' \in (\mu, \infty) \text{ such that } T(\mu') > t\}$$

defines a continuous function which is the maximal solution to this equation. With little effort it can be seen that this definition is equivalent to

$$\tilde{x}(t;\mu) = \begin{cases} x(t;\mu) & \text{for } t < T(\mu) \\ \tilde{x}(t;0) & \text{for } t \geq T(\mu). \end{cases}$$

Hence,

$$\tilde{x}(t;\mu_1) \leq \tilde{x}(t;\mu_2) \quad \text{for all} \quad t \in [0,\infty) \quad \text{if} \quad 0 \leq \mu_1 < \mu_2 < \infty,$$

and

$$\tilde{x}(t;\mu) \downarrow \tilde{x}(t;0) \quad \text{as } \mu \downarrow 0 \quad \text{for all} \quad t \in [0,\infty).$$

Now, for any $\mu \geq 0$, let $[0, \tilde{T}(\mu))$ denote the maximal interval of existence of $\tilde{x}(t;\mu)$ as a solution of (24) with the convention that $\tilde{T}(0) = 0$ if (1) has no solution. Plainly $\tilde{T}(\mu) \geq T(\mu)$, and by Theorem 4

$$\tilde{x}(\tilde{T}(\mu), \mu) = 0 \quad \text{if} \quad \tilde{T}(\mu) < \infty$$

for every $\mu \geq 0$. Furthermore, by Lemma 10, $\tilde{T}$ is monotonic increasing on $[0,\infty)$. Subsequently, for any $\mu \geq 0$ we may define

$$\tilde{S}(\mu) := \lim_{\mu' \downarrow \mu} \tilde{T}(\mu').$$

We can now state necessary and sufficient conditions for the existence of a solution of (1).

Theorem 5. *Equation* (1) *has a solution if and only if*

$$\tilde{S}(0) > 0 \tag{43}$$

and there exists a $\sigma \in (0, \tilde{S}(0))$ *such that*

$$g(t)/\tilde{x}(t;0) \in L^1_{\mathrm{loc}}(0,\sigma). \tag{44}$$

Moreover, in this event, $\tilde{T}(0) = \sup\{\sigma \in [0, \tilde{S}(0)) : (44)$ *holds*$\}$.

Proof. Suppose firstly that (1) admits a solution, i.e., $\tilde{T}(0) > 0$. Then the necessity of (43) and of (44) for any $\sigma \in (0, \tilde{T}(0))$ follow immediately from Lemma 10 and the definition of a solution. On the other hand, suppose that (43) holds. Then, since $\tilde{x}(t;\mu)$ is a solution of (24) on $[0, \tilde{T}(\mu)) \supseteq [0, \tilde{S}(0))$ for every $\mu > 0$,

$$\tilde{x}(t^+;\mu) = \tilde{x}(t^-;\mu) + f(t^+) - f(t^-) + \int_{t^-}^{t^+} g(s)/\tilde{x}(s;\mu)\, ds \tag{45}$$

for any $0 < t^- < t^+ < \tilde{S}(0)$. In addition, if (44) holds for some $\sigma \in (0, \tilde{S}(0))$ and if $0 < t^- < t^+ < \sigma$, we may take the limit $\mu \downarrow 0$ in (45). Hereafter, letting $t^- \downarrow 0$ it can be deduced that $\tilde{x}(t;0)$ solves (1) on $[0,\sigma)$ as in the completion of the proof of Lemma 10. $\square$

If the coefficient g is nonnegative almost everywhere or is nonpositive almost everywhere in a right neighborhood of zero, Theorem 5 can be improved upon.

Theorem 6. *Suppose that* $\operatorname{ess\,inf}\{g(t) : 0 < t < \tau\} \geq 0$ *for some* $0 < \tau \leq \infty$. *Then equation* (1) *has a solution if and only if* $\tilde{S}(0) > 0$. *Moreover, in this event,* $\min\{\tilde{T}(0), \tau\} = \min\{\tilde{S}(0), \tau\}$.

Proof. This theorem is actually no more than a corollary of the previous one. Under the additional hypothesis,

$$\int_0^\sigma |g(s)/\tilde{x}(s;\mu)|\, ds = \int_0^\sigma g(s)/\tilde{x}(s;\mu)\, ds = \tilde{x}(\sigma;\mu) - \mu - f(\sigma)$$

for any $\sigma \in (0, \min\{\tilde{T}(\mu), \tau\})$ and $\mu > 0$. Thus, letting $\mu \downarrow 0$ and invoking the monotone convergence theorem, (44) is automatically satisfied for every $\sigma \in (0, \min\{\tilde{S}(0), \tau\})$. $\square$

Theorem 7. *Suppose that* $\operatorname{ess\,sup}\{g(t) : 0 < t < \tau\} \leq 0$ *for some* $0 < \tau \leq \infty$. *Then equation* (1) *has a solution if and only if* $S(0) > 0$. *Moreover, in this event,* $\min\{\tilde{T}(0), \tau\} = \min\{S(0), \tau\}$.

Recall that $S(0)$ is defined as the limit of $T(\mu)$ as $\mu \downarrow 0$ where $T(\mu) = \sup\{t \in [0, \tilde{T}(\mu)) : \tilde{x}(s; \mu) > 0 \text{ for all } s \in [0, t)\}$.

Proof of Theorem 7. If $\tilde{T}(0) > 0$, then we have

$$\begin{aligned} \tilde{x}(t; \mu) &= \mu + f(t) + \int_0^t g(s)/\tilde{x}(s; \mu)\, ds \\ &\geq \mu + f(t) + \int_0^t g(s)/\tilde{x}(s; 0)\, ds \\ &= \mu + \tilde{x}(t; 0) \geq \mu \end{aligned}$$

for all $t \in [0, \min\{\tilde{T}(0), \tau\})$ and $\mu > 0$. Thus $S(0) \geq \min\{\tilde{T}(0), \tau\}$. On the other hand, if $S(0) > 0$, then

$$\begin{aligned} \int_0^\sigma |g(s)/\tilde{x}(s; \mu)|\, ds &= -\int_0^\sigma g(s)/x(s; \mu)\, ds \\ &= \mu + f(\sigma) - x(\sigma; \mu) \end{aligned}$$

for any $\sigma \in (0, \min\{T(\mu), \tau\})$ and $\mu > 0$. Hence, arguing as in the proof of Theorem 6, $\tilde{T}(0) \geq \min\{S(0), \tau\}$. $\square$

Under the assumption in Theorem 7, we can also provide an alternative criterion for the existence of a solution of (1).

Theorem 8. *Suppose that* $\operatorname{ess\,sup}\{g(t) : 0 < t < \tau\} \leq 0$ *for some* $0 < \tau \leq \infty$. *Set*

$$\sigma_0 := \tau$$

and

$$x_0(t) := f(t) \quad \text{for all} \quad t \in [0, \sigma_0).$$

Subsequently, using induction define

$$\sigma_{k+1} := \sup\{\delta \in [0,\sigma_k) : x_k(s) \geq 0 \quad \textit{for all } s \in [0,\delta] \textit{ and } g/x_k \in L^1(0,\delta)\}$$

and

$$x_{k+1}(t) := f(t) + \int_0^t g(s)/x_k(s)\,ds \quad \textit{for all} \quad t \in [0,\sigma_{k+1}) \tag{46}$$

for every $k \geq 0$. *Then equation* (1) *has a solution if and only if* $\sigma_\infty := \inf\{\sigma_k : 0 \leq k < \infty\} > 0$. *Moreover, in this event,* $\min\{\tilde{T}(0),\tau\} = \sigma_\infty$ *and* $\tilde{x}(t;0) = \inf\{x_k(t) : 0 \leq k < \infty\}$ *for all* $t \in [0,\sigma_\infty)$.

Proof. We observe, to begin with, that by definition the sequence of values σ_k is decreasing. Moreover, $x_1(t) \leq x_0(t)$ for all $t \in [0,\sigma_1)$. Subsequently substituting in (46),

$$x_{k+1}(t) \leq x_k(t) \quad \text{for all} \quad t \in [0,\sigma_{k+1}),$$

for every $k \geq 0$. Furthermore,

$$\int_0^t |g(s)/x_k(s)|\,ds = -\int_0^t g(s)/x_k(s)\,ds \leq f(t)$$

for any $t \in (0,\sigma_k)$. It follows that, if $\sigma_\infty > 0$, then

$$x_\infty(t) := \inf\{x_k(t) : 0 \leq k < \infty\}$$

is well defined for every $t \in [0,\sigma_\infty)$, and $g/x_\infty \in L^1(0,\delta)$ for every $\delta \in (0,\sigma_\infty)$. Moreover, letting $k \uparrow \infty$ in (46), x_∞ solves (1) on $[0,\sigma_\infty)$. On the other hand, if (1) is supposed to have a solution $\tilde{x}(t;0)$ on an interval $[0,\tilde{T}(0))$, then from (1) itself follows simply $x_0(t) = f(t) \geq \tilde{x}(t;0)$ for all $t \in [0,\min\{\tau,\tilde{T}(0)\})$. Whence, using induction, this infers that

$$x_{k+1}(t) \geq f(t) + \int_0^t g(s)/\tilde{x}(s;0)\,ds = \tilde{x}(t;0)$$

and

$$\int_0^t |g(s)/x_k(s)|\, ds \le -\int_0^t g(s)/\tilde{x}(s;0)\, ds < \infty$$

for all $t \in (0, \min\{\tau, \tilde{T}(0)\})$ and any $k \ge 0$. Thus, we deduce that $\sigma_\infty \ge \min\{\tau, \tilde{T}(0)\}$ and $x_\infty(t) \ge \tilde{x}(t;0)$ for all $t \in [0, \min\{\tau, \tilde{T}(0)\})$. $\square$

6. Comparison principles. The objective of this section is to present some results indicating how the solvability of one equation of the type (1) may be used to deduce the solvability of another.

Our first result along this line is the following.

Theorem 9. *Suppose that, for some f_1 and g_1, the equation*

$$x_1(t) = f_1(t) + \int_0^t g_1(s)/x_1(s)\, ds \tag{47}$$

has a solution x_1 in an interval $[0,\tau)$ with $0 < \tau < \infty$. Suppose furthermore that

$$f(0) \ge f_1(0), \tag{48}$$

$$(f - f_1) \quad \textit{is nondecreasing on} \quad [0,\tau), \tag{49}$$

$$g(t) \ge g_1(t) \quad \textit{for almost all} \quad t \in (0,\tau)$$

and

$$g/x_1 \in L^1_{\mathrm{loc}}(0,\tau). \tag{50}$$

Then $\tilde{x}(t;0)$ solves (1) on $[0,\tau)$ and

$$\tilde{x}(t;0) \ge x_1(t) \quad \textit{for all} \quad t \in [0,\tau). \tag{51}$$

Moreover, if

$$(f - f_1)(s) < (f - f_1)(t) \quad \textit{for all} \quad s \in [0,t)$$

for some $t \in (0,\tau)$, *then*

$$\tilde{x}(t;0) = x_1(t) \quad \textit{if and only if} \quad \tilde{x}(t;0) = 0.$$

Proof. The primary conclusions of this theorem are contained in Lemmata 9 and 10. Furthermore, if $\tilde{x}(t;0) = 0$ for some point $t \in [0,\tau)$, then by (51), plainly, $\tilde{x}(t;0) = x_1(t)$. The proof of the theorem therefore boils down to the establishment of the impossibility of

$$\tilde{x}(t^*;0) = x_1(t^*) > 0 \tag{52}$$

for some $t^* \in (0,\tau)$, for which

$$(f - f_1)(t^*) > (f - f_1)(t) \quad \text{for all} \quad t \in [0,t^*). \tag{53}$$

This we achieve by contradiction.

If there is a $t^* \in (0,\tau)$ such that (52) and (53) hold, then there must be a $0 \le t^- < t^* < t^+ < \infty$ such that $\tilde{x}(t;0)$ and x_1 are defined as solutions of (1) and (47), respectively, on $[0,t^+)$ and

$$\tilde{x}(t;0) \ge x_1(t) > 0 \quad \text{for all} \quad t \in [t^-,t^+).$$

Should, though, $\tilde{x}(t;0) = x_1(t)$ for all $t \in [t^-,t^*]$, then

$$\begin{aligned}
0 &= \tilde{x}(t^*;0) - x_1(t^*) \\
&= \left\{ \tilde{x}(t^-;0) + f(t^*) - f(t^-) + \int_{t^-}^{t^*} g(s)/\tilde{x}(s;0)\,ds \right\} \\
&\quad - \left\{ x_1(t^-) + f_1(t^*) - f_1(t^-) + \int_{t^-}^{t^*} g_1(s)/x_1(s)\,ds \right\} \\
&= (f - f_1)(t^*) - (f - f_1)(t^-) + \int_{t^-}^{t^*} \{g(s) - g_1(s)\}/x_1(s)\,ds \\
&\ge (f - f_1)(t^*) - (f - f_1)(t^-)
\end{aligned}$$

which contradicts (53). Thus, if (53) holds, there is a $t_0 \in (t^-,t^*)$ for which $\tilde{x}(t_0;0) > x_1(t_0)$. However, in this event, Lemma 4 implies $\tilde{x}(t;0) > x_1(t)$ for all $t \in [t_0,t^+)$. This provides the sought-after contradiction. $\square$

Remark. If $g \equiv g_1$ in Theorem 9, then the assumption (50) is automatically satisfied. Moreover, in general, (50) can be replaced by the weaker hypothesis $g(t)/\tilde{x}(t;0) \in L^1_{\mathrm{loc}}(0,\tau)$.

Note that, in general, even if the coefficients in (1) and (47) are smooth and only positive solutions of these equations are considered, hypotheses (48) and (49) in Theorem 9 cannot be replaced by the weaker assumption

$$f(t) \geq f_1(t) \quad \text{for all} \quad t \in [0,\tau). \tag{54}$$

As a counter-example, consider

$$f_1(t) := 1, \qquad f(t) := 1 + (1+5t)^{-2}(1+7t+t^2)$$

and

$$g_1(t) := g(t) := 6(1+5t)^{-3}(1+11t).$$

It can be checked that, with this set of coefficients, $x_1(t) = (1+5t)^{-1}(1+11t)$ is the unique solution of (47) on $[0,\infty)$ whilst $x(t) = 2$ is the unique solution of (1) on $[0,\infty)$. However, whereas

$$f(t) > f_1(t) \quad \text{for all} \quad t \in [0,\infty)$$

there holds

$$x(t) \geq x_1(t) \quad \text{only if} \quad t \leq 1.$$

Notwithstanding, if $g_1 \leq 0$ almost everywhere in a right neighborhood of zero, we may replace (48) and (49) in Theorem 9 by (54). Moreover, in this event we can also drop hypothesis (50).

Theorem 10. *Suppose that, for some f_1 and g_1, the equation* (47) *has a solution x_1 in an interval $[0,\tau)$ with $0 < \tau < \infty$. Suppose, furthermore, that* (54) *holds and*

$$\min\{0, g(t)\} \geq g_1(t) \quad \textit{for almost all} \quad t \in (0,\tau). \tag{55}$$

Then $\tilde{x}(t;0)$ solves (1) *on $[0,\tau)$ and*

$$\tilde{x}(t;0) \geq x_1(t) + f(t) - f_1(t) \quad \textit{for all} \quad t \in [0,\tau). \tag{56}$$

Proof. For every $\mu > 0$,

$$x(t;\mu) > x_1(t) + f(t) - f_1(t) + \int_0^t \max\{0, g(s)\}/x(s;\mu)\,ds \tag{57}$$

for all t in a small enough neighborhood of zero. Supposing, however, that there is a $t^* < \min\{T(\mu), \tau\}$ which demarcates the supremum of all such t, we compute

$$\begin{aligned}
x(t^*;\mu) &= \mu + f(t^*) + \int_0^{t^*} \max\{0, g(s)\}/x(s;\mu)\,ds \\
&\quad + \int_0^{t^*} \min\{0, g(s)\}/x(s;\mu)\,ds \\
&\geq \mu + f(t^*) + \int_0^{t^*} \max\{0, g(s)\}/x(s;\mu)\,ds \\
&\quad + \int_0^{t^*} g_1(s)/x_1(s)\,ds \\
&= \mu + x_1(t^*) + f(t^*) - f_1(t^*) + \int_0^{t^*} \max\{0, g(s)\}/x(s;\mu)\,ds.
\end{aligned}$$

We must therefore conclude that (57) actually holds for all $t \in [0, \tau)$ and moreover, as a consequence, $T(\mu) \geq \tau$. This yields (56) and the observation that $S(0) \geq \tau$. Recalling Theorem 5, it subsequently suffices to show that

$$g(t)/\tilde{x}(t;0) \in L^1_{\mathrm{loc}}(0,\tau) \tag{58}$$

to prove the present theorem. From (55) and (56), though, we have

$$\int_0^t \min\{0, g(s)\}/x(s;\mu)\,ds \geq \int_0^t g_1(s)/x_1(s)\,ds = x_1(t) - f_1(t)$$

for any $t \in (0, \tau)$, whilst by (57)

$$\int_0^t \max\{0, g(s)\}/x(s;\mu)\,ds \leq x(t;\mu)$$

for any $t \in (0, \tau)$. This yields

$$\int_0^t |g(s)/x(s;\mu)| \, ds \leq x(t;\mu) + f_1(t)$$

for all $t \in (0, \tau)$. Whence, in the limit $\mu \downarrow 0$, (58) is obtained. $\square$

7. Negative kernel. As has been indicated by a number of theorems in the previous sections, if the function g does not effectively change sign in a right neighborhood of zero, then certain inferences of the general theory for equation (1) can be sharpened. In this and the next section, we shall present a number of further results specifically concerning (1) under this assumption. Since if $f(0) > 0$ then we definitely know that the equation has a unique positive solution in a right neighborhood of zero (cf. Theorem 1), we shall concentrate on the more open case of

$$f(0) = 0. \tag{59}$$

The results we obtain serve to illustrate the three possibilities regarding the number of solutions of the equation mentioned in Theorem 2.

In this section we consider the option

$$\operatorname{ess\,sup}\{g(t) : 0 < t < \tau\} \leq 0 \quad \text{for some} \quad 0 < \tau \leq \infty. \tag{60}$$

For convenience, we define

$$G(t) = \left| 2 \int_0^t g(s) \, ds \right|^{1/2} \tag{61}$$

for all $t \in [0, \tau)$. Since when $g = 0$ almost everywhere in a right neighborhood of zero, (1) reduces to the trivial identity $x = f$, we shall generally henceforth assume that

$$G(t) > 0 \quad \text{for all} \quad t \in (0, \tau). \tag{62}$$

The next two theorems illustrate that, under the conditions (59), (60) and (62), equation (1) may or may not admit a solution. Moreover, this is dependent upon the behavior of $f(t)$ as $t \downarrow 0$.

Theorem 11. *Suppose that* (60) *and* (62) *hold. Set*

$$L(t) := |\ln G(t)|^{-1} \tag{63}$$

and

$$J(t) := |\ln L(t)|^{-1}. \tag{64}$$

Then if

$$f(t) \geq (2G - GL^2\{1 + J^2\}/4)(t) \quad \textit{for all} \quad t \in (0, \tau),$$

equation (1) *has a* (*maximal*) *solution* $\tilde{x}(t;0)$ *on an interval* $[0, \delta)$ *with* $0 < \delta \leq \tau$ *such that*

$$-(G+GL\{1+J\}^2/2)(t) \leq \tilde{x}(t;0) - f(t) \leq 0 \quad \textit{for all} \quad t \in (0, \delta). \tag{65}$$

Proof. Define

$$D := 2 + L(1 + J + J^2) + L^2(1 + J)(1 + 2J^2)$$

and

$$\begin{aligned} E := 1 - J\{2 + LJ^{-4}(1 + J + J^2)(3 + 2J + 5J^2 + 4J^3) \\ + L^2J^{-4}(1 + J)(1 + J^2)(1 + 2J^2)\}/4 \end{aligned}$$

as functions of t. Then it can be verified that, when

$$f_1 := 2G - GL^2\{1 + J^2\}/4 - GL^2J^3E/D$$

and $g_1 := g$, the function

$$x_1 := 2G/D$$

is a solution of (47) in any interval $[0, \delta) \subseteq [0, \tau)$ for which $G(t) < \exp(-1)$ for all $t \in [0, \delta)$. Hence, when δ is chosen so small that $G(t) < \exp(-1)$ and $E(t) > 0$ for all $t \in (0, \delta)$, the assertion that

(1) has a solution on $[0,\delta)$ is a corollary of Theorem 10. Moreover, this theorem supplies the estimate

$$\tilde{x}(t;0) \geq x_1(t) + f(t) - f_1(t) = f(t) - (G + GL\{1 + J + J^2\}/2)(t)$$

for $t \in (0,\delta)$, which gives the left-hand inequality in (65). The right-hand inequality in (65) is evident. □

Theorem 12. *Suppose that the introductory assumptions of Theorem* 11 *hold. Then if*

$$f(t) \leq (2G - \alpha GL^2)(t) \quad \text{for all} \quad t \in (0,\tau) \tag{66}$$

for some $\alpha > 1/4$, *equation* (1) *has no solution.*

Proof. Let us suppose that (1) does have a solution x on an interval $[0,\delta) \subseteq [0,\tau)$. Without loss of generality, we may take δ to be so small that

$$L(t) \leq 8 \quad \text{for all} \quad t \in (0,\delta). \tag{67}$$

Set

$$H(t) := G^{-1}(t)L^{-1}(t) \quad \text{for} \quad t \in (0,\delta)$$

and note that H is locally absolutely continuous on $(0,\delta)$ with

$$H'(t) = G^{-2}(t)H(t)\{1 + L(t)\}g(t) \leq 0 \tag{68}$$

for almost all $t \in (0,\delta)$. Next, set

$$Y(t) := -\int_0^t g(s)/x(s)\,ds. \tag{69}$$

Our goal is to obtain an estimate of Y which is absurd.

By (1) and (66), $x(t) \leq f(t) \leq 2G(t)$ for all $t \in [0,\delta)$. Hence, we can define

$$A := \sup\{x(t)/G(t) : 0 < t < \delta\}$$

with the assurance that $0 \leq A \leq 2$. By the definition of A, though, for any $\varepsilon > 0$ one can find a $t \in (0,\delta)$ such that $(A - \varepsilon)G(t) \leq x(t)$ whilst

$x(s) \le (A+\varepsilon)G(s)$ for all $s \in (0,t)$. Substituting these inequalities in (1) gives

$$\begin{aligned}(A-\varepsilon)G(t) &\le f(t) + \int_0^t g(s)/\{(A+\varepsilon)G(s)\}\,ds \\ &= f(t) - G(t)/(A+\varepsilon) \\ &\le 2G(t) - G(t)/(A+\varepsilon).\end{aligned}$$

Whence, multiplying by $(A+\varepsilon)/G(t)$ and thereafter letting $\varepsilon \downarrow 0$, we obtain $(A-1)^2 \le 0$. This implies $A = 1$ and, thus, $x(s) \le G(s)$ for any $s \in [0,\delta)$. So, for a start, we have the estimate

$$Y(\varepsilon) \ge -\int_0^\varepsilon g(s)/G(s)\,ds = G(\varepsilon) \tag{70}$$

for any $\varepsilon \in (0,\delta)$.

To sharpen (70), we utilize the formula for integration by parts:

$$Y(t)H(t) - Y(\varepsilon)H(\varepsilon) = \int_\varepsilon^t \{Y'(s)H(s) + Y(s)H'(s)\}\,ds \tag{71}$$

for any $0 < \varepsilon < t < \delta$. Applying (69) to eliminate Y' and (1) to eliminate Y, (71) becomes

$$\begin{aligned}Y(t)H(t) &= \int_\varepsilon^t \{-gH/x + (f-x)H'\}\,ds + Y(\varepsilon)H(\varepsilon) \\ &= \int_\varepsilon^t [\{|xH'|^{1/2} - |Hg/x|^{1/2}\}^2 + 2|HH'g|^{1/2} + H'f]\,ds \\ &\quad + Y(\varepsilon)H(\varepsilon) \\ &\ge \int_\varepsilon^t \{2|HH'g|^{1/2} + H'f\}\,ds + Y(\varepsilon)H(\varepsilon).\end{aligned}$$

Hence, inserting (68), (66) and (70) in this expression,

$$\begin{aligned}Y(t)H(t) &\ge \int_\varepsilon^t G^{-1}H\{2(1+L)^{1/2} - (1+L)(2-\alpha L^2)\}|g|\,ds + L^{-1}(\varepsilon) \\ &\ge \int_\varepsilon^t G^{-1}H\{2(1+L)^{1/2} - 2 - 2L + \alpha L^2\}|g|\,ds + L^{-1}(\varepsilon).\end{aligned}$$

In the last integral we now substitute the inequality $2(1+L)^{1/2} \geq 2+L-L^2/4$ which can be verified under the constraint (67) by taking the square of both its sides. This yields

$$\begin{aligned} Y(t)H(t) &\geq \int_\varepsilon^t G^{-2}\{(\alpha-1/4)L-1\}|g|\,ds + L^{-1}(\varepsilon) \\ &= (\alpha-1/4)\ln\{L(t)/L(\varepsilon)\} + L^{-1}(t). \end{aligned}$$

Subsequently, letting $\varepsilon \downarrow 0$ gives $Y(t)=\infty$ for any $t \in (0,\delta)$.

Thus, supposing that when (66) holds, (1) does have a solution on an interval $[0,\delta)$, we have indeed obtained an absurdity. By means of this contradiction, the theorem is proved. $\square$

An alternative necessary condition for the existence of a solution of (1) is contained below.

Theorem 13. *Suppose that* (60) *holds. Then, if*

$$f(s) \leq f(t) + \beta\{G^2(t) - G^2(s)\}^{1/2} \quad \text{for all} \quad 0 \leq s < t$$

and

$$f(t) < AG(t) \tag{72}$$

where

$$A = 2/(\sqrt{\beta^2+4}+\beta) \tag{73}$$

for some $t \in (0,\tau)$ *and* $-\infty < \beta \leq \infty$, *equation* (1) *has no solution on* $[0,\tau)$.

The next lemma will be used for the proof of this theorem.

Lemma 11. *Suppose that* $\operatorname{ess\,inf}\{g(t) : 0 < t < \tau\} \geq 0$ *and* $f(t) \leq \beta G(t)$ *for all* $t \in [0,\tau)$ *where* $G(t)$ *is defined by* (61), *for some* $0 < \tau < \infty$ *and* $-\infty < \beta \leq \infty$. *Then, given any* $\mu > 0$, *there exists a* $\rho > 0$ *such that*

$$x(t;\mu) \geq \rho + f(t) + AG(t) \quad \text{for all} \quad t \in [0, \min\{T(\mu),\tau\}) \tag{74}$$

where A is defined by (73).

Proof. Since $\operatorname{ess\,inf}\{g(t) : 0 < t < \tau\} \geq 0$, equation (24) immediately gives $x(t;\mu) \geq \mu + f(t)$ for all $t \in [0, \min\{T(\mu), \tau\})$. This yields (74) with $\rho = \mu$ in the case $\beta = \infty$. To confirm (74) when $\beta < \infty$, set

$$Y(t) := \mu + \int_0^t g(s)/x(s;\mu)\,ds = x(t;\mu) - f(t)$$

and

$$\delta := \sup\{t \in [0, \min\{T(\mu), \tau\}) : x(s;\mu) > f(s) \\ + AG(s) \text{ for all } s \in [0,t)\}.$$

Multiplying (24) by Y' this equation can be rewritten as

$$\begin{aligned} 0 &= Y(t)Y'(t) + f(t)Y'(t) - G(t)G'(t) \\ &\leq Y(t)Y'(t) + \beta G(t)Y'(t) - G(t)G'(t) \end{aligned}$$

for almost all $t \in (0, \delta)$. Whence, multiplying by $(A + 1/A)^{-1}(Y + G/A)^{A-1}(Y - AG)^{1/A-1}$ and integrating from 0 to t we obtain

$$\mu^{A+1/A} \leq (Y + G/A)^A (Y - AG)^{1/A}(t)$$

or

$$\{x(t;\mu) - f(t) + G(t)/A\}^A \{x(t;\mu) - f(t) - AG(t)\}^{1/A} \geq \mu^{A+1/A}$$

for every $t \in [0, \delta)$. If, though, $x(t;\mu) - f(t) - AG(t) \to 0$ as $t \uparrow \delta$ with $\delta < \infty$, this last inequality is self-contradictory. We are therefore forced to conclude that $\delta \geq \min\{T(\mu), \tau\}$ and there exists a $\rho > 0$ such that (74) holds. This completes the proof in the case $\beta < \infty$. $\square$

Proof of Theorem 13. Suppose that (1) has a solution on $[0, \tau)$. Pick $\mu > 0$. By Theorem 10, necessarily $T(\mu) > t$. Subsequently setting $\mu^* := x(t;\mu)$, $f^*(s) := f(t - s) - f(t)$ and $g^*(s) := -g(t - s)$, it can be verified that $x^*(s) := x(t - s;\mu)$ is the solution on $[0, t)$ of equation (24) with μ, f and g replaced by μ^*, f^* and g^*, respectively. Applying Lemma 11 to this variant of (24) yields

$$x(s;\mu) \geq \rho + f(s) - f(t) + A\{G^2(t) - G^2(s)\}^{1/2}$$

for all $s \in (0,t)$ for some $\rho > 0$. Whence letting $s \downarrow 0$, we obtain

$$f(t) > AG(t) - \mu.$$

Passing to the limit $\mu \downarrow 0$ delivers a contradiction of (72). $\square$

Corollary. *Suppose that* (60) *holds. Then* (1) *has a solution on* $[0,\tau)$ *only if* $f(t) \geq 0$ *for all* $t \in [0,\tau)$. *Moreover, if* f *is nondecreasing on* $[0,\tau)$, *then* (1) *has a solution on* $[0,\tau)$ *only if* $f(t) \geq G(t)$ *for all* $t \in [0,\tau)$.

Proof. The first assertion is trivial but can be obtained from the theorem by taking $\beta = \infty$. The second assertion is deducible by letting $\beta = 0$. $\square$

Having established that if (59) and (60) hold equation (1) may or may not have a solution, we turn now to a discussion of the possible number of solutions in the first event. Our main result in this direction will call upon the following simple corollary of Theorem 12.

Lemma 12. *Suppose that* (60) *holds and* $f(t) \leq \alpha G(t)$ *for all* $t \in [0,\tau)$ *for some* $\alpha < 2$. *Then* $\tilde{x}(t;0)$ *solves* (1) *on* $[0,\tau)$ *only if* $\tilde{x}(t;0) = f(t) = G(t) = 0$ *for all* $t \in [0,\tau)$.

The main result itself is the following.

Theorem 14. *Suppose that* (59), (60), *and* (62) *hold, and furthermore, given any* $t \in (0,\tau)$, *there exist a* $\delta > 0$ *and an* $\alpha < 2$ *such that*

$$f(s) - f(t) \leq \alpha\{G^2(s) - G^2(t)\}^{1/2} \tag{75}$$

for all $s \in (t, t+\delta)$. *Then either* (1) *has no solution or it has an uncountable number of solutions.*

Proof. Since (62) holds, if $\tilde{T}(0) > 0$ we can pick an arbitrarily small $t^* \in (0, \min\{\tilde{T}(0), \tau\})$ such that $\tilde{x}(t^*;0) > 0$. For fixed $\rho \in (0, \tilde{x}(t^*;0))$

consider the integral equation (12). By Lemma 3 there is a unique positive solution x^* to this equation, and when (t^-, t^+) denotes its maximal interval of existence, (13)–(16) hold. Furthermore, by Lemma 4, we have

$$\tilde{x}(t;0) > x^*(t) \quad \text{for all} \quad t \in (t^-, t^+). \tag{76}$$

Applying Lemma 12 to (16), though, the possibility that $t^- > 0$ is excluded by (75). So $t^- = 0$; by (76) necessarily $x^*(0) = 0$; and, subsequently, x^* is a solution of (1) on $[0, t^*)$. In view of the arbitrariness of t^* and ρ, this yields the theorem. □

Remark. As much as if (1) is solvable, then it has a maximal solution by Theorem 4, the proof of Theorem 14 shows that, in general, the equation has no complementary minimal solution.

In the last section we discuss a specific example of (1) in which this phenomenon is overtly apparent.

8. Positive kernel. This section comprises a similar study to that in the previous section for equation (1) when

$$\operatorname{ess\,inf}\{g(s) : 0 < s < \tau\} \geq 0 \quad \text{for some} \quad 0 < \tau \leq \infty. \tag{77}$$

As in the previous section, it is convenient to define G by (61). Moreover, in order that (1) does not reduce to the trivial identity $x = f$, we shall again generally assume that (62) holds. On the one hand, this case is easier to analyze than the previous one. This is because, by Theorem 3, the equation is now known to have at most one solution in $[0, \tau)$. On the other hand, we are handicapped in that instead of Theorem 10 we now have to fall back on Theorem 9 for comparison arguments. This specifically means that, given equations (1) and (47) with $g_1 \equiv g$ we can only deduce the existence of a solution of (1) from the existence of a solution of (47) when (48) and (49) hold. Thus, for instance, although for $f_1 := -\gamma G$ for some constant $\gamma > 0$ and $g_1 := g$ we can compute that

$$x_1(t) := \{(\sqrt{\gamma^2 + 4} - \gamma)/2\}G(t)$$

is a solution of (47), this only infers the existence of a solution of (1) when $(f+\gamma G)$ is nondecreasing. When this information is not available, the next result can be of help.

Theorem 15. *Suppose that* (77) *and* (62) *hold and that*

$$\alpha G(t) \le f(t) \le \beta G(t) \quad \text{for all} \quad t \in [0,\tau) \tag{78}$$

for some $-\infty < \alpha \le \beta \le \infty$. *Set*

$$A = 2/(\sqrt{\beta^2+4}+\beta) \quad \text{and} \quad B = 2/(\sqrt{\alpha^2+4}+\alpha).$$

Then, if

$$\alpha \ge -A \tag{79}$$

equation (1) has a (*unique*) *solution* $\tilde{x}(t;0)$ *on* $[0,\tau)$ *such that*

$$AG(t) \le \tilde{x}(t;0) - f(t) \le BG(t) \quad \text{for all} \quad t \in [0,\tau). \tag{80}$$

Proof. Without loss of generality, we may assume that $\tau < \infty$. Referring to the machinery set up in Sections 4 and 5, for every $\mu > 0$ the equation (24) has a unique positive solution on a maximal interval of existence $[0, T(\mu))$. Furthermore, by Lemma 11, there is a $\rho > 0$ such that (74) holds. Whence, by (78) and (79), $x(t;\mu) \ge \rho > 0$ for all $t \in [0, \min\{T(\mu),\tau\})$. However, seeing that $x(t;\mu) \to 0$ as $t \uparrow T(\mu)$ if $T(\mu) < \infty$ this is only possible if $T(\mu) \ge \tau$. Consequently, $T(\mu) \ge \tau$ and

$$x(t;\mu) > f(t) + AG(t) \quad \text{for all} \quad t \in [0,\tau). \tag{81}$$

Theorem 6 now states that $\tilde{x}(t;0)$ is a solution of (1) on $[0,\tau)$. Letting $\mu \downarrow 0$ in (81) yields the left-hand inequality in (80).

It remains to establish the right-hand inequality in (80). Supposing though that this is not true, there must be a $t_0 \in [0,\tau)$ and a $t_1 \in (t_0,\tau)$ such that

$$\tilde{x}(t_0;0) = f(t_0) + BG(t_0)$$

and

$$\tilde{x}(t;0) > f(t) + BG(t) \quad \text{for all} \quad t \in (t_0, t_1]. \tag{82}$$

Setting

$$Y(t) := \int_0^t g(s)/\tilde{x}(s;0)\, ds,$$

we may rewrite (1) as

$$\begin{aligned} 0 &= Y(t)Y'(t) + f(t)Y'(t) - G(t)G'(t) \\ &\geq Y(t)Y'(t) + \alpha G(t)Y'(t) - G(t)G'(t) \end{aligned}$$

for almost all $t \in (0,\tau)$. Whence, multiplying by $(B+1/B)^{-1}(Y + G/B)^{B-1}(Y - BG)^{1/B-1}$, integrating from t_0 to t_1, and finally using (1) to eliminate Y, we compute

$$\begin{aligned} &\{\tilde{x}(t_1;0) - f(t_1) + G(t_1)/B\}^B\{\tilde{x}(t_1;0) - f(t_1) - BG(t_1)\}^{1/B} \\ &\leq \{\tilde{x}(t_0;0) - f(t_0) + G(t_0)/B\}^B\{\tilde{x}(t_0;0) - f(t_0) - BG(t_0)\}^{1/B} = 0, \end{aligned}$$

which contradicts (82). □

Our final result in this section basically infers that if g is positive in a right neighborhood of zero and f is relatively smooth, then equation (1) has a solution no matter how rapidly f may become negative away from zero.

Theorem 16. *Suppose that* (77) *holds, and that given any $t \in (0,\tau)$ there exists a $\delta \in (0,\tau]$ and an $\alpha < 2$ such that*

$$f(s) - f(t) \leq \alpha\{G^2(t) - G^2(s)\}^{1/2}$$

for all $s \in (t-\delta, t)$. Then equation (1) *has a* (*unique*) *solution on* $[0,\tau)$.

For the proof of this theorem, we use a lemma. Note that in this lemma we do not necessarily need any of the assumptions (59), (60), (77) or (62).

Lemma 13. *Let $\mu \geq 0$. Suppose that $\tilde{x}(t;\mu) = 0$ for some point $t \in (0,\infty)$ such that*

$$g(s) \geq 0 \quad \textit{for almost all} \quad s \in (t-\delta, t)$$

and

$$f(s) - f(t) \leq \alpha\{G^2(t) - G^2(s)\}^{1/2} \quad \textit{for all} \quad s \in (t-\delta, t)$$

for some $\delta \in (0,t]$ and $\alpha < 2$. Then if $t \leq \tilde{T}(\mu)$, there holds $\tilde{x}(s;\mu) = 0$ for all $s \in [t-\delta, t]$.

Proof. Set $f_1(s) := f(t-s) - f(t)$ and $g_1(s) := -g(t-s)$ for $s \in (0,\delta)$. Then, invoking Lemma 2 if need be, the function $x_1(s) := \tilde{x}(t-s;\mu)$ can be verified to be a solution of the equation (47) on $[0,\delta)$. Applying Lemma 12 to (47) yields the stated result. □

Proof of Theorem 16. If $T(\mu) < \tau$ for any $\mu > 0$, then by Lemma 13 necessarily $\tilde{x}(t;\mu) = 0$ for all $t \in [0, T(\mu))$. However, this is at odds with $\tilde{x}(0;\mu) = \mu$. So $T(\mu) \geq \tau$ for all $\mu > 0$. The present theorem now follows from Theorem 6. □

9. A special case. In this final section we examine equation (1) when

$$\int_0^t g(s)\,ds < 0 \quad \text{for all} \quad t \in (0,\infty) \tag{83}$$

and

$$f(t) = \alpha G(t) \quad \text{for all} \quad t \in [0,\infty) \tag{84}$$

with $\alpha \geq 2$ and G defined by (61). Because of the peculiar structure of the equation in this instance, we are able to show that it admits an uncountable number of solutions which can be precisely characterized by their behavior as $t \downarrow 0$.

Theorem 17. *Suppose that* (83) *and* (84) *hold. Let L and J be defined by* (63) *and* (64). *Set*

$$\beta_1 = (\alpha - \sqrt{\alpha^2 - 4})/2, \qquad \beta_2 = (\alpha + \sqrt{\alpha^2 - 4})/2$$

and

$$\kappa = \begin{cases} \exp(-1) & \text{for } \alpha = 2 \\ \{\beta_1^{\beta_1}(1-\beta_1/\beta_2)^{\beta_2}\}^{1/(\beta_2-\beta_1)} & \text{for } \alpha > 2. \end{cases}$$

(i) *If $\alpha = 2$ equation* (1) *admits the maximal solution*

$$\tilde{x}(t;0) = G(t),$$

for each $\gamma \in \mathbf{R}$ a unique solution x_γ such that

$$x_\gamma(t) = (G - GL + GL^2J^{-1} + \gamma GL^2)(t) + \mathcal{O}((GL^3J^{-2})(t)) \quad \text{as } t \downarrow 0$$

with maximal interval of existence of $[0, T_\gamma)$ where $T_\gamma = \sup\{t \in [0,\infty) : G(s) \le \kappa \exp(\gamma)$ for all $s \in [0,t)\}$, and no other solutions.

(ii) *If $\alpha > 2$ equation* (1) *admits the maximal solution*

$$\tilde{x}(t;0) = \beta_2 G(t),$$

for each $\gamma \in \mathbf{R}$ a unique solution x_γ such that

$$x_\gamma(t) = (\beta_1 G + \gamma G^{\beta_2/\beta_1})(t) + \mathcal{O}(G^{(2\beta_2-\beta_1)/\beta_1}(t)) \quad \text{as } t \downarrow 0$$

with maximal interval of existence $[0, T_\gamma)$ where $T_\gamma = \infty$ if $\gamma \ge 0$ and $T_\gamma = \sup\{t \in [0,\infty) : G(s) \le \kappa|\gamma|^{-\beta_1/(\beta_2-\beta_1)}$ for all $s \in [0,t)\}$ if $\gamma < 0$, and no other solutions.

In both cases, $\tilde{x}(t;0) > x_{\gamma'}(t) > x_\gamma(t)$ for all $t \in (0, T_\gamma)$ for any $\infty > \gamma' > \gamma > -\infty$.

Proof. Suppose, to begin with, that x is a solution of (1) on $[0,\delta)$ for some $0 < \delta \le \infty$. Define Y by (69). Then multiplying (1) by Y' this equation may be reformulated as

$$\{Y(t) - \alpha G(t)\}Y'(t) + G(t)G'(t) = 0 \tag{85}$$

for almost all $t \in (0,\delta)$. The trick is now to observe that (85) becomes exact using the integrating factor

$$X(t) := |Y(t) - \beta_1 G(t)|^{-\alpha\beta_2}.$$

Consequently, if Y is a solution of (85), either $Y = \beta_1 G$ or we may multiply (85) by X and integrate to obtain

$$\ln|Y - G| + G/(Y - G) = C \quad \text{when} \quad \alpha = 2 \tag{86}$$

or

$$(Y - \beta_1 G)|Y - \beta_1 G|^{-\alpha\beta_2}(\beta_2 G - Y) = C \quad \text{when} \quad \alpha > 2 \tag{87}$$

for some constant C. Analysis of (86) and (87), which is tedious and will therefore be skipped, reveals that if $Y \to 0$ as $G \to 0$ in these relationships, then necessarily $Y > \beta_1 G$. Consequently, using (1) to eliminate Y we may conclude that either

$$x(t) = \beta_2 G(t), \tag{88}$$

or $x(t) < \beta_2 G(t)$ and

$$\{\ln(G - x) + G/(G - x)\}(t) = C \quad \text{for} \quad \alpha = 2 \tag{89}$$

$$\{(\beta_2 G - x)^{-\beta_2/\beta_1}(x - \beta_1 G)\}(t) = C \quad \text{for} \quad \alpha > 2 \tag{90}$$

for some real constant C. Retracing the above argument, one may furthermore ascertain that any function x which satisfies (88), (89), or (90) on $(0, \delta)$ for some $\delta > 0$ is a solution of (1). The theorem now follows from analysis of the relations (88)–(90). Incidentally, this may also be used to obtain the corollary below. We omit further details. $\square$

Corollary. *Suppose that* $\alpha > 2$. *Then* $x_0 \equiv \beta_1 G$. *Furthermore, if* $G(t) \to \infty$ *as* $t \to \infty$, *then for every* $\gamma > 0$ *there holds* $x_\gamma(t) \sim \beta_2 G(t)$ *as* $t \to \infty$.

Inspiration for Theorem 17 was obtained from the results of de Pablo and Vazquez [**6**] on the occurrence of 'finite' travelling-wave solutions of the reaction-diffusion equation (4) with $a(s) \equiv s^m$, $b(s) \equiv 0$ and $c(s) \equiv s^p$ for some real parameters $m > 1$ and $p = 2 - m$, and the correspondence between travelling-wave solutions of (4) and the integral equation (1) outlined in the introduction.

We remark that the method of reformulating (1) as an ordinary differential equation with the integral in (1) as the unknown and then identifying an integrating factor for the ensuing differential equation may also be applied when (83) holds and $f(t) = \mu + \alpha G(t)$ for any $\mu \geq 0$ and $\alpha \in \mathbf{R}$. This technique can furthermore be applied when the inequality sign in (83) is reversed and f has the form just mentioned. However, for these cases the analysis yields little particularly noteworthy information which has not been covered by the preceding results.

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