JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 15, Number 4, Winter 2003

EXPLICIT SOLUTION OF A DIRICHLET-NEUMANN WEDGE DIFFRACTION PROBLEM WITH A STRIP

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ABSTRACT. A rectangular wedge diffraction problem with a strip is studied from an operator theoretical point of view. The problem is formulated as a boundary-value problem with first and second kind conditions for the Helmholtz equation on the exterior of the wedge. The proposed operator approach and the use of certain representation formulas lead to an equation characterized by a Wiener-Hopf-Hankel operator with oscillating Fourier symbols. As a consequence of the construction of some operator matrix identities, the latter operator is related to a matrix Wiener-Hopf operator. In a first stage, these relations allow a detailed study of the Fourier symbol of the Wiener-Hopf operator that was derived from the present problem. Secondly, by factorization of a semi-almost periodic matrix function and a combination of operator relations, the inverses of the Wiener-Hopf and Wiener-Hopf-Hankel operators are represented in explicit form. Those results can be extended to the problem formulated in a scale of Bessel potential spaces. This leads to the well-posedness of the proposed problem as well as its closed form solution in a convenient space setting.

1. Introduction. Since the pioneering work of Sommerfeld [**19**] about a canonical boundary value problem for time-harmonic waves governed by the Helmholtz equation, several different approaches have been considered in the applied mathematics literature studying canonical problems of plane wave diffraction. The probably best known and efficient methods to study such kind of problems are the classical Wiener-Hopf technique and the Maliushinets method [**6, 17**]. Despite these advances, some of those investigations still have gaps in their presentation. For instance, the pertinent question of what are the *most appropriate spaces* to consider such problems is a good example for justifying innovative, and sometimes incompatible, approaches.

2000 AMS *Mathematics Subject Classification.* 35C05, 35J05, 35J25, 45E10, 47A20, 47B35.
Key words and phrases. Wedge diffraction problem, Helmholtz equation, Wiener-Hopf operator, Wiener-Hopf-Hankel operator/equation.
Received by the editors on April 30, 2003, and in revised form on July 14, 2003.

We will consider here a Sommerfeld type problem where the geometry comprises a rectangular wedge formed by a half-plane and a strip.

Unlike other works, we want to understand better what are the operators behind such a problem. Thus, one of the main purposes of the present work is the use of an operator theoretical machinery that will translate the problem to the study of properties of well-known types of operators.

In particular, this will lead us to operators of Wiener-Hopf and Wiener-Hopf-Hankel type with semi-almost periodic Fourier symbol matrices. For practical reasons, we start with a formulation in terms of Bessel potential spaces that allow a certain freedom in the smoothness orders.

The methods involved combine algebraic, operator and function theoretic features in a constructive way. In fact, for instance, several explicit Δ-relations after extension [**3**] and toplinear [**10**, Chapter IV] equivalence (after extension) relations [**1, 4**], between corresponding operators, will be presented. These will allow a transparent transfer of properties between the operators obtained and, therefore, lead us to a complete solution of the problem including regularity results.

2. Formulation of the problem. We shall consider the following diffraction problem in the context of Bessel potential spaces $H^s(\Omega)$, where s is a real number and Ω is a special Lipschitz domain [**21**] of $\mathbf{R}^n$, $n = 1, 2$. A Bessel potential space can be defined as the linear space of distributions, $\phi = r_{\mathbf{R}^n \to \Omega}\, \varphi$, that are obtained by restriction to Ω of the elements in the space

$$H^s(\mathbf{R}^n) = \Big\{ \varphi \in \mathcal{S}'(\mathbf{R}^n) : \|\varphi\|_{H^s(\mathbf{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}\varphi\|_{L^2(\mathbf{R}^n)} < +\infty \Big\},$$

where $\mathcal{F}$ denotes the Fourier transformation. Moreover, the $H^s(\Omega)$ space endowed with the norm

$$\|\phi\|_{H^s(\Omega)} = \inf \Big\{ \|\varphi\|_{H^s(\mathbf{R}^n)} : \varphi \in H^s(\mathbf{R}^n), r_{\mathbf{R}^n \to \Omega}\, \varphi = \phi \Big\}$$

becomes a Banach space. For $\Omega \subset \mathbf{R}^n$ we will denote by $\widetilde{H}^s(\Omega)$ the closed subspace of $H^s(\mathbf{R}^n)$ of distributions with support contained in $\overline{\Omega}$.

In what follows we shall work with functions $u^{\pm} \in H^s(\Omega_{\pm})$ which are weak solutions, in the sense of the Schwartz distribution space $\mathcal{S}'(\Omega_{\pm})$, of the Helmholtz equation in the first quadrant Ω_+ or in the interior of its complement Ω_-, respectively. We need the Cauchy data on the banks of the common boundary defined by

$$u_0^{l,r} = \Big[u_{|\mathbf{R}^2_{l,r}}(x_1, \cdot)\Big]_{|x_1=\mp 0}, \qquad u_1^{\pm} = \left[\frac{\partial u}{\partial x_2}(\cdot, x_2)\right]_{|x_2=\pm 0}$$

in a classical way for $u^{\pm} \in \mathcal{D}(\overline{\Omega_{\pm}})$, e.g. It turns out that the corresponding trace operators given by $u^{\pm} \mapsto u_0^{\pm}$, $u^{\pm} \mapsto u_1^{\pm}$ have continuous extensions as operators from the solution space of the Helmholtz equation equipped with the $H^s(\Omega_{\pm})$ norm onto the trace spaces $H^{s-1/2}(\mathbf{R}_+)$ and $H^{s-3/2}(\mathbf{R}_+)$, respectively, where $\mathbf{R}_+$ is a copy of the closure of a bank of $\Gamma_1 = \{(0, x_2) : x_2 > 0\}$ or $\Gamma_2 \cup \overline{\Gamma_3} = \{(x_1, 0) : 0 < x_1 < a\} \cup \{(x_1, 0) : x_1 \geq a\}$, $0 < a < +\infty$. This fact is not a consequence of the trace theorem but of the representation formulas presented later, and it will make the following formulation mathematically consistent.

Problem $\mathcal{P}$. Find $u \in L^2(\mathbf{R}^2)$ with $u_{|\Omega_{\pm}} \in H^s(\Omega_{\pm})$ being solutions of the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k^2\right) u = 0 \quad \text{in} \quad \Omega_{\pm} \tag{2.1}$$

and such that

$$u_0^r = u_0^l = f_+ \quad \text{on } \Gamma_1 \tag{2.2}$$

$$u_1^+ = u_1^- = -g_a \quad \text{on } \Gamma_2 \tag{2.3}$$

where $s - 1/2 \in]0, 1[$, $k \in \mathbb{C}$, $\Im(k) > 0$ and $f_+ \in H^{s-1/2}(\mathbf{R}_+)$, $g_a \in H^{s-3/2}(]0, a[)$ are given distributions.

Although Problem $\mathcal{P}$ does not belong to the class of diffraction problems that were recognized to be solvable by the Wiener-Hopf technique [**24**, Chapter 8] or other analytical methods [**6**], a convenient interpretation of this problem will allow a reduction to a Wiener-Hopf-Hankel equation. It is known already that certain rectangular wedge diffraction problems without a strip reduce to the solution of Wiener-Hopf-Hankel equations [**12–13, 16, 22**].

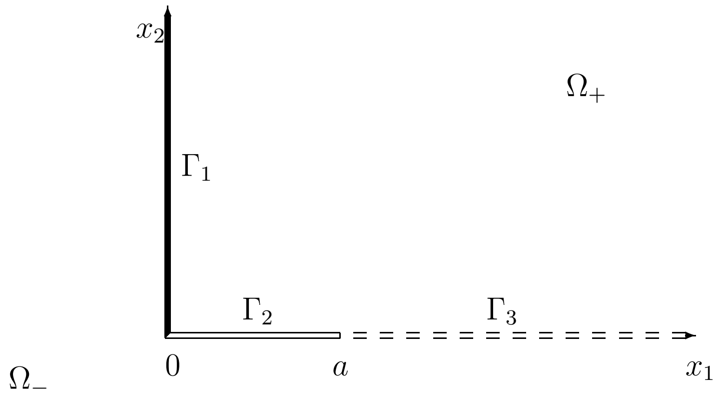

FIGURE 1.1. The boundary in Problem $\mathcal{P}$.

In this sense, Problem $\mathcal{P}$ represents a prototype of a class of diffraction problems which cannot be treated "directly" by the Wiener-Hopf technique but can be reduced to the inversion of matrix Wiener-Hopf operators in a more sophisticated way based upon a modified approach via operator relations.

3. Reduction to a Wiener-Hopf-Hankel equation. Throughout the paper, we always assume that $s \in]1/2, 3/2[$. This includes the usual case of the "energy space", $s = 1$, and some regularity results (as well as the possibility of normalization).

Within this context, we will often use the even and odd extension operators, respectively defined by

$$l^e : H^{s-1/2}(\mathbf{R}_\pm) \longrightarrow H^{s-1/2}(\mathbf{R})$$
$$l^e \varphi(\xi) = \begin{cases} \varphi(\xi), & \xi \in \mathbf{R}_\pm \\ \varphi(-\xi), & \xi \in \mathbf{R}_\mp \end{cases}$$

and

$$l^o : H^{s-3/2}(\mathbf{R}_\pm) \to H^{s-3/2}(\mathbf{R})$$
$$l^o \varphi(\xi) = \begin{cases} \varphi(\xi), & \xi \in \mathbf{R}_\pm \\ -\varphi(-\xi), & \xi \in \mathbf{R}_\mp \end{cases}$$

(in a distributional sense in this last case).

Lemma 3.1. *The above even and odd operators, l^e and l^o, are well-defined and bounded in the corresponding spaces.*

Proof. We will give a direct proof only for the operator(s)

$$l^e : H^{s-1/2}(\mathbf{R}_\pm) \longrightarrow H^{s-1/2}(\mathbf{R});$$

the case of l^o follows by duality arguments analogous to the case $s = 1$ [**14**].

First, although we do not directly need the result for $s = 1/2$ and $s = 3/2$, these limit cases will be helpful in our proof. For $\varphi \in L^2(\mathbf{R}_+)$, it is clear that

$$\|(l^e\varphi)_{|\mathbf{R}_\mp}\|_{L^2(\mathbf{R}_\mp)} = \|\varphi\|_{L^2(\mathbf{R}_\pm)}$$

and therefore

$$\|l^e\varphi\|_{L^2(\mathbf{R})} = 2\|\varphi\|_{L^2(\mathbf{R}_\pm)}.$$

For H^1, we take into account the well known fact [**23**] that if $\sigma \in \mathbf{N}$ then

$$H^\sigma(\mathbf{R}) = W_2^\sigma(\mathbf{R})$$

are the *classical Sobolev spaces*, equipped with the equivalent norm that incorporates the derivatives of the functions

$$\|\psi\|_{W_2^\sigma(\mathbf{R})} = \left(\sum_{j\leq\sigma} \|D^j\psi\|^2_{L^2(\mathbf{R})}\right)^{1/2}.$$

Consequently, if $\varphi \in H^1(\mathbf{R}_\pm)$, one has for $x \in \mathbf{R}_\mp$ that $Dl^e\varphi(x) = -D\varphi(-x)$ and

$$\|Dl^e\varphi_{|\mathbf{R}_\mp}\|_{L^2(\mathbf{R}_\mp)} = \|D\varphi\|_{L^2(\mathbf{R}_\pm)}.$$

Hence, $l^e : H^\sigma(\mathbf{R}_\pm) \to H^\sigma(\mathbf{R})$ is bounded for $\sigma = 0, 1$.

For $0 < \sigma < 1$, with the knowledge of the limit cases, we will make use of the complex interpolation methods introduced by Lions, Kreĭn and Calderón, see, e.g., [**23**]. Among other properties, these allow us to rewrite our spaces by the use of the interpolation couples $\{H^1(\mathbf{R}_\pm), L^2(\mathbf{R}_\pm)\}$ and $\{H^1(\mathbf{R}), L^2(\mathbf{R})\}$:

$$H^\sigma(\mathbf{R}_\pm) = \left[H^1(\mathbf{R}_\pm), L^2(\mathbf{R}_\pm)\right]_{1-\sigma} \tag{3.1}$$

$$H^\sigma(\mathbf{R}) = \left[H^1(\mathbf{R}), L^2(\mathbf{R})\right]_{1-\sigma}. \tag{3.2}$$

Thus, because we already obtained the boundedness of the l^e operator for the limit cases $l^e : L^2(\mathbf{R}_\pm) \to L^2(\mathbf{R})$, $l^e : H^1(\mathbf{R}_\pm) \to H^1(\mathbf{R})$, from the identities (3.1) and (3.2), we derive that l^e is also a bounded operator when acting between the corresponding intermediate interpolated spaces in the left-hand side of (3.1) and (3.2). More precisely, there is a positive constant C_σ such that

$$\|l^e\|_{\mathcal{L}(H^\sigma(\mathbf{R}_\pm),H^\sigma(\mathbf{R}))} \le C_\sigma \, \|l^e\|^\sigma_{\mathcal{L}(L^2(\mathbf{R}_\pm),L^2(\mathbf{R}))} \, \|l^e\|^{1-\sigma}_{\mathcal{L}(H^1(\mathbf{R}_\pm),H^1(\mathbf{R}))}. \quad \square$$

Remark 3.2. From the above proof we also have that for the limit case, e.g., for $l^e : H^\sigma(\mathbf{R}_\pm) \to H^\sigma(\mathbf{R})$ with $\sigma = 0, 1$, the corresponding operators are still bounded (with $\|l^e\varphi\| = 2\|\varphi\|$). We do not pay special attention to these, and other, cases in the statement of Lemma 3.1 because we are only interested in the above values of indices s due to the nature of our problem.

The proof of Lemma 3.1 can be also based on the study of extension operators of Fichtenholz-Hestenes type, see [**23**, Section 2.9].

Theorem 3.3 (Representation formulas). *An element $u \in L^2(\mathbf{R}^2)$ with $u_{|\Omega_\pm} \in H^s(\Omega_\pm)$ is a solution of Problem $\mathcal{P}$ if and only if it is represented by*

(3.3)
$$u(x_1, x_2) = \mathcal{F}^{-1}_{\xi\to x_2} \exp[x_1 t(\xi)] \left(\mathcal{F}(l^e f_+ + f)\right)(\xi), \qquad x_1 < 0, \quad x_2 \in \mathbf{R}$$

(3.4)
$$\begin{aligned} u(x_1, x_2) = {} & \mathcal{F}^{-1}_{\xi\to x_2} \exp[-x_1 t(\xi)] \left(\mathcal{F} l^e f_+\right)(\xi) \\ & + \mathcal{F}^{-1}_{\xi\to x_1} t^{-1}(\xi) \exp[-x_2 t(\xi)] \left(\mathcal{F} l^o g_+\right)(\xi), \quad x_1 > 0,\ x_2 > 0 \end{aligned}$$

(3.5)
$$\begin{aligned} u(x_1, x_2) = {} & \mathcal{F}^{-1}_{\xi\to x_2} \exp[-x_1 t(\xi)] \left(\mathcal{F}(l^e f_+ + l^e f_{|\mathbf{R}_-})\right)(\xi) \\ & - \mathcal{F}^{-1}_{\xi\to x_1} t^{-1}(\xi) \exp[x_2 t(\xi)] \left(\mathcal{F} l^o g_+\right)(\xi), \quad x_1 > 0,\ x_2 < 0, \end{aligned}$$

where $t(\xi) = (\xi^2 - k^2)^{1/2}$ *denotes the usual square root function with branch cuts along* $\xi = \pm k \pm i\eta$, $\eta \geq 0$; $f \in \tilde{H}^{s-1/2}(\mathbf{R}_-)$ *and* $g_+ = lg_a \in H^{s-3/2}(\mathbf{R}_+)$ *is an extension of the datum* g_a *given in Problem* $\mathcal{P}$.

Proof. Let us assume that we have a solution of Problem $\mathcal{P}$. From (2.2), it is clear that $u(0, x_2) = f_+(x_2)$ for $x_2 > 0$. Moreover,

$$\frac{\partial u}{\partial x_2}(x_1, 0) = -g_+(x_1), \qquad x_1 > 0, \tag{3.6}$$

for $g_+ = lg_a \in H^{s-3/2}(\mathbf{R}_+)$. Therefore (3.3) and (3.4) are obtained, cf., [**15, 20**]. In addition,

$$\begin{aligned} u(0, x_2) &= (l^e f_+ + f)(x_2), \qquad x_2 < 0 \\ &= \left(l^e f_+ + l^e f_{|\mathbf{R}_-}\right)(x_2), \qquad x_2 < 0 \end{aligned}$$

lead us to (3.5).

Conversely, a substitution process shows that u given by (3.3), (3.4) and (3.5) is a solution of Problem $\mathcal{P}$. □

Now we will work out transmission conditions given implicitly in the formulation of Problem $\mathcal{P}$. They reflect the fact that the traces on Γ_3 of u in the first and fourth quadrants coincide as well as the normal derivatives of u on the two banks of the half-line $x_2 < 0$, $x_1 = 0$. We obtain from (3.3) and (3.5), respectively,

(3.7)
$$-\frac{\partial u}{\partial x_1}(0, x_2) = -\mathcal{F}^{-1}_{\xi \to x_2} t(\xi) \mathcal{F}\left(l^e f_+ + f\right)(\xi), \quad x_2 < 0,\ x_1 = 0$$

(3.8)
$$\begin{aligned} -\frac{\partial u}{\partial x_1}(0, x_2) &= \mathcal{F}^{-1}_{\xi \to x_2} t(\xi) \mathcal{F}\left(l^e f_+ + l^e r_{\mathbf{R} \to \mathbf{R}_-} f\right)(\xi) \\ &\quad + i(2\pi)^{-1} \int_{\mathbf{R}} \xi t^{-1}(\xi) \exp[x_2 t(\xi)] \mathcal{F} l^o g_+(\xi) d\xi, \\ &\qquad x_2 < 0,\ x_1 = 0 \end{aligned}$$

On the other hand, for $x_1 > a$ and $x_2 = 0$, we derive

$$(2\pi)^{-1}\int_{\mathbf{R}} \exp[-x_1 t(\xi)]\,(\mathcal{F}l^e f)\,(\xi)\,d\xi + \mathcal{F}^{-1}t^{-1}\mathcal{F}l^o g_+(\xi) \tag{3.9}$$
$$= (2\pi)^{-1}\int_{\mathbf{R}} \exp[-x_1 t(\xi)]\,\big(\mathcal{F}(l^e f_+ + l^e r_{\mathbf{R}\to\mathbf{R}_-} f)\big)\,(\xi)\,d\xi - \mathcal{F}^{-1}t^{-1}\mathcal{F}l^o g_+(\xi),$$

cf., (3.4) and (3.5).

From (3.7) and (3.8), we have

$$\mathcal{F}^{-1}t\cdot\mathcal{F}\left(2l^e f_+ + l^e r_{\mathbf{R}\longrightarrow\mathbf{R}_-} f + f\right)(\xi) + C_1 l^o g_+(\xi) = 0, \qquad \xi \in \mathbf{R}_- \tag{3.10}$$

where $t\cdot$ denotes the multiplication operator due to the function t and

$$C_1 : H^{s-3/2}(\mathbf{R}) \to H^{s-3/2}(\mathbf{R}_-)$$
$$C_1\varphi(x) = i(2\pi)^{-1}\int_{\mathbf{R}} \xi t^{-1}(\xi)\exp[xt(\xi)]\mathcal{F}\varphi(\xi)\,d\xi, \qquad x < 0.$$

On $]a, +\infty[$, from (3.9), we obtain

$$2\mathcal{F}^{-1}t^{-1}\cdot\mathcal{F}l^o g_+ = C_2 l^e r_{\mathbf{R}\to\mathbf{R}_-} f \tag{3.11}$$

with

$$C_2 : H^{s-1/2}(\mathbf{R}) \to H^{s-1/2}(\mathbf{R}_+)$$
$$C_2\varphi(x) = (2\pi)^{-1}\int_{\mathbf{R}} \exp[-xt(\xi)]\mathcal{F}\varphi(\xi)\,d\xi, \qquad x > 0.$$

Substituting ξ by $-\xi$ in (3.10), we have

$$C_3 l^o g_+(\xi) + \mathcal{F}^{-1}t\cdot\mathcal{F}\left(2l^e f_+ + l^e r_{\mathbf{R}\to\mathbf{R}_-} f + f\right)(-\xi) = 0, \qquad \xi > 0,$$

where

$$C_3 : H^{s-3/2}(\mathbf{R}) \to H^{s-3/2}(\mathbf{R}_+)$$
$$C_3\varphi(y) = i(2\pi)^{-1}\int_{\mathbf{R}} \xi t^{-1}(\xi)\exp[-yt(\xi)]\mathcal{F}\varphi(\xi)\,d\xi, \qquad y > 0.$$

Using the reflection operator $J\psi(\xi) = \psi(-\xi)$, we get

$$\mathcal{F}^{-1}t \cdot \mathcal{F}\left(2l^e f_+ + l^e r_{\mathbf{R}\longrightarrow\mathbf{R}_-} f + Jf\right)(\xi) + C_3 l^o g_+(\xi) = 0, \qquad \xi > 0$$

and therefore the following equations are valid on $\mathbf{R}_+$

(3.12)

$$\begin{aligned} l^e r_{\mathbf{R}\to\mathbf{R}_-} f + Jf &= -2l^e f_+ - \mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} l^e C_3 l^o g_+ \\ r_{\mathbf{R}\to\mathbf{R}_-} f &= -2r_{\mathbf{R}\to\mathbf{R}_-} l^e f_+ - r_{\mathbf{R}\to\mathbf{R}_-} \mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} l^e C_3 l^o g_+ \\ l^e r_{\mathbf{R}\to\mathbf{R}_-} f &= -2l^e f_+ - \mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} l^e C_3 l^o g_+. \end{aligned}$$

Finally, from (3.11) and (3.12), on $]a, +\infty[$ we obtain

$$(3.13) \qquad 2\mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} l^o g_+ = C_2\left(-2l^e f_+ - \mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} l^e C_3 l^o g_+\right).$$

Lemma 3.4. *The following operator identification holds*

$$C_2\mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} l^e C_3 l^o = -2r_{\mathbf{R}\to\mathbf{R}_+} \mathcal{F}^{-1}t^{-1} \cdot \mathcal{F} J l_0,$$

as operators acting from $r_{\mathbf{R}\to\mathbf{R}_+}\tilde{H}^{s-3/2}(\mathbf{R}_+)$ *to* $H^{s-1/2}(\mathbf{R}_+)$ *and where* l_0 *denotes the zero extension operator from the positive half-line to the full line in corresponding spaces.*

Proof. For $s = 1$, the result was announced in [**12**] within the framework of similar identities for so-called *operators around the corner* (the notation changed for convenience; for $s = 1$, our operators C_1, C_2, C_3 here correspond with $r_{\mathbf{R}\to\mathbf{R}_+} J l^o C_1$, C_0, C_1 in [**12**], respectively). Here, the knowledge of Lemma 3.1 leads us to the present statement by using the same arguments as in Theorem 5.2 and Theorem 6.2 of [**12**] and performing corresponding restrictions or continuous extensions depending on the scales $s \in]1, 3/2[$ or $s \in]1/2, 1[$, respectively. ◻

Theorem 3.5. *Problem* $\mathcal{P}$ *is uniquely solvable if and only if the equation*

$$(3.14) \qquad \mathcal{H}\varphi = h$$

is uniquely solvable, where $\mathcal{H}$ denotes the Wiener-Hopf-Hankel operator defined by

$$\mathcal{H} = r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_{-a}\left(t^{-1}\cdot -2t^{-1}\cdot J\right)\tau_a\cdot\mathcal{F}: \tilde{H}^{s-3/2}(\mathbf{R}_+)\to H^{s-1/2}(\mathbf{R}_+), \tag{3.15}$$

with $\tau_c(\xi) = \exp[ic\xi]$, $\xi\in\mathbf{R}$.

Proof. Choosing a particular known extension $lg_a \in H^{s-3/2}(\mathbf{R}_+)$ of g_a, we are able to use the representation

$$g_+ = lg_a + \psi, \tag{3.16}$$

for some $\psi \in r_{\mathbf{R}\to\mathbf{R}_+}\tilde{H}^{s-3/2}(]a,+\infty[)$. Therefore, from (3.13) and (3.16), one obtains

$$2r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_{-a}t^{-1}\cdot\mathcal{F}l^o\psi + r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}lC_2\mathcal{F}^{-1}t^{-1}\cdot\mathcal{F}l^eC_3l^o\psi = 2h, \tag{3.17}$$

where

$$\begin{aligned} h = &-r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_{-a}t^{-1}\cdot\mathcal{F}l^o lg_a - r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}lC_2l^ef_+ \\ &-\frac{1}{2}r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}lC_2\mathcal{F}^{-1}t^{-1}\cdot\mathcal{F}l^eC_3l^o lg_a, \end{aligned} \tag{3.18}$$

for some extension $l : H^{s-1/2}(\mathbf{R}_+)\to H^{s-1/2}(\mathbf{R})$ whose particular choice does not change the identity (3.17).

By Lemma 3.4 and the identity $l^o\psi = (I-J)l_0\psi$, we have, equivalently to (3.17),

$$\mathcal{H}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}l_0\psi = h. \tag{3.19}$$

Therefore, if φ is a solution of (3.14) for h defined by (3.18), then

$$\psi = r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}\varphi$$

will define the value of g_+, as in (3.16), and for $f\in\tilde{H}^{s-1/2}(\mathbf{R}_-)$ resulting from (3.12), the representation formulas (3.3)–(3.5) give us the solution of Problem $\mathcal{P}$. Conversely, by the identity (3.6), a solution u of Problem $\mathcal{P}$ will provide us with the value of g_+ and, therefore, for each particular extension lg_a, we will obtain an element ψ so that (3.16) holds. This leads us to a solution $\varphi = \mathcal{F}^{-1}\tau_{-a} \cdot \mathcal{F} l_0 \psi$ of (3.14), for h in (3.18). □

Remark 3.6. Note that (3.15) is in fact a Wiener-Hopf-Hankel operator because one can rewrite

$$\mathcal{H} = r_{\mathbf{R}\to\mathbf{R}_+}(A + BJ)_{|\tilde{H}^{s-3/2}(\mathbf{R}_+)},$$

where

$$\begin{aligned} A &= \mathcal{F}^{-1} t^{-1} \cdot \mathcal{F}, \\ B &= -2\mathcal{F}^{-1}\tau_{-2a}\, t^{-1} \cdot \mathcal{F} \in \mathcal{L}\left(H^{s-3/2}(\mathbf{R}), H^{s-1/2}(\mathbf{R})\right). \end{aligned}$$

In particular, from Theorem 3.5 and its proof, we have:

Corollary 3.7. *If for some extension* $lg_a \in H^{s-3/2}(\mathbf{R}_+)$ *of* g_a*,* φ *is a solution of equation* (3.14)*, for* h *given as in* (3.18) *and where* f_+ *and* g_a *are the data exhibited in Problem* $\mathcal{P}$*, then the element* u *in* (3.3)–(3.5)*, for*

$$g_+ = lg_a + r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\tau_a \cdot \mathcal{F}\varphi, \tag{3.20}$$

$$f = l_0\left(-2r_{\mathbf{R}\to\mathbf{R}_-} l^e f_+ - r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1} t^{-1} \cdot \mathcal{F} l^e C_3 l^o g_+\right), \tag{3.21}$$

is a solution of Problem $\mathcal{P}$*.*

Remark 3.8. If we *associate* [**15, 20**] an operator

$$L : D(L) \longrightarrow H^{s-1/2}(\mathbf{R}_+) \times H^{s-3/2}(]0, a[)$$

to our Problem $\mathcal{P}$ in such a way that the action of this operator will be defined by the transformation given by the nonhomogeneous conditions (2.2)–(2.3) and where $D(L)$ is the subspace of $H^s(\Omega_+) \times H^s(\Omega_-)$ whose elements fulfill the Helmholtz equation (2.1), then the above identities (and, in particular, Theorem 3.5) can be translated by a toplinear *equivalence after extension* operator relation [**1, 4, 10**]. That means that we have an identity of the following type between the operators L and $\mathcal{H}$:

$$\begin{bmatrix} L & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} \mathcal{H} & 0 \\ 0 & I_{Z_2} \end{bmatrix} F,$$

for certain Banach spaces Z_1 and Z_2 and invertible bounded linear operators E and F.

A general framework devoted to the identification of toplinear extensions between convolution type operators and operators associated to canonical wedge diffraction problems will be part of a forthcoming paper.

4. A Wiener-Hopf operator related to $\mathcal{H}$. Here we will work with other operators that allow us to transfer regularity properties for $\mathcal{H}$. For this purpose, let us first introduce the following auxiliary notation.

Definition 4.1 [**3**]. Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators acting between Banach spaces. We shall say that $\mathcal{T}$ is Δ*-related after extension to* $\mathcal{W}$ if there are a bounded linear operator acting between Banach spaces $\mathcal{T}_\Delta : X_{1\Delta} \to X_{2\Delta}$, an additional Banach space Z and invertible bounded linear operators

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} : Y_2 \oplus Z \to X_2 \oplus X_{2\Delta}, \tag{4.1}$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} : X_1 \oplus X_{1\Delta} \to Y_1 \oplus Z, \tag{4.2}$$

such that

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix} = E \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_Z \end{bmatrix} F. \tag{4.3}$$

Theorem 4.1. *The operator $\mathcal{H}$ is Δ-related after extension to the Wiener-Hopf operator*

$$\mathcal{W} = r_{\mathbf{R}\to\mathbf{R}_+} \mathcal{F}^{-1} \Phi_{\mathcal{W}} \cdot \mathcal{F} : [L^2_+(\mathbf{R})]^2 \to [L^2(\mathbf{R}_+)]^2, \tag{4.4}$$

where

$$\Phi_{\mathcal{W}} = \begin{bmatrix} 3\lambda_-^{s-1/2} t^{-1} \lambda_+^{-s+3/2} & 2\lambda_-^{s-1/2} \tau_{-2a} \, (\widetilde{\lambda_-})^{-s+1/2} \\ 2(\widetilde{\lambda_+})^{s-3/2} \tau_{2a} \, \lambda_+^{-s+3/2} & (\widetilde{\lambda_+})^{s-3/2} t (\widetilde{\lambda_-})^{-s+1/2} \end{bmatrix},$$

with $\lambda_\pm(\xi) = \xi \pm i$ *and* $(\widetilde{\lambda_\pm})(\xi) = \lambda_\pm(-\xi)$, $\xi \in \mathbf{R}$.

Proof. Let $P_\pm = l_0 r_{\mathbf{R}\to\mathbf{R}_\pm} : L^2(\mathbf{R}) \to L^2_\pm(\mathbf{R})$ be the complementary projectors onto the subspaces of $L^2(\mathbf{R})$ functions supported on $\overline{\mathbf{R}_+}$ and $\overline{\mathbf{R}_-}$, respectively.

In a first step we will perform a lifting [**23**] of $\mathcal{H}$ to L^2 spaces. This can be done with the help of convenient Bessel potential operators. Let us consider

$$E_1 = r_{\mathbf{R}\to\mathbf{R}_+} \mathcal{F}^{-1} \lambda_-^{-s+1/2} \cdot \mathcal{F} : L^2_+(\mathbf{R}) \to H^{s-1/2}(\mathbf{R}_+)$$

and

$$F_1 = P_+ \mathcal{F}^{-1} \lambda_+^{s-3/2} \cdot \mathcal{F} : \widetilde{H}^{s-3/2}(\mathbf{R}_+) \to L^2_+(\mathbf{R}).$$

These operators are bounded with respective inverses

$$E_1^{-1} = P_+ \mathcal{F}^{-1} \lambda_-^{s-1/2} \cdot \mathcal{F} l : H^{s-1/2}(\mathbf{R}_+) \to L^2_+(\mathbf{R}),$$

and

$$F_1^{-1} = P_+ \mathcal{F}^{-1} \lambda_+^{-s+3/2} \cdot \mathcal{F} : L^2_+(\mathbf{R}) \to \widetilde{H}^{s-3/2}(\mathbf{R}_+),$$

where $l : H^{s-1/2}(\mathbf{R}_+) \to H^{s-1/2}(\mathbf{R})$ is an operator of extension to the full line which particular choice does not change the definition of E_1^{-1} (one can choose, e.g., l^e).

Due to the structure of the above operators E_1 and F_1, we obtain

$$\mathcal{H} = E_1 \mathcal{H}_1 F_1, \tag{4.5}$$

where we are already dealing with a Wiener-Hopf-Hankel operator acting between L^2 spaces

$$\mathcal{H}_1 = P_+ \mathcal{F}^{-1} \left(\Phi_1 \cdot + \Phi_2 \cdot J \right) \mathcal{F} : L^2_+(\mathbf{R}) \to L^2_+(\mathbf{R}),$$

with

$$\Phi_1 = \lambda_-^{s-1/2} t^{-1} \lambda_+^{-s+3/2}, \tag{4.6}$$

$$\Phi_2 = -2\lambda_-^{s-1/2} \tau_{-2a} t^{-1} (\widetilde{\lambda_+})^{-s+3/2}, \tag{4.7}$$

where $\widetilde{\lambda_+} = J\lambda_+$. In particular, the identity (4.5) shows that $\mathcal{H}$ and $\mathcal{H}_1$ are toplinear equivalent.

Now we will extend $\mathcal{H}_1$ to $L^2(\mathbf{R})$, by the use of the identity operator $I_{L^2_-(\mathbf{R})}$ in $L^2_-(\mathbf{R})$ and, next, we *double the space* by use of the *coupling operator*

$$\mathcal{H}_2 = \mathcal{F}^{-1} (\Phi_1 \cdot - \Phi_2 \cdot J) \mathcal{F} P_+ + P_- : L^2(\mathbf{R}) \to L^2(\mathbf{R}).$$

(note the different sign in front of Φ_2). Therefore, we arrive at the paired operator

$$\mathcal{W}_0 = \begin{bmatrix} \mathcal{F}^{-1}\Phi_1 \cdot \mathcal{F} & 0 \\ \mathcal{F}^{-1}\widetilde{\Phi_2} \cdot \mathcal{F} & 1 \end{bmatrix} P_+ + \begin{bmatrix} 1 & \mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\Phi_1} \cdot \mathcal{F} \end{bmatrix} P_- : \left[L^2(\mathbf{R})\right]^2 \longrightarrow \left[L^2(\mathbf{R})\right]^2$$

(putting $\widetilde{\Phi_j}(\xi) = \Phi_j(-\xi)$) that appears in the identity

$$\begin{bmatrix} \mathcal{H}_1 P_+ + P_- & 0 \\ 0 & \mathcal{H}_2 \end{bmatrix} = E_2 \mathcal{W}_0 F_2, \tag{4.8}$$

where E_2 and F_2 are bounded invertible operators defined by

$$E_2 = \frac{1}{2} \begin{bmatrix} I_{L^2(\mathbf{R})} & J \\ I_{L^2(\mathbf{R})} & -J \end{bmatrix},$$

$$F_2 = \begin{bmatrix} I_{L^2(\mathbf{R})} & I_{L^2(\mathbf{R})} \\ J & -J \end{bmatrix} \times \begin{bmatrix} I_{L^2(\mathbf{R})} - P_- \mathcal{F}^{-1} (\Phi_1 \cdot + \Phi_2 \cdot J) \mathcal{F} P_+ & 0 \\ 0 & I_{L^2(\mathbf{R})} \end{bmatrix}.$$

One can rewrite $\mathcal{W}_0$ in the form

$$\begin{aligned} \mathcal{W}_0 &= \begin{bmatrix} 1 & \mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\Phi_1} \cdot \mathcal{F} \end{bmatrix} \left(\mathcal{F}^{-1}\Phi_3 \cdot \mathcal{F} P_+ + P_- \right) \\ &= \begin{bmatrix} 1 & \mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\Phi_1} \cdot \mathcal{F} \end{bmatrix} \left(P_+ \mathcal{F}^{-1}\Phi_3 \cdot \mathcal{F} P_+ + P_- \right) \\ &\qquad \left(I_{[L^2(\mathbf{R})]^2} + P_- \mathcal{F}^{-1}\Phi_3 \cdot \mathcal{F} P_+ \right), \end{aligned} \tag{4.9}$$

where the last factor is invertible by $I_{[L^2(\mathbf{R})]^2} - P_- \mathcal{F}^{-1}\Phi_3 \cdot \mathcal{F} P_+$ and

$$\Phi_3 = \begin{bmatrix} \Phi_1 - \Phi_2\left(\widetilde{\Phi_1}\right)^{-1}\widetilde{\Phi_2} & -\Phi_2\left(\widetilde{\Phi_1}\right)^{-1} \\ \left(\widetilde{\Phi_1}\right)^{-1}\widetilde{\Phi_2} & \left(\widetilde{\Phi_1}\right)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} -3\Phi_1 & -\Phi_2\left(\widetilde{\Phi_1}\right)^{-1} \\ \left(\widetilde{\Phi_1}\right)^{-1}\widetilde{\Phi_2} & \left(\widetilde{\Phi_1}\right)^{-1} \end{bmatrix}.$$

The identity (4.9) tells us that $\mathcal{W}_0$ is toplinear equivalent after extension to

$$r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\Phi_3 \cdot \mathcal{F} : \left[L^2_+(\mathbf{R})\right]^2 \to \left[L^2(\mathbf{R}_+)\right]^2. \tag{4.10}$$

Thus, noticing that this last operator (4.10) coincides with the operator $\mathcal{W}$ of (4.4) up to a sign in the first column, we have a Δ-relation after extension in the form of (4.3) with $\mathcal{T} = \mathcal{H}$, $\mathcal{W}$ given by (4.4) and

$$\mathcal{T}_\Delta = \begin{bmatrix} I_{L^2_-(\mathbf{R})} & 0 \\ 0 & \mathcal{H}_2 \end{bmatrix}. \qquad \square$$

In the last result we already reached the goal of this section: we obtained a Wiener-Hopf operator that is explicitly related to $\mathcal{H}$. This will allow us several conclusions about the transfer of regularity properties [**3**]. The next corollary provides an example of these conclusions.

Corollary 4.2. *If the Wiener-Hopf operator* $\mathcal{W}$ *is* (*left, right*) *invertible, or Fredholm, then the Wiener-Hopf-Hankel operator* $\mathcal{H}$ *is also* (*left, right*) *invertible, or a Fredholm operator, respectively. In these cases*

$$\dim\ker\mathcal{H} \le \dim\ker\mathcal{W}$$
$$\operatorname{codim}\operatorname{im}\mathcal{H} \le \operatorname{codim}\operatorname{im}\mathcal{W}.$$

Proof. The statement is a direct consequence of Theorem 4.1 and of the structure of Δ-relations after extension, cf., Definition 4.1 and, in particular, the form of the identity (4.3). $\square$

5. Analyzing the symbol of $\mathcal{W}$. After the last section, we are in a position to study the Fourier symbols of the constructed operators. For that we will concentrate, in the first place, on the Wiener-Hopf operators related (in the above sense) to our problem.

In what follows, similarly as in the representation formulas of Theorem 3.3, we are taking branch cuts as straight lines from i to $i\infty$ and $-i$ to $-i\infty$ for

$$\lambda_-^{-s+1/2}(\xi) = (\xi - i)^{-s+1/2}, \qquad \lambda_+^{s-3/2}(\xi) = (\xi + i)^{s-3/2}$$

with $-3\pi/2 < \arg(\xi - i) \le \pi/2$, $-\pi/2 < \arg(\xi + i) \le 3\pi/2$ and according to

$$\left(\widetilde{\lambda_+}\right)^{s-3/2} = (-1)^{s-3/2}\,\lambda_-^{s-3/2}, \qquad \left(\widetilde{\lambda_-}\right)^{-s+1/2} = (-1)^{-s+1/2}\,\lambda_+^{-s+1/2}.$$

We start by observing that the Fourier symbol $\Phi_{\mathcal{W}}$ of $\mathcal{W}$ can be, in a sense, simplified. Due to the fact that we are dealing with a wave number k so that $\Im(k) > 0$, $\mathcal{W}$ has the same regularity properties [**3**] as the operator

$$\mathcal{W}_{i,s} = r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\Phi_{\mathcal{W}_{i,s}}\cdot\mathcal{F} : \left[L^2_+(\mathbf{R})\right]^2 \longrightarrow \left[L^2(\mathbf{R}_+)\right]^2 \tag{5.1}$$

where

$$\Phi_{\mathcal{W}_{i,s}} = \zeta_i^{s-1}\begin{bmatrix} 3 & 2(-1)^{-s+1/2}\zeta_k^{1/2}\tau_{-2a} \\ 2(-1)^{s-1/2}\zeta_k^{-1/2}\tau_{2a} & 1 \end{bmatrix}, \tag{5.2}$$

with $\zeta_\mu(\xi) = (\xi - \mu)/(\xi + \mu)$, $\xi \in \mathbf{R}$ and $\Im(\mu) > 0$. In fact, we have the following more general result.

Proposition 5.1. *The Wiener-Hopf operators $\mathcal{W}$ and $\mathcal{W}_{i,s}$ are toplinear equivalent.*

Proof. We will use the notation $\lambda_{k\pm}(\xi) = \xi \pm k$, $\xi \in \mathbf{R}$. Considering the bounded invertible operators

$$E_3 = r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\begin{bmatrix}\lambda_{k-}^{1/2}\lambda_{-}^{-1/2} & 0\\ 0 & -\lambda_{k-}^{-1/2}\lambda_{-}^{1/2}\end{bmatrix}$$
$$\cdot\,\mathcal{F}l_0 : [L^2(\mathbf{R}_+)]^2 \to [L^2(\mathbf{R}_+)]^2,$$
$$F_3 = P_+\mathcal{F}^{-1}\begin{bmatrix}\lambda_{k+}^{1/2}\lambda_{+}^{-1/2} & 0\\ 0 & \lambda_{k+}^{-1/2}\lambda_{+}^{1/2}\end{bmatrix}$$
$$\cdot\,\mathcal{F} : [L^2_+(\mathbf{R})]^2 \to [L^2_+(\mathbf{R})]^2,$$

we obtain $\mathcal{W}_{i,s} = E_3\mathcal{W}F_3$, due to the above choice of the corresponding branch cuts and to the lifting properties of the present Bessel potential operators E_3 and F_3, see, e.g., [**23**, Section 2.3.10]. □

In particular, from the relations constructed in the proofs of Theorem 4.1 and Proposition 5.1, we derive that $\mathcal{H}$ is Δ-related after extension to $\mathcal{W}_{i,s}$ in the following explicit way (that will be of fundamental importance for the results in Section 6):

$$\begin{bmatrix}\mathcal{H} & 0 & 0\\ 0 & I_{L^2_-(\mathbf{R})} & 0\\ 0 & 0 & \mathcal{H}_2\end{bmatrix} = E_4\begin{bmatrix}\mathcal{W}_{i,s} & 0\\ 0 & I_{[L^2_-(\mathbf{R})]^2}\end{bmatrix}F_4 \tag{5.3}$$

where

$$E_4 = \frac{1}{2}\begin{bmatrix}E_1 & 0 & 0\\ 0 & I_{L^2_-(\mathbf{R})} & 0\\ 0 & 0 & I_{L^2(\mathbf{R})}\end{bmatrix}\begin{bmatrix}P_+ & 0\\ P_- & 0\\ 0 & I_{L^2(\mathbf{R})}\end{bmatrix}$$
$$\begin{bmatrix}I_{L^2(\mathbf{R})} & \mathcal{F}^{-1}\Phi_2\cdot\mathcal{F} + J\mathcal{F}^{-1}\widetilde{\Phi_1}\cdot\mathcal{F}\\ I_{L^2(\mathbf{R})} & \mathcal{F}^{-1}\Phi_2\cdot\mathcal{F} - J\mathcal{F}^{-1}\widetilde{\Phi_1}\cdot\mathcal{F}\end{bmatrix}$$
$$\begin{bmatrix}P_+ & 0 & P_- & 0\\ 0 & P_+ & 0 & P_-\end{bmatrix}$$
$$\begin{bmatrix}\operatorname{diag}[-l_0 r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\lambda_{k-}^{-\frac12}\lambda_{-}^{\frac12}\cdot\mathcal{F}l_0,\ -l_0 r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\lambda_{k-}^{\frac12}\lambda_{-}^{-\frac12}\cdot\mathcal{F}l_0] & 0\\ 0 & I_{[L^2_-(\mathbf{R})]^2}\end{bmatrix}$$

with $l_0 : L^2(\mathbf{R}_+) \to L^2_+(\mathbf{R})$ standing again for the zero extension operator and

$$F_4 = \begin{bmatrix} \operatorname{diag}[-l_0 r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\lambda_{k+}^{-1/2}\lambda_+^{1/2}\cdot\mathcal{F},\, l_0 r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\lambda_{k+}^{\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\cdot\mathcal{F}] & 0 \\ 0 & I_{[L^2_-(\mathbf{R})]^2} \end{bmatrix}$$
$$\begin{bmatrix} P_+ & 0 \\ 0 & P_+ \\ P_- & 0 \\ 0 & P_- \end{bmatrix}$$
$$\begin{bmatrix} I_{L^2(\mathbf{R})} - 3l_0 r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1}\Phi_1\cdot\mathcal{F}l_0 r_{\mathbf{R}\to\mathbf{R}_+} & -l_0 r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1}\Phi_2\left(\widetilde{\Phi_1}\right)^{-1}\cdot\mathcal{F}l_0 r_{\mathbf{R}\to\mathbf{R}_+} \\ l_0 r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1}\left(\widetilde{\Phi_1}\right)^{-1}\widetilde{\Phi_2}\cdot\mathcal{F}l_0 r_{\mathbf{R}\to\mathbf{R}_+} & I_{L^2(\mathbf{R})} + l_0 r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1}\left(\widetilde{\Phi_1}\right)^{-1}\cdot\mathcal{F}l_0 r_{\mathbf{R}\to\mathbf{R}_+} \end{bmatrix}$$
$$\begin{bmatrix} I_{L^2(\mathbf{R})} - l_0 r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1}\left(\Phi_1\cdot+\Phi_2\cdot J\right)\mathcal{F}l_0 r_{\mathbf{R}\to\mathbf{R}_+} & I_{L^2(\mathbf{R})} \\ J - Jl_0 r_{\mathbf{R}\to\mathbf{R}_-}\mathcal{F}^{-1}\left(\Phi_1\cdot+\Phi_2\cdot J\right)\mathcal{F}l_0 r_{\mathbf{R}\to\mathbf{R}_+} & -J \end{bmatrix}$$
$$\begin{bmatrix} P_+ & P_- & 0 \\ 0 & 0 & I_{L^2(\mathbf{R})} \end{bmatrix} \begin{bmatrix} F_1 & 0 & 0 \\ 0 & I_{L^2_-(\mathbf{R})} & 0 \\ 0 & 0 & I_{L^2(\mathbf{R})} \end{bmatrix}.$$

Please observe that from the form of the last two formulas, presented for defining E_4 and F_4, we directly derive that, in fact, these are invertible bounded operators.

The following facts are immediate consequences of (5.3), cf., [**3**].

Proposition 5.2. *Let us consider our Wiener-Hopf-Hankel and Wiener-Hopf operators $\mathcal{H}$ and $\mathcal{W}_{i,s}$, see* (3.15) *and* (5.1)*, respectively, as well as the additional Wiener-Hopf-Hankel operator*

$$\mathcal{H}_3 = r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\left(\Phi_1\cdot-\Phi_2\cdot J\right)\mathcal{F} : L^2_+(\mathbf{R}) \to L^2(\mathbf{R}_+), \tag{5.4}$$

for Φ_1 and Φ_2 introduced in (4.6) *and* (4.7)*.*

(i) *The operators $\mathcal{H}$ and $\mathcal{H}_3$ are both* (*left, right, generalized*) *invertible if and only if $\mathcal{W}_{i,s}$ is* (*left, right, generalized*) *invertible.*

(ii) *$\mathcal{H}$ and $\mathcal{H}_3$ are both Fredholm operators if and only if $\mathcal{W}_{i,s}$ is also a Fredholm operator.*

(iii) *If $\mathcal{W}_{i,s}$ is a Fredholm operator, then the Fredholm indices of these three operators satisfy the identity*

$$\operatorname{Ind}\mathcal{H} = \operatorname{Ind}\mathcal{W}_{i,s} - \operatorname{Ind}\mathcal{H}_3.$$

Remark 5.3. Similar results were obtained already for a more restricted class of Fourier symbols and assuming the generalized invertibility of corresponding Wiener-Hopf operators, cf., Theorem 3.2 of [**11**]. For different Fourier symbols and also based on certain operator matrix identities, the Fredholm property of corresponding Wiener-Hopf-Hankel operators was discussed in the recent works [**7, 8**].

We observe that $\Phi_{\mathcal{W}_{i,s}}$ belongs to the C^*-algebra of the semi-almost periodic (SAP) two by two matrix functions on the real line, see [**18**], i.e., $\Phi_{\mathcal{W}_{i,s}}$ belongs to the smallest closed subalgebra of $[L^\infty(\mathbf{R})]^{2\times 2}$ that contains the, classical, algebra of, two by two, *almost periodic elements* and the (two by two) continuous matrices with possible jumps at infinity.

Thus we can choose a continuous function on the real line, say γ, so that $\gamma(-\infty) = 0$, $\gamma(+\infty) = 1$ and

$$\Phi_{\mathcal{W}_{i,s}} = (1-\gamma)\big(\Phi_{\mathcal{W}_{i,s}}\big)_l + \gamma\big(\Phi_{\mathcal{W}_{i,s}}\big)_r + \big(\Phi_{\mathcal{W}_{i,s}}\big)_0$$

where $\big(\Phi_{\mathcal{W}_{i,s}}\big)_l$ and $\big(\Phi_{\mathcal{W}_{i,s}}\big)_r$ are matrices with almost periodic elements, uniquely determined by $\Phi_{\mathcal{W}_{i,s}}$,

$$\big(\Phi_{\mathcal{W}_{i,s}}\big)_l = \begin{bmatrix} 3 & 2(-1)^{-s+1/2}\tau_{-2a} \\ 2(-1)^{s-1/2}\tau_{2a} & 1 \end{bmatrix}$$
$$\big(\Phi_{\mathcal{W}_{i,s}}\big)_r = \exp[2\pi i(s-1)] \begin{bmatrix} 3 & -2(-1)^{-s+1/2}\tau_{-2a} \\ -2(-1)^{s-1/2}\tau_{2a} & 1 \end{bmatrix}$$

and $\big(\Phi_{\mathcal{W}_{i,s}}\big)_0$ is a continuous two by two matrix function with zero limit at infinity.

Definition 5.1 (See, e.g., [**9**]). An invertible almost periodic matrix function $\Phi \in \mathcal{G}[AP]^{2\times 2}$ admits a *right canonical* AP-*factorization* if

$$\Phi = \Phi^- \Phi^+,$$

where $\Phi^\pm \in \mathcal{G}[AP^\pm]^{2\times 2}$, with $AP^\pm$ denoting the intersection of AP with the non-tangential limits of functions in $H^\infty(\mathbf{C}_\pm)$ (the set of all bounded and analytic functions in $\mathbf{C}_\pm$).

Proposition 5.4. *The matrices* $\left(\Phi_{\mathcal{W}_{i,s}}\right)_l$ *and* $\left(\Phi_{\mathcal{W}_{i,s}}\right)_r$ *admit right canonical* AP-*factorizations given by*

$$\left(\Phi_{\mathcal{W}_{i,s}}\right)_l = \begin{bmatrix} i & 2(-1)^{-s+1/2}\tau_{-2a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 2(-1)^{s-1/2}\tau_{2a} & 1 \end{bmatrix}$$

$$\left(\Phi_{\mathcal{W}_{i,s}}\right)_r = \exp[2\pi i(s-1)] \times \begin{bmatrix} i & -2(-1)^{-s+1/2}\tau_{-2a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ -2(-1)^{s-1/2}\tau_{2a} & 1 \end{bmatrix}.$$

Proof. The result follows from a direct computation of the matrix products; the factor properties are evident. □

The preceding result, based on a certain symmetry of the matricial structure of our operators and on its behavior at infinity, yields explicit representations of the inverses of the two (limit) operators corresponding to +/− infinity (in terms of the factors) and it allows the following main result for the full operator (5.1):

Theorem 5.5. *The operator* $\mathcal{W}_{i,s}$*, defined in* (5.1)*, is a Fredholm operator with zero Fredholm index.*

Proof. Taking profit of the above factorizations, namely those in Proposition 5.4, we can apply Theorem 3.2 of [**9**], see also [**2**], and obtain the statement. For that it remains to compute the *geometric mean values* of the matrices $\left(\Phi_{\mathcal{W}_{i,s}}\right)_l$ and $\left(\Phi_{\mathcal{W}_{i,s}}\right)_r$, denoted respectively by $d\left[\left(\Phi_{\mathcal{W}_{i,s}}\right)_l\right]$ and $d\left[\left(\Phi_{\mathcal{W}_{i,s}}\right)_r\right]$, see [**9**] for the definitions, and evaluate the eigenvalues $\alpha_1 = \alpha_2 = \exp[2\pi i(s-1)]$ of the matrix

$$M = \left(d\left[\left(\Phi_{\mathcal{W}_{i,s}}\right)_l\right]\right)^{-1} d\left[\left(\Phi_{\mathcal{W}_{i,s}}\right)_r\right].$$

The latter can be computed also with the use of the *mean values* of the factors of the elements $\left(\Phi_{\mathcal{W}_{i,s}}\right)_l$ and $\left(\Phi_{\mathcal{W}_{i,s}}\right)_r$, presented in Proposition 5.4.

The Fredholm index is obtained from the formula, see [**9**],

$$\operatorname{Ind}\mathcal{W}_{i,s} = -\operatorname{ind}(\det \mathcal{W}_{i,s}) - \frac{1}{2\pi}\sum_{j=1}^{2}\arg\alpha_j,$$

if we choose $\arg\alpha_j \in (-\pi,\pi)$. □

Remark 5.6. For the particular case of $s=1$, our operator $\mathcal{W}_{i,1}$ has the same regularity properties as the operator

$$\mathcal{W}_1 = r_{\mathbf{R}\to\mathbf{R}_+}\mathcal{F}^{-1}\Phi_{\mathcal{W}_1}\cdot\mathcal{F}l_0 : [L^2(\mathbf{R}_+)]^2 \to [L^2(\mathbf{R}_+)]^2$$

where

$$\Phi_{\mathcal{W}_1} = \begin{bmatrix} 3 & -2i\,\zeta_i^{1/2}\tau_{-2a} \\ 2i\,\zeta_i^{-1/2}\tau_{2a} & 1 \end{bmatrix}$$

Moreover, $\mathcal{W}_1$ is a self-adjoint operator. This immediately yields

$$\operatorname{Ind}\mathcal{W}_1 = \dim\ker\mathcal{W}_1 - \dim\ker\mathcal{W}_1^* = 0$$

and corroborates what was stated above (for the case $s=1$).

Corollary 5.7. *The Wiener-Hopf-Hankel operator $\mathcal{H}$ is a Fredholm operator and its Fredholm index fulfills the identity*

$$\operatorname{Ind}\mathcal{H} + \operatorname{Ind}\mathcal{H}_3 = 0,$$

where $\mathcal{H}_3$ is the operator presented in (5.4).

Proof. The statement is a direct consequence of Proposition 5.2 and Theorem 5.5. □

6. Invertibility of the related operators and explicit solution of the problem. Besides the Fredholm property presented in the last section by the help of our operator relations and the theory of semi-almost periodic matrix functions, we now prove the invertibility of the above operators. This will lead us to the closed form solution of the problem through the constructed operator relations.

We begin by noticing that the elements $\tau_{-2a}\zeta_k^{1/2}$ and $\tau_{2a}\zeta_k^{-1/2}$ are semi-almost periodic functions with negative, respectively positive, Bohr almost periodic indices at infinity [**2, 18**]. In addition, one can factorize $\Phi_{\mathcal{W}_{i,s}}$, see (5.2), in the following way

$$\begin{aligned}\Phi_{\mathcal{W}_{i,s}} &= \begin{bmatrix} 1 & 2(-1)^{-s+1/2}\zeta_k^{1/2}\tau_{-2a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_i^{s-1} & 0 \\ 0 & \zeta_i^{s-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} -1 & 0 \\ 2(-1)^{s-1/2}\zeta_k^{-1/2}\tau_{2a} & 1 \end{bmatrix}. \end{aligned} \tag{6.1}$$

Since the middle matrix function on the right-hand side of (6.1) admits a canonical generalized factorization, we obtain a *canonical generalized* SAP-*factorization* [**2, 4**] of $\Phi_{\mathcal{W}_{i,s}}$ in the form

$$\begin{aligned}\Phi_{\mathcal{W}_{i,s}} &= \left(\begin{bmatrix} 1 & 2(-1)^{-s+1/2}\zeta_k^{1/2}\tau_{-2a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_-^{s-1} & 0 \\ 0 & \lambda_-^{s-1} \end{bmatrix} \right) \\ &\quad \times \left(\begin{bmatrix} \lambda_+^{-s+1} & 0 \\ 0 & \lambda_+^{-s+1} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2(-1)^{s-1/2}\zeta_k^{-1/2}\tau_{2a} & 1 \end{bmatrix} \right) \\ &= \left(\Phi_{\mathcal{W}_{i,s}}\right)_- \left(\Phi_{\mathcal{W}_{i,s}}\right)_+ . \end{aligned} \tag{6.2}$$

Please observe that although $\zeta_k^{\pm 1/2}$ are not holomorphically extendible into the upper or lower half-planes, when they are multiplied with the remaining exponential functions $\tau_{\mp 2a}$ we end up with the appropriate properties for the factors of the above generalized SAP-factorization. This is due to the fact that, as mentioned above, those functions have negative and positive Bohr almost periodic indices, respectively, cf., e.g., Theorem 1 in [**18**].

Formula (6.2) immediately allows the inversion of $\mathcal{W}_{i,s}$ in the following explicit form.

Theorem 6.1. *The operator* $\mathcal{W}_{i,s}$*, see* (5.1)*, is invertible by*

$$\mathcal{W}_{i,s}^{-1} = P_+\mathcal{F}^{-1}\left(\Phi_{\mathcal{W}_{i,s}}\right)_+^{-1} \cdot \mathcal{F}P_+\mathcal{F}^{-1}\left(\Phi_{\mathcal{W}_{i,s}}\right)_-^{-1} \cdot \mathcal{F}l_0 : \left[L^2(\mathbf{R}_+)\right]^2 \to \left[L_+^2(\mathbf{R})\right]^2.$$

We are now in a position to provide the explicit solution of Problem $\mathcal{P}$.

Theorem 6.2. *Problem $\mathcal{P}$ is well-posed and its solution is given by* (3.3)–(3.5), *where g_+ and f are obtained from* (3.20)–(3.21), *φ is determined by*

$$\begin{aligned}\varphi &= \mathcal{H}^{-1}h \\ &= R\left(F_4^{-1}\begin{bmatrix}\mathcal{W}_{i,s}^{-1} & 0 \\ 0 & I_{[L_-^2(\mathbf{R})]^2}\end{bmatrix}E_4^{-1}\right)h,\end{aligned}$$

and where h is given by (3.18) *and R denotes the restriction to the first block of the* (*inverse of the*) *matrix in the left-hand side of* (5.3).

Proof. The statement is a combination of Theorem 3.5, Corollary 3.7, the operator relations presented in Section 4, the identity (5.3) and Theorem 6.1. □

7. Final remarks and conclusions. We note that arguments of strong ellipticity like those used in [**5**] cannot be applied for the present case essentially because the "middle factors" are not of this kind even when we choose convenient symmetric lateral factors.

We also observe that procedures for obtaining explicit factorizations, like (6.2), of semi-almost periodic matrix functions are still unknown for most classes of matrix functions. In fact, nowadays the research on factorization theory of almost periodic matrix functions is a field of research by itself and a considerable amount of problems on this area are still open, see [**2**].

In our case, this kind of factorization together with all the presented operator relations allowed the solution of the Problem $\mathcal{P}$ in closed analytic form for the above considered spaces. In particular, due to the continuity of the used operators, we notice that the obtained solution shows continuous dependence on the given data. Such results are also relevant for a qualitative study of the possible solutions in dependence of the smoothness space orders [**13**] as well as for numerical treatments of solutions, of corresponding noncanonical problems, near geometrically critical points.

Acknowledgments. The authors would like to acknowledge the financial support of *Centro de Matemática e Aplicações* of Instituto Superior Técnico and *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro, through *Programa Operacional "Ciência, Tecnologia, Inovação"* (POCTI) of the *Fundação para a Ciência e a Tecnologia* (FCT), co-financed by the European Community fund FEDER.

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