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MULTILEVEL METHODS FOR THE APPROXIMATION OF SINGULAR SOLUTIONS OF COMPLETELY CONTINUOUS OPERATOR EQUATIONS

IGOR MORET

ABSTRACT. The direct techniques for approximating singular solutions of nonlinear equations depending on parameters transform the original problem into that of solving a suitable augmented system. Dealing with equations involving completely continuous operators, a multilevel approach to the solution of these larger systems is presented.

1. Introduction. We are concerned with the approximation of singular points in branches of solutions of nonlinear parameter-dependent equations having the form

$$F(u,\beta_1,\dots,\beta_{p-1},\gamma) = u - K(u,\beta_1,\dots,\beta_{p-1},\gamma) = 0, \tag{1.1}$$

where $u \in U$, with U a real Banach space, $\gamma \in \mathbf{R}$ and the β_i's are $(p-1)$-additional real parameters, for some $p \geq 1$. From now on, we assume that F is a C^ν-mapping, with $\nu \geq 3$, from an open set $D \subset U \times \mathbf{R}^p$ into U and that the operator K is completely continuous.

As is well known, the direct techniques for the approximation of a singular solution are based on the construction of an augmented system having it as a regular solution (cf. [**9**] for a general discussion). Then, moving from a suitable starting point, obtained for instance by a continuation procedure, the arising system is solved by an iterative method. In several practical cases, this procedure is carried out using operator approximations, on which the accuracy of the computed solution depends (cf. [**11, 20**]).

In this paper we deal with singular solutions of (1.1), such as turning points and, when unfolded, bifurcation points, which fall within the unifying theory recently developed by Griewank and Reddien in [**10, 11**], where the corresponding augmented system is built up through the minimum number of additional scalar equations characterizing the singularity. For solving this larger system, we present an approach based

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on the use of iterative multilevel methods, with special attention to the case where collectively compact approximations of K are employed.

The basic approximation scheme is presented and studied in Section 2. In Section 4 we illustrate how it can be suitably combined with classical linear multigrid algorithms, briefly outlined in Section 3.

2. The basic approximation scheme. From now on, for ease of notation, we set $X = U \times \mathbf{R}^p$ and $Z = U \times \mathbf{R}^{p-1}$ ($Z = U$, if $p = 1$). Moreover, we denote by Q_U and Q_Z the coordinate projections from X into U and into Z, respectively. That is, we set $Q_U\mathbf{x} = u$, $Q_Z\mathbf{x} = (u, \beta_1, \dots, \beta_{p-1})^T$, for $\mathbf{x} = (u, \beta_1, \dots, \beta_{p-1}, \gamma)^T \in X$. Accordingly, we rewrite equation (1.1) in the form

$$F(\mathbf{x}) = Q_U\mathbf{x} - K(\mathbf{x}) = 0, \quad \mathbf{x} \in X. \tag{2.1}$$

Throughout, the first and second derivatives of F with respect to $\mathbf{x}$ will be indicated by F' and F'', respectively, and we will use subscripts for indicating the partial derivatives with respect to each variable. In particular, F_z will denote the derivative of F with respect to $Q_Z\mathbf{x}$. As usual, for $i = 1, \dots, p$, e_i will be the i^{th} coordinate vector in $\mathbf{R}^p$; any product space $U \times \mathbf{R}^m$ will be endowed with a product norm; $|\cdot|_E$ will stand for the norm in a Banach space E, except for the space $\mathbf{L}(E, E')$ (E and E' Banach spaces) of all bounded linear operators from E into E' whose norm will be denoted by $||\cdot||_{E\to E'}$; E^* will stand for the dual space of E and B^* will be the adjoint of a linear operator B. The notation $S(\mathbf{x}, \delta)$ will indicate the closed ball in X of center $\mathbf{x}$ and radius δ. Finally, $\mathbf{e} \in X^*$ will be such that

$$\mathbf{ex} = \gamma, \quad \text{for } \mathbf{x} = (u, \beta_1, \dots, \beta_{p-1}, \gamma)^T \in X.$$

Let $\mathbf{x}^s \in D$ be a singular point of (2.1) in the sense that:

$$F(\mathbf{x}^s) = 0; \tag{h$_1$}$$

$$\operatorname{Ker}(F_z(\mathbf{x}^s)) = \operatorname{span}\{Q_Z\mathbf{y}_1^s, \dots, Q_Z\mathbf{y}_p^s\}; \tag{h$_2$}$$

with $e_i^T M Q_Z \mathbf{y}_j^s = \delta_{ij}$ (the Kronecker's delta), for $i, j = 1, \dots, p$, where M is a suitable linear mapping acting from Z into $\mathbf{R}^p$;

$$\begin{gathered} F_\gamma(\mathbf{x}^s) \notin \operatorname{Range} F_z(\mathbf{x}^s) \quad \text{and} \\ \operatorname{Range}(F_z(\mathbf{x}^s)) = \{u \in U : \psi^s u = 0\}, \end{gathered} \tag{h$_3$}$$

for some $\psi^s(\neq 0) \in U^*$. We also assume that

(h$_4$) the $p \times p$ matrix $R^s = [r^s_{ij}]$ having entries
$$r^s_{ij} = \psi^s F_{zz}(\mathbf{x}^s) Q_Z \mathbf{y}^s_i Q_Z \mathbf{y}^s_j, \quad \text{for } i, j = 1, \dots, p,$$

is nonsingular.

These conditions characterize various types of singular solutions (see [**7, 10–12, 15, 18, 20**]), which Griewank and Reddien in [**10, 11**] called *generalized turning points*. As mentioned in the introduction, for characterizing $\mathbf{x}^s$ as a regular solution of a suitable augmented system, we adopt here the following approach proposed in [**10, 11**].

For any $\mathbf{x} \in D$, we consider the mapping $A(\mathbf{x}) : X \to X$ defined by

$$A(\mathbf{x})\mathbf{y} = (F'(\mathbf{x})\mathbf{y}, MQ_Z\mathbf{y})^T, \quad \text{for } \mathbf{y} \in X. \tag{2.2}$$

Then, the following result can be proved (see [**11**]).

Proposition 2.1. *There exists δ^* such that, for every $\mathbf{x} \in S(\mathbf{x}^s, \delta^*)$, $A(\mathbf{x})$ has a bounded inverse. Moreover, $\mathbf{x}^s$ is a regular solution of the following augmented system* (*from X into itself*)

$$F(\mathbf{x}) = 0 \tag{2.3}$$

$$\mathbf{e}\mathbf{y}_i(\mathbf{x}) = 0, \quad \textit{for } i = 1, \dots, p, \tag{2.4}$$

where, for each $i = 1, \dots, p$, $\mathbf{y}_i(\mathbf{x}) \in X$ solves the equation

$$A(\mathbf{x})\mathbf{y}_i(\mathbf{x}) = (0, e_i)^T, \quad 0 \in U.$$

From now on, let $\{F^n\}$, $n = 0, 1, \dots,$ be a sequence of nonlinear operators from D into U satisfying the following assumptions:

(k$_1$) F^n is pointwise convergent to F on D, i.e., for each $\mathbf{x} \in D$,

$$F^n(\mathbf{x}) \to F(\mathbf{x}), \quad \text{as } n \to \infty;$$

(k$_2$)
$\{Q_U - F^n\}$ is a family of collectively compact (c.c.) operators (cf. [**1**]);

k_3 for every n, F^n is of class C^ν, $\nu \geq 3$, on $S(\mathbf{x}^s, \delta^*)$ and the derivatives $F^{n'}(\mathbf{x}), F^{n''}(\mathbf{x})$ and $F^{n'''}(\mathbf{x})$ are uniformly bounded with respect to $\mathbf{x}$ and n.

Accordingly, we define the sequence of operators $\{A^n\}$ as

$$A^n(\mathbf{x})\mathbf{v} = (F^{n\prime}(\mathbf{x})\mathbf{v}, MQ_Z\mathbf{v})^T, \quad \text{for } \mathbf{v} \in X.$$

We assume that for all n sufficiently large:

(k_4) for every $\mathbf{x} \in S(\mathbf{x}^s, \delta^*)$, the operators $A^n(\mathbf{x})^{-1}$ exist and they are uniformly bounded with respect to $\mathbf{x}$ and n.

Then for solving (2.3)–(2.4), we consider the following approximation scheme:

Scheme 1. Select n^* and take a subsequence of $\{F^n : n \geq n^*\}$, we relabel here as $\{F^k\}$, for $k = 0, 1, \ldots$. Starting from $\mathbf{x}^0 \in D$, for $k = 0, 1, \ldots$, do the following:

1. Select an approximation B^k of $A^k(\mathbf{x}^k)^{-1}$.
2. Take
$$\mathbf{s}^k = -B^k(F^k(\mathbf{x}^k), 0)^T.$$
3. For each $i = 1, \ldots, p$, take
$$\mathbf{y}_i^k = B^k(0, e_i)^T.$$
4. Construct the $p \times p$ matrix R^k having entries
$$\rho_{ij}^k = [(B^k)^*\mathbf{e}](-F^{k\prime\prime}(\mathbf{x}^k)\mathbf{y}_i^k\mathbf{y}_j^k, 0)^T, \quad \text{for } i, j = 1, \ldots, p,$$

and the p-dimensional vector c^k having entries

$$\eta_i^k = -\mathbf{e}\mathbf{y}_i^k + [(B^k)^*\mathbf{e}][(F^{k\prime\prime}(\mathbf{x}^k)\mathbf{s}^k\mathbf{y}_i^k, 0)^T + A^k(\mathbf{x}^k)\mathbf{y}_i^k - (0, e_i)^T],$$

for $i = 1, \ldots, p$.

5. Find $a^k = (\alpha_1^k, \ldots, \alpha_p^k)^T$ which solves

$$\mathbf{R}^k a^k = c^k. \tag{2.5}$$

6. Set
$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{s}^k + \Sigma_{1 \leq i \leq p}\alpha_i^k\mathbf{y}_i^k.$$

In order to simplify the notation, we set
$(\Delta F)^k := |F^k(\mathbf{x}^s)|_U$.
$(\Delta A)^k = \max_{1\le i\le p} |(A^k(\mathbf{x}^s) - A(\mathbf{x}^s))\mathbf{y}_i^s|_X$,
$\vartheta^k = ||I - B^k A^k(\mathbf{x}^k)||_{X\to X}$,
$\mathbf{e}_{\mathbf{x}}^k = |\mathbf{x}^s - \mathbf{x}^k|_X$,
$\mathbf{e}_{\mathbf{y}}^k = \max_{1\le i\le p} |\mathbf{y}_i^s - \mathbf{y}_i^k|_X$.

Theorem 2.1. *Let* (h_1)–(h_4) *and* (k_1)–(k_4) *hold. There exist constants* δ *and* ϑ *such that, if* $\mathbf{x}^0 \in S(\mathbf{x}^s, \delta)$, $\vartheta^k \le \vartheta$ *for every* k, *and* n^* *is sufficiently large, then Scheme* 1 *is well defined and* $\mathbf{x}^k \in S(\mathbf{x}^s, \delta)$, *for every* k. *Moreover, there are two constants* $c_{\mathbf{y}}$ *and* $c_{\mathbf{x}}$ *such that, for every* k,

$$e_{\mathbf{y}}^k \le c_{\mathbf{y}}[\vartheta^k + e_{\mathbf{x}}^k + (\Delta A)^k], \tag{2.6}$$

and

$$e_{\mathbf{x}}^{k+1} \le c_{\mathbf{x}}[(e_{\mathbf{x}}^k + \vartheta^k)(e_{\mathbf{x}}^k + e_{\mathbf{y}}^k) + (\Delta F)^k + (\Delta A)^k]. \tag{2.7}$$

Proof. If n^* is sufficiently large, then, by (k_4), there is a constant μ such that, for every k, if $\mathbf{x}^k \in S(\mathbf{x}^s, \delta^*)$, then $A^k(\mathbf{x}^k)$ is invertible and

$$||(A^k(\mathbf{x}^k))^{-1}||_{X\to X} \le \mu.$$

Then

$$B^k - (A^k(\mathbf{x}^k))^{-1} = -[I - B^k A^k(\mathbf{x}^k)](A^k(\mathbf{x}^k))^{-1},$$

and, since $\vartheta^k \le \vartheta$, it follows that

$$||B^k||_{X\to X} \le (1+\vartheta)\mu.$$

By (h_1)–(h_4), using well-known results about approximations of completely continuous operators (see [**1, 17**]), since for each $i = 1, \dots, p$, $\mathbf{y}_i(\mathbf{x}^s) = \mathbf{y}_i^s$ (cf. (2.4)), we realize that, if $\mathbf{x}^k \in S(\mathbf{x}^s, \delta)$, if δ and ϑ are sufficiently small and if n^* is sufficiently large, then the coefficient matrix R^k of (2.5) is nonsingular and there is a constant τ, independent of k, such that

$$||(R^k)^{-1}|| \le \tau. \tag{2.8}$$

(Here $|| \quad ||$ denotes the matrix norm induced by the 1-norm on $\mathbf{R}^p$). Hence, relation (2.6) easily follows and, moreover, by (2.8), $\mathbf{x}^{k+1}$ is well defined. Now let us prove (2.7).

Let $\alpha_i'^k$, for $i = 1, \dots, p$, be such that

$$\Sigma_{1\le i\le p}\alpha_i'^k e_i = MQ_Z(\mathbf{x}^s - \mathbf{x}^k). \tag{2.9}$$

Clearly, the $\alpha_i'^k$'s are uniquely determined.

Then we define $\mathbf{x}'^k$ by

$$\mathbf{x}'^k = \mathbf{x}^k + \mathbf{s}^k + \Sigma_{1\le i\le p}\alpha_i'^k \mathbf{y}_i^k.$$

By (2.9), we have

$$\begin{aligned}\mathbf{x}'^k - \mathbf{x}^k &= -B^k[(F^k(\mathbf{x}^k),0)^T - \Sigma_{1\le i\le p}\alpha_i'^k(0,e_i)^T] \\ &= B^k(F^k(\mathbf{x}^s) - F^k(\mathbf{x}^k) - F^{k\prime}(\mathbf{x}^k)(\mathbf{x}^s-\mathbf{x}^k),0)^T \\ &\quad + B^kA^k(\mathbf{x}^k)(\mathbf{x}^s-\mathbf{x}^k) - B^k(F^k(\mathbf{x}^s)),0)^T.\end{aligned}$$

Hence, we easily get

$$|x'^k - \mathbf{x}^s|_X \le c[|\mathbf{x}^s - \mathbf{x}^k|_X^2 + (\Delta F)^k + \vartheta^k|\mathbf{x}^s - \mathbf{x}^k|_X]. \tag{2.10}$$

Observing that the $\mathbf{y}_i^k$'s are uniformly bounded, we have, for some constant c_0, independent of k,

$$|\mathbf{x}^s - \mathbf{x}^{k+1}|_X \le c_0[|\mathbf{x}^s - \mathbf{x}'^k|_X + \Sigma_{1\le i\le p}|\alpha_i^k - \alpha_i'^k|]. \tag{2.11}$$

Now we provide an upper bound for $\Sigma_{1\le i\le p}|\alpha_i^k - \alpha_i'^k|$. From the identities

$$\Sigma_{1\le j\le p}\rho_{ij}^k\alpha_j^k = \mathbf{e}[-\mathbf{y}_i^k + B^k((F^{k\prime\prime}(\mathbf{x}^k)\mathbf{s}^k\mathbf{y}_i^k,0)^T + A^k(\mathbf{x}^k)\mathbf{y}_i^k - (0,e_i)^T)]$$

and

$$\Sigma_{1\le j\le p}\rho_{ij}^k\alpha_j'^k = \Sigma_{1\le j\le p}\alpha_j'^k\mathbf{e}B^k(-F^{k''}(\mathbf{x}^k)\mathbf{y}_i^k\mathbf{y}_j^k,0)^T,$$

for every $i = 1, \dots, p$, we get

(2.12)

$$\begin{aligned}&\Sigma_{1\le j\le p}\rho_{ij}^k(\alpha_j^k - \alpha_j'^k) \\ &\quad = e[-\mathbf{y}_i^k + B^k((F^{k\prime\prime}(\mathbf{x}^k)(\mathbf{x}^{k'} - \mathbf{x}^k)\mathbf{y}_i^k,0)^T + A^k(\mathbf{x}^k)\mathbf{y}_i^k - (0,e_i)^T)].\end{aligned}$$

Now, recalling that, by (h_3), $\mathbf{e}\mathbf{y}_i^s = 0$ and that $A(\mathbf{x}^s))\mathbf{y}_i^s = (0, e_i)^T$, for every $i = 1, \ldots, p$, we obtain the following inequality

$$
\begin{aligned}
&|\mathbf{e}[-\mathbf{y}_i^k + B^k((F^{k''}(\mathbf{x}^k)(\mathbf{x}^s - \mathbf{x}^k)\mathbf{y}_i^k, 0)^T + A^k(\mathbf{x}^k)\mathbf{y}_i^k - (0, e_i)^T)]| \\
&\quad \le |\mathbf{e}B^k(-F^{k'}(\mathbf{x}^s)\mathbf{y}_i^k + F^{k''}(\mathbf{x}^k)(\mathbf{x}^s - \mathbf{x}^k)\mathbf{y}_i^k + F^{k'}(\mathbf{x}^k)\mathbf{y}_i^k, 0)^T| \\
&\quad + |\mathbf{e}[(\mathbf{y}_i^s - \mathbf{y}_i^k) - B^k(A^k(\mathbf{x}^s)(\mathbf{y}_i^s - \mathbf{y}_i^k) + B^k(A^k(\mathbf{x}^s) - A(\mathbf{x}^s))\mathbf{y}_i^s]|.
\end{aligned}
$$

Hence, we easily get, for every i,

(2.13)

$$
\begin{aligned}
&|\mathbf{e}[-\mathbf{y}_i^k + B^k((F^{k''}(\mathbf{x}^k)(\mathbf{x}^{k'} - \mathbf{x}^k)\mathbf{y}_i^k)0)^T + A^k(\mathbf{x}^k)\mathbf{y}_i^k - (0, e_i)^T)]| \\
&\qquad\qquad \le c_1[(e_{\mathbf{x}}^k)^2 + \vartheta^k e_{\mathbf{y}}^k + e_{\mathbf{x}}^k e_{\mathbf{y}}^k + (\Delta A)^k + |\mathbf{x}'^k - \mathbf{x}^s|_X],
\end{aligned}
$$

for some constant c_1 independent of k. Thus, by (2.10)–(2.13), we obtain (2.7). Clearly, $\mathbf{x}^{k+1} \in S(\mathbf{x}^s, \delta)$, for a sufficiently large n^* and for sufficiently small δ and ϑ. □

3. A review of linear multilevel methods. In this section we will consider some linear multilevel (here we will use the term multigrid) procedures which can be employed for constructing the operators B^k and $(B^k)^*$ in Scheme 1.

At first, we give a general formulation of the linear multigrid methods which includes various classical procedures, like for instance those proposed by Hackbusch [**12**] and Hemker and Schippers [**13**].

Let $\{X_i\}$, $i = 0, 1, \ldots$, be a sequence of Banach spaces, let I_i denote the identity operator on X_i, and let us suppose that:

1) $\{\pi_i\}$, $\pi_i : X_{i-1} \to X_i$, $i = 1, 2, \ldots$, is a sequence of linear operators (prolongations), such that

$$||\pi_i||_{X_{i-1} \to X_i} \le c_\pi, \quad \text{for every } i;$$

2) $\{\rho_i\}$, $\rho_i : X_i \to X_{i-1}$, $i = 1, 2, \ldots$, is a sequence of linear operators (restrictions) such that, for every i,

$$||\rho_i||_{Xi \to Xi-1} \le c_\rho$$

and

$$\rho_i \pi_i = I_{i-1};$$

3) $\{A_i\}$, $\{\Gamma_i\}$, $\{\Psi_i\}$, $\{\Phi_i\}$, for $i = 0, 1, \dots$, are uniformly bounded sequences of linear operators, with $A_i, \Gamma_i, \Psi_i, \Phi_i : X_i \to X_i$. We assume that the inverses of the operators A_i exist and are uniformly bounded.

Then, for every $i = 1, 2, \dots$, we consider the approximation B_i of A_i^{-1} generated by the following

General linear multigrid scheme. Let an approximation B_0 of A_0^{-1} and an integer $\gamma \geq 1$ be given. For $i = 1, 2, \dots$, set

$$S_{i-1} = I_{i-1} - B_{i-1} A_{i-1}, \tag{3.1}$$

$$Q_{i-1} = \Sigma_{0 \leq j \leq \gamma - 1} (S_{i-1})^j B_{i-1}, \tag{3.2}$$

$$B_i = \Gamma_i + \Psi_i \pi_i Q_{i-1} \rho_i \Phi_i. \tag{3.3}$$

For the study of this scheme, the following operators will also be considered:

$$R_i = I_i - \Gamma_i A_i - \Psi_i \pi_i A_{i-1}^{-1} \rho_i \Phi_i A_i,$$
$$W_i = I_i - \Gamma_i A_i + (I_i - \Psi_i) \pi_i A_{i-1}^{-1} \rho_i \Phi_i A_i.$$

Straightforward computations show that, since $Q_{i-1} = (I_{i-1} - (S_{i-1})^\gamma) \cdot A_{i-1}^{-1}$ and $\rho_i \pi_i = I_{i-1}$, we have

$$\begin{aligned} S_i &= R_i + \Psi_i \pi_i (S_{i-1})^\gamma A_{i-1}^{-1} \rho_i \Phi_i A_i \\ &= R_i + \Psi_i \pi_i (S_{i-1})^\gamma \rho_i (W_i - R_i). \end{aligned} \tag{3.4}$$

Remark. The above scheme contains the classical methods described in [**12**], where the coarse-grid corrections are combined with some pre-smoothing and post-smoothing steps. Indeed, for every i, let M_i and N_i be linear operators, from X_i into itself, such that $M_i = (I_i - N_i A_i)$. Then, for some integers $m \geq q > 0$, set

$$\Gamma_i = \Sigma_{0 \leq j \leq m-1} M_i^j N_i,$$
$$\Psi_i = M_i^{m-q},$$
$$\Phi_i = I_i - A_i \Sigma_{0 \leq j \leq q-1} M_i^j N_i.$$

Accordingly, we have

$$R_i = M_i^{m-q}[I_i - \pi_i A_{i-1}^{-1}\rho_i A_i]M_i^q,$$
$$W_i = M_i^{m-q} - [(I_i - M_i^{m-q})\pi_i A_{i-1}^{-1}\rho_i A_i]M_i^q.$$

In particular, choosing, for each i, $M_i = I_i$, $m = q = 1$, we obtain certain multigrid methods for equations of the second kind discussed in [**13**].

We come back to the study of the scheme defined by (3.1)–(3.3). Let us assume that $\{v_i\}$, $i = 0, 1, \dots$, is a sequence such that, for each $i = 1, 2, \dots$,

$$||R_i||_{X_i \to X_i} \le v_i$$

and

$$||S_0||_{X_0 \to X_0} \le v_0.$$

Thus, setting

$$\kappa = c_\pi c_\rho \operatorname{Sup}_i ||\Psi_i||_{X_i \to X_i}$$

and

$$s_i = ||S_i||_{X_i \to X_i}, \qquad w = \operatorname{Sup}_i ||W_i||_{X_i \to X_i},$$

from (3.4) it follows that

$$s_i \le v_i + \kappa (s_{i-1})^\gamma (w + v_i), \quad \text{for } i = 1, 2, \dots, \tag{3.5}$$

with

$$s_0 \le v_0. \tag{3.6}$$

Arguing as in [**13**, Lemma 3.3], one proves the following result.

Proposition 3.1. *Let* (3.5) *and* (3.6) *hold. Assume that* $v_i \le v$ *for every* i*, set* $d_i = v_i / v_{i-1}$ *and then set* $d = \inf_i(d_i)$*. If either,*

$$\gamma \ge 1, \quad 0 < \kappa w < d < 1, \quad \textit{and} \quad v \le (d - \kappa w)/(d + \kappa d)$$

or

$$\gamma = 2, \quad \textit{and} \quad 4\kappa v_{i-1}(w/d_i + v_{i-1}) \le 1, \quad \textit{for every } i, \tag{3.7}$$

then there is a $c > 0$ *such that, for every* i*, we have*

$$s_i \leq cv_i.$$

More precisely, we can take, in the first case, $c = 1 + \kappa(w + d)/(d - \kappa w)$ *and, in the second case,* $c = 2$.

The above proposition contains the results given in [**13**], where the sequence $\{v_i\}$ is assumed to be nonincreasing and $\kappa = 1$. Moreover, in the case $\gamma = 2$, here the ratios d_i are allowed to become arbitrarily small, provided that their rate of decay is controlled by (3.7).

In the next section, we will combine Scheme 1 of Section 2 with linear multigrid methods where

$$X_i = X, \qquad \pi_i = \rho_i = I, \qquad \Psi_i = I. \tag{3.8}$$

Precisely, given B_0, we will consider two pairs of methods of the type (3.1)–(3.3), defined by (3.8) and by particular choices of Γ_i and Φ_i at the i-th step.

The first pair, we call *linear multigrid methods* 1, is defined as follows (together with (3.8)):

Method 1a. $\Gamma_i = I$, $\Phi_i = I - A_i$.

Method 1b. $\Gamma_i = 0$, $\Phi = I - A_i + A_{i-1}$.

Procedures of this type were proposed and discussed in [**13**]. In particular, they also yield the classical methods of Atkinson [**2**] and Brakhage [**6**] (cf. also Kelley [**14**]). For both Methods 1a and 1b, we get

$$R_i = (A_{i-1})^{-1}[A_{i-1} - A_i](I - A_i). \tag{3.9}$$

The second pair of multigrid methods, we call here *linear multigrid methods* 2, are defined (together with (3.8)) by:

Method 2a. $\Gamma_i = I$, $\Phi_i = I - A_{i-1}$.

Method 2b. $\Gamma_i = 0$, $\Phi_i = I$.

Now, for both Methods 2a and 2b, we have

$$R_i = (A_{i-1})^{-1}[A_{i-1} - A_i]. \tag{3.10}$$

In any case $\kappa = 1$. For Methods 1a and 2a we can take in (3.5) $w = \operatorname{Sup}_i ||I - A_i||$ and for Methods 1b and 2b, we take $w = 1$. Accordingly (cf. Proposition 3.1), we can choose for Methods 1a and 2a even the value $\gamma = 1$, provided that w is sufficiently small, while for Methods 1b and 2b, we must take $\gamma = 2$.

It is easy to verify that, for each case, the corresponding adjoint operators B_i^*, for $i = 1, 2, \ldots$, are defined through the following multigrid scheme:

Starting from B_0^*, for $i = 1, 2, \ldots$, set

$$\begin{aligned} S_{i-1}^* &= I - A_{i-1}^* B_{i-1}^* \\ Q_{i-1}^* &= \Sigma_{0 \le j \le \gamma} B_{i-1}^* (S_{i-1}^*)^j, \\ B_i^* &= \Gamma_i^* + \Phi_i^* Q_{i-1}^*, \end{aligned}$$

where now I is the identity on X^*.

4. Nonlinear multilevel procedures. In this section we will examine some procedures obtained combining Scheme 1 with the two pairs of linear multigrid methods described in the previous section. The use of such procedures will be motivated in the light of the following result which is an immediate consequence of Theorem 2.1 (to which we refer for the notation).

Proposition 4.1. *Let $\{\Delta_k\}$, $k = 0, 1, \ldots$, be a nonincreasing null sequence of positive numbers such that, for some $0 < d < 1$,*

$$\Delta_{k+1} \ge d\Delta_k, \quad \text{for every } k = 0, 1, \ldots .$$

Assume that there exist constants C_0 and C_1 such that, for every $k = 0, 1, \ldots$,

$$(\Delta A)^k + (\Delta F)^k \le C_0 \Delta_{k+1},$$

and

$$\vartheta^k \le C_1 \Delta_k.$$

If Δ_0 and $e_{\mathbf{x}}^0$ are sufficiently small, then Scheme 1 is well defined and there exists a null sequence $\{q_k\}$, $k = 0, 1, \ldots$, such that, for every k we have

$$e_{\mathbf{x}}^{k+1} \le c_{\mathbf{x}} (C_0 + q_k) \Delta_{k+1}. \tag{4.1}$$

Proof. Proceed by induction, using estimates (2.6) and (2.7).

Let us return to the equation

$$Q_U\mathbf{x} - K(\mathbf{x}) = 0.$$

In this case the operator A, defined in (2.2), is given by

$$A(\mathbf{x})\mathbf{v} = Q_U\mathbf{v} - K'(\mathbf{x})\mathbf{v}, MQ_Z\mathbf{v}), \quad \text{for } \mathbf{v} \in X.$$

Here we propose two different approaches for constructing the approximating operators involved in Scheme 1. The first approach is based only on the use of collectively compact (c.c. for brevity) approximations of K; the second one also employs projection methods.

a) *Approximations through c.c. operators.* Let $\{K_n\}$ be a sequence of completely continuous nonlinear operators from D into U, which satisfies the following assumptions:

(i_1) *$\{K_n\}$ is a c.c. family on D;*

(i_2) *K_n is pointwise convergent to K on D, i.e., for each $\mathbf{x} \in D$,*

$$K_n(\mathbf{x}) \to K(\mathbf{x}), \quad \text{as } n \to \infty;$$

(i_3) *for every n, K_n is of class C^ν, $\nu \geq 3$, on $S(\mathbf{x}^s, \delta^*)$, and the derivatives $K_n'(\mathbf{x})$, $K_n''(\mathbf{x})$ and $K_n'''(\mathbf{x})$ are uniformly bounded with respect to $\mathbf{x}$ and n.*

Then, we consider the sequence of operators $\{A^n\}$, $A^n : X \to X$, defined by

$$A^n(\mathbf{x})\mathbf{v} = (Q_U\mathbf{v} - K_n'(\mathbf{x})\mathbf{v}, MQ_Z\mathbf{v}), \quad \text{for } \mathbf{v} \in X,\ \mathbf{x} \in S(\mathbf{x}^s, \delta^*). \tag{4.2}$$

As well-known (cf. [**1, 2**]) assumptions (i_1)–(i_3) ensure that:

(j_1) $\{I - A^n\}$ is a c.c. family of linear operators on $S(\mathbf{x}^s, \delta^*)$;

(j_2) $A^n(\mathbf{x})\mathbf{v} \to A(\mathbf{x})\mathbf{v}$, for every $\mathbf{x} \in S(\mathbf{x}^s, \delta^*)$ and $\mathbf{v} \in X$;

(j_3) there is a constant C_A such that, for every n and for every $\mathbf{u}, \mathbf{v} \in S(\mathbf{x}^s, \delta^*)$,

$$||A^n(\mathbf{u}) - A^n(\mathbf{v})||_{X\to X} \leq C_A|\mathbf{u} - \mathbf{v}|_X;$$

(j_4) the sequence $\{A^n(\mathbf{x})\}$ is uniformly bounded, with respect to $\mathbf{x}$ and n;

(j_5) for any compact operator $T : X \to X$, $\lim_{n\to\infty} ||(A(\mathbf{x}) - A^n(\mathbf{x}))T||_{X\to X} = 0$, for each $\mathbf{x} \in S(\mathbf{x}^s, \delta^*)$;

(j_6) for all sufficiently large n and for every $\mathbf{x} \in S(\mathbf{x}^s, \delta^*)$, $A^n(\mathbf{x})$ is invertible and the inverses are uniformly bounded with respect to $\mathbf{x}$ and n. Then, we consider a suitable subsequence of $\{K_n\}$, we relabel as $\{K_k\}$, for $k = 0, 1, \dots$. Accordingly, in Scheme 1, we take

$$F^k(\mathbf{x}) = Q_U\mathbf{x} - K_k(\mathbf{x}).$$

Assumptions (k_1)–(k_4) of Theorem 2.1 are clearly fulfilled.

Now we discuss the construction of B^k (and $(B^k)^*$) in Scheme 1, by the linear multigrid methods of Section 3. Referring to (4.2), a first procedure could be defined by taking

$$B_0 = A^0(\mathbf{x}^0)^{-1} \tag{4.3}$$

and, for each $k = 1, 2, \dots,$

$$A_i = A^i(\mathbf{x}^k), \quad \text{for } i = 0, 1, \dots, k. \tag{4.4}$$

Then we employ one of the two *linear multigrid schemes* 1 considered in Section 3. We set, in Scheme 1, $B^k = B_k$, for $k = 0, 1, \dots$. Accordingly, for each $k = 1, 2, \dots,$ from (3.9) we have

$$R_i = [A^{i-1}(\mathbf{x}^k)]^{-1}[A^{i-1}(\mathbf{x}^k) - A^i(\mathbf{x}^k)](I - A^i(\mathbf{x}^k)), \quad \text{for } i = 1, \dots, k, \tag{4.5}$$

and

$$S_0 = [A^0(\mathbf{x}^0)]^{-1}[A^0(\mathbf{x}^0) - A^0(\mathbf{x}^k)]. \tag{4.6}$$

A second procedure could be defined taking B_0 as in (4.3) and, for each $k = 1, 2, \dots,$

$$A_i = A^i(\mathbf{x}^i), \quad \text{for } i = 0, 1, \dots, k. \tag{4.7}$$

Again, we use one of the *linear multigrid schemes* 1 considered in Section 3, setting $B^k = B_k$, for each k. In this case, we have, for $k = 1, 2, \dots,$

$$R_k = [A^{k-1}(\mathbf{x}^{k-1})]^{-1}[A^{k-1}(\mathbf{x}^{k-1}) - A^k(\mathbf{x}^k)](I - A^k(\mathbf{x}^k)), \tag{4.8}$$

and

$$S_0 = 0. \tag{4.9}$$

Clearly, using (4.7) instead of (4.4), one has the advantage of constructing each A_i once and for all.

Now we show how Proposition 4.1 can be applied to the cases considered above. In the following c_j, for $j = 1, 2, 3, 4, 5$, will be constants independent of the index k.

Assume (i_1)–(i_3) and (j_1)–(j_6), and suppose that

$$||[K'(\mathbf{x}^s) - K'_{k-1}(\mathbf{x}^s)][I - A^k(\mathbf{x}^s)]||_{X\to U} \leq c_1\Delta_k,$$

$$|K(\mathbf{x}^s) - K_k(\mathbf{x}^s)|_U + \operatorname*{Max}_{1\leq i\leq p} |[K'(\mathbf{x}^s) - K'_k(\mathbf{x}^s)]\mathbf{y}_i^s|_U \leq c_2\Delta_{k+1};$$

for every $k = 1, 2, \ldots$, where the sequence $\{\Delta_k\}$ is chosen as in Proposition 4.1. Then, for both choices (4.3)–(4.4) and (4.3)–(4.7), using (4.5)–(4.6), or (4.8)–(4.9), and Proposition 3.1, one can easily prove by induction that (4.1) holds provided that Δ_0 and $e^0_{\mathbf{x}}$ are sufficiently small.

b) *Approximations through c.c. operators and projections.* Let $\{\mathbf{P}_n\}$ be a sequence of linear projections from X into itself which is *pointwise convergent* to I. Let the sequence $\{K_n\}$ be given as before. As is well-known (cf. [**4**]), the family of operators $\{K_n(\mathbf{P}_n\cdot)\}$ satisfies (i_1)–(i_3). Assume that $\{K_k\}$ and $\{\mathbf{P}_k\}$ are (relabelled) subsequences of $\{K_n\}$ and $\{\mathbf{P}_n\}$, respectively, such that:

(m_1) for every $\mathbf{x} \in S(\mathbf{x}^s, \delta^*)$, the operators $A^k(\mathbf{x})$ defined by

$$A^k(\mathbf{x})\mathbf{v} = (Q_U\mathbf{v} - K'_k(\mathbf{P}_k\mathbf{x})\mathbf{P}_k\mathbf{v}, MQ_Z\mathbf{v}), \quad \text{for } \mathbf{v} \in X, \tag{4.10}$$

are invertible and their inverses are uniformly bounded with respect to $\mathbf{x}$ and k.

Then, in Scheme 1 we take

$$F^k(\mathbf{x}) = Q_U\mathbf{x} - K_k(\mathbf{P}_k\mathbf{x}).$$

Moreover, referring to the operators defined in (4.10), we can use one of the *linear multigrid schemes* 1, taking B_0 as in (4.3), A_i as in (4.7) and $B^k = B_k$. Accordingly, (4.8) and (4.9) follow.

Let $\{\Delta_k\}$ be a sequence as in Proposition 4.1 and suppose that, for every $k = 1, 2, \dots,$

$$||(K'(\mathbf{x}^s) - K'_{k-1}(\mathbf{P}_{k-1}\mathbf{x}^s)\mathbf{P}_{k-1})[I - A^k(\mathbf{x}^s)]||_{X\to U} \leq c_3\Delta_k,$$

(4.11)
$$|K(\mathbf{x}^s) - K_k(\mathbf{P}_k\mathbf{x}^s)|_U + \underset{1\leq i\leq p}{\text{Max}}|[K'(\mathbf{x}^s) - K'_k(\mathbf{P}_k\mathbf{x}^s)\mathbf{P}_k]\mathbf{y}_i^s|_U \leq c_4\Delta_{k+1}.$$

Then, by assumption (m_1) and invoking again Propositions 3.1 and 4.1, the usual inductive argument shows that relation (4.1) holds for every k, provided that Δ_0 and e_x^0 are sufficiently small.

Let us consider the particular case where $K_k = K$, for every k, and therefore

$$F^k(\mathbf{x}) = Q_U\mathbf{x} - K(\mathbf{P}_k\mathbf{x})$$

and

$$A^k(\mathbf{x})\mathbf{v} = (Q_U\mathbf{v} - K'(\mathbf{P}_k\mathbf{x})\mathbf{P}_k\mathbf{v}, MQ_Z\mathbf{v}), \quad \text{for } \mathbf{v} \in X.$$

If

$$||(K'(\mathbf{x}^s)(I - \mathbf{P}_k)||_{X\to U} \to 0, \tag{4.12}$$

we could also use the *linear multigrid schemes* 2 of Section 3, taking, as before, $B^0 = B_0 = A^0(\mathbf{x}^0)^{-1}$, for each k, $A_i = A^i(\mathbf{x}^i)$, for $i = 0, 1, \dots, k$ and $B^k = B_k$. In this case, from (3.10), we get

$$R_k = [A^{k-1}(\mathbf{x}^{k-1})]^{-1}[A^{k-1}(\mathbf{x}^{k-1}) - A^k(\mathbf{x}^k)]$$

and $S_0 = 0$. Then, assuming that, for every k, relation (4.11) (with $K_k = K$) holds and that

$$||(K'(\mathbf{x}^s) - K'(\mathbf{P}_{k-1}\mathbf{x}^s)\mathbf{P}_{k-1})||_{X\to U} \leq c_5\Delta_k,$$

the conclusions of Proposition 4.1 easily follow.

We notice that, for all the cases considered here, it was understood that, in Scheme 1, the adjoint B^{k*} is obtained by the same multigrid method which yields B^k, as shown at the end of Section 3.

There are several results, on the approximation of integral operators, which can be used for checking the validity of the various assumptions above considered as well as for estimating the values of $\{\Delta_k\}$. Among the more recent papers, we refer to [**2, 3, 4, 5, 8**].

Remarks. The more natural way for constructing the sequence of projections $\{\mathbf{P}_n\}$ in X is that of taking a sequence $\{P_n\}$ of linear projections in U, and then defining $\mathbf{P}_n$ as $\mathbf{P}_n\mathbf{v} = (P_n Q_U \mathbf{v}, \mathbf{v}_p)$. Here, for conciseness, we have expressed any $\mathbf{v} \in X$ in the form $\mathbf{v} = (Q_U\mathbf{v}, \mathbf{v}_p)$, with $\mathbf{v}_p \in \mathbf{R}^p$. In order to illustrate how in this case the application of $B_0 = A^0(\mathbf{x}^0)^{-1}$, with A^0 given by (4.10), can be carried out, we assume, for simplicity, that the operator MQ_Z has the form $MQ_Z\mathbf{v} = (M_1Q_U\mathbf{v}, M_2\mathbf{v}_p)$, with $M_1 \in U^*$, $M_2 \in L(\mathbf{R}^p, \mathbf{R}^{p-1})$. This is the case usually occurring in practice. Accordingly, an application of B_0 consists of solving a system of the type

$$\begin{aligned} Q_U\mathbf{v} - K_0'(\mathbf{P}_0\mathbf{x}^0)\mathbf{P}_0\mathbf{v} &= f_1, \\ M_1Q_U\mathbf{v} &= f_2, \\ M_2\mathbf{v}_p &= f_3, \end{aligned}$$

with $f_1 \in U$, $f_2 \in \mathbf{R}$ and $f_3 \in \mathbf{R}^{p-1}$. This is equivalent to solve, at first, the projected system

$$\begin{aligned} P_0(Q_U\mathbf{v} - K_0'(\mathbf{P}_0\mathbf{x}^0)\mathbf{P}_0\mathbf{v} - f_1) &= 0, \\ M_1(K_0'(\mathbf{P}_0\mathbf{x}^0)\mathbf{P}_0\mathbf{v} + f_1) &= f_2, \\ M_2\mathbf{v}_p &= f_3, \end{aligned} \tag{4.13}$$

with respect to the unknowns $\mathbf{P}_0\mathbf{v}$ and $\mathbf{v}_p$ and then to get $Q_U\mathbf{v}$ as the iterated solution of (4.13) (in the sense of Sloan [**19**]), namely,

$$Q_U\mathbf{v} = K_0'(\mathbf{P}_0\mathbf{x}^0)\mathbf{P}_0\mathbf{v} + f_1.$$

Of course, the coefficient matrix of the above linear system will be factorized once and for all.

In the particular case when equation (2.1) has the classical Hammerstein form

$$Q_U\mathbf{w} - KG(\mathbf{w}) = 0, \tag{4.14}$$

where now K is a linear compact operator into U and G is a nonlinear mapping from X into U, for an efficient use of projection methods, it is convenient to consider, instead of (4.14), the equation

$$Q_U\mathbf{x} - G(KQ_U\mathbf{x}, \mathbf{x}_p) = 0,$$

obtained through the change of variable $\mathbf{x} = (G(\mathbf{w}), \mathbf{w}_p)$ that is $\mathbf{w} = (KQ_U\mathbf{x}, \mathbf{x}_p)$. For more details on this point, we refer to [**15, 16**].

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Dipartimento di Scienze Matematiche, Università di Trieste, Piazzale Europa 1, 34127 Trieste, Italia