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GLOBAL EXISTENCE OF SMOOTH SHEARING MOTIONS OF A NONLINEAR VISCOELASTIC FLUID

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Dedicated to John Nohel on the occasion of his sixty-fifth birthday.

1. Introduction. The aim of this note is to establish global existence of smooth rectilinear shearing motions of incompressible nonlinear viscoelastic fluids of $\mathrm{K-BKZ}$ type. These fluids, which are described by constitutive relations of integral type, were introduced independently by Kaye [**7**] and Bernstein, Kearsley and Zapas [**1**]. Global existence theorems have been obtained previously by Kim [**8**] and by Renardy, Hrusa and Nohel [**10**, Section IV.5]. Kim discusses a situation in which the fluid occupies all of $\mathbf{R}^3$ and the nonlinearity in the constitutive equation has a special form. Renardy, Hrusa and Nohel study spatially periodic three-dimensional motions with a general nonlinearity in the constitutive equation. In [**8**] and [**10**] the initial data are assumed to be smooth and small and the kernel of the constitutive relation is assumed to be smooth on $[0,\infty)$.

The equation of motion that we shall consider is

(1.1)

$$u_{tt}(x,t) = \int_0^\infty a'(s) g(u_x(x,t) - u_x(x,t-s))_x \, ds + f(x,t), \quad x \in B, \ t \geq 0,$$

where subscripts x and t indicate partial derivatives and a' denotes the derivative of a. Here the unknown u is a component of the displacement, f is a forcing function, $a : [0,\infty) \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are smooth constitutive functions, and $B \subset \mathbf{R}$ is an interval. It follows from symmetry considerations that g is an odd function, i.e., $g(-\xi) = -g(\xi)$ for all $\xi \in \mathbf{R}$. We refer to Coleman and Noll [**4**] and to Sections 2 and 3 of Coleman and Gurtin [**3**] for a general discussion of shearing motions of incompressible viscoelastic fluids. The monograph [**10**] contains relevant information on $\mathrm{K-BKZ}$ fluids as well as derivation of (1.1) for rectilinear shearing motions.

If the displacement u is sufficiently regular then we may rewrite (1.1) in the form

$$v_t(x,t) = \int_0^\infty a'(s)g(\overline{v}_x^t(x,s))_x\,ds + f(x,t), \quad x \in B,\ t \geq 0, \tag{1.2}$$

where $v := u_t$ is the velocity and $\overline{v}^t$ is the summed history up to time t of v, i.e.,

$$\overline{v}^t(x,s) := \int_{t-s}^t v(x,\lambda)\,d\lambda. \tag{1.3}$$

For our purposes, it will be convenient to work with (1.2) in place of (1.1). We assume that the fluid has been at rest prior to time $t = 0$ and that an initial velocity is prescribed at $t = 0^+$. We treat in detail the case when $B = [0,1]$ and nonslip boundary conditions are imposed:

$(1.4)_1$
$$v_t(x,t) = \int_0^\infty a'(s)g(\overline{v}_x^t(x,s))_x\,ds + f(x,t), \quad x \in [0,1], t \geq 0,$$

$(1.4)_2$ $$v(0,t) = v(1,t) = 0, \quad t \geq 0,$$

$(1.4)_3$ $$v(x,\tau) = 0 \quad x \in [0,1], \tau < 0.$$

$(1.4)_4$ $$v(x,0) = v_0(x), \quad x \in [0,1].$$

We assume that the constitutive functions a and g satisfy

$$a, a', a'' \in L^1(0,\infty), \quad a \textit{ is strongly positive,} \tag{1.5}$$

(1.6)
$$g \in C^3(\mathbf{R}), \quad g \textit{ is odd, } \exists\,\gamma > 0 \textit{ such that } g'(\xi) \leq -\gamma \quad \forall\,\xi \in \mathbf{R}.$$

The derivatives appearing in (1.5) and throughout the remainder of the paper should be interpreted in the sense of distributions. The definition of a strongly positive kernel is given in the next section. For now, we note that (1.5) implies

$$a \in C^1[0,\infty), \quad a(0) > 0, \quad a'(0) < 0; \tag{1.7}$$

moreover, if a satisfies

$$a \in C^2[0,\infty), \quad a \geq 0, \quad a' \leq 0, \quad a'' \geq 0, \quad a'(0) < 0, \tag{1.8}$$

then a is strongly positive. Regarding the smoothness of v_0 and f, we require

$$v_0 \in H^2(0,1), \quad \text{i.e.,} \quad v_0, v_0', v_0'' \in L^2(0,1), \tag{1.9}$$

$$\begin{gathered} f, f_x, f_t \in C_b([0,\infty); L^2(0,1)) \cap L^2([0,\infty); L^2(0,1)), \\ f_{tt} \in L^2([0,\infty); L^2(0,1)), \end{gathered} \tag{1.10}$$

where $C_b([0,\infty); L^2(0,1))$ denotes the set of all $w : [0,1] \times [0,\infty) \to \mathbf{R}$ such that the mapping $t \mapsto w(\cdot,t)$ is bounded and continuous from $[0,\infty)$ to $L^2(0,1)$. We also assume that v_0 and f are compatible with the boundary conditions in the sense that

$$v_0(0) = v_0(1) = f(0,0) = f(1,0) = 0. \tag{1.11}$$

Assumptions (1.5) and (1.6) ensure that equation $(1.4)_1$ is of hyperbolic type and that the memory has a dissipative effect. However, we cannot expect (1.4) to have a globally defined smooth solution unless some restrictions are placed on the "sizes" of v_0 and f. (Compare with Coleman and Gurtin [**3**] who show that, in shearing motions of a general nonlinear viscoelastic fluid, acceleration waves of small amplitude decay, but waves of large amplitude can explode in finite time.)

To "measure" v_0 and f we define

$$V_0(v_0) := \int_0^1 \{v_0(x)^2 + v_0'(x)^2 + v_0''(x)^2\}\,dx \tag{1.12}$$

$$\begin{aligned} F(f) := {}& \sup_{t\geq 0} \int_0^1 \{f^2 + f_x^2 + f_t^2\}(x,t)\,dx \\ & + \int_0^\infty \int_0^1 \{f^2 + f_x^2 + f_t^2 + f_{tt}^2\}(x,t)\,dx\,dt. \end{aligned} \tag{1.13}$$

There are some superfluous terms in (1.12) and (1.13) that can be eliminated because of Poincaré's inequality and the Sobolev embedding theorem. However, for our proof of global existence, it is convenient to define V_0 and F as above.

We shall not obtain a time-independent bound for $\overline{v}_x^t$. Consequently, for technical reasons, we need to make an assumption concerning the rate of growth of g at infinity relative to the rate of decay of a. Precisely, we assume that there are constants $K > 0$ and $k > 1$ such that

$$|g^{(j)}(\xi) - g^{(j)}(0)| \le K(|\xi| + |\xi|^k), \quad j = 1, 2, 3, \ \forall \xi \in \mathbf{R}, \tag{1.14}$$

and

$$\int_0^\infty |a'(z)| z^{\frac{k}{2}+1}\, dz, \quad \int_0^\infty |a''(z)| z^{\frac{k}{2}}\, dz < \infty. \tag{1.15}$$

THEOREM 1.1. *Assume that* (1.5), (1.6), (1.14) *and* (1.15) *hold. Then there is a number* $\delta > 0$ *such that, for every* v_0 *and* f *satisfying* (1.9), (1.10), (1.11) *and*

$$V_0(v_0) + F(f) \le \delta, \tag{1.16}$$

the problem (1.4) *has a unique solution* v *with*

(1.17)

$$v, v_x, v_t, v_{xx}, v_{xt}, v_{tt} \in C_b([0,\infty); L^2(0,1)) \cap L^2([0,\infty); L^2(0,1)).$$

REMARK 1.1. *It follows from* (1.17) *and standard embedding theorems that* $v \in C^1([0,1] \times [0,\infty))$ *and* $v, v_x, v_t \to 0$ *uniformly on* $[0,1]$ *as* $t \to \infty$.

A similar theorem holds when $B = \mathbf{R}$, i.e., for the problem

$(1.18)_1$

$$v_t(x,t) = \int_0^\infty a'(s) g(\overline{v}_x^t(x,s))_x\, ds + f(x,t), \quad x \in \mathbf{R}, \ t \ge 0,$$

$$v(x,\tau) = 0, \quad x \in \mathbf{R}, \tau < 0, \tag*{$(1.18)_2$}$$

$$v(x,0) = v_0(x), \quad x \in \mathbf{R}. \tag*{$(1.18)_3$}$$

In place of (1.9), (1.10) and (1.11), we assume that

$$v_0 \in H^2(\mathbf{R}), \tag{1.19}$$

$$
\begin{aligned}
&f, f_x, f_t \in C_b([0,\infty); L^2(\mathbf{R})),\\
&f_x, f_t, f_{tt} \in L^2([0,\infty); L^2(\mathbf{R})),\\
&f \in L^1([0,\infty); L^2(\mathbf{R})),
\end{aligned}
\tag{1.20}
$$

and we use

$$
V_0^*(v_0) := \int_{-\infty}^{\infty} \{v_0(x)^2 + v_0'(x)^2 + v_0''(x)^2\}\, dx, \tag{1.21}
$$

$$
\begin{aligned}
F^*(f) : = &\sup_{t\geq 0} \int_{-\infty}^{\infty} \{f^2 + f_x^2 + f_t^2\}(x,t)\, dx\\
&+ \int_0^{\infty} \int_{-\infty}^{\infty} \{f_x^2 + f_t^2 + f_{tt}^2\}(x,t)\, dx\, dt\\
&+ \left(\int_0^{\infty} \left(\int_{-\infty}^{\infty} f(x,t)^2\, dx \right)^{\frac{1}{2}} dt \right)^2
\end{aligned}
\tag{1.22}
$$

to measure the data.

THEOREM 1.2. *Assume that* (1.5), (1.6), (1.14) *and* (1.15) *hold. Then there is a number* $\delta^* > 0$ *such that, for every* v_0 *and* f *satisfying* (1.19), (1.20) *and*

$$
V_0^*(v_0) + F^*(f) \leq \delta^*, \tag{1.23}
$$

the problem (1.18) *has a unique solution* v *with*

$$
\begin{aligned}
&v, v_x, v_t, v_{xx}, v_{xt}, v_{tt} \in C_b([0,\infty); L^2(\mathbf{R})),\\
&v_x, v_t, v_{xx}, v_{xt}, v_{tt} \in L^2([0,\infty); L^2(\mathbf{R})).
\end{aligned}
\tag{1.24}
$$

REMARK 1.2. It follows from (1.24) and standard embedding theorems that $v \in C^1(\mathbf{R} \times [0,\infty))$ and that $v, v_x, v_t \to 0$ uniformly, $v_x, v_t \to 0$ in $L^2(\mathbf{R})$ as $t \to \infty$.

Similar existence theorems for viscoelastic solids have been established by various authors (cf., e.g., [**5, 10**] and the references cited therein). (For solids, one generally can also obtain precise information on the behavior of the displacement as $t \to \infty$.) Moreover, for nonlinear constitutive equations, several authors have shown that smooth solutions can develop singularities in finite time if the data are too large; such results have been obtained both for fluids and solids (cf., e.g., [**10, 11**]). In the theorems concerning formation of singularities it is assumed that the kernel is smooth on $[0,\infty)$. We refer to the paper of Engler [**6**] for some interesting results concerning global existence of weak solutions for equations with singular kernels and data of unrestricted size.

The next section contains some preliminary material concerning strong positivity of the kernel. In Section 3 we prove Theorem 1.1 and point out the modifications needed to prove Theorem 1.2.

2. Preliminaries. This section contains some preliminary material concerning the kernel a. Since the notion of strong positivity plays an important role in our analysis, we briefly recall a few basic concepts. Let $b \in L^1_{\text{loc}}[0,\infty)$ be given. We say that b is a kernel of positive type if

$$\int_0^t y(s) \int_0^s b(s-\tau) y(\tau)\, d\tau\, ds \geq 0, \quad \forall\, t \geq 0, \tag{2.1}$$

for every $y \in C[0,\infty)$; we say that b is of strongly positive type if there exists $\varepsilon > 0$ such that the kernel $t \mapsto b(t) - \varepsilon e^{-t}$ is of positive type. As the terminology suggests, strong positivity of b implies positivity of b.

The above definitions generally are not easy to check directly. We note that if

$$b \in C^2[0,\infty), \quad b \geq 0, \quad b' \leq 0, \quad b'' \geq 0, \quad b' \not\equiv 0, \tag{2.2}$$

then b is strongly positive (cf., e.g., Corollary 2.2 of [**9**]). Strong positivity does not imply any global sign conditions (e.g., e^{-t} cost is strongly positive). However, if a strongly positive function is sufficiently regular, then statements can be made regarding its pointwise behavior near zero. In particular, (1.5) implies

$$a(0) > 0, \quad a'(0) < 0. \tag{2.3}$$

For each $T>0$ and $w \in C([0,T]; L^2(0,1))$, let us put

(2.4)
$$Q(w,t,a) = \int_0^t \int_0^1 w(x,s) \int_0^s a(s-\tau) w(x,\tau)\, d\tau\, dx\, ds \quad \forall\, t \in [0,T].$$

Strong positivity of a implies some very useful estimates of coercive type for $Q(w,t,a)$. Our first lemma, which was established by Brandon (cf. Lemma 2.4 of [**2**]), generalizes an inequality that was used by Dafermos and Nohel [**5**] to obtain global estimates for a one-dimensional nonlinear viscoelastic solid. The basic idea is that a time-independent bound for $Q(w,t,a) + Q(w_t,t,a)$ yields a time-independent bound for

$$\int_0^1 w(x,t)^2\, dx + \int_0^t \int_0^1 w(x,s)^2\, dx\, ds.$$

For technical reasons, we use difference operators

$$(\Delta_h w)(x,t) := w(x,t+h) - w(x,t) \tag{2.5}$$

in place of derivatives.

LEMMA 2.1. *Assume that* (1.5) *holds. Then there is a constant* $L>0$ *such that, for every* $T>0$ *and every* $w \in C([0,T]; L^2(0,1))$*, we have*

(2.6)
$$\int_0^1 w(x,t)^2\, dx + \int_0^t \int_0^1 w(x,s)^2\, dx\, ds \\ \leq L\left\{ \int_0^1 w(x,0)^2\, dx + Q(w,t,a) + \liminf_{h\downarrow 0} \frac{1}{h^2}\, Q(\Delta_h w, t, a)\right\} \\ \forall\, t \in [0,T].$$

Our next lemma is due to Staffans (cf. Lemma 4.2 of [**12**]).

LEMMA 2.2. *If a satisfies* (1.5)*, then, for every* $T>0$ *and every* $w \in C([0,T]; L^2(0,1))$*, we have*

$$\int_0^1 \left(\int_0^t a(t-\tau) w(x,\tau)\, d\tau \right)^2 dx \leq 2a(0) Q(w,t,a) \quad \forall\, t \in [0,T]. \tag{2.7}$$

Lemmas 2.1 and 2.2 will be used to obtain a priori bounds for the (local) solution v of (1.4) directly from energy integrals. We shall need an additional bound that can be obtained by expressing v_{xx} in terms of v_{tt} through an inverse Volterra operator.

If (1.5) holds, then, for each $h \in L^1_{\mathrm{loc}}[0,\infty)$, the Volterra equation

$$a(0)y(t) + \int_0^t a'(t-\tau)y(\tau)\,d\tau = h(t), \quad t \geq 0, \tag{2.8}$$

has a unique solution $y \in L^1_{\mathrm{loc}}[0,\infty)$; the solution can be expressed in the form

$$y(t) = \frac{1}{a(0)}\left(h(t) + \int_0^t r(t-\tau)\,h(\tau)\,d\tau\right), \tag{2.9}$$

where r is the resolvent kernel associated with (2.8), i.e., the unique solution of

$$a(0)r(t) + \int_0^t a'(t-\tau)r(\tau)\,d\tau = -\,a'(t). \tag{2.10}$$

It follows from the Paley-Wiener theorem that $r \notin L^1(0,\infty)$. However, we do have the following result.

LEMMA 2.3. *Assume that* (1.5) *holds and let r be the resolvent kernel associated with* (2.8)*, i.e., the solution of* (2.10)*. Then r is locally absolutely continuous on* $[0,\infty)$ *and* $r' \in L^1(0,\infty)$.

See, for example, Lemma 2.3 of [**2**] for a proof. (In [**2**] it is also assumed that $a''' \in L^1(0,\infty)$. However, this assumption is not needed for the proof of Lemma 2.3.)

REMARK 2.1. For a viscoelastic solid the analog of equation (2.8) has a resolvent kernel that belongs to $L^1(0,\infty)$. Nonintegrability of the resolvent kernel is one of the key reasons why it is generally more difficult to obtain estimates for a viscoelastic fluid than for a viscoelastic solid.

REMARK 2.2. Lemmas 2.1 and 2.2 remain valid if the spatial interval $[0,1]$ is replaced by $(-\infty,\infty)$ throughout.

3. Proof of Theorem 1.1. Our proof is based on essentially the same line of argument as the one employed in Section 3 of [**2**]. Therefore, some of the steps that are virtually identical will not be repeated here.

The local existence of a smooth solution can be established by a routine contraction-mapping argument. The relevant result is recorded below without proof. We refer the reader to Chapter III of [**10**] and Lemma 2.1 of [**2**] for proofs of similar results.

PROPOSITION. *Assume that* (1.5), (1.6), (1.9)–(1.11), (1.14) *and* (1.15) *hold. Then the initial-value problem* (1.4) *has a unique solution* v*, defined on a maximal time interval* $[0,T_0), T_0>0$*, satisfying*

$$v, v_x, v_t, v_{xx}, v_{xt}, v_{tt} \in C([0,T_0); L^2(0,1)). \tag{3.1}$$

Moreover, if

$$\sup_{t\in[0,T_0)} \int_0^1 (v^2+v_x^2+v_t^2+v_{xx}^2+v_{xt}^2+v_{tt}^2)(x,t)\,dx < \infty, \tag{3.2}$$

then $T_0=\infty$.

In order to prove that (1.4) has a solution which is defined globally in time, it suffices to show that if (1.16) holds for $\delta>0$ sufficiently small, then the local solution satisfies (3.2). For this purpose it is convenient to introduce the quantities

$$\begin{aligned} \mathcal{E}\,(t) := \sup_{s\in[0,t]} &\int_0^1 (v^2+v_x^2+v_t^2+v_{xx}^2+v_{xt}^2+v_{tt}^2)(x,s)\,dx \\ &+ \int_0^t\int_0^1 (v^2+v_x^2+v_t^2+v_{xx}^2+v_{xt}^2+v_{tt}^2)(x,s)\,dx\,ds, \\ &\qquad t\in[0,T_0) \end{aligned} \tag{3.3}$$

and
(3.4)

$$\nu(t) := \sup_{\substack{x\in[0,1]\\ s\in[0,t]}} (v^2 + v_x^2 + v_t^2)^{\frac{1}{2}}(x,t) + \left(\int_0^t (\sup_{x\in[0,1]} |v_x(x,s)|)^2 \, ds \right)^{\frac{1}{2}}, \quad t \in [0, T_0).$$

It is also convenient to rewrite $(1.4)_1$ as

$$\begin{aligned} &v_t(x,t) + g'(0) \int_0^t a(t-s) v_{xx}(x,s)\, ds \\ &\quad = f(x,t) + \int_0^t v_{xx}(x,s) \int_{t-s}^\infty a'(z)[g'(\overline{v}_x^t(x,z)) - g'(0)]\, dz\, ds, \\ &\qquad x \in [0,1],\ t \in [0, T_0). \end{aligned} \tag{3.5}$$

Our aim is to establish inequality (3.28) below; to do so we use energy methods. Most of the estimates are derived from energy integrals; some additional ones are obtained from equation (3.5) by inverting the Volterra operator. In the energy integrals, the left-hand side of (3.5) yields positive definite terms, and the right-hand side leads to expressions which are under control near equilibrium.

In the calculations below we frequently use the inequalities

$$\left(\sum_{i=1}^n A_i \right)^2 \le n \sum_{i=1}^n A_i^2, \quad A_1, \dots, A_n \in \mathbf{R}, \tag{3.6}$$

$$|AB| \le \frac{A^2}{4\lambda} + \lambda B^2, \quad A, B \in \mathbf{R}, \quad \lambda > 0, \tag{3.7}$$

and

$$||A * B||_{L^p((0,T);L^2(0,1))} \le ||A||_{L^1(0,\infty)} ||B||_{L^p((0,T);L^2(0,1))} \tag{3.8}$$

for every $T > 0$, $A \in L^1(0,\infty)$ and $B \in L^p((0,T); L^2(0,1))$, where $p \in [1, \infty]$ and $*$ denotes convolution in the time variable. We use Γ to denote a (possibly large) positive generic constant, independent of v_0, f and T_0.

The first energy integral is obtained by multiplying (3.5) by v and integrating over $[0,1]\times[0,t], t\in[0,T_0)$. After integration by parts, we obtain the identity

$$\begin{aligned}
&\frac{1}{2}\int_0^1 v^2(x,t)\,dx - g'(0)Q(v_x,t,a)\\
&\quad= \frac{1}{2}\int_0^1 v_0^2(x)\,dx + \int_0^t\int_0^1 v(x,s)f(x,s)\,dx\,ds\\
&\qquad+ \int_0^t\int_0^1 v(x,s)\int_0^s v_{xx}(x,y)\int_{s-y}^\infty a'(z)[g'(\overline{v}_x^s(x,z))\\
&\qquad- g'(0)]\,dz\,dy\,dx\,ds, \quad t\in[0,T_0).
\end{aligned} \tag{3.9}$$

We now differentiate (3.5) with respect to t to obtain

$$\begin{aligned}
&v_{tt}(x,t) + g'(0)a(0)v_{xx}(x,t) + g'(0)\int_0^t a'(t-s)v_{xx}(x,s)\,ds\\
&\quad= f_t(x,t) + \frac{\partial}{\partial t}\bigg\{\int_0^t v_{xx}(x,s)\int_{t-s}^\infty a'(z)[g'(\overline{v}_x^t(x,z)) - g'(0)]\,dz\,ds\bigg\},\\
&\qquad\qquad x\in[0,1], t\in[0,T_0).
\end{aligned} \tag{3.10}$$

Multiplying (3.10) by v_t and then integrating over $[0,1]\times[0,t], t\in[0,T_0)$, obtains the relation

$$\begin{aligned}
&\frac{1}{2}\int_0^1 v_t^2(x,t)\,dx - g'(0)Q(v_{xt},t,a)\\
&\quad= \frac{1}{2}\int_0^1 v_t^2(x,0)\,dx - g'(0)\int_0^t\int_0^1 v_t(x,s)a(s)v_0''(x)\,dx\,ds\\
&\qquad+ \int_0^t\int_0^1 v_t(x,s)f_t(x,s)\,dx\,ds\\
&\qquad+ \int_0^t\int_0^1 v_t(x,s)\,\frac{\partial}{\partial s}\bigg\{\int_0^s v_{xx}(x,y)\int_{s-y}^\infty a'(z)[g'(\overline{v}_x^s(x,z))\\
&\qquad\qquad - g'(0)]\,dz\,dy\bigg\}\,dx\,ds, \quad t\in[0,T_0).
\end{aligned} \tag{3.11}$$

Observe that

$$v_t(x,0) = f(x,0), \quad x\in[0,1], \tag{3.12}$$

by virtue of (3.5).

We next apply the forward difference operator Δ_h to (3.10). We multiply the resulting equation by $\Delta_h v_t$ and integrate over $[0,1] \times [0,t], t \in [0,T_0)$. We integrate several terms by parts, divide both sides by h^2, and let $h \downarrow 0$ to get

(3.13)

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 v_{tt}^2(x,t)\,dx - g'(0)\lim_{h\downarrow 0}\frac{1}{h^2}\,Q(\Delta_h v_{xt},t,a)\\
&\quad = \frac{1}{2}\int_0^1 v_{tt}^2(x,0)\,dx - g'(0)\int_0^t\int_0^1 v_{tt}(x,s)a'(s)v_0''(x)\,dx\,ds\\
&\qquad - g'(0)\int_0^t\int_0^1 v_{xt}(x,s)a'(s)v_{xt}(x,0)\,dx\,ds\\
&\qquad\qquad + g'(0)\int_0^1 a(t)v_{xt}(x,t)v_{xt}(x,0)\,dx\\
&\qquad - g'(0)\int_0^1 a(0)v_{xt}^2(x,0)\,dx + \int_0^t\int_0^1 v_{tt}(x,s)f_{tt}(x,s)\,dx\,ds\\
&\qquad + \int_0^t\int_0^1 v_{tt}(x,s)\int_0^s v_{xx}(x,y)\int_{s-y}^\infty a'(z)\frac{\partial}{\partial s}\{g''(\overline{v}_x^s(x,z))\}[v_x(x,s)\\
&\qquad\qquad - v_x(x,s-z)]\,dz\,dy\,dx\,ds\\
&\qquad + 2\int_0^t\int_0^1 v_{tt}(x,s)\int_0^\infty a'(z)\frac{\partial}{\partial s}\{g'(\overline{v}_x^s(x,z))\}[v_{xx}(x,s)\\
&\qquad\qquad - v_{xx}(x,s-z)]\,dz\,dx\,ds\\
&\qquad - \frac{1}{2}\int_0^1 v_{xt}^2(x,t)\int_0^\infty a'(s)[g'(\overline{v}_x^t(x,s) - g'(0))]\,ds\,dx\\
&\qquad + \frac{1}{2}\int_0^t\int_0^1 v_{xt}^2(x,s)\frac{\partial}{\partial s}\left\{\int_0^\infty a'(z)[g'(\overline{v}_x^s(x,z)) - g'(0)]\,dz\right\}dx\,ds\\
&\qquad + \int_0^1 v_{xt}(x,t)\int_0^t a'(s)[g'(\overline{v}_x^t(x,s)) - g'(0)]v_{xt}(x,t-s)\,ds\,dx\\
&\qquad - \int_0^t\int_0^1 v_{xt}(x,s)\int_0^s a''(z)[g'(\overline{v}_x^s(x,z))\\
&\qquad\qquad - g'(0)]v_{xt}(x,s-z)\,dz\,dx\,ds
\end{aligned}
$$

$$-\int_0^t\int_0^1 v_{xt}(x,s)v_x(x,s)\int_0^s a'(z)g''(\overline{v}_x^s(x,z)) \cdot v_{xt}(x,s-z)\,dz\,dx\,ds, \quad t\in[0,T_0).$$

The initial values of v_{tt} and v_{xt} can be expressed in terms of f and v_0. In particular, (3.10) and (3.12) imply

$$v_{tt}(x,0)=f_t(x,0)-g'(0)a(0)v_0''(x), \quad x\in[0,1], \tag{3.14}$$

and

$$v_{xt}(x,0)=f_x(x,0), \quad x\in[0,1]. \tag{3.15}$$

We note that $\lim_{h\downarrow 0}(1/h^2)Q(\Delta_h v_{xt},t,a)$ exists for $t\in[0,T_0)$, since the limit of each of the other terms in the derivation of (3.13) exists. Furthermore, the limit under consideration is nonnegative.

We add (3.9), (3.11) and (3.13) and use Lemma 2.2 to obtain a lower bound for the left-hand side. After making some routine estimations we arrive at

$$\begin{aligned}\int_0^1(v^2+v_x^2+v_t^2+v_{xt}^2+v_{tt}^2)(x,t)\,dx+\int_0^t\int_0^1(v_x^2+v_{xt}^2)(x,s)\,dx\,ds\\ \le\ \Gamma(V_0+F)+\Gamma(\sqrt{V_0}+\sqrt{F})\sqrt{\mathcal{E}\,(t)}+\Gamma(\nu(t)+\nu^{k+2}(t))\mathcal{E}\,(t),\\ \forall\, t\in[0,T_0).\end{aligned} \tag{3.16}$$

We will now give an indication of how (3.16) was derived. We show the details of the estimation of several terms that arise in (3.9), (3.11) and (3.13). Many of the terms can be estimated in a simple way, e.g.,

$$\begin{aligned}&\left|\int_0^t\int_0^1 v_t(x,s)f_t(x,s)\,dx\,ds\right|\\ &\quad\le\left(\int_0^t\int_0^1 v_t^2(x,s)\,dx\,ds\right)^{\frac12}\left(\int_0^t\int_0^1 f_t^2(x,s)\,dx\,ds\right)^{\frac12}\\ &\quad\le\Gamma\sqrt{F}\sqrt{\mathcal{E}\,(t)}, \quad \forall\, t\in[0,T_0)\end{aligned} \tag{3.17}$$

and

(3.18)

$$\begin{aligned}
&\left| g'(0) \int_0^t \int_0^1 v_{tt}(x,s) a'(s) v_0''(x)\, dx\, ds \right| \\
&\quad \le \Gamma \left(\int_0^t \int_0^1 v_{tt}^2(x,s)\, dx\, ds \right)^{\frac{1}{2}} \left(\int_0^t a'(s)^2\, ds \int_0^1 v_0''(x)^2\, dx \right)^{\frac{1}{2}} \\
&\quad \le \Gamma \sqrt{V_0} \sqrt{\mathcal{E}\,(t)}, \quad \forall\, t \in [0, T_0).
\end{aligned}$$

The following two estimations are much more involved than (3.17) and (3.18). Recall here that $g''(0) = 0$. From (3.13), we consider

(3.19)

$$\begin{aligned}
&\left| \int_0^t \int_0^1 v_{xt}(x,s) v_x(x,s) \int_0^s a'(z) g''(\overline{v}_x^s(x,z)) v_{xt}(x,s-z)\, dz\, dx\, ds \right| \\
&\quad \le \sup_{\substack{x\in[0,1]\\ s\in[0,t]}} |v_x(x,s)| \int_0^t \int_0^1 |v_{xt}(x,s)| \int_0^s |a'(z)| \cdot |g''(\overline{v}_x^s(x,z)) - g''(0)| \\
&\qquad\qquad \cdot\ |v_{xt}(x,s-z)|\, dz\, dx\, ds \\
&\quad \le \nu(t) \int_0^t \int_0^1 |v_{xt}(x,s)| \int_0^s |a'(z)| K[|\overline{v}_x^s(x,z)| + |\overline{v}_x^s(x,z)|^k] \\
&\qquad\qquad \cdot\ |v_{xt}(x,s-z)|\, dz\, dx\, ds \\
&\quad \le \Gamma\nu(t) \int_0^t \int_0^1 |v_{xt}(x,s)| \int_0^s |a'(z)| \Bigg[\sqrt{z} \left(\int_{s-z}^s v_x^2(x,\xi)\, d\xi \right)^{\frac{1}{2}} \\
&\qquad\qquad + (\sqrt{z})^k \left(\int_{s-z}^s v_x^2(x,\xi)\, d\xi \right)^{\frac{k}{2}} \Bigg] |v_{xt}(x,s-z)|\, dz\, dx\, ds \\
&\quad \le \Gamma\nu(t) \left(\int_0^t \int_0^1 v_{xt}^2(x,s)\, dx\, ds \right)^{\frac{1}{2}} \Bigg(\int_0^t \int_0^1 \bigg(\int_0^s |a'(z)| [\sqrt{z}\, \nu(t) \\
&\qquad\qquad + (\sqrt{z})^k \nu^k(t)] |v_{xt}(x,s-z)|\, dz \bigg)^2 dx\, ds \Bigg)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le \Gamma\nu(t)\sqrt{\mathcal{E}\,(t)}\left(\int_0^t\int_0^1 v_{xt}^2(x,s)\,dx\,ds\right)^{\frac{1}{2}}\left\{\nu(t)\int_0^\infty |a'(z)|\sqrt{z}\,dz\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.+\,\nu^k(t)\int_0^\infty |a'(z)|(\sqrt{z})^k\,dz\right\}\\
&\le \Gamma\nu(t)(\nu(t)+\nu^k(t))\mathcal{E}\,(t)\\
&=\ \Gamma(\nu^2(t)+\nu^{k+1}(t))\mathcal{E}\,(t),\quad \forall\, t\in[0,T_0).
\end{aligned}
$$

The following expression, arising after differentiation with respect to s, is carried out in the last term of (3.11):

(3.20)

$$
\begin{aligned}
&\left|\int_0^t\int_0^1 v_t(x,s)\int_0^s v_{xx}(x,y)\int_{s-y}^\infty a'(z)g''(\overline{v}_x^s(x,z))v_x(x,s-z)\,dz\,dy\,dx\,ds\right|\\
&\quad\le \sup_{\substack{x\in[0,1]\\ s\in[0,t]}}|v_x(x,s)|\int_0^t\int_0^1|v_t(x,s)|\int_0^s|v_{xx}(x,y)|\int_{s-y}^\infty|a'(z)|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot|g''(\overline{v}_x^s(x,z))-g''(0)|dz\,dy\,dx\,ds\\
&\quad\le \nu(t)\int_0^t\int_0^1|v_t(x,s)|\int_0^s|v_{xx}(x,y)|\int_{s-y}^\infty|a'(z)|K[|\overline{v}_x^s(x,z)|\\
&\qquad\qquad\qquad\qquad\qquad\qquad+|\overline{v}_x^s(x,z)|^k]\,dz\,dy\,dx\,ds\\
&\quad\le \Gamma\nu(t)\left(\int_0^t\int_0^1 v_t^2(x,s)\,dx\,ds\right)^{\frac{1}{2}}\\
&\qquad\qquad\cdot\left(\int_0^t\int_0^1\left(\int_0^s|v_{xx}(x,y)|\int_{s-y}^\infty|a'(z)|[\sqrt{z}\;\nu(t)\right.\right.\\
&\qquad\qquad\qquad\qquad\left.\left.+(\sqrt{z})^k\nu^k(t)]\,dz\,dy\right)^2dx\,ds\right)^{\frac{1}{2}}\\
&\quad\le \Gamma(\nu^2(t)+\nu^{k+1}(t))\sqrt{\mathcal{E}\,(t)}\left(\int_0^t\int_0^1 v_{xx}^2(x,s)\,dx\,ds\right)^{\frac{1}{2}}\\
&\qquad\qquad\qquad\qquad\int_0^\infty\int_s^\infty|a'(z)|(\sqrt{z}+(\sqrt{z})^k)\,dz\,ds\\
&\quad\le \Gamma(\nu^2(t)+\nu^{k+1}(t))\mathcal{E}\,(t),\quad\forall\,t\in[0,T_0).
\end{aligned}
$$

In the derivation of (3.20) we have used (3.8) with $A(s) := \int_s^\infty |a'(z)|\cdot(\sqrt{z}+(\sqrt{z})^{k/2})\,dz$. The estimations (3.17)–(3.20) are typical of the calculations necessary to obtain (3.16). All the terms appearing on the right-hand side of (3.9), (3.11) and (3.13) can be estimated in the same spirit as (3.17), (3.18), (3.19) or (3.20).

A temporal L^2 bound for v_t follows from Poincaré's inequality and (3.16). However, since Poincaré's inequality cannot be applied in the proof of Theorem 1.2, we shall derive an additional energy integral. We multiply (3.5) by v_{xx} and integrate over $[0,1]\times[0,t], t\in[0,T_0)$, to obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 v_x^2(x,t)\,dx - g'(0)Q(v_{xx},t,a)\\
&\quad= \frac{1}{2}\int_0^1 v_0'(x)^2\,dx + \int_0^t\int_0^1 v_{xx}(x,s)f(x,s)\,dx\,ds\\
&\qquad+\int_0^t\int_0^1 v_{xx}(x,s)\int_0^s v_{xx}(x,y)\int_{s-y}^\infty a'(z)[g'(\overline{v}_x^s(x,z))\\
&\qquad\qquad - g'(0)]\,dz\,dy\,dx\,ds, \quad t\in[0,T_0),
\end{aligned}\tag{3.21}
$$

which implies

$$
Q(v_{xx},t,a)\le \Gamma V_0 + \Gamma\sqrt{F}\sqrt{\mathcal{E}\,(t)} + \Gamma(\nu(t)+\nu^k(t))\mathcal{E}\,(t), \quad \forall\, t\in[0,T_0).\tag{3.22}
$$

We now square (3.5) and integrate over $[0,1]\times[0,t], t\in[0,T_0)$. The terms resulting from the right-hand side of (3.5) can be controlled near equilibrium; Lemma 2.1 and (3.22) can be employed to estimate the square of the convolution term from the left-hand side of (3.5). Thus we have

$$
\begin{aligned}
&\int_0^t\int_0^1 v_t^2(x,s)\,dx\,ds\\
&\quad\le \Gamma(V_0+F) + \Gamma\sqrt{F}\sqrt{\mathcal{E}\,(t)} + \Gamma(\nu(t)+\nu^{2k}(t))\mathcal{E}\,(t), \quad \forall\, t\in[0,T_0).
\end{aligned}\tag{3.23}
$$

If we set G equal to the right-hand side of (3.5), we can invert (3.10) to express v_{xx} in terms of v_t, v_{tt}, G and G_t (see (2.8) et seq.), i.e.,

$$\begin{aligned} &g'(0)a(0)v_{xx}(x,t)\\ &\quad = G_t(x,t)-v_{tt}(x,t)- \frac{a'(0)}{a(0)}(G(x,t)-v_t(x,t))\\ &\qquad +\int_0^t r'(t-s)(G(x,s)-v_t(x,s))\,ds, \quad x\in[0,1],\ t\in[0,T_0). \end{aligned} \tag{3.24}$$

We now square (3.24) and integrate over $[0,1]$. Using (3.16) and Lemma 2.3 we arrive at

$$\begin{aligned} &\int_0^1 v_{xx}^2(x,t)\,dx\\ &\quad\le \Gamma(V_0+F)+\Gamma(\sqrt{V_0}+\sqrt{F})\sqrt{\mathcal{E}\,(t)}+\Gamma(\nu(t)+\nu^{2k+2}(t))\mathcal{E}\,(t),\\ &\qquad \forall\, t\in[0,T_0). \end{aligned} \tag{3.25}$$

We next multiply (3.24) by v_{tt} and integrate over $[0,1]\times[0,t], t\in[0,T_0)$ to obtain

$$\begin{aligned} &\int_0^t\int_0^1 v_{tt}^2(x,s)\,dx\,ds\\ &\quad\le \Gamma(V_0+F)+\Gamma(\sqrt{V_0}+\sqrt{F})\sqrt{\mathcal{E}\,(t)}+\Gamma(\nu(t)+\nu^{k+2}(t))\mathcal{E}\,(t),\\ &\qquad \forall\, t\in[0,T_0). \end{aligned} \tag{3.26}$$

Here, we made crucial use of Lemma 2.3 and inequality (3.7). We again square (3.24), but now we integrate over $[0,1]\times[0,t], t\in[0,T_0)$, to establish a temporal L^2 estimate for v_{xx}. We employ (3.16), (3.23), (3.26) and Lemma 2.3 to get

$$\begin{aligned} &\int_0^t\int_0^1 v_{xx}^2(x,s)\,dx\,ds\\ &\quad\le \Gamma(V_0+F)+\Gamma(\sqrt{V_0}+\sqrt{F})\sqrt{\mathcal{E}\,(t)}\\ &\qquad +\Gamma(\nu(t)+\nu^{2k+2}(t))\mathcal{E}\,(t), \quad \forall\, t\in[0,T_0). \end{aligned} \tag{3.27}$$

Therefore, by combining (3.16), (3.23), (3.25), (3.26), (3.27) and using Poincaré's inequality to obtain a temporal L^2 bound for v, we arrive at

$$\begin{aligned} \mathcal{E}\,(t) &\le \Gamma(V_0+F)+\Gamma(\sqrt{V_0}+\sqrt{F})\sqrt{\mathcal{E}\,(t)}\\ &\quad +\Gamma(\nu(t)+\nu^{2k+2}(t))\mathcal{E}\,(t), \quad \forall\, t\in[0,T_0). \end{aligned} \tag{3.28}$$

It follows from (3.28) and (3.7) (with λ sufficiently small) that

$$\mathcal{E}(t) \le \overline{\Gamma}(V_0 + F) + \overline{\Gamma}(\nu(t) + \nu^{2k+2}(t))\mathcal{E}(t), \quad \forall\, t \in [0, T_0), \tag{3.29}$$

where $\overline{\Gamma}$ denotes a fixed positive constant which is independent of v_0, f and T_0. We choose $\overline{\mathcal{E}}, \delta > 0$, such that

$$\overline{\Gamma}(\sqrt{2\overline{\mathcal{E}}} + (\sqrt{2\overline{\mathcal{E}}})^{2k+2}) \le \frac{1}{4}, \quad \overline{\Gamma}\delta \le \frac{1}{4}\,\overline{\mathcal{E}}. \tag{3.30}$$

Suppose that (1.16) holds for the above choice of δ. By the Sobolev embedding theorem,

$$\nu(t) \le \sqrt{2\mathcal{E}(t)}, \quad \forall\, t \in [0, T_0). \tag{3.31}$$

Hence (3.29) implies that, for any $t \in [0, T_0)$ with $\mathcal{E}(t) \le \overline{\mathcal{E}}$, we actually have $\mathcal{E}(t) \le \overline{\mathcal{E}}/2$; thus, by continuity, if $\mathcal{E}(0) \le \overline{\mathcal{E}}/2$, then

$$\mathcal{E}(t) \le \frac{1}{2}\,\overline{\mathcal{E}}, \quad \forall\, t \in [0, T_0). \tag{3.32}$$

Clearly, one can choose a smaller $\delta > 0$ (if necessary) so that (1.16) yields $\mathcal{E}(0) \le \overline{\mathcal{E}}/2$. Therefore, for $\delta > 0$ sufficiently small, (3.32) holds and $T_0 = \infty$. Furthermore, (1.17) follows directly from (3.32).

The proof of Theorem 1.1 is now complete.

The proof of Theorem 1.2 is almost identical. The only significant difference is that we do not obtain a bound for v in $L^2([0,\infty); L^2(\mathbf{R}))$ because Poincaré's inequality fails on unbounded spatial intervals. Since we do obtain a bound for $v \in L^\infty([0,\infty); L^2(\mathbf{R}))$, the assumption $f \in L^1([0,\infty); L^2(\mathbf{R}))$ allows us to control the term $\int_0^t \int_{-\infty}^{\infty} fv$. The last term on the right-hand side of (3.9) also requires special attention. To treat this integral we integrate by parts with respect to z and then with respect to x.

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