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THE SOLUTION OF INTEGRAL EQUATIONS WITH DIFFERENCE KERNELS

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ABSTRACT. This paper investigates integral equations with difference kernels posed on finite intervals. Formulae relating the solutions of second kind equations corresponding to particular free terms, including one of the "imbedding" variety, are derived using straightforward operator manipulation. These lead to an explicit expression for the solution of the second kind equation with a general free term. Some attention is given to the practically important logarithmically singular kernel for both first and second kind equations.

1. Introduction. Suppose that the integral equation

$$\mu\phi(x) = f(x) + \int_{-\infty}^{\infty} k(x-t)\phi(t)\,dt, \quad -\infty < x < \infty, \tag{1.1}$$

has a solution ϕ for "suitable" given functions f and k. It is not difficult to verify that this solution is given by

$$\mu\phi(x) = f(x) + \int_{-\infty}^{\infty} r(x-t)f(t)\,dt, \quad -\infty < x < \infty,$$

where r satisfies the integral equation

$$\mu r(x) = k(x) + \int_{-\infty}^{\infty} k(x-t)r(t)\,dt, \quad -\infty < x < \infty.$$

This conclusion is valid if, for example, the functions involved are in $L_2(-\infty,\infty)$.

One of the objectives of this paper is to derive a corresponding result for the integral equation

$$\mu\phi(x) = f(x) + \int_{0}^{1} k(x-t)\phi(t)\,dt, \quad 0 \le x \le 1. \tag{1.2}$$

That is, an explicit formula for the solution of (1.2) is sought in the form

$$\mu\phi(x) = f(x) + \int_0^1 r(x,t)f(t)\,dt, \quad 0 \le x \le 1, \tag{1.3}$$

where the resolvent kernel r is determined by versions of (1.2) with particular free terms f. Since (1.1) can be generally solved by Fourier transform methods, the expression given above for its solution is of little practical interest. There is no comparable solution method for (1.2), however, and (1.3) is consequently of some value. Even if it does not lead to the exact solution of (1.2) in a particular case, (1.3) gives a useful insight into the application of approximation methods, by clearly revealing the structure of the unknown ϕ.

From a practical point of view, the importance of (1.2) is due mainly to its association with boundary value problems for partial differential equations. In this case the kernel k has a particular form which we shall give some attention to in due course. For the moment the only assumptions we make are to provide a convenient setting in which to investigate (1.2).

Let $k : [-1,1] \to \mathbf{C}$ be such that $k \in L_2(-1,1)$, in which case the operator K defined by

$$(K\phi)(x) = \int_0^1 k(x-t)\phi(t)\,dt, \quad 0 \le x \le 1, \tag{1.4}$$

is a compact operator on $L_2(0,1)$. The compactness follows if $k \in L_1(-1,1)$ but we shall need the stronger condition on k for other reasons. If we further suppose that $f \in L_2(0,1)$, we can therefore consider (1.2) *via* the equation $(\mu I - K)\phi = f$ in $L_2(0,1)$. The parameter $\mu \in \mathbf{C}$ can be regarded as assigned and such that $\mu I - K$ is invertible, so that there is a unique solution $\phi \in L_2(0,1)$.

An important part is played in the proceedings by the operator V_α, where

$$(V_\alpha\phi)(x) = \int_0^x e^{-i\alpha(x-t)}\phi(t)\,dt, \quad 0 \le x \le 1,\ \alpha \in \mathbf{R}, \tag{1.5}$$

by the "reflection operator" U, which is such that

$$(U\phi)(x) = \phi(1-x), \quad 0 \le x \le 1, \tag{1.6}$$

and by f_α, where

$$f_\alpha(x) = e^{-i\alpha x}, \quad 0 \le x \le 1,\ \alpha \in \mathbf{R}. \tag{1.7}$$

Obviously V_α and U are bounded operators on $L_2(0,1)$, and it easily follows that $V_{-\alpha}^* = UV_\alpha U$ and that

$$V_\alpha \phi + V_\alpha^* \phi = (\phi, f_\alpha) f_\alpha, \quad \phi \in L_2(0,1), \tag{1.8}$$

where V_α^*, the adjoint of V_α, is given by

$$(V_\alpha^* \phi)(x) = \int_x^1 e^{-i\alpha(x-t)} \phi(t)\, dt, \quad 0 \le x \le 1,$$

and $(\ ,\ )$ denotes the inner product on $L_2(0,1)$. We shall also need to use the adjoint of K, defined by

$$(K^* \phi)(x) = \int_0^1 l(x-t)\phi(t)\, dt, \quad 0 \le x \le 1,$$

where

$$l(x) = \overline{k(-x)}, \quad -1 \le x \le 1. \tag{1.9}$$

It is not difficult to show that, for $\phi \in L_2(0,1)$,

$$\begin{aligned}(V_\alpha K\phi)(x) + (KV_\alpha^*\phi)(x) &= \int_0^1 \phi(t)\, dt \int_{-t}^x e^{-i\alpha(x-t-s)} k(s)\, ds \\ &= \int_0^1 \phi(t)\, dt \biggl\{ e^{i\alpha t} \int_0^x e^{-i\alpha(x-s)} k(s)\, ds \\ &\qquad\qquad + e^{-i\alpha x} \int_0^t e^{i\alpha(t-s)} k(-s)\, ds \biggr\} \\ &= (\phi, f_\alpha)(V_\alpha k)(x) + (\phi, V_\alpha l) f_\alpha(x),\ 0 \le x \le 1,\end{aligned}$$

using the notation of (1.7) and (1.9). Thus, $V_\alpha K\phi + KV_\alpha^*\phi = (\phi, f_\alpha)V_\alpha k + (\phi, V_\alpha l) f_\alpha$ for $\phi \in L_2(0,1)$, which, combined with (1.8), gives

$$\begin{gathered} V_\alpha A\phi + AV_\alpha^*\phi = \{\mu(\phi, f_\alpha) - (\phi, V_\alpha l)\} f_\alpha - (\phi, f_\alpha) V_\alpha k, \\ \phi \in L_2(0,1), \end{gathered} \tag{1.10}$$

where $A = \mu I - K$. This is the central relationship in what follows. Before proceeding further, however, an explanation is needed concerning the use of the symbols k and l. In (1.10) and in certain places elsewhere in this account, k and l are to be regarded as elements of $L_2(0,1)$ and they are, therefore, restrictions of the kernel elements k and l used previously and belonging to $L_2(-1,1)$. As it is always clear from the context which interpretation of k and l is required, the introduction of alternative symbols, or of a restriction operator, is not warranted.

The identity (1.10) shows that $V_\alpha A + AV_\alpha^*$ is a rank two operator, mapping each $\phi \in L_2(0,1)$ onto the subspace of $L_2(0,1)$ spanned by f_α and $V_\alpha k$. This feature can be advantageously used in a number of ways. Here we effectively regard (1.10) as an equation for $V_\alpha^* \phi$. Inevitably, the adjoint operator $A^* = \bar{\mu} I - K^*$ plays a significant part but we do not need to build it explicitly into a relationship like (1.10) because an equation involving A^* can be restated in terms of A. Specifically, if ϕ satisfies $A^*\phi = f$ then, using (1.4), (1.6) and (1.9), we see that $UAU\bar{\phi} = \bar{f}$, whence $A(U\bar{\phi}) = U\bar{f}$.

The first kind equation given by setting $\mu = 0$ in (1.2) arises often in practical problems but our results do not apply directly to this case as the operator K is not invertible. When we examine the first kind equations in Section 5, it turns out that we need to remove the limitations of working in the space $L_2(0,1)$. This is not so serious a step as it may seem for, having derived specific results about equations in $L_2(0,1)$, it is not a difficult matter to extend them, where appropriate, to a larger class of equations. This viewpoint, which removes the need to consider the whole theory in a more general and possibly more opaque setting, is illustrated by reference to an example.

The equation (1.2) has, of course, been the subject of previous investigations, prominent among these being the work of Mullikin and his coauthors. Leonard and Mullikin ([**3**] and [**4**]) considered an equation essentially the same as (1.2) with the kernel having the particular form

$$k(x) = \int_\nu^\infty \psi(t) t^{-1} e^{-|x|t}\, dt, \tag{1.11}$$

where $0 \le \nu < \infty$ and ψ is nonnegative on $[\nu, \infty)$ and such that

$$2\int_\nu^\infty \psi(t) t^{-2}\, dt = 1.$$

For different functions ψ the resulting versions of (1.2) arise variously in neutron transport theory, radiative transfer and other areas. Leonard and Mullikin derived a method for determining the resolvent corresponding to the kernel (1.11) in terms of the solutions of auxiliary integral equations. The latter are particularly suited to iterative methods, having kernels different from (1.11). While this analysis is far-reaching for the many applications in which the kernel is of the form (1.11), it produces information more specific than we seek here.

Gohberg and Feldman [**1**] produce an expression for the resolvent of (1.2) for any $k \in L_1(-1,1)$. They achieve this by considering a finite dimensional counterpart of (1.2) which involves a Toeplitz matrix. Having derived a method for calculating the inverse of such a matrix, they are able to conjecture a corresponding method for finding its continuous analogue, that is, the inverse of the operator $A = \mu I - K$, in the notation of (1.4). To verify that the correct resolvent does actually emerge from this process requires considerable intricate manipulation. The Gohberg and Feldman formula for the resolvent, which extends to (1.2) the simple structural result given earlier for (1.1), has itself been extended to the case of matrix-valued kernels by Mullikin and Victory [**5**], whose derivation is reminiscent of the Wiener-Hopf solution method in that it hinges on the use of the Fourier transform.

So far as the basic approach is concerned, the work of Sakhnovich [**7**] is most closely related to the material presented here. As part of a substantial investigation of (1.2), Sakhnovich, using an identity like (1.10) but with $\alpha = 0$, derived an explicit solution formula, different from that of Gohberg and Feldman [**1**]. Although the present account has a similar starting point, it proceeds along another, more direct, route.

The presence of the parameter α provides a natural equation to consider first, namely, $(\mu I - K)\phi = f$ with $f = f_\alpha$. This equation is of interest in its own right, being of a type which frequently arises in wave scattering problems. However, the free term f_α can also be employed to generate any $f \in L_2(0,1)$ using Fourier series, and this is the means by which we construct the Gohberg and Feldman formula referred to above. This tactic is evidently new even though it is the direct counterpart for (1.2) of the standard Fourier transform solution method for (1.1).

The technique used here is elementary, being based on straightforward operator manipulation which may well be capable of adaptation to other integral equations. In addition to the direct, constructive nature of the method, the Fourier series approach reveals a variety of ways for finding the two functions which together determine the resolvent kernel of (1.2). In particular, it transpires that the solution of (1.2) is usually given explicitly for any $f \in L_2(0,1)$ in terms of its solution with $f = f_\alpha$, for two distinct values of α. In the case of a kernel which is an even function of its argument, such as (1.11), the solution of (1.2) with $f = f_\alpha$ for only one value of α is normally sufficient to determine the associated resolvent.

2. The equation $(\boldsymbol{\mu}\mathbf{I} - \mathbf{K})\boldsymbol{\phi} = \mathbf{f}_\alpha$. Let $\alpha \in \mathbf{R}$ and denote by ϕ_α the solution of $(\mu I - K)\phi = f$ with $f = f_\alpha$, so that $A\phi_\alpha = f_\alpha$ where $A = \mu I - K$. It is assumed throughout that A does not depend on the parameter α. A form of reciprocal principle for $A\phi_\alpha = f_\alpha$ is required before we tackle the first main result.

Lemma 1. *Let $A\phi_\alpha = f_\alpha$ in $L_2(0,1)$, where α is a real parameter, and let $\beta \in \mathbf{R}$. Then $e^{i\alpha}(\phi_\alpha, f_\beta) = e^{i\beta}(\phi_\beta, f_\alpha)$.*

Proof. It was noted in Section 1 that $A^*\phi = f$ implies $AU\bar{\phi} = U\bar{f}$, where U is the operator defined by (1.6). Suppose that $A^*\psi_\alpha = f_\alpha$. Then $AU\bar{\psi}_\alpha = U\bar{f}_\alpha = e^{i\alpha}f_\alpha$, and so $A(U\bar{\psi}_\alpha - e^{i\alpha}\phi_\alpha) = 0$. Since $A\phi L = 0$ has only the trivial solution, by hypothesis, $U\bar{\psi}_\alpha = e^{i\alpha}\phi_\alpha$ and, therefore, $\psi_\alpha = e^{-i\alpha}U\bar{\phi}_\alpha$.

Now $(\phi_\alpha, f_\beta) = (\phi_\alpha, A^*\psi_\beta) = (A\phi_\alpha, \psi_\beta) = (f_\alpha, \psi_\beta)$. Thus,

$$e^{i\alpha}(\phi_\alpha, f_\beta) = e^{i\alpha}(f_\alpha, e^{-i\beta}U\bar{\phi}_\beta) = e^{i\alpha+i\beta}(\phi_\beta, U\bar{f}_\alpha) = e^{i\beta}(\phi_\beta, f_\alpha).$$

□

Theorem 1. *Let $A\phi_\alpha = f_\alpha$ in $L_2(0,1)$, where α is a real parameter, let β, γ and δ be real and denote*

$$G_{\alpha,\beta} = e^{i\alpha}(\alpha - \beta)(\phi_\alpha, f_\beta).$$

Then

(i)
$$G_{\beta,\gamma}\phi_\alpha = e^{i(\gamma-\alpha)}G_{\beta,\alpha}\{I + i(\gamma-\alpha)V_\alpha\}\phi_\gamma \\ - e^{i(\beta-\alpha)}G_{\gamma,\alpha}\{I + i(\beta-\alpha)V_\alpha\}\phi_\beta,$$

(ii)
$$G_{\alpha,\beta}G_{\gamma,\delta} + G_{\alpha,\delta}G_{\beta,\gamma} = G_{\beta,\delta}G_{\alpha,\gamma}.$$

Proof. Suppose that α, β and γ are distinct real numbers, that at least one of (ϕ_β, f_α) and (ϕ_γ, f_α) is nonzero, and let

$$\Phi = (\phi_\gamma, f_\alpha)\phi_\beta - (\phi_\beta, f_\alpha)\phi_\gamma. \tag{2.1}$$

Since

$$(\Phi, f_\alpha) = 0, \tag{2.2}$$

applying (1.10) to Φ gives

$$V_\alpha A\Phi + AV_\alpha^*\Phi = -(\Phi, V_\alpha l)f_\alpha, \tag{2.3}$$

which we solve for $V_\alpha^*\Phi$.

A straightforward calculation reveals that

$$V_\alpha f_\beta = i(\beta-\alpha)^{-1}(f_\beta - f_\alpha) = i(\beta-\alpha)^{-1}A(\phi_\beta - \phi_\alpha), \quad \alpha \neq \beta. \tag{2.4}$$

Therefore,

$$\begin{aligned} V_\alpha A\Phi &= (\phi_\gamma, f_\alpha)V_\alpha f_\beta - (\phi_\beta, f_\alpha)V_\alpha f_\gamma \\ &= i(\beta-\alpha)^{-1}(\phi_\gamma, f_\alpha)A(\phi_\beta - \phi_\alpha) \\ &\quad - i(\gamma-\alpha)^{-1}(\phi_\beta, f_\alpha)A(\phi_\gamma - \phi_\alpha), \end{aligned}$$

so that (2.3) can be rewritten in the form

$$i(\beta-\alpha)^{-1}(\phi_\gamma, f_\alpha)A\phi_\beta - i(\gamma-\alpha)^{-1}(\phi_\beta, f_\alpha)A\phi_\gamma + AV_\alpha^*\Phi = CA\phi_\alpha,$$

where C is some constant. As we are assuming that $A\phi = 0$ implies $\phi = 0$, we deduce that

$$C\phi_\alpha = i(\beta-\alpha)^{-1}(\phi_\gamma, f_\alpha)\phi_\beta - i(\gamma-\alpha)^{-1}(\phi_\beta, f_\alpha)\phi_\gamma + V_\alpha^*\Phi. \tag{2.5}$$

To determine C, note that if $\delta \in \mathbf{R}$ and $\delta \neq \alpha$ then, by (2.2) and the first equality in (2.4),

$$(V_\alpha^*\Phi, f_\delta) = (\Phi, V_\alpha f_\delta) = -i(\delta-\alpha)^{-1}(\Phi, f_\delta).$$

Using this identity in the equation formed by taking the inner product of both sides of (2.5) with f_δ gives, after substituting for Φ from (2.1) and gathering together like terms,

$$\begin{aligned} C(\phi_\alpha, f_\delta) = i(\delta-\beta)(\beta-\alpha)^{-1}(\delta-\alpha)^{-1}(\phi_\gamma, f_\alpha)(\phi_\beta, f_\delta) \\ - i(\delta-\gamma)(\gamma-\alpha)^{-1}(\delta-\alpha)^{-1}(\phi_\beta, f_\alpha)(\phi_\gamma, f_\delta). \end{aligned} \tag{2.6}$$

By an earlier assumption and Lemma 1, at least one of (ϕ_α, f_β) and (ϕ_α, f_γ) is nonzero. Taking either $\delta = \beta$ or $\delta = \gamma$ in (2.6) we find, using Lemma 1 again, that

$$C = -i(\beta-\gamma)(\gamma-\alpha)^{-1}(\beta-\alpha)^{-1}e^{i(\alpha-\gamma)}(\phi_\beta, f_\gamma). \tag{2.7}$$

Substituting for C in (2.5) leads to

$$\begin{aligned} (\beta-\gamma)e^{i(\alpha-\gamma)}(\phi_\beta, f_\gamma)\phi_\alpha = -(\gamma-\alpha)(\phi_\gamma, f_\alpha)\phi_\beta + (\beta-\alpha)(\phi_\beta, f_\alpha)\phi_\gamma \\ + i(\gamma-\alpha)(\beta-\alpha)V_\alpha^*\Phi \end{aligned}$$

from which the required expression (i) follows on noting that $V_\alpha^*\Phi = -V_\alpha\Phi$, a consequence of (1.8) and (2.2), replacing Φ by means of (2.1) and introducing the $G_{\alpha,\beta}$ notation. The result in (ii) is immediate on eliminating C between (2.6) and (2.7) and converting to the given notation.

We now have to remove the various restrictions which have been introduced. Suppose first that (with α, β and γ distinct and $\delta \neq \alpha$) both (ϕ_β, f_α) and (ϕ_γ, f_α) are zero. The sequence $(f_{2n\pi})$, $n \in \mathbf{Z}$, is complete in $L_2(0,1)$ and $\phi_\alpha \neq 0$, so there is certainly a $\beta' \in \mathbf{R}$ such that $(\phi_\alpha, f_{\beta'}) \neq 0$, Hence $(\phi_{\beta'}, f_\alpha) \neq 0$, by Lemma 1. Following through the first part of the proof with β' replacing β and $(\phi_\gamma, f_\alpha) = 0$, one readily finds, on taking $\delta = \beta$ in the equation replacing (2.6), that $(\phi_\gamma, f_\beta) = 0$. Therefore, $(\phi_\beta, f_\alpha) = 0$ and $(\phi_\gamma, f_\alpha) = 0$ imply that $(\phi_\beta, f_\gamma) = 0$, showing that (i) and (ii) are identically satisfied in this case. Now observe that $G_{\alpha,\beta} + G_{\beta,\alpha} = 0$ is implied by Lemma 1 and that $G_{\alpha,\alpha} = 0$. It follows that the results established are identically satisfied if $\alpha = \beta$ or $\alpha = \gamma$ or $\beta = \gamma$ or $\alpha = \beta = \gamma$; the result in (ii) also holds identically for $\delta = \alpha$. $\square$

Except in degenerate cases, the formula for ϕ_α given in (i) of the theorem is of the "invariant imbedding" variety in that it determines ϕ_α for any $\alpha \in \mathbf{R}$ in terms of ϕ_β and ϕ_γ, where β and γ are any distinct real numbers, each different from α. Wave diffraction theory provides a direct application of the theorem, as the example in Section 5 illustrates. There, and in other cases, the result in (ii) is also found to be useful.

The feature which makes Theorem 1 immediately useful is that it relates solutions of the same equation, corresponding to different values of a parameter. There are ways of expressing the solution of $A\phi_\alpha = f_\alpha$ in terms of the solutions of two other equations involving the same operator but different free terms, and some of these can be deduced from Theorem 1. For example, differentiating the expression (i) with respect to γ and then setting $\gamma = \beta$ results in

(2.8)
$$\begin{aligned}(\phi_\beta, f_\beta)\phi_\alpha = e^{i(\beta-\alpha)}\{&(\phi_\beta, f_\alpha)\phi_\beta \\ &+ i(\beta-\alpha)(I + i(\beta-\alpha)V_\alpha)((\phi_\beta, f_\alpha)\chi_\beta - (\chi_\beta, f_\alpha)\phi_\beta)\},\end{aligned}$$

where $\chi_\beta = i\partial\phi_\beta/\partial\beta$. Note that $A\phi_\beta = f_\beta$ implies $A\chi_\beta = i\partial f_\beta/\partial\beta$ and that $\partial f_\beta(x)/\partial\beta = -ixf_\beta(x)$. Therefore, χ_β is the unique solution of $A\chi_\beta = g_\beta$ where $g_\beta = xe^{-i\beta x}$, $0 \le x \le 1$. In the case $\beta = 0$, (2.8) gives ϕ_α for any $\alpha \in \mathbf{R}$ in terms of the solutions of $A\phi_0 = f_0$ and $A\chi_0 = g_0$, the free terms in these equations being $f_0(x) = 1$ and $g_0(x) = x$, $0 \le x \le 1$.

A formula complementary to (2.8), in which χ_α is expressed in terms of ϕ_α and ϕ_β, provided $\alpha \ne \beta$, also follows from (i) of Theorem 1, on differentiating with respect to γ and then putting $\gamma = \alpha$. Other variants of the theorem can be produced in this fashion or directly, along the lines used in the proof. However, the most far-reaching result of this sort, relating solutions of $A\phi = f$ associated with different free terms f, requires a fresh approach.

Theorem 2. *Let $A\psi = k$ and $A^*\chi = l$ in $L_2(0,1)$, where $A = \mu I - K$ and $\mu \neq 0$, and let $A\phi_\alpha = f_\alpha$, where α is a real parameter. Then*

(i)
$$\begin{aligned}\mu\phi_\alpha &= e^{i\alpha} b_\alpha f_\alpha - V_\alpha^*(a_\alpha\psi - b_\alpha U\bar{\chi}) \\ &= a_\alpha f_\alpha + V_\alpha(a_\alpha\psi - b_\alpha U\bar{\chi}),\end{aligned}$$

where $a_\alpha = 1 + (f_\alpha, \chi) = 1 + (\phi_\alpha, l)$ and $b_\alpha = e^{-i\alpha}(1 + (\psi, f_\alpha)) = e^{-i\alpha} + (\phi_\alpha, U\bar{k})$;

(ii) *$p_\alpha = \phi_\alpha' + i\alpha\phi_\alpha \in L_2(0,1)$, where $\phi_\alpha'(x) = d\phi_\alpha(x)/dx$, and*

$$\begin{aligned}(\beta - \alpha)(\phi_\alpha, f_\beta)\psi &= ie^{i\beta}(b_\beta p_\alpha - b_\alpha p_\beta), \\ (\beta - \alpha)(\phi_\alpha, f_\beta)U\bar{\chi} &= ie^{i\beta}(a_\beta p_\alpha - a_\alpha p_\beta),\end{aligned}$$

where $\beta \in \mathbf{R}$.

Proof. (i). First note that

$$(f_\alpha, \chi) = (A\phi_\alpha, \chi) = (\phi_\alpha, A^*\chi) = (\phi_\alpha, l)$$

and, referring to the proof of Lemma 1, that

$$(\psi, f_\alpha) = (\psi, A^* e^{-i\alpha} U\bar{\phi}_\alpha) = e^{i\alpha}(A\psi, U\bar{\phi}_\alpha) = e^{i\alpha}(k, U\bar{\phi}_\alpha) = e^{i\alpha}(\phi_\alpha, U\bar{k}).$$

These equalities confirm that the two expressions given for a_α and for b_α do indeed coincide.

Now let

$$\Phi = a_\alpha\psi - b_\alpha U\bar{\chi} \tag{2.9}$$

and observe that, since $(U\bar{\chi}, f_\alpha) = (U\bar{f}_\alpha, \chi) = e^{i\alpha}(f_\alpha, \chi)$,

$$(\Phi, f_\alpha) = e^{i\alpha} b_\alpha - a_\alpha. \tag{2.10}$$

From $A^*\chi = l$ we deduce that $AU\bar{\chi} = U\bar{l}$. Therefore, $A\Phi = a_\alpha k - b_\alpha U\bar{l}$ and, applying (1.10) to Φ, yields

$$V_\alpha(a_\alpha k - b_\alpha U\bar{l}) + AV_\alpha^*\Phi = \{\mu(\Phi, f_\alpha) - (\Phi, V_\alpha l)\} f_\alpha - (\Phi, f_\alpha)V_\alpha k.$$

This reduces to

$$b_\alpha V_\alpha(e^{i\alpha}k - U\bar{l}) + AV_\alpha^*\Phi = \{\mu(\Phi, f_\alpha) - (V_\alpha^*\Phi, l)\}f_\alpha \tag{2.11}$$

on using (2.10) and the property $(\Phi, V_\alpha l) = (V_\alpha^*\Phi, l)$.

A direct calculation shows that $Kf_\alpha = V_\alpha k + e^{-i\alpha}V_\alpha^* U\bar{l}$ and, by (1.8), $V_\alpha U\bar{l} + V_\alpha^* U\bar{l} = (U\bar{l}, f_\alpha)f_\alpha = e^{i\alpha}(f_\alpha, l)f_\alpha$. Thus

$$Kf_\alpha = V_\alpha k - e^{-i\alpha}V_\alpha U\bar{l} + (f_\alpha, l)f_\alpha,$$

which may be used to form Af_α, giving

$$V_\alpha(e^{i\alpha}k - U\bar{l}) + e^{i\alpha}Af_\alpha = e^{i\alpha}\{\mu - (f_\alpha, l)\}f_\alpha.$$

Combining this last equation with (2.11) we find that

$$A\Psi = -\{a_\alpha\mu + (\Psi, l)\}f_\alpha,$$

after using (2.10) and writing

$$\Psi = V_\alpha^*\Phi - e^{i\alpha}b_\alpha f_\alpha. \tag{2.12}$$

We deduce that

$$\Psi = -\{a_\alpha\mu + (\Psi, l)\}\phi_\alpha, \tag{2.13}$$

whence $\{1 + (\phi_\alpha, l)\}(\Psi, l) = -a_\alpha\mu(\phi_\alpha, l)$. Now $a_\alpha = 1 + (\phi_\alpha, l)$ and, therefore, $(\Psi, l) = -\mu(a_\alpha - 1)$, showing that (2.13) reduces to $\Psi = -\mu\phi_\alpha$. Using (2.9) and (2.12) we thus arrive at

$$\mu\phi_\alpha = e^{i\alpha}b_\alpha f_\alpha - V_\alpha^*(a_\alpha\psi - b_\alpha U\bar{\chi}), \tag{2.14}$$

which is one of the required formulae. The alternative expression given for $\mu\phi_\alpha$ follows at once from (2.14) because $V_\alpha\Phi + V_\alpha^*\Phi = (e^{i\alpha}b_\alpha - a_\alpha)f_\alpha$, according to (1.8) and (2.10).

(ii). Since

$$\left(\frac{d}{dx} + i\alpha\right)(V_\alpha^*\phi)(x) = -\phi(x)$$

almost everywhere in $[0,1]$, for $\phi \in L_2(0,1)$, it follows from (2.14) that

$$\mu(\phi'_\alpha + i\alpha\phi_\alpha) = a_\alpha\psi - b_\alpha U\bar{\chi}.$$

Now ϕ_α, ψ and χ are members of $L_2(0,1)$ by hypothesis and so, therefore, are ϕ'_α and $p_\alpha = \phi'_\alpha + i\alpha\phi_\alpha$.

Using (2.4), (2.9) and (2.12) we find that

$$\begin{aligned}(\Psi, f_\beta) &= (\Phi, V_\alpha f_\beta) - e^{i\alpha} b_\alpha (f_\alpha, f_\beta) \\ &= -i(\beta-\alpha)^{-1} e^{i\beta}(a_\alpha b_\beta - a_\beta b_\alpha), \quad \alpha \neq \beta.\end{aligned}$$

Therefore, since we know that $\Psi = -\mu\phi_\alpha$, we have

$$\mu(\beta-\alpha)(\phi_\alpha, f_\beta) = ie^{i\beta}(a_\alpha b_\beta - a_\beta b_\alpha), \tag{2.15}$$

which is identically satisfied for $\alpha = \beta$.

Finally, we deduce from $\mu p_\alpha = a_\alpha\psi - b_\alpha U\bar{\chi}$ and $\mu p_\beta = \alpha_\beta\psi - b_\beta U\bar{\chi}$ that

$$(a_\alpha b_\beta - a_\beta b_\alpha)\psi = \mu(b_\beta p_\alpha - b_\alpha p_\beta)$$

and

$$(a_\alpha b_\beta - a_\beta b_\alpha)U\bar{\chi} = \mu(a_\beta p_\alpha - a_\alpha p_\beta).$$

The expressions given in the theorem result from these, on using (2.15) to replace $a_\alpha b_\beta - a_\beta b_\alpha$. □

We thus see that the pair ψ, χ and the pair ϕ_α, ϕ_β are usually interchangeable, allowing formulae given in terms of one pair to be rewritten in terms of the other pair. More precisely, ϕ_α and ϕ_β can always be replaced by ψ and χ; the reciprocal transfer requires $(\phi_\alpha, f_\beta) \neq 0$, as well as $\alpha \neq \beta$.

We saw that for equation (1.1) the resolvent kernel satisfies the given equation with the free term replaced by the kernel. The significance of Theorem 2 is that it takes us closer to our aim of finding the parallel construction for (1.2), by introducing functions ψ and χ which satisfy equations in which the free terms are the kernel and its adjoint. The next section fulfills this aim.

3. The equation $(\boldsymbol{\mu}\mathbf{I} - \mathbf{K})\boldsymbol{\phi} = \mathbf{f}$. The solution of $A\phi = f$ can be constructed from the solution of $A\phi_\alpha = f_\alpha$ by a superposition method suggested by problems in wave scattering theory, which can often be formulated as integral equations of the form (1.2). There the free term f_α represents a monochromatic incident wave and ϕ_α is the "response" of the system to that wave. The response to a more general wave is the appropriate linear combination of the individual modal responses.

To deal with any free term $f \in L_2(0,1)$, we use the fact that the orthonormal sequence $(f_{2n\pi})$, $n \in \mathbf{Z}$, is complete in $L_2(0,1)$. Therefore, we can write $f = \sum_{-\infty}^{\infty}(f, f_{2n\pi})f_{2n\pi}$, and the solution of $A\phi = f$ is given by $\phi = \sum_{-\infty}^{\infty}(f, f_{2n\pi})\phi_{2n\pi}$, where $A\phi_{2n\pi} = f_{2n\pi}$. This construction explains the emphasis given to the equation $A\phi_\alpha = f_\alpha$ with $\alpha \in \mathbf{R}$. In particular, using the first of the representations of ϕ_α in Theorem 2, we see that the solution of $A\phi = f$ can be written in the form

$$\mu\phi = \sum_{-\infty}^{\infty}(f, f_{2n\pi})\{b_{2n\pi}f_{2n\pi} - V_{2n\pi}^*(a_{2n\pi}\psi - b_{2n\pi}U\bar{\chi})\}, \tag{3.1}$$

the principal virtue of which is that the series can be summed to provide a closed expression for ϕ.

To see how the summation is achieved, note that, because $a_{2n\pi} = 1 + (f_{2n\pi}, \chi) = 1 + (U\bar{\chi}, f_{2n\pi})$ and $b_{2n\pi} = 1 + (\psi, f_{2n\pi})$, (3.1) can be arranged as $\mu\phi = f + u + v$, where

$$u = \sum_{-\infty}^{\infty}(f, f_{2n\pi})\{(\psi, f_{2n\pi})f_{2n\pi} - V_{2n\pi}^*(\psi - U\bar{\chi})\}$$

and

$$v = \sum_{-\infty}^{\infty}(f, f_{2n\pi})V_{2n\pi}^*\{(\psi, f_{2n\pi})U\bar{\chi} - (U\bar{\chi}, f_{2n\pi})\psi\}.$$

Now $(V_{2n\pi} + V_{2n\pi}^*)\psi = (\psi, f_{2n\pi})f_{2n\pi}$, by (1.8), and so $u = \sum_{-\infty}^{\infty}$ $(f, f_{2n\pi})\{V_{2n\pi}\psi + V_{2n\pi}^*U\bar{\chi}\}$. Therefore

$$\begin{aligned} u(x) = \sum_{-\infty}^{\infty}(f, f_{2n\pi})\bigg\{ \int_0^x e^{-2n\pi i(x-t)}\psi(t)\,dt \\ + \int_x^1 e^{-2n\pi i(x-t)}\overline{\chi(1-t)}\,dt\bigg\} \\ = \sum_{-\infty}^{\infty}(f, f_{2n\pi})\bigg\{ \int_0^x \psi(x-t)e^{-2n\pi it}\,dt \\ + \int_x^1 \overline{\chi(t-x)}e^{-2n\pi i(t-1)}\,dt\bigg\}, \end{aligned}$$

giving

$$u(x) = \int_0^x \psi(x-t)f(t)\,dt + \int_x^1 \overline{\chi(t-x)}f(t)\,dt$$

for almost all x in $[0, 1]$.

To reduce the corresponding step for v to a concise form, let

$$F(x,t) = \psi(t)(U\bar{\chi})(x) - \psi(x)(U\bar{\chi})(t),$$

in terms of which we have

$$v(x) = \sum_{-\infty}^{\infty}(f, f_{2n\pi})\int_x^1 ds \int_0^1 F(s,t)e^{-2n\pi i(x-s-t)}\,dt.$$

It is not difficult to see that

$$\int_x^1 ds \int_{1+x-s}^1 F(s,t)e^{2n\pi i(s+t)}\,dt = 0$$

because the integration domain is symmetric about the line $s = t$ and $F(s,t) = -F(t,s)$. Hence,

$$\begin{aligned} v(x) &= \sum_{-\infty}^{\infty}(f, f_{2n\pi})\int_x^1 ds \int_0^{1+x-s} F(s,t)e^{-2n\pi i(x-s-t)}\,dt \\ &= \sum_{-\infty}^{\infty}(f, f_{2n\pi})\int_x^1 d\sigma \int_0^{\sigma} F(1+x-\sigma, \sigma-\tau)e^{-2n\pi i(\tau-1)}\,d\tau, \end{aligned}$$

on making the variable changes $\sigma = 1+x-s$ and $\tau = \sigma - t$ in succession. The summation is now trivial and

$$v(x) = \int_x^1 ds \int_0^s F(1+x-s, s-t) f(t)\, dt.$$

Reversing the integration order gives

$$v(x) = \int_0^1 f(t)\, dt \int_{\max(x,t)}^1 F(1+x-s, s-t)\, ds \tag{3.2}$$

almost everywhere in $[0,1]$.

Notice here that, with the aid of (1.8), the original expression for v can be arranged in the form

$$v = -\sum_{-\infty}^{\infty} (f, f_{2n\pi}) V_{2n\pi} \{(\psi, f_{2n\pi}) U\bar{\chi} - (U\bar{\chi}, f_{2n\pi})\psi\}.$$

This alternative version of v also results if the second representation of ϕ_α in Theorem 2 is used at the outset. Summing it gives

$$v(x) = \int_0^x ds \int_s^1 F(1+s-t, x-s) f(t)\, dt,$$

that is,

$$v(x) = \int_0^1 f(t)\, dt \int_0^{\min(x,t)} F(1+s-t, x-s)\, ds \tag{3.3}$$

almost everywhere in $[0,1]$. It is not difficult to verify that the inner integrals in (3.2) and (3.3) are indeed equal.

Gathering together the expressions for u and v and recalling that $\mu\phi = f + u + v$ shows that we have established the following result.

Theorem 3. *Let $A\psi = k$ and $A^*\chi = l$ in $L_2(0,1)$, where $A = \mu I - K$ and $\mu \neq 0$, and let $w : [-1,1] \to \mathbf{C}$ be defined by*

$$w(x) = \begin{cases} \psi(x), & 0 \le x \le 1, \\ \overline{\chi(-x)}, & -1 \le x < 0. \end{cases}$$

Then the unique solution of $A\phi = f$ in $L_2(0,1)$ is given by

$$\mu\phi(x) = f(x) + \int_0^1 r(x,t)f(t)\,dt$$

for almost all x in $[0,1]$ where $r : [0,1]\times[0,1] \to \mathbf{C}$ is defined by

$$\begin{aligned} r(x,t) &= w(x-t) + \int_{\max(x,t)}^1 \{\psi(s-t)\overline{\chi(s-x)} \\ &\qquad - \psi(1-s+x)\overline{\chi(1-s+t)}\}\,ds \\ &= w(x-t) + \int_0^{\min(x,t)} \{\psi(x-s)\overline{\chi(t-s)} \\ &\qquad - \psi(1-t+s)\overline{\chi(1-x+s)}\}\,ds. \qquad \square \end{aligned}$$

The derivation of this result can be set aside at this stage as a direct verification is possible, if rather intricate. Gohberg and Feldman [**1**] of necessity carry out this verification for the second version of r above; the first version of r has apparently not been given before.

Alternative forms of the solution of $A\phi = f$ which are perhaps more revealing as to structure follow from Theorem 3 by noticing that the operator generated by the kernel r can be expressed in terms of operators defined by convolutions. By using the two equivalent forms of r we find that

$$\mu\phi = f + (S + T^* + T^*S - V^*W)f$$

and

$$\mu\phi = f + (S + T^* + ST^* - WV^*)f,$$

where $S\phi = \psi * \phi$, $T\phi = \chi * \phi$, $V\phi = (U\bar{\psi}) * \phi$, $W\phi = (U\bar{\chi}) * \phi$ and the convolution $*$ is defined by

$$(\psi * \phi)(x) = \int_0^x \psi(x-t)\phi(t)\,dt, \quad 0 \le x \le 1.$$

S is a bounded operator on $L_2(0,1)$ with $||S|| = ||\psi||$ and so on. The presence of the operator U in V and W means that these operators represent "convolutions about $x = 1$."

Theorem 2 indicates that the functions ψ and χ needed to construct the solution of $A\phi = f$ can be found indirectly using ϕ_α and ϕ_β (provided $\alpha \neq \beta$ and $(\phi_\alpha, f_\beta) \neq 0$) rather than by solving $A\psi = k$ and $A^*\chi = l$. The solution of $A\phi = f$ is, therefore, determined for any f by a "suitable" pair ϕ_α, ϕ_β. Returning to the relationship with wave scattering theory, the response of a system to an arbitrary incident wave can be calculated once its response to two "independent" individual wave modes is known, if the problem can be represented in the form (1.2). This connection between the key elements ψ and χ and the "wave responses" ϕ_α and ϕ_β has not previously been given.

There are other indirect ways of finding ψ and χ, of which one needs to be mentioned since it provides a link with the different approach of Sakhnovich [**7**], referred to in Section 1.

Applying (1.10) to ψ, where $A\psi = k$ as before, gives

$$AV_\alpha^*\psi = \{\mu(\psi, f_\alpha) - (\psi, V_\alpha l)\}f_\alpha - (1 + (\psi, f_\alpha))V_\alpha k$$

from which we deduce that

$$V_\alpha^*\psi = \{\mu(\psi, f_\alpha) - (\psi, V_\alpha l)\}\phi_\alpha - (1 + (\psi, f_\alpha))\omega_\alpha \tag{3.4}$$

where ω_α satisfies $A\omega_\alpha = V_\alpha k$ and $A\phi_\alpha = f_\alpha$ as usual. Taking the inner product with l and solving the resulting equation for $(\psi, V_\alpha l)$ enables (3.4) to be expressed in the form

$$a_\alpha V_\alpha^*\psi = e^{i\alpha}b_\alpha(\mu + (\omega_\alpha, l))\phi_\alpha - \mu\phi_\alpha - e^{i\alpha}a_\alpha b_\alpha\omega_\alpha \tag{3.5}$$

where the notation of Theorem 2 has been employed. Part (i) of that theorem also combines with (3.5) to produce

$$V_\alpha^*U\bar{\chi} = e^{i\alpha}(\mu + (\omega_\alpha, l))\phi_\alpha - e^{i\alpha}(f_\alpha + a_\alpha\omega_\alpha).$$

Since a_α and b_α are determined by ϕ_α, we see that ψ and χ are given once ϕ_α and ω_α are known for any $\alpha \in \mathbf{R}$; more exactly, ϕ_α, ω_α and their first derivatives are required to provide ψ and χ explicitly.

This approach to the solution of $A\phi = f$ through ϕ_α and ω_α generalizes Sakhnovich's result. His solution formula is based solely on ϕ_0 and ω_0, in the present notation, and the connection of these functions with others, such as ψ and χ, is not explored in his paper.

Although we are concerned here with integral equations posed on finite intervals, we can deduce from Theorem 3 the corresponding resolvent kernel construction for the Wiener-Hopf type integral equation

$$\mu\phi(x) = f(x) + \int_0^\infty k(x-t)\phi(t)\,dt, \quad x \geq 0. \tag{3.6}$$

By transforming (1.2) so that the interval $[0,1]$ on which that equation holds maps onto $[0,a]$, redefining the dependent variables suitably and formally taking the limit $a \to \infty$, Theorem 3 shows that if (3.6), interpreted as an equation in $L_2(0,\infty)$, has a solution, it is given by

$$\mu\phi(x) = f(x) + \int_0^\infty r(x,t)f(t)\,dt$$

(for almost all $x \geq 0$). In this case,

$$r(x,t) = \int_0^{\min(x,t)} \psi(x-s)\overline{\chi(t-s)}\,ds + \left\{ \begin{matrix} \psi(x-t), & x > t \\ \overline{\chi(t-x)}, & t > x \end{matrix} \right\},$$

ψ and χ satisfying the counterparts in $L_2(0,\infty)$ of

$$\mu\psi(x) = k(x) + \int_0^\infty k(x-t)\psi(t)\,dt, \quad x \geq 0,$$

and

$$\bar{\mu}\chi(x) = \overline{k(-x)} + \int_0^\infty \overline{k(t-x)}\chi(t)\,dt, \quad x \geq 0,$$

respectively. This representation of the solution of (3.6), which can be confirmed directly, was given by Krein [**2**].

4. The kernel $\mathbf{g}(|\mathbf{x}-\mathbf{t}|)$.

We now consider the case in which $k(x) = k(-x)$ so that we can write

$$k(x) = g(|x|), \quad -1 \leq x \leq 1, \tag{4.1}$$

where $g : [0,1] \to \mathbf{C}$ is such that $g \in L_2(0,1)$, thus obtaining information about a commonly occurring equation of the type (1.2), namely,

$$\mu\phi(x) = f(x) + \int_0^1 g(|x-t|)\phi(t)\,dt, \quad 0 \leq x \leq 1. \tag{4.2}$$

To avoid a wholesale change of notation we continue to use the operator K, defined for the purposes of this section by

$$(4.3) \qquad (K\phi)(x) = \int_0^1 g(|x-t|)\phi(t)\,dt, \quad 0 \le x \le 1.$$

Even though $A = \mu I - K$ is not self-adjoint in these new circumstances, as μ is not necessarily real nor is g necessarily real-valued, some significant simplifications of the earlier theory follow from (4.1). This is because A now satisfies $A = UAU$, so that $A\phi = f$ implies $A^*\bar{\phi} = \bar{f}$, and because $l = \bar{k}$, by (1.9). An immediate consequence of these relationships is that the elements ψ and χ of $L_2(0,1)$ arising in Theorems 2 and 3 are now related by

$$(4.4) \qquad \chi = \bar{\psi}.$$

Further, $A^*\psi_\alpha = f_\alpha$ implies $A\bar{\psi}_\alpha = \bar{f}_\alpha = f_{-\alpha}$ and, therefore, $\bar{\psi}_\alpha = \phi_{-\alpha}$. But from the proof of Lemma 1 we recall that $\psi_\alpha = e^{i\alpha}U\bar{\phi}_\alpha$, and so the solutions of $A\phi_\alpha = f_\alpha$ are such that

$$(4.5) \qquad \phi_{-\alpha} = e^{i\alpha}U\phi_\alpha.$$

Modified versions of Theorems 1, 2 and 3 follow if (4.1) applies. Choosing $\gamma = -\beta$ in Theorem 1(i), and using (4.5), leads to

$$(4.6) \qquad \begin{aligned} 2\beta e^{i\alpha}(\phi_\beta, f_{-\beta})\phi_\alpha &= (\beta-\alpha)(\phi_\beta, f_\alpha)(I - i(\beta+\alpha)V_\alpha)U\phi_\beta \\ &\quad + (\beta+\alpha)(U\phi_\beta, f_\alpha)(I + i(\beta-\alpha)V_\alpha)\phi_\beta, \end{aligned}$$

giving ϕ_α for any $\alpha \in \mathbf{R}$ in terms of ϕ_β only, provided $\beta \neq 0$ and $(\phi_\beta, f_{-\beta}) \neq 0$. However, ϕ_α cannot be determined for $\alpha \neq 0$ in terms of ϕ_0 alone; a knowledge of χ_0 is also required, in the notation of (2.8), even if (4.1) holds.

Letting $\beta = -\alpha$ in Theorem 2(ii) and using (4.1) and its consequences leads to

$$(4.7) \qquad 2i\alpha(\phi_\alpha, f_{-\alpha})\psi = a_\alpha p_\alpha + b_\alpha U p_\alpha,$$

where $p_\alpha = \phi_\alpha' + i\alpha\phi_\alpha$, $a_\alpha = 1 + (\phi_\alpha, \bar{g})$ and $b_\alpha = e^{-i\alpha} + (\phi_\alpha, U\bar{g})$. Thus, ψ can be determined using any ϕ_α, other than ϕ_0, if $(\phi_\alpha, f_{-\alpha}) \neq 0$.

Conversely, from Theorem 2(i) and (4.4), ϕ_α is expressed in terms of ψ for any $\alpha \in \mathbf{R}$ by

$$\mu\phi_\alpha = a_\alpha f_\alpha + V_\alpha(a_\alpha\psi - b_\alpha U\psi), \tag{4.8}$$

where the alternative versions $a_\alpha = 1 + e^{-i\alpha}(U\psi, f_\alpha)$ and $b_\alpha = e^{-i\alpha}(1 + (\psi, f_\alpha))$ are now required.

Adapting Theorem 3 to (4.2), regarded as defining an equation in $L_2(0,1)$, produces it solution in the form

$$\mu\phi(x) = f(x) + \int_0^1 r(x,t)f(t)\,dt$$

(almost everywhere in $[0,1]$), where

$$\begin{aligned} r(x,t) &= \psi(|x-t|) + \int_{\max(x,t)}^1 \{\psi(s-t)\psi(s-x) \\ &\qquad\qquad - \psi(1+x-s)\psi(1+t-s)\}\,ds \\ &= \psi(|x-t|) + \int_0^{\min(x,t)} \{\psi(x-s)\psi(t-s) \\ &\qquad\qquad - \psi(1-x+s)\psi(1-t+s)\}\,ds. \end{aligned} \tag{4.9}$$

We have shown, therefore, that $\mu\phi = f + K\phi$, with K defined by (4.3), is solved for any $f \in L_2(0,1)$ once ψ is known. Several ways of obtaining ψ present themselves, in addition to the direct one of solving $\mu\psi = g + K\psi$. One alternative is offered by (4.7) which requires that ϕ_α and ϕ'_α be determined from $\mu\phi_\alpha = f_\alpha + K\phi_\alpha$ for some $\alpha \neq 0$ such that $(\phi_\alpha, f_{-\alpha}) \neq 0$. Other routes to ψ are less attractive because they require solutions of two auxiliary equations rather than just one. Thus, (2.8), with $\beta = 0$, and (4.7) show that ψ is given in terms of ϕ_0 and χ_0, and (3.5) relates ψ to ϕ_α and ω_α.

Having derived this collection of methods for solving $\mu\phi = f + K\phi$ in $L_2(0,1)$, we address the issue of translating our results into more concrete terms, applicable to (4.2).

First note that, as $(V_\alpha\phi)(x)$ is the convolution of the continuous function f_α with ϕ, it is continuous for $x \in [0,1]$, for any $\phi \in L_2(0,1)$. It follows from (4.8) (assuming $\mu \neq 0$) that $\phi_\alpha(x)$ is also continuous for $x \in [0,1]$. Moreover, the quantities a_α and b_α are related to particular values of $\phi_\alpha(x)$, for $a_\alpha = 1 + (\phi_\alpha, \bar{g}) = \mu\phi_\alpha(0)$ and $b_\alpha = e^{-i\alpha} + (\phi_\alpha, U\bar{g}) = \mu\phi_\alpha(1)$. (These properties of ϕ_α hold whether or not the kernel k satisfies (4.1).)

We also deduce from (4.8) that

$$\mu(\phi'_\alpha(x) + i\alpha\phi_\alpha(x)) = a_\alpha\psi(x) - b_\alpha\psi(1-x), \tag{4.10}$$

the equality holding almost everywhere in $[0,1]$. More useful information about ϕ'_α and ψ only follows if we are more precise about the kernel g. To give an example, suppose that

$$g(x) = \log x + h(x), \tag{4.11}$$

where h is a continuous function in $[0,1]$. As mentioned in Section 1, integral equations of the type under consideration here often arise in connection with boundary value problems and (4.3) with (4.11) gives the typical structure of the operator in such a case, when the underlying boundary value problem is two-dimensional.

It is not difficult to show that, with g given by (4.11), the function $(K\phi)(x)$ defined in (4.3) is continuous for $x \in [0,1]$, for each $\phi \in L_2(0,1)$. If $\mu \neq 0$, therefore, $\mu\phi = f + K\phi$ implies that "ϕ is as continuous as f." In particular, the solution ψ of

$$\mu\psi(x) = g(x) + \int_0^1 g(|x-t|)\psi(t)\,dt, \quad 0 \le x \le 1, \tag{4.12}$$

is continuous for $x \in (0,1]$ and behaves like $\mu^{-1}\log x$ near $x = 0$. Thus, the resolvent kernel r given by (4.9) is logarithmically singular at $x = t$; in fact,

$$r(x,t) = \mu^{-1}\log|x-t| + m(x,t),$$

where m is a continuous kernel. This implies that the solution of (4.2) has the form $\mu\phi = f + \mu^{-1}Kf + Tf$, T being generated by a continuous kernel.

Returning to (4.10), the properties of ψ deduced from (4.11) mean that $\phi'_\alpha(x)$ is continuous for $x \in (0,1)$ and behaves like $\mu^{-1}a_\alpha\log x = \phi_\alpha(0)\log x$ near $x = 0$ and like $-\mu^{-1}b_\alpha\log(1-x) = -\phi_\alpha(1)\log(1-x)$ near $x = 1$.

Deductions such as those we have made on the basis of (4.11), which are valuable when approximation methods have to be implemented, also follow for other forms of g (for example, $g(x) = x^{-\nu} + h(x)$ where $0 < \nu < 1/2$) and for the more general equation (1.2) if the structure of the kernel k is given.

5. First kind equations. Under the assumptions made in Section 1, the operator K defined by (1.4) is not invertible, and it is not possible to discuss the first kind equation $K\phi + f = 0$ in general terms on the basis of the results derived so far. A specific example helps to focus attention on the issues raised by first kind equations.

Let ϕ_α satisfy

$$(K_0\phi_\alpha)(x) = -e^{-i\kappa x\cos\theta_\alpha}, \quad 0 \le x \le 1, \tag{5.1}$$

where

$$(K_0\psi)(x) = \frac{1}{2}\pi i\int_0^1 H_0^{(1)}(\kappa|x-t|)\phi(t)\,dt, \quad 0 \le x \le 1,$$

$H_0^{(1)}$ denoting the zero order Hankel function. The equation (5.1) arises, for instance, in connection with the diffraction of a plane water wave through a gap in a straight, purely reflecting breakwater. The nondimensionalized wavenumber $\kappa > 0$ may be regarded as fixed, and $\theta_\alpha \in [0,\pi]$ is the angle which the incident wave makes with the breakwater.

The operator K_0 is an example of a type considered in Section 4, namely,

$$(K\phi)(x) = \int_0^1 \{\log(|x-t|) + h(|x-t|)\}\phi(t)\,dt, \tag{5.2}$$

where h is continuous in $[0,1]$. In this section we shall restrict attention to an operator K of the form (5.2), having already noted its importance in practical problems. As remarked in the last section, $K\phi$ is continuous in $[0,1]$ for any $\phi \in L_2(0,1)$. Therefore, $K\phi + f = 0$ certainly has no solution in $L_2(0,1)$ if f is not continuous in $[0,1]$, and, in particular, the first kind counterpart of (4.12), $K\psi + g = 0$, has no solution in $L_2(0,1)$.

This rules out an attempt to construct a formula for the resolvent of a first kind equation along the lines previously employed, at any rate in $L_2(0,1)$.

One remedy is to recast the whole development in a wider setting with the prospect of producing a theory of greater generality than we have given here. There are, however, more immediate ways of salvaging from existing material results of interest in practical problems, where the extra generality required is usually of a quite specific nature and can be accommodated without undue sophistication. Having used $L_2(0,1)$ to provide a straightforward, secure framework in which to generate results, we can extend these by *ad hoc* means.

Equation (5.1) offers a means of illustrating this point of view. It is an example of $K\phi_\alpha + f_\alpha = 0$ if we make the identification $\alpha = \kappa\cos\theta_\alpha$, so that $|\alpha| \le \kappa$ and varying α corresponds to varying the incident angle θ_α with κ fixed. (Note that $\alpha = 0$ corresponds to the incident angle $\theta_0 = \pi/2$.) The question of whether Theorem 1 applies to (5.1) is, therefore, a matter of practical interest. From the underlying wave diffraction problem one can show that (5.1) has one and only one physically acceptable solution which is of the form

$$\phi_\alpha(x) = x^{-1/2}(1-x)^{-1/2}\tilde{\phi}_\alpha(x), \quad 0 < x < 1, \tag{5.4}$$

where $\tilde{\phi}_\alpha$ is continuous in $[0,1]$ and is nonvanishing at the ends of this interval. Obviously, ϕ_α is not in $L_2(0,1)$ and we cannot use the results of Theorem 1 without further investigation.

The structure (5.4) is typical of the solution of $K\phi_\alpha + f_\alpha = 0$, when K is given by (5.2), suggesting that we should explore the validity of the formulae in Theorem 1 for such first kind equations, considering functions of the form

$$\phi(x) = x^{-1/2}(1-x)^{-1/2}\tilde{\phi}(x), \quad 0 < x < 1,$$

where $\tilde{\phi}$ is continuous in $[0,1]$. We let E denote the space of such functions and adopt the understanding that V_α and K, defined by (1.5) and (5.2), respectively, now denote operators on E; it is a straightforward matter to show that both $V_\alpha\phi$ and $K\phi$ are continuous in $[0,1]$ for $\phi \in E$. We continue to use the notation (ϕ,ψ), now merely as a shorthand for $\int_0^1 \phi(x)\overline{\psi(x)}\,dx = 0$.

Let $\phi \in E$ be such that $(\phi, f_\alpha) = 0$ and write $\Psi = V_\alpha \phi$. Then Ψ is continuous in $[0,1]$, $\Psi(0) = 0$ and (since $(\phi, f_\alpha) = 0$) $\Psi(1) = 0$. Also, $\Psi' + i\alpha\Psi = \phi$, so $\Psi' \in E$. An integration by parts using (5.2) shows that $K\Psi' = (K\Psi)'$. Hence,

$$\begin{aligned} V_\alpha K\phi &= V_\alpha K(\Psi' + i\alpha\Psi) \\ &= V_\alpha\{(K\Psi)' + i\alpha(K\Psi)\} \\ &= K\Psi + Cf_\alpha, \end{aligned}$$

where $C(= -(K\Psi)(0))$ is a constant. We have, therefore, established that

$$V_\alpha K\phi - KV_\alpha\phi = Cf_\alpha \tag{5.5}$$

for all $\phi \in E$ such that $(\phi, f_\alpha) = 0$. The identity (5.5) is a special case of (1.10), applying to a different class of functions and a particular type of operator.

Now suppose that the solution of $K\phi_\alpha + f_\alpha = 0$ is in E, and let

$$\Phi = (\phi_\gamma, f_\alpha)\phi_\beta - (\phi_\beta, f_\alpha)\phi_\gamma,$$

where β and γ are distinct real numbers, so that $\Phi \in E$ and $(\Phi, f_\alpha) = 0$. Starting from (5.5) applied to Φ, we can now derive the two formulae of Theorem 1, the required construction being almost identical to that given in the proof of that theorem.

Therefore, both elements (i) and (ii) of Theorem 1 hold for the first kind equation $K\phi_\alpha + f_\alpha = 0$ for K of the form (5.2), if the equation has a solution in E. In particular, the solution of (5.1) for any incident angle θ_α is given in terms of the solutions for any two different angles θ_β and θ_γ, provided $(\phi_\alpha, f_\beta) \neq 0$.

Further, it was shown in Section 4 that the solution of the second kind equation $\mu\phi_\alpha = f_\alpha + K\phi_\alpha$ satisfies $\phi_{-\alpha} = e^{i\alpha}U\phi_\alpha$ when K has the form (4.3). This property holds with $\mu = 0$ and, in particular, applies to (5.1). Thus, the solution of (5.1) also satisfies the special version (4.6) of Theorem 1(i). We, therefore, require the solution of (5.1) for only one incident angle, θ_β, in order to determine its solution for any other incident angle, as long as $\theta_\beta \neq \theta_0 (= \pi/2)$ and $(\theta_\beta, f_{-\beta}) \neq 0$. This generalizes "imbedding formulae" obtained previously for (5.1) by

Williams [**8**] and by Porter and Chu [**6**] and constructed by methods which produced only the $\theta_\beta = 0$ (that is, $\beta = \kappa$) case.

The solution of (5.1) for $\theta_\beta = \theta_0$ does not generate the solution for any other angle on its own. As noted after (4.6), two auxiliary equations have to be solved in this case. This makes sense in terms of the diffraction problem because $\theta_\beta = \theta_0$ corresponds to normally incident waves, and the associated solution ϕ_0 of (5.1) contains no information about the behavior of a transverse wave component which is present for every other incident angle.

In the context of (5.1) the quantity (ϕ_γ, f_δ) is sometimes called the diffraction coefficient; it is a measure of the far-field amplitude and phase of the diffracted wave field at an angle θ_γ for a wave incident at angle θ_δ. From this interpretation we infer that the vanishing of (ϕ_γ, f_δ) is exceptional and that formula (ii) of Theorem 1 can also be usefully employed in relation to (5.1). Putting $\alpha = -\beta$ in that formula and using $\phi_{-\beta} = e^{i\beta} U \phi_\beta$, we find that

(5.6)
$$\begin{aligned} 2\beta(\gamma-\delta)e^{i\gamma}(\phi_\beta, f_{-\beta})(\phi_\gamma, f_\delta) &= (\beta-\gamma)(\beta+\delta)e^{i\gamma}(\phi_\beta, f_\gamma)(\phi_\beta, f_{-\delta}) \\ &\quad - (\beta-\delta)(\beta+\gamma)e^{i\delta}(\phi_\beta, f_\delta)(\phi_\beta, f_{-\gamma}). \end{aligned}$$

Hence, the diffraction coefficient (ϕ_γ, f_δ) may be calculated for any field angle θ_γ and any different incident angle θ_δ using only the solution of (5.1) for one angle θ_β, provided $\theta_\beta \neq \theta_0$ and $(\phi_\beta, f_{-\beta}) \neq 0$. Formula (5.6) generalizes one given by Porter and Chu [**6**]. The saving which it and the formula (4.6) offer when numerical solutions of (5.1) are determined is clearly significant.

We finally note that the adaptation of Theorem 1 to (5.1) can be carried further. Hardly any extra effort is needed to deal with equations whose solutions are known to be continuous in (0,1), integrable on $[0, 1]$ and which have end-point singularities stronger than inverse square roots.

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