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ON INFINITE DELAY INTEGRAL EQUATIONS HAVING NONLINEAR PERTURBATIONS

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ABSTRACT. The existence of bounded solutions and periodic solutions is studied for a system of infinite delay integral equations having nonlinear perturbations. An equivalent system of equations is obtained in terms of the resolvent kernel. Then the existence results are shown for the equivalent equations. Contraction principle, Schauder's fixed point theorem, and monotone method are used in the study.

1. Introduction. Let n be a positive integer. In this paper we have studied the existence of bounded solutions and periodic solutions of

$$x(t) = f(t) + \int_{-\infty}^{t} a(t,s)[x(s) + g(t, x(s))]\, ds, \quad t \in \mathbf{R} = (-\infty, \infty), \tag{1}$$

where we assume that $f : \mathbf{R} \to \mathbf{R}^n$, $g : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ are basically continuous and bounded functions, a is an n-by-n matrix function with elements in $\mathbf{R}$, $a(t,s)$ is continuous for $-\infty < s \le t < \infty$, and $a(t,s) = 0$ for $s > t$. We also assume that

$$\sup_{-\infty<t<\infty} \int_{-\infty}^{t} |a(t,s)|\, ds \le A < \infty, \tag{A1}$$

where $|\cdot|$ denotes the matrix norm induced by a vector norm, also denoted $|\cdot|$, on vectors in $\mathbf{R}^n$.

If $a(t,s) = \bar{a}(t-s)$, then (1) is known as convolution type. Convolution forms of (1) are used in studying various types of problems in physics and engineering. For example, in the study of hereditary response in continuum physics for a material with large memory [**6**], or in the study of the response of nonlinear feedback systems to periodic input signals [**11**], constitutive equations can have the form of (1).

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Leitman and Mizel have studied scalar convolution forms of (1) in [**7, 8**]. The particular form $[x(s) + g(s, x(s))]$ of (1) instead of simply $g(s, x(s))$ as used in [**7, 8**] enables us to use the resolvent equation

$$r(t,s) = -a(t,s) + \int_s^t a(t,u) r(u,s)\,du, \quad -\infty < s \le u \le t. \tag{2}$$

The advantage of using (2) is that we can derive an equivalent (to (1)) equation which is easier to study than the equation (1) itself. All the existence and uniqueness results for solutions of (1) are obtained by showing the existence and uniqueness of solutions of the equivalent equation which we have derived in Lemma 2. In Theorem 1 we have proved the existence and uniqueness of bounded solutions by using the contraction principle as the basic tool, whereas in Theorems 2 and 3 we have shown the existence of periodic solutions and the existence of bounded solutions, respectively. Schauder's fixed point theorem is used in Theorem 2 and a monotone method is used in Theorem 3.

One can view (1) as an infinite delay equation having nonlinear perturbations. Some perturbation results for infinite delay integral and integrodifferential equations are available in [**5**]. Studies on perturbation problems in Volterra equations can be found in [**1–4, 9, 12–15**].

2. Bounded solutions and periodic solutions. It is known [**5**, Theorem 1] that for each continuous $a(t,s)$ there exists a continuous $r(t,s)$ satisfying (2). Suppose

$$\sup_{-\infty<t<\infty} \int_{-\infty}^t |r(t,s)|\,ds \le L < \infty. \tag{A2}$$

Assume that $g(t,0) = 0$ for all real t, and that for each real $\alpha > 0$ there exists a real $\eta > 0$ such that

$$|g(t,x) - g(t,y)| \le \alpha |x-y| \tag{A3}$$

uniformly in t, whenever $|x|$, $|y| \le \eta$. Also assume that

$$\lim_{h\to 0} \int_{-\infty}^t |a(t+h,s) - a(t,s)|\,ds = 0 \tag{A4}$$

for each real t.

Lemma 1. *If* (A1), (A2) *and* (A4) *hold, then*

$$\lim_{h\to 0}\int_{-\infty}^{t} |r(t+h,s)-r(t,s)|\,ds = 0$$

for each real t.

The proof of Lemma 1 involves the use of (2) and the application of Fubini's theorem. We omit its proof.

Lemma 2. *Assume* (A1) *and* (A2) *hold. Then* $x(t)$ *is a bounded solution of* (1) *if and only if* $x(t)$ *is a bounded solution of*

$$x(t) = f(t) - \int_{-\infty}^{t} r(t,s)[f(s)+g(s,x(s))]\,ds. \tag{3}$$

Proof. Let $x(t)$ be a bounded solution of (1) for any real t. Multiplying both sides of (1) by $r(t,s)$, integrating from $-\infty$ to t, interchanging the order of integrations, and then applying (2) yields (3). In this process we have used the fact that

$$\int_{s}^{t} r(t,u)a(u,s)\,du = \int_{s}^{t} a(t,u)r(u,s)\,du$$

(see [**10**, p. 193]). The converse is also true since all the above steps are reversible. $\square$

We are motivated by [**12**, Theorem 1] for our next theorem. We have shown how Theorem 1 of [**12**] can be extended to the integral equations with infinite delay.

Theorem 1. *Assume* (A1)–(A4) *hold. For each* $\lambda \in (0,1)$ *there exists a real* $\beta > 0$ *such that if* $||f|| = \sup\{|f(t)| : t \in \mathbf{R}\} \leq \beta$, *then* (1) *has a unique bounded solution.*

Proof. Fix $\lambda \in (0,1)$. Then choose $\alpha > 0$ such that $1-\alpha L = \lambda$ where L is the constant of (A2). Clearly, $\alpha L < 1$. For this α, choose $\eta > 0$ of (A3). Let β be such that $0 < \beta \leq \eta\lambda/(1+L)$. Let

$$S(\eta) = \{\varphi \in \mathrm{BC}\,(-\infty,\infty) : ||\varphi|| \leq \eta\}$$

where $\mathrm{BC}\,(-\infty,\infty)$ is the Banach space of bounded continuous functions on $(-\infty,\infty)$ with the usual sup norm. For each $\varphi \in S$, let

$$(F\varphi)(t) = f(t) - \int_{-\infty}^{t} r(t,s)[f(s) + g(s,\varphi(s))]\,ds. \tag{4}$$

It follows from the continuity of $r(t,s)$ that $\int_t^{t+h} |r(t+h,s)|\,ds \to 0$ as $h \to 0$. Using this property and Lemma 1, one can easily verify that $(F\varphi)(t)$ is continuous in t. It also follows from the given assumptions that $||F\varphi|| \leq \eta$. Therefore, F maps from $S(\eta)$ into itself. Finally, (A2) and (A3) imply that, for each φ and Ψ in $S(\eta)$, $||F\varphi - F\Psi|| \leq \alpha L||\varphi - \Psi||$. Since $\alpha L < 1$, F is a contraction. Therefore, there exists a unique solution $x(t)$ of (3) (and hence of (1) by Lemma 2) with $||x|| \leq \eta$. $\square$

Assume that, for some $T > 0$,

$$f(t+T) = f(t) \quad \text{for all real } t; \tag{A5}$$

$$a(t+T, s+T) = a(t,s) \quad \text{for } -\infty < s \leq t < \infty; \tag{A6}$$

$$g(t+T, x) = g(t,x) \quad \text{for all } x. \tag{A7}$$

Corollary 1. *Assume* (A1)–(A7) *hold. For each* $\lambda \in (0,1)$ *there exists a* $\beta > 0$ *such that if* $||f|| \leq \beta$ *then* (1) *has a unique* T*-periodic solution.*

Proof. It follows from (A5)–(A7) that if $x(t)$ is a solution of (1) then $x(t+T)$ is also a solution of (1). By the uniqueness property of Theorem

1, $x(t+T) = x(t)$ for all real t. This completes the proof of Corollary 1. $\square$

Remark. From Theorem 1 and Corollary 1, it follows that the T-periodic solution $x(t)$ of (1) is also the only bounded solution of (1) under the assumed hypotheses.

Theorem 2. *Assume* (A1), (A2), (A4)–(A7) *hold. Also assume that for each $\alpha > 0$ there exists a real $\eta > 0$ such that $|g(t,x) - g(t,y)| \leq \eta|x-y|$ uniformly in t whenever $|x|, |y| \leq \alpha$. Then there exists a continuous T-periodic solution of* (1).

Proof. Let $|f(t)| \leq F$ for all real t and $|g(t,x)| \leq G$ for all (t,x) in $\mathbf{R} \times \mathbf{R}^n$. Let $F + F(L+G) = K$, where L is the constant of (A2). Consider

$$B = \{x \in P_T(-\infty, \infty) : ||x|| \leq K\}$$

where $P_T(-\infty,\infty)$ is the Banach space of continuous T-periodic functions with the sup norm. The space B is obviously convex and closed. Let us define $F : B \to B$ by (4) for each φ in B. Assumption (A6) implies that $r(t+T, s+T) = r(t,s)$ for $-\infty < s \leq t < \infty$ [**5**, Theorem 7]. Now one easily shows that $(F\varphi)(t+T) = (F\varphi)(t)$. Clearly, $||F\varphi|| \leq K$, and for φ and Ψ from B, one obtains $||F\varphi - F\Psi|| \leq L\eta||\varphi - \Psi||$. Therefore, $F : B \to B$ is a continuous map. Since all the functions are T-periodic, we can use the Arzela-Ascoli theorem to show that $\overline{F(B)}$, the closure of $\{Fx : x \in B\}$ is compact. The continuity of $r(t,s)$ and Lemma 1 are used in the arguments of the equicontinuity property. Therefore, by Schauder's fixed point theorem there exists a continuous T-periodic solution $x(t)$ of (1). $\square$

Lemma 3. *In addition to the basic assumptions that g is continuous and bounded, we assume that there exists a real $M > 0$ and a continuous bounded function φ with $\varphi(t) \to 0$ as $t \to -\infty$ such that $|g(t,x)| \leq \varphi(t)M$. Then for each $\varepsilon > 0$ there exists a real w such that*

$$\left| \int_{-\infty}^{w} r(t,s) g(s, x(s))\, ds \right| < \varepsilon$$

for every continuous $x(s)$ defined on $\mathbf{R}$.

The proof of this lemma is a trivial exercise and hence we omit its proof.

Definition 1. For x, y in $\mathbf{R}^n$, $x \le y$ if and only if $x_i \le y_i$ for all $i = 1, 2, 3, \dots, n$.

Definition 2. $g(t, x)$ is nonincreasing in x if $x \le y$ implies $g_i(t, x) \ge g_i(t, y)$ for all $i = 1, 2, 3, \dots, n$.

Theorem 3. *Let $r(t, s) \ge 0$, i.e., each element $r_{ij}(t, s) \ge 0$ for $i, j = 1, 2, 3, \dots, n$. Assume* (A1), (A2), (A4), *and the hypotheses of Lemma* 3 *hold. Also, assume $g(t, x) \ge 0$, i.e., $g_i(t, x) \ge 0$ for $i = 1, 2, 3, \dots, n$ and $g(t, x)$ is nonincreasing in x. Then there exists a continuous bounded solution of* (1).

Proof. Let $y(t) = f(t) - \int_{-\infty}^{t} r(t, s) f(s)\, ds$. Clearly, $y(t)$ is continuous and bounded. Let $||y|| \le Y$. Substituting $y(t)$ in (3) yields

$$x(t) = y(t) - \int_{-\infty}^{t} r(t, s) g(s, x(s))\, ds. \tag{5}$$

Let $x_1(t) = y(t)$, and for $n = 1, 2, 3, \dots,$

$$x_{n+1}(t) = y(t) - \int_{-\infty}^{t} r(t, s) g(s, x_n(s))\, ds.$$

It follows from the hypotheses that

$$x_{n+1}(t) \le x_n(t)$$

for $n = 1, 2, 3, \dots$. Also, one can easily verify that the sequence $\{x_n(t)\}$ is equicontinuous and uniformly bounded on every compact subset of R. Lemma 1 is used for the equicontinuity property. Therefore, there exists a continuous function $x^*(t)$ such that $x_n(t) \to x^*(t)$, and the convergence is uniform on every compact subset of $\mathbf{R}$. Clearly, $x^*(t)$ is bounded.

We shall now prove that $x^*(t)$ is a solution of (1). Let $\varepsilon > 0$ be arbitrary. Pick any real t. Then choose $k > 0$ such that $-k < t$ and

$$\left| \int_{-\infty}^{-k} r(t, s) g(s, x^*(s))\, ds \right| < \varepsilon/4.$$

Choosing such k is possible by Lemma 3. Since $x_n(s) \to x^*(s)$ uniformly on $[-k,t]$ we can choose a large n such that

$$|x_{n+1}(s) - x^*(s)| < \varepsilon/4 \quad \text{for all } s \in [-k,t],$$

and

$$\left| \int_{-k}^{t} r(t,s)[g(s,x^*(s) - g(s,x_n(s))]\,ds \right| < \varepsilon/4.$$

Now,

$$\begin{aligned}
&\left| x^*(t) - \left\{ y(t) - \int_{-\infty}^{t} r(t,s)g(s,x^*(s))\,ds \right\} \right| \\
&\le |x^*(t) - x_{n+1}(t)| + \left| x_{n+1}(t) - y(t) + \int_{-\infty}^{t} r(t,s)g(s,x_n(s))\,ds \right| \\
&\quad + \left| \int_{-\infty}^{t} r(t,s)[g(s,x^*(s)) - g(s,x_n(s))]\,ds \right| \\
&\le \varepsilon/4 + 0 + \left| \int_{-\infty}^{-k} r(t,s)g(s,x^*(s))\,ds \right| + \left| -\int_{-\infty}^{-k} r(t,s)g(s,x_n(s))\,ds \right| \\
&\quad + \left| \int_{-k}^{t} r(t,s)[g(s,x^*(s)) - g(s,x_n(s))]\,ds \right| \\
&\le \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4.
\end{aligned}$$

This proves by Lemma 2 that $x^*(t)$ is a continuous bounded solution of (1). $\square$

Remark. Theorem 3 holds if we assume $r(t,s) \le 0$, $g(t,x) \le 0$, and $g(t,x)$ is nondecreasing in x. The proof is similar to the proof of Theorem 3.

Theorem 4. *If $a(t,s) > 0$, then $r(t,s) < 0$ for $-\infty < s \le t$.*

Proof. From (2) it is clear that $r(t,t) = -a(t,t) < 0$. Suppose $r(t,s_0) = 0$ and $r(t,u) < 0$ for all $s_0 < u \le t$. Then, from (2),

$$r(t,s_0) = -a(t,s_0) + \int_{s_0}^{t} r(t,u)a(u,s_0)\,du.$$

Clearly, $r(t, s_0) < 0$. This contradicts that $r(t, s_0) = 0$. This proves that $r(t, s) < 0$ for all $s \le t$.

It remains an open question whether $a(t, s) < 0$ implies $r(t, s) > 0$. However, suppose $a(t, s) = -\tilde{a}(t-s)$ is of convolution type. Then $r(t, s) = \tilde{r}(t-s)$ is also of convolution type, and the resolvent equation becomes

$$\tilde{r}(t) = \tilde{a}(t) - \int_0^t \tilde{a}(t-s)\tilde{r}(s)\, ds.$$

Let $\tilde{a}(t) > 0$, $\tilde{a}(t)$ is nonincreasing in $t > 0$, $\tilde{a}(t) \in L^1(R^+)$, and for each $w > 0$, $\tilde{a}(t)/\tilde{a}(t+w)$ is nonincreasing in $t > 0$. Then $\tilde{r}(t)$ satisfies (i) $0 < \tilde{r}(t) < \tilde{a}(t)$ for all $t > 0$, and (ii) $\int_0^\infty \tilde{r}(t)\, dt < 1$. (See [**10**, Theorem 6.2, p. 212]). $\square$

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