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FORCED VIBRATIONS IN ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY

EDUARD FEIREISL

ABSTRACT. We prove the existence of time-periodic weak solutions to the integro-differential equation

$$U_{tt}(x,t) = f(U_x(x,t))_x + \int_0^\infty \dot{a}(s) f(U_x(x,t-s))_x \, ds + g(x,t),$$

where g is a time-periodic function.

The main idea of the proof is to construct invariant regions for a parabolic system arising as a viscosity regularization of the original problem. Consequently, we are able both to find a sequence of approximate (viscosity) time-periodic solutions via the Schauder fixed point technique and to pass to the limit using the ideas of compensated compactness.

1. Introduction and statement of results. We consider the motion of a one-dimensional body (string or bar) with undistorted reference configuration $J = (0,l)$, a connected open subset of $\mathbf{R}^1$. Denoting by $U = U(x,t)$ the displacement at the instant t of the point with reference position $x \in J$, we make the following assumption relating the stress $\mathcal{G}$ to the motion of the form

$$\text{(C)} \qquad \mathcal{G}(x,t) = f(U_x(x,t)) + \int_0^\infty \dot{a}(s) f(U_x(x,t-s)) \, ds,$$

where a dot indicates differentiation with respect to time.

Assumptions concerning the kernel a are motivated by a simple model for a material with fading memory (see Hrusa, Nohel, Renardy [**13**] and also Renardy, Hrusa, Nohel [**21**] for an excellent survey):

(A1) $a : [0,\infty) \to [0,\infty)$ is smooth, $a(0) < 1$, a, da/dt, $d^2a/dt^2 \in L_1(0,\infty)$, a is strongly positive (see Nohel, Shea [**19**]),

(A2) $d^k a/dt^k \in L_1(0,\infty)$, $k = 0, \dots, 4$.

We assume, for simplicity, that the body is homogeneous with unit density. Under these circumstances the motion is governed by the equation

$$\text{(E)} \qquad U_{tt}(x,t) = \mathcal{G}_x(x,t) + g(x,t), \quad x \in J, \ t \in \mathbf{R}^1,$$

g representing the prescribed body force.

For the sake of definiteness, both faces are held fixed, i.e.,

$$U(0,t) = U(l,t) = 0, \quad t \in \mathbf{R}^1. \tag{B}$$

Our interest lies in studying the problem (E), (C), (B) when g is a smooth time-periodic function with a period ω, that is to say

$$g(x,t+\omega) = g(x,t) \quad \text{for all } x \in J, \quad t \in \mathbf{R}^1. \tag{G}$$

Our main result given below states that there is at least one weak solution U that is periodic in t with the period ω, i.e.,

$$U(x,t+\omega) = U(x,t), \quad x \in J, \ t \in \mathbf{R}^1. \tag{P}$$

Definition 1. A function $U = U(x,t)$, $u \in C(\bar{J} \times \mathbf{R}^1)$ is called a weak solution to the problem (E), (B), (P) if U satisfies (B), (P),

U_x, $U_t \in L_\infty(J \times \mathbf{R}^1)$ and the integral identity

$$\int_T \int_J (-U_t \varphi_t + \mathcal{G}\varphi_x - g\varphi)\, dx\, dt = 0 \tag{1.1}$$

holds for any smooth function φ obeying (B), (P) and for any time interval T of the length ω.

To motivate the study of weak solutions, observe that, if $a \equiv$ const, (E) becomes a quasilinear wave equation and smooth solutions generally develop singularities in a finite time. If $\dot{a} \not\equiv 0$, the same feature occurs when the body force g is allowed to be large (see Dafermos [**4**], Nohel, Renardy [**17**]).

By contrast, there is a dissipative mechanism, hidden in (C), which is able to damp out small singularities and provides existence of smooth global solutions whenever the data are, vaguely speaking, small and smooth enough (see Mac Camy [**14**], Staffans [**23**], Hrusa [**11**], Dafermos, Nohel [**6**], etc.).

The technique used in obtaining global smooth solutions is insufficient to cope with a large periodic traction g. Indeed, all the results quoted above employ energy methods and assume global integrability (in t) of g in a certain sense, a condition that fails in the problem in question.

Our main results may be stated as

Theorem 1. *Let the kernel a satisfy* (A_1), (A_2). *Suppose that the function f complies with the following hypotheses*:

(F_1) *$f = f(z)$, $f : \mathbf{R}^1 \to \mathbf{R}^1$ is a smooth function satisfying the growth condition*

$$f'(z) \geq f_0 > 0 \quad \textit{for all } z \in \mathbf{R}^1, \qquad f' \stackrel{\text{def}}{=} \frac{df}{dz},$$

(F_2) $\lim_{|z|\to\infty} f'(z) = \infty$,

(F_3) $f''(z)z > 0$ *for all* $z \neq 0$.

Then, for any smooth function g satisfying (G), *there exists at least one weak solution U to the problem* (E), (B), (P).

Similar to Nohel, Rogers, Tzavaras [**18**], we try to attack the problem via the compensated compactness theory (see, e.g., Murat [**15**], Tartar [**24**]) that has proved to be useful in the case of (2×2)-systems of conservation laws (see DiPerna [**7**], Serre [**22**], Rascle [**20**]), as well as for a single equation with "memory" (Dafermos [**5**]).

Let us remark that, in contrast to the initial-value problem, it is not clear if the time-periodic solution actually contains shocks. Unfortunately, the final answer to the question of regularity seems to be far from being settled.

An essential difference between our approach and that of Nohel, Rogers, Tzavaras [**18**], where the Cauchy problem is treated, is that we need and actually do construct invariant regions for a parabolic system arising as a viscosity regularization of the problem (see Sections 2, 3). More specifically, using the trick of Mac Camy (see Mac Camy [**14**]), we start (as in [**18**]) by inverting the linear Volterra operator in (C) and thus computing the nonlinear term $f(U_x)_x$. Then setting $u = U_x$, $v = U_t + dU$, $d > 0$, and integrating by parts we obtain a parabolic system where the memory effect reduces to lower order terms.

Artificial as this transformation may appear, the resulting problem is tractable. As we will see, the technique of Chueh, Conley, Smoller [**3**] can (and actually will) be applied to get the existence of invariant regions by using essential a priori estimates of the sequence of approximate solutions (cf. also [**8**]).

Once this has been accomplished, we are able, to begin with, to construct a sequence of approximate solutions via the Schauder fixed point theorem (Sections 4, 5), and, since we have L_∞ a priori estimates, to pass to the zero-viscosity limit invoking the compensated compactness theory coupled with fundamental results of DiPerna [**7**], Rascle [**20**] on hyperbolic systems of conservation laws (see Section 6). Note that this last step is almost identical with the corresponding part in [**18**] and, consequently, the ideas are only sketched.

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2. Preliminary results and transformation to a system. The method of transforming (E) to a system rests upon the following result concerning the solution K of the Volterra integral equation

$$\dot{a}(t) + K(t) + \int_0^t \dot{a}(t-s)K(s)\,ds = 0. \tag{I}$$

Lemma 1. *Let the condition* (A_1) *hold. Then there exists a unique smooth solution* K *of* (I), K, $dK/dt \in L_1(0,\infty)$, *and*

$$K(0) = -\dot{a}(0) > 0. \tag{2.1}$$

If, in addition, (A_2) *is true, then*

$$\frac{d^k K}{dt^k} \in L_1(0,\infty), \quad k = 0,\dots,3. \tag{2.2}$$

Proof. (according to Nohel [**16**]). The smoothness of K results from the equation (I). Moreover, the results of Hrusa, Nohel [**12**] imply (2.1).

For the integrability, one observes easily that this is a consequence of Lemma 3.2 in Dafermos, Nohel [**6**]. The integrability of the derivatives follows by differentiating the equation (I). □

In view of the difficulties connected with boundary conditions we prefer to work within the class of double periodic functions, i.e., the functions determined on a torus $T^2 = \{(x,t)|x \in S^1, t \in S^2\}$, with $S^1 = [-l,l]/\{-l,l\}$, $S^2 = [0,\omega]/\{0,\omega\}$.

Given such a function w, we are at liberty to choose and will actually make no distinction between $w = w(x,t)$ as a double periodic function on $\mathbf{R}^2$, $w = w(x,t)$ as a function defined on T^2, and $w : t \mapsto w(\cdot,t)$ as a vector function ranging, say, in $L_2(S^1)$.

Next, we introduce a pair of spaces

$$\begin{aligned} Y_1 &= \{u \mid u \in L_2(S^1),\ u(-x) = u(x),\ x \in S^1,\ \int_0^l u(x)\,dx = 0\}, \\ Y_2 &= \{v \mid v \in L_2(S^1),\ v(-x) = -v(x),\ x \in S^1\}. \end{aligned}$$

Note that there is an alternative definition by means of the Fourier coefficients b_k, $k \in \mathbf{Z}$, related to the orthogonal system

$$e_k(x) = \begin{cases} \sin(k\pi l^{-1}x), & k = 1,2,\dots \\ \cos(k\pi l^{-1}x), & k = 0,-1,-2,\dots, \end{cases}$$

viz.

$$\begin{aligned} Y_1 &= \{u \mid b_k(u) = 0,\ k = 0,1,\dots\}, \\ Y_2 &= \{v \mid b_k(v) = 0,\ k = 0,-1,\dots\}. \end{aligned}$$

Now, the relevant hypothesis for the function U to obey (B), (P) takes the form

$$U \in C(T^2),\ U(\cdot,t) \in Y_2 \quad \text{for all } t \in S^2. \tag{2.3}$$

To comply with smoothness of the data, we modify the function g, setting

$$g(x,t) = \begin{cases} \chi^\varepsilon(x)g(x,t), & x \in [0,l],\ t \in S^2, \\ -g^\varepsilon(-x,t), & x \in [-l,0),\ t \in S^2, \end{cases}$$

where χ^ε, $\varepsilon > 0$, is a smooth function satisfying

$$\chi^\varepsilon(x) = \begin{cases} 1, & \text{for } x \in (\varepsilon, l-\varepsilon) \\ \in [0,1], & \text{for } x \in [0,\varepsilon] \cup [l-\varepsilon, l] \\ 0, & \text{for } x \in \mathbf{R}^1 \backslash [0,l]. \end{cases}$$

Thus, g^ε may be viewed as a smooth function on T^2 such that $g^\varepsilon(\cdot, t) \in Y_2$, $t \in S^2$.

We transform the equation (E) by using the resolvent kernel K and the variation-of-constants formula for Volterra equations, obtaining

$$f(U_x(t))_x = U_{tt}(t) + \int_0^\infty K(s) U_{tt}(t-s)\, ds - h^\varepsilon(t),$$

where

$$h^\varepsilon(t) = \int_0^\infty K(s) g^\varepsilon(t-s)\, ds + g^\varepsilon(t), \tag{2.4}$$

the argument x being omitted. Using (2.2) and integrating by parts enables one to rewrite the above equation in the form

$$\begin{aligned} (U_t + dU)_t(t) + d(U_t + dU)(t) - f(U_x(t))_x + bU(t) \\ + \int_0^\infty \ddot{K}(s) U(t-s)\, ds = h^\varepsilon(t), \end{aligned} \tag{2.5}$$

with

$$2d = K(0) > 0, \qquad b = \dot{K}(0) - d^2. \tag{2.6}$$

Setting $u = U_x$, $v = U_t + dU$, we arrive at the system

$$u_t(t) - v_x(t) + du(t) = 0, \tag{S$_1$}$$

$$v_t(t) - f(u(t))_x + dv(t) + bU(t) + \int_0^\infty \ddot{K}(s) U(t-s)\, ds = h^\varepsilon(t), \tag{S$_2$}$$

where U is determined by the relation

$$U(x,t) = \int_0^x u(z,t)\, dz. \tag{2.7}$$

For a vector function $w = w(t)$, $w \in L_\infty(S^2; Y_1)$, we define a linear operator $\mathcal{K}$,

$$\mathcal{K}(w)(x,t) = \int_0^x \int_0^\infty \ddot{K}(s) w(z, t-s)\, ds\, dz, \tag{2.8}$$

w being identified with a function on $\mathbf{R}^2$, $2l$-periodic in x and ω-periodic in t.

Following the ideas of the vanishing viscosity method we look for approximate solutions u, $v \in C^2(T^2)$ satisfying the regularized system

$$u_t - v_x + du = \varepsilon u_{xx}, \tag{S$_1^\varepsilon$}$$

$$v_t - f(u)_x + dv + bU + \mathcal{K}(u) = \varepsilon v_{xx} + h^\varepsilon \tag{S$_2^\varepsilon$}$$

in conjunction with the additional requirement

$$u(\cdot, t) \in Y_1,\ v(\cdot, t) \in Y_2 \quad \text{for all } t \in S^2. \tag{2.9}$$

At this stage, it is worth remarking on the connection between the function U in (2.7) and the original equation.

Proposition 1. *Let $u, v \in C^2(T^2)$ be a pair of functions satisfying* (S$_1^\varepsilon$), (S$_2^\varepsilon$) *together with* (2.9).

Then the function U assigned to u by (2.7) *satisfies*

$$U \in C^2(T^2), \quad U(\cdot, t) \in Y_2 \quad \textit{for all } t \in S^2, \tag{2.10}$$

$$U_x = u, \qquad U_t + dU = v + \varepsilon u_x, \tag{2.11}$$

and, finally,

$$U_{tt}(x,t) = \mathcal{G}_x(x,t) + g^\varepsilon(x,t) + R^\varepsilon(x,t) \quad \textit{on } T^2, \tag{2.12}$$

where $\mathcal{G}$ is defined by (C) *and*

$$\begin{aligned} R^\varepsilon(x,t) = \varepsilon\{&(v_{xx} + du_x + u_{xt})(x,t) \\ &+ \int_0^\infty \dot{a}(s)(v_{xx} + du_x + u_{xt})(x, t-s)\, ds\}. \end{aligned}$$

Proof. First, (2.10) follows from (2.9). Second, it is clear that $U_x = u$. The second assertion in (2.11) requires us to integrate (S_1^ε):

$$U_t(x,t) - \int_0^x v_x(z,t)\,dz + dU(x,t) = \varepsilon \int_0^x u_{xx}(z,t)\,dz.$$

Seeing that, as a consequence of (2.9), $v(0,t) = u_x(0,t) = 0$, we obtain (2.11).

By virtue of (2.11), (S_2^ε) may be rewritten as

$$\begin{aligned} U_{tt}(t) &+ K(0)U_t(t) - f(U_x(t))_x \\ &+ \int_0^\infty \ddot{K}(s)U(t-s)\,ds + \dot{K}(0)U(t) \\ &= h^\varepsilon(t) + \varepsilon(v_{xx} + u_{xt} + du_x)(t). \end{aligned}$$

Thus, integrating by parts, we obtain

$$\begin{aligned} f(U_x(t))_x = U_{tt}(t) &+ \int_0^\infty K(s)U_{tt}(t-s)\,ds - h^\varepsilon(t) \\ &+ \varepsilon(v_{xx} + u_{xt} + du_x)(t) \quad \text{on } T^2, \end{aligned}$$

and a straightforward application of the variation-of-constants formula results in (2.12). □

3. Invariant regions. From now on we make use of the symbol c to denote strictly positive constants that arise in the computations with no particular regard to distinguishing one from another.

Our first goal is to remove the term $\mathcal{K}(u)$ from (S_2^ε). To this end, consider a modified system ($\tilde{\mathrm{S}}_1^\varepsilon$), ($\tilde{\mathrm{S}}_2^\varepsilon$), where the former equation remains unchanged, while the equation (S_2^ε) is replaced by

$$(\tilde{\mathrm{S}}_2^\varepsilon) \qquad v_t - f(u)_x + dv + bU + \mathcal{K}(w) = \varepsilon v_{xx} + h^\varepsilon,$$

where w is a fixed function belonging to the space $L_\infty(S^2; Y_1)$.

We begin with an auxiliary assertion concerning the regularity of $\mathcal{K}(w)$.

Lemma 2. *For any fixed* $w \in L_\infty(S^2; Y_1 \cap L_\infty(S_1))$, *the function* $\mathcal{K}(w) = \mathcal{K}(w)(x,t)$ *is Lipschitz continuous on the compact set* T^2. *Moreover,*

$$\mathcal{K}(w)(\cdot,t) \in Y_2 \quad \textit{for all } t \in S^2. \tag{3.1}$$

Proof. The relation $w(\cdot,t) \in Y_1$ for a.e. t yields (3.1). Consequently, $\mathcal{K}$ is $2l$-periodic in x and ω-periodic in t, and it suffices to prove the Lipschitz continuity on the rectangle $[-l,l] \times [0,\omega]$. Toward this end, consider $-l \le x_1 < x_2 \le l$, $0 \le t_1 < t_2 \le \omega$ and estimate

$$\begin{aligned} &|\mathcal{K}(w)(x_2,t_2) - \mathcal{K}(w)(x_1,t_1)| \\ &\qquad \le \left| \int_{x_1}^{x_2} \int_{-\infty}^{t_2} \ddot{K}(t_2 - s) w(z,s)\, ds\, dz \right| \\ &\qquad + \left| \int_0^{x_1} \int_{-\infty}^{t_1} \ddot{K}(t_2 - s) - \ddot{K}(t_1 - s) w(z,s)\, ds\, dz \right| \\ &\qquad + \left| \int_0^{x_1} \int_{t_1}^{t_2} \ddot{K}(t_2 - s) w(z,s)\, ds\, dz \right|. \end{aligned}$$

Since $\ddot{K}(t_2 - s) - \ddot{K}(t_1 - s) = \int_{t_1}^{t_2} \dddot{K}(y - s)\, dy$, the application of Lemma 1 completes the proof. □

Consider now the Cauchy problem related to $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$ with

$$u(\cdot,0) = u^0, \qquad v(\cdot,0) = v^0, \tag{$\tilde{\mathrm{I}}$}$$

where

$$u^0 \in C(S^1) \cap Y_1, \qquad v^0 \in C(S^1) \cap Y_2. \tag{3.2}$$

We introduce the concept of invariant region.

Definition 2. (invariant region). A set $M \subset R^2$ is called an invariant region for the system $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$ if, satisfying

$$[u^0(x), v^0(x)] \in M \quad \text{for all } x \in S^1, \tag{3.3}$$

any solution (u,v) of $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$, $(\tilde{\mathrm{I}})$ is bound to remain in M on the whole existence interval $[0,t_0)$, more specifically,

$$[u(x,t),v(x,t)] \in M \quad \text{for all } x \in S^1,\ t \in [0,t_0). \tag{3.4}$$

Remark. The solution in question is supposed to be a classical one, i.e., $u,v \in C(S^1 \times [0,t_0))$ having all the derivatives appearing in $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$ continuous on $S^1 \times (0,t_0)$.

In view of (F_1) it is permissible to introduce the quantities

$$F(z) = \int_0^z \sqrt{f'}(y)\,dy,$$
$$r(u,v) = v + F(u), \qquad s(u,v) = v - F(u),$$

the latter pair being the standard Riemann invariants for the quasilinear wave equation.

Pursuing the line of arguments delineated in Chueh, Conley, Smoller [**3**], we are going to establish a result which is the key to the proof of Theorem 1. Observe that, in the course of the proof, the conditions (F_2), (F_3) will be exploited to their full extent.

Proposition 2. *There is a (sufficiently large) constant $C > 0$, independent of $\varepsilon > 0$, such that the set*

$$M_C = \{[u,v] \mid -C \le s(u,v), r(u,v) \le C\}$$

represents an invariant region for the system $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$ whenever

$$F(w(x,t)) \in [-C,C] \quad \textit{for a.e. } x,t. \tag{3.5}$$

Proof. Consider the pair $r = r(u(x,t),v(x,t))$, $s = s(u(x,t),v(x,t))$, where (u,v) is a solution of $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$ on $[0,t_0)$ satisfying (3.3).

For fixed $\tilde{t} \in (0,t_0)$, let

$$D = \max\left\{ \max_{\substack{x\in S^1\\ t\in[0,\tilde{t}\,]}} |r(x,t)|,\ \max_{\substack{x\in S^1\\ t\in[0,\tilde{t}\,]}} |s(x,t)| \right\}.$$

Since the set $S^1\times[0,\tilde t\,]$ is compact, the value D must be attained at some point (x_1,t_1). Consequently, four different cases are to be considered:

$$r(x_1,t_1)=D,\quad r(x_1,t_1)=-D,\quad s(x_1,t_1)=D,\quad s(x_1,t_1)=-D.$$

Let us examine, for instance, the first possibility, the other cases being treated in a similar fashion.

If $D>C$, then necessarily $t_1>0$, $u^1=u(x_1,t_1)\geq 0$, $v^1=v(x_1,t_1)\geq 0$; hence, it is permissible to differentiate with respect to x,t to obtain

(a) $r_x(x_1,t_1)=0,\ r_{xx}(x_1,t_1)\leq 0,$

(b) $r_t(x_1,t_1)\geq 0.$

We show that (b) is impossible unless C is small. To this end, compute

$$r_x=\sqrt{f'}(u)u_x+v_x,$$
$$r_{xx}=\sqrt{f'}(u)u_{xx}+v_{xx}+\frac{1}{2}(\sqrt{f'}(u))^{-1}f''(u)u_x^2,$$

and, taking advantage of $(\tilde{\mathrm{S}}_1^\varepsilon)$, $(\tilde{\mathrm{S}}_2^\varepsilon)$,

$$\begin{aligned}r_t=\sqrt{f'}(u)(\sqrt{f'}(u)u_x+v_x)+\varepsilon(\sqrt{f'}(u)u_{xx}+v_{xx})\\ -d(\sqrt{f'}(u)u+v)-bU-\mathcal{K}(w)+h^\varepsilon.\end{aligned}$$

Thus, (a) in conjunction with (F_3) implies that

$$r_t(x_1,t_1)\leq -d(\sqrt{f'}(u^1)u^1+v^1)+(-bU+\mathcal{K}(w)+h^\varepsilon)(x_1,t_1).$$

Our aim is to demonstrate

(c) $\quad d(\sqrt{f'}(u^1)u^1+v^1)>|(-bU+\mathcal{K}(w)+h^\varepsilon)(x_1,t_1)|$

in conflict with (b). As to the left-hand side, $u^1\geq 0$, $v^1\geq 0$ together with (F_3) yield the inequality

$$d(\sqrt{f'}(u^1)u^1+v^1)\geq dr(x_1,t_1)=dD.$$

On the other hand, in view of (2.7), we have

$$|U(x,t)|\leq l\max\{F^{-1}(D),-F^{-1}(-D)\},\quad t\leq t_1.$$

Lemma 1 together with (2.4) implies $|h^\varepsilon| \le c$ while, combined with (3.5), the same assertion gives rise to

$$|\mathcal{K}w(x,t)| \le c \operatorname*{ess\,sup}_{\substack{x\in S^1\\ t\in S^2}} |w(x,t)| \le c \max\{F^{-1}(C), -F^{-1}(-C)\}.$$

According to (F_2), the inverse function F^{-1} is sublinear for large D and, consequently, the right-hand side of (c) cannot exceed the value $(d/2)D + c$, where c may be large but independent of $\varepsilon > 0$. ◻

4. Approximate solutions—global existence. To begin with, we introduce a class of functions

$$W = \{w \mid w \in L_\infty(S^2; Y_1 \cap C(S^1)) \cap C([0,\omega]; Y_1 \cap C(S^1)),\\ F(w(x,t)) \in [-C, C] \text{ for a.e. } x \in S^1,\ t \in S^2\}.$$

In other words, W contains functions ω-periodic in t that are allowed to be discontinuous at $k\omega$, $k \in \mathbf{Z}$, and that satisfy the condition (3.5) of Proposition 2. Note that, as a consequence of (F_1), (F_3), the class W may (and actually will) be viewed as a bounded closed convex subset of the Banach space $C([0,\omega], C(S^1))$.

We turn now to the problem ($\tilde{\mathrm{S}}_1^\varepsilon$), ($\tilde{\mathrm{S}}_2^\varepsilon$), ($\tilde{\mathrm{I}}$) and look for global in time solutions. Our approach is standard and appeals frequently to the geometric theory of parabolic equations (see Henry [**10**]) combined with some results of Amann [**1, 2**].

To comply with "mild" setting of the problem, we introduce an operator

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad D(\mathcal{A}) \subset Y = Y_1 \times Y_2,$$

where A generates the one-dimensional diffusion semigroup on the space $L_2(S^1)$, i.e.,

$$Ay = \sum_{k\in\mathbf{Z}} \lambda_k b_k(y) e_k, \qquad \lambda_k = \frac{k^2\pi^2}{l^2} + d$$

is a self-adjoint extension of $Ay = -\varepsilon y_{xx} + dy$ on $L_2(S^1)$.

It is customary to work within the scale of spaces

$$\mathcal{X}_\alpha = D(\mathcal{A}^\alpha), \qquad \mathcal{X}_\alpha = (D(A^\alpha)\cap Y_1)\times(D(A^\alpha)\cap Y_2), \quad \alpha \ge 0$$

where $D(A^\alpha)$ is provided with a Hilbert structure by means of the norm

$$||y||_\alpha = \left(\sum_{k\in\mathbf{Z}} \lambda_k^{2\alpha} b_k^2(y)\right)^{\frac{1}{2}}.$$

We quote, from Amann [**2**, Proposition (4.1)], a fundamental embedding relation, namely,

$$D(A^\alpha) \curvearrowright C^{1+\lambda}(S^1), \quad 0 < \lambda < 2\alpha - \frac{3}{2}. \tag{4.1}$$

As a matter of fact, Amann stated (4.1) in the case of the Dirichlet boundary conditions. On the face of it, one observes easily that the same arguments apply to our situation as well.

Taking advantage of the well-known procedure we rewrite ($\tilde{\mathrm{S}}_1^\varepsilon$), ($\tilde{\mathrm{S}}_2^\varepsilon$), ($\tilde{\mathrm{I}}$) to the integral form

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = \mathcal{T}_t \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} + \int_0^t \mathcal{T}_{t-s} \begin{pmatrix} \mathcal{F}_1(s,u(s),v(s)) \\ \mathcal{F}_2(s,u(s),v(s)) \end{pmatrix} ds, \tag{IE}$$

where $\mathcal{T}_t = \exp(-\mathcal{A}t)$,

$$\mathcal{F}_1(t,u,v) = v_x,$$
$$\mathcal{F}_2(t,u,v) = f'(u)u_x - b\int_0^x u(z)\,dz + h^\varepsilon(\cdot,t) - \mathcal{K}(w)(\cdot,t).$$

Lemma 3. *If $w \in W$, then the nonlinear mapping $\mathcal{F} = \binom{\mathcal{F}_1}{\mathcal{F}_2}$: $\mathbf{R}^1 \times D(\mathcal{A}^\beta) \to Y$ is locally Hölder continuous in t and locally Lipschitz in (u,v) whenever $\beta \in (3/4,1)$. Moreover, $\mathcal{F}$ is sublinear in the following sense*:

$$\sum_{i=1}^{2} ||\mathcal{F}_i(t,u,v)||_0 \le c(1 + ||u||_\beta + ||v||_\beta), \tag{4.2}$$

where c depends exclusively on ε, $||u||_{C(S^1)}$, $||v||_{C(S^1)}$.

Proof. Since $g^\varepsilon(\cdot, t) \in Y_2$ (see Section 2), the relations (2.4), (3.1) imply that the image of $\mathbf{R}^1 \times \mathcal{X}_\beta$ is contained in Y.

Next, we claim that $\mathcal{F}$ is locally Hölder continuous in t whenever h^ε, $\mathcal{K}(w)$ are. The former function is smooth together with g^ε. As for the latter quantity, we refer to Lemma 2.

The proof of the Lipschitz continuity in (u, v) rests upon (4.1) combined with some standard Sobolev embedding relations. Estimating the hardest term we obtain

$$\begin{aligned} &||f'(u^1)u_x^1 - f'(u^2)u_x^2||_0 \\ &\qquad \leq ||f'(u^1)(u_x^1 - u_x^2)||_0 \\ &\qquad\quad + ||u_x^2(f'(u^1) - f'(u^2))||_0 \qquad \text{(in view of (4.1))} \\ &\qquad \leq c(||u^1||_\beta, ||u^2||_\beta)||u^1 - u^2||_{C^1(S^1)} \\ &\qquad \leq c(||u^1||_\beta, ||u^2||_\beta)||u^1 - u^2||_\beta. \end{aligned}$$

Similarly,

$$\begin{aligned} ||U^1 - U^2||_0 &\leq c||U^1 - U^2||_{C(S^1)} \leq c||u^1 - u^2||_{C(S^1)} \\ &\leq c||u^1 - u^2||_\beta. \end{aligned}$$

The proof of (4.2) follows the line of the same arguments. Recall, for instance, that

$$||f'(u)u_x||_0 \leq c(\max_{x \in S^1} |u(x)|)||u||_\beta. \quad \square$$

We proceed to assemble several auxiliary facts to achieve, eventually, the global existence result.

Step 1. (local existence). *Let* $w \in W$, $(u^0, v^0) \in \mathcal{X}_\beta$, $\beta \in (3/4, 1)$. *Then there is a unique solution pair* $(u, v) \in C([0, t_0]; \mathcal{X}_\beta)$ *of the problem* (IE) *on a time interval* $[0, t_0]$, $t_0 > 0$.

With Lemma 3 in mind, the proof is standard and may be found in Henry [**10**, Theorem 3.3.3].

Step 2. (regularity). In [**1**, Theorem (4.1)] Amann proved (in a more general case) the following:

Any local solution (u, v) *appearing in Step* 1 *belongs to the class*

$$(u, v) \in C^{\nu}([0, t_0]; \mathcal{X}_{\alpha}) \quad \textit{for } \alpha \in (3/4, \beta),\ \nu \in (0, \beta - \alpha),$$

where the Hölder constant depends exclusively on the norm of (u, v) *in* $C([0, t_0]; \mathcal{X}_{\beta})$.

Combining the fact together with (4.1) and with the regularity result (concerning $\mathcal{K}(w)$) achieved in Lemma 2, we see that u, v are solutions of the one-dimensional heat equation with the right-hand side $\mathcal{F}_1, \mathcal{F}_2$, respectively, lying in $C^{\gamma}(S^1 \times [0, t_0])$ for certain $\gamma > 0$. Consequently, the solution is, in fact, a classical one and solves $(\tilde{\mathrm{S}}_1^{\varepsilon})$, $(\tilde{\mathrm{S}}_2^{\varepsilon})$, $(\tilde{\mathrm{I}})$.

Step 3. (global solutions). According to Step 2, we are allowed to apply Proposition 2 to ensure L_{∞} a priori estimates of the local solutions. Combining this fact with (4.2) (cf. Henry [**10**, Theorem 3.3.4]) we arrive at the following essential result.

Proposition 3. *Given any fixed* $w \in W$, $(u^0, v^0) \in \mathcal{X}_{\beta}$, $\beta \in (3/4, 1)$, *there is a unique classical solution of the problem* $(\tilde{\mathrm{S}}_1^{\varepsilon})$, $(\tilde{\mathrm{S}}_2^{\varepsilon})$, $(\tilde{\mathrm{I}})$ *defined for all* $t \geq 0$ *and satisfying* (2.9).

Moreover, if the data satisfy (3.3) *for* $M = M_C$, M_C *independent of* $\varepsilon > 0$, *we have*

$$[u(x, t), v(x, t)] \in M_C \quad \textit{for all } x \in S^1, \quad t \geq 0. \tag{4.3}$$

Finally, $u, v \in C^{\nu}([0, \omega]; D(A^{\alpha}))$, $\alpha \in (3/4, \beta)$, $\nu \in (0, \beta - \alpha)$, *where the corresponding Hölder constant depends on the norm* (u, v) *in* $C([0, \omega]; \mathcal{X}_{\beta})$.

5. Approximate periodic solutions. Our goal is to find at least one solution of the time-periodic problem $(\mathrm{S}_1^{\varepsilon})$, $(\mathrm{S}_2^{\varepsilon})$ satisfying the estimate (4.3).

Consider a set

$$\mathcal{M} = \{(u^0, v^0; w) | (u^0, v^0) \in \mathcal{X}_{\beta},\ [u^0(x), v^0(x)] \in M_C \text{ for all } x \in S^1; w \in W\}$$

along with a mapping

$$\mathcal{P}: \qquad \mathcal{P}(u^0, v^0; w) = (u(\omega), v(\omega); u|_{[0,\omega]}),$$

where (u,v) is the unique solution corresponding to the data $(u^0, v^0; w)$, the existence of which is guaranteed by Proposition 3.

Moreover, according to (4.3), $\mathcal{P}$ maps $\mathcal{M}$ into itself.

Observe that *a fixed point* of $\mathcal{P}$ corresponds to a time-periodic solution of $(\mathrm{S}_1^\varepsilon)$, $(\mathrm{S}_2^\varepsilon)$. Besides, the classical regularity results (see, e.g., Friedman [**9**]) will guarantee smoothness of such a solution whenever g^ε is smooth.

Proposition 4. *For any fixed $\varepsilon > 0$, there is at least one pair $(u^\varepsilon, v^\varepsilon) \in [C^2(T^2)]^2$, a classical time-periodic solution of the system $(\mathrm{S}_1^\varepsilon)$, $(\mathrm{S}_2^\varepsilon)$, $(u^\varepsilon, v^\varepsilon)$ satisfying* (2.9).

Moreover, we have the estimate

$$||u^\varepsilon||_{C(T^2)} + ||v^\varepsilon||_{C(T^2)} \le c, \tag{5.1}$$

where c does not depend on $\varepsilon > 0$.

Proof. As claimed above, it suffices to find a fixed point of the mapping $\mathcal{P}: \mathcal{M} \to \mathcal{M}$, (5.1) being an easy consequence of (4.3).

(a) ($\mathcal{P} : \mathcal{M} \to \mathcal{M}$ is a continuous mapping with respect to the $\mathcal{X}_\beta \times C([0,\omega], C(S^1))$ topology induced on $\mathcal{M}$.) To see this, we need but prove $\mathcal{K}(w^n)(\cdot,t) \to \mathcal{K}(w)(\cdot,t)$ in Y_2 locally uniformly in t whenever $w^n \to w$ in $C([0,\omega], C(S^1))$, the remaining part of the proof being a consequence of the standard results on continuous dependence of solutions to parabolic problems (see Henry [**10**, Theorem 3.4.1]). But the first assertion is easy to verify due to (2.2).

(b) (The image $\mathcal{P}(\mathcal{M})$ is bounded.) It suffices to prove $(u(\omega), v(\omega))$ is bounded in $\mathcal{X}_\beta$ whenever $(u^0(x), v^0(x)) \in M_C$, $x \in S^1$. But here we can say more. Namely, boundedness of (u^0, v^0) in $\mathcal{X}_0$ is enough for the quantity $(u(\omega), v(\omega))$ to be bounded in $\mathcal{X}_\gamma$ for any $\gamma < 1$. The proof takes advantage of the generalized Gronwall lemma, and we refer to Henry ([**10**, Theorem 3.3.6]) or Amann ([**1**, Theorem (5.3)]) for details.

(c) (The image of any bounded subset $\mathcal{B} \subset \mathcal{M}$ is relatively compact in $\mathcal{M}$.) As we have already seen, mere boundedness in $\mathcal{X}_0$ of (u^0, v^0)

gives rise to boundedness of $(u(\omega), v(\omega))$ in $\mathcal{X}_\gamma$, $\gamma \in (\beta, 4)$ and, thus, to compactness in $\mathcal{X}_\beta$.

Let us turn to the function $u|_{[0,\omega]}$. According to Proposition 3, $u \in C^\nu([0,\omega]; D(A^\alpha))$, $\alpha \in (3/4, \beta)$, $\nu \in (0, \beta - \alpha)$, where the Hölder constant depends on the norm of (u, v) in $C([0,\omega]; \mathcal{X}_\beta)$. Consequently, since u^0, v^0 are supposed to be bounded in $\mathcal{X}_\beta$, the global estimates resulting from (4.2), (4.3) imply boundedness of $(u(t), v(t))$ in $\mathcal{X}_\beta$ on the whole compact interval $[0, \omega]$.

According to the generalized version of the Arzéla-Ascoli theorem, we can see that u belongs to a compact set in $C([0,\omega]; D(A^{\alpha_1}))$, $\alpha_1 \in (3/4, \alpha)$. Thus, the result (4.1) of Amann [**1**] completes the proof of compactness in $C([0,\omega]; C(S^1))$.

Since $\mathcal{M}$ is a closed convex subset of the space $\mathcal{X}_\beta \times C([0,\omega]; C(S^1))$, the Schauder fixed point theorem completes the proof. $\square$

6. A limit passage. Our eventual task will be to pass to the limit in the sequence of approximate solutions $\{u^\varepsilon, v^\varepsilon\}_{\varepsilon>0}$, the existence of which is guaranteed by Proposition 4. Note that the only estimate we have at hand, i.e., the estimate (5.1), is clearly insufficient to cope with the nonlinear terms.

Further arguments rest upon the concept of entropy-flux (e-f) pairs $\eta = \eta(u, v)$, $q = q(u, v)$, a C^2-solution of the linear system

$$\begin{aligned} \eta_u + q_v &= 0 \\ f'(u)\eta_v + q_u &= 0 \quad \text{on } \mathbf{R}^2. \end{aligned} \tag{6.1}$$

As $u^\varepsilon, v^\varepsilon$ solves (S_1^ε), (S_2^ε), we can repeat a (slightly modified) procedure of Nohel, Rogers, Tzavaras [**18**] to obtain the estimate

$$\sqrt{\varepsilon}(||u_x^\varepsilon||_{L_2(T^2)} + ||v_x^\varepsilon||_{L_2(T^2)}) \leq c, \tag{6.2}$$

c independent of ε.

Applying the Murat lemma (see Tartar [**24**, Lemma 28]), we get

$$\eta(u^\varepsilon, v^\varepsilon)_t + q(u^\varepsilon, v^\varepsilon)_x \in \text{ a compact subset of } H^{-1}(Q), \tag{6.3}$$

for any fixed (e-f) pair η, q (see Nohel, Rogers, Tzavaras [**18**] for details).

It is a famous result of the compensated compactness theory that (5.1), (6.2) together with (6.3) yield, in fact, the strong convergence of the sequence $\{u^\varepsilon, v^\varepsilon\}_{\varepsilon>0}$ (see DiPerna [**7**], Rascle [**20**]). Just for completeness, let us pause to delineate that procedure.

According to (5.1) we may suppose that $u^\varepsilon \to u$, $v^\varepsilon \to v$ with respect to the weak-star topology on $L_\infty(Q)$ (we pass to a subsequence as the case may be).

It is convenient to characterize the weak limits via the Young measures. We refer to Tartar [**24**] for detailed description of this procedure, the result of which is:

There is a subsequence (not relabelled again) such that the limit

$$\lim_{\varepsilon\to 0+} G(u^\varepsilon, v^\varepsilon) = \overline{G} \quad \text{in } L_\infty(Q) \text{ weak-star}$$

exists for all continuous functions G with a compact support in $\mathbf{R}^2$ containing the range of the functions $(u^\varepsilon, v^\varepsilon)$ which is uniformly bounded in view of (5.1).

Moreover, there is a family of probability measures (the Young measures) $\nu_{x,t}$, $(x,t) \in Q$ such that

$$\langle \nu_{x,t}, G\rangle = \overline{G}(x,t) \quad \text{for a.e. } (x,t) \in Q.$$

It is a matter of routine to verify that the convergence is strong if (and only if) $\nu_{x,t}$ reduces to a Dirac mass for a.e. $(x,t) \in Q$.

A classical result of the compensated compactness theory, the div-curl lemma (see Murat [**15**, Theorem 7.1]), applied to (6.3) leads to the Tartar equation

$$\text{(T)} \qquad \langle \nu_{x,t}, \eta^1 q^2 - \eta^2 q^1\rangle = \langle \nu_{x,t}, \eta^1\rangle\langle \nu_{x,t}, q^2\rangle - \langle \nu_{x,t}, \eta^2\rangle\langle \nu_{x,t}, q^1\rangle$$

for any (e-f) pairs (η^i, q^i), $i = 1, 2$ and a.e. $(x,t) \in Q$.

Analyzing (T), DiPerna [**7**, Section 5] succeeded in proving that $\nu_{x,t}$ reduces to a Dirac mass whenever (F_3) holds. Recently, Rascle [**20**] proved the same assertion under quite general hypotheses.

Summing up the results we conclude that

$$\text{(6.4)} \qquad u^\varepsilon \to u,\ v^\varepsilon \to v \quad \text{a.e. on } Q \text{ (and thus on } T^2).$$

Returning to (2.11), we have

$$U^\varepsilon \to U \quad \text{strongly in } H^1(T^2), \tag{6.5}$$

since $U_t^\varepsilon, U^\varepsilon$ are orthogonal in $L_2(T^2)$ and $\varepsilon u_x^\varepsilon \to 0$ strongly in $L_2(T^2)$ in view of (6.2).

Thus

$$U_x = u, \qquad U_t + dU = v \quad \text{on } T^2, \tag{6.6}$$

and, as a consequence of (2.10),

$$U(0,t) = U(l,t) = 0 \tag{6.7}$$

in the sense of traces.

We can multiply (2.12) by a test function $\varphi \in C^\infty(T^2)$, $\varphi(\cdot,t) \in Y_2$ and pass to the limit for $\varepsilon \searrow 0$. Taking advantage of the symmetry properties (2.9) we obtain (1.1). Indeed, the term $R^\varepsilon\varphi$, being integrated by parts, tends to zero due to (5.1).

Finally, we have $U_x = u \in L_\infty(T^2)$, $U_t = v - dU \in L_p(T^2)$, $p > 2$ which implies $U \in C(T^2)$, U satisfying (B) (and, of course, (P)) pointwise.

Theorem 1 has been proved. $\square$

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Institute of Mathematics of the Czechoslovak Academy of Sciences, Žitná 25, 115 67 Praha 1, Czechoslovakia