JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 14, Number 2, Summer 2002

# ON SOLVABILITY OF URYSOHN-VOLTERRA EQUATIONS WITH HYSTERESIS IN WEIGHTED SPACES

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ABSTRACT. This paper concerns the unique solvability of the nonlinear integral equations of the second kind with hysteresis of the form

$$y(t) = f(t) + \int_{-\infty}^{t} F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds, \quad 0 \le t \le T$$

in weighted spaces. Also we have treated the case of nonlinear integral equations of the first kind with hysteresis.

**1. Introduction.** There are various ways in which hysteretic behavior of a system can be related to an integral equation. One particular setting, which has been studied by many authors, is using a convolution integral to describe the memory of a given system. The memory is characterized by the convolution kernel and thus the evolution depends on all past values of the state; typically, as one goes back in time, the influence of the past values of the present evolution decreases. There are, however, several hysteretic phenomena which cannot be treated by this method; in particular, it cannot be used to describe a hysteretic system whose hysteresis loops do not depend on the speed with which they are traversed. This property is called rate independence and is inherently nonlinear. In [**2**]–[**4**] we discuss systems where a Urysohn-Volterra integral equation is coupled to a rate independent hysteretic process. For more information about hysteresis, for instance, see [**1**], [**6**], [**9**].

In this paper we consider a nonlinear integral equation of the second kind with hysteresis, namely,

$$y(t) = f(t) + \int_{-\infty}^{t} F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds, \quad 0 \le t \le T, \tag{1.1}$$

AMS *Mathematics Subject Classification.* 45G10, 47H10, 47H30.
*Key words and phrases.* Existence and uniqueness, integral equations, Urysohn-Volterra, weighted spaces.

where $\mathcal{W}$ and $S$ denote a hysteresis operator and a superposition operator of the form $S[y](t) = g(y(t))$, respectively. More precisely, we assume $f$ and $F$ to be given $n$-vector valued functions, while $y$ is the unknown $n$-vector function. Equation (1.1) is known as a Urysohn-Volterra equation (see [**5**]).

Equation (1.1) is history-dependent so, in general, this problem requires that one give an initial condition on $(-\infty, 0]$ and it may then be treated with the techniques of standard Urysohn-Volterra equations with hysteresis, see Darwish [**2**]–[**4**]. Therefore, the nonuniqueness of solutions of equation (1.1) is an intrinsic feature which occur even in the case of linear Volterra integral equations without hysteresis (see [**7**]).

The main object of this paper is to give sufficient conditions in order to guarantee the existence of the unique solutions of equation (1.1) in the weighted space $C_w$ which were introduced in [**8**] and references therein. Also we have treated the case of nonlinear integral equations of the first kind with hysteresis.

**2. Preliminaries.** Let $I \subset \mathbf{R}$ and consider a weight function $w : I \to \mathbf{R}_+$ be continuous and nondecreasing, $\mathbf{R}_+ = (0, +\infty)$. Define $C_w \equiv C_w(I; \mathbf{R}^n) := \{\phi \mid \phi : I \to \mathbf{R}^n \text{ continuous}\}$ with the following norm

$$\|\phi\|_w = \sup_{t\in I} \frac{\|\phi(t)\|_{\mathbf{R}^n}}{w(t)}, \quad \forall\, \phi \in C_w$$

to be the underlying space for our problem. Then $C_w$ is a Banach space.

**Definition 2.1** (Rate independent functionals). A functional $\mathcal{H} : C([0,T] : \mathbf{R}^n) \to \mathbf{R}$ is called rate independent if and only if $\mathcal{H}[u \circ \psi] = \mathcal{H}[u]$ holds for all $u \in C([0,T]; \mathbf{R}^n)$ and all admissible time transformations, i.e., continuous increasing functions $\psi : [0,T] \to [0,T]$ satisfying $\psi(0) = 0$ and $\psi(T) = T$.

**Definition 2.2** (Volterra-operator). Let $X$ be a Banach space. An operator $F : C([0,T]; X) \to C([0,T])$ is called a Volterra-operator if, for all $s \in [0,T]$ and for all $u, v \in C([0,T]; X)$ with $u(\sigma) = v(\sigma)$ for all $\sigma \in [0,s]$, $(Fu)(\sigma) = (Fv)(\sigma)$ for all $\sigma \in [0,s]$.

Recall that an operator $\mathcal{W} : C(I;\mathbf{R}^n) \to C(I)$ is *hysteresis* if it has both the Volterra property and the rate independence property. For more information about the hysteresis operator, see [**1**] and the references therein.

*Remark* 2.1. By definition hysteresis operators possess the Volterra property. This is actually what is needed here; the rate independence itself does not play any role.

**Lemma 2.1** [**4**]. *Let $F : C(I;\mathbf{R}^n) \to C(I)$ be a Volterra operator. Assume that $F$ is Lipschitz continuous on every bounded subset of $C(I;\mathbf{R}^n)$. Then for every $C > 0$, there exists $L > 0$ such that*

$$|(Fy_2)(s) - (Fy_1)(s)| \le L \sup_{\substack{\tau\in I\\ \tau\le s}} \|y_2(\tau) - y_1(\tau)\|_{\mathbf{R}^n}, \tag{2.1}$$

*holds for all $s \in I$ and all $y_i \in C(I;\mathbf{R}^n)$ with $\|y_i\| \le C$, $i = 1, 2$.*

**3. The unique solvability.** Let $I$ be a (bounded or unbounded) closed subinterval of $\mathbf{R}$ and define $C_w$ as above. To facilitate our discussion, let us first state the following assumptions:

(H1) $f \in C_w$,

(H2) $F : I \times I \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ continuous and

$$F(t, s, 0, 0) = 0 \quad \text{for all } (t, s) \in I^2, \tag{3.1}$$

(H3) There exists a measurable function $m(t, s)$ defined on $I^2$, such that

$$\|F(t, s, y_2, w_2) - F(t, s, y_1, w_1)\| \le m(t, s)\{\|y_2 - y_1\| + |w_2 - w_1|\},$$

(H4) $\mathcal{W} \circ S : C(I;\mathbf{R}^n) \to C(I)$ satisfies the Lipschitz condition

$$|\mathcal{W}[S[y_2]](s) - \mathcal{W}[S[y_1]](s)| \le L \sup_{s\in I} \|y_2(s) - y_1(s)\|_{\mathbf{R}^n},$$

for some $L > 0$, and

$$\mathcal{W}[S[0]](s) = 0 \quad \text{in } 0 \le s \le T, \tag{3.2}$$

$$E = (1+L)\sup_{t\in I}\int_I \frac{w(s)}{w(t)}\, m(t,s)\, ds \le \frac{1}{2} \tag{H5}$$

and

$$\tilde{E} = (1+L)\int_I \sup_{t\in I}\left\{\frac{w(s)}{w(t)}\, m(t,s)\right\} ds \le \delta < 1. \tag{H6}$$

**Theorem 3.1** (Existence). *Let assumptions* (H1)–(H5) *be satisfied. Then the equation*

$$y(t) = f(t) + \int_I F(t,s,y(s),\mathcal{W}[S[y]](s))\, ds, \quad t \in I, \tag{3.3}$$

*has at least one solution in* $C_w$.

*Proof.* Let $D = \{y \in C_w : \|y - f\|_w \le b\}$ be a closed subset of $C_w$, where $b$ is a number such that $\|f\|_w \le b$, and define the operator $\mathcal{F}$ on $D$ by

$$(\mathcal{F}y)(t) = f(t) + \int_I F(t,s,y(s),\mathcal{W}[S[y]](s))\, ds, \quad t \in I, \tag{3.4}$$

which enjoys the property that any fixed point of $\mathcal{F}$ is a solution of (3.3). We shall prove that

(i) $\mathcal{F}$ maps $D$ into itself,

(ii) $\mathcal{F}$ is a contraction mapping on $D$.

To show (i) we have the estimate

(3.5)

$$\begin{aligned}\|\mathcal{F}y - f\|_w \le &\left\|\int_I [F(t,s,y(s),\mathcal{W}[S[y]](s)) - F(t,s,f(s),\mathcal{W}[S[f]](s))]\, ds\right\|_w \\ &+ \left\|\int_I F(t,s,f(s),\mathcal{W}[S[f]](s))\, ds\right\|_w.\end{aligned}$$

By the aid of (H2)–(H4), we have

$$
\begin{aligned}
&\Big\| \int_I F(t,s,f(s),\mathcal{W}[S[f]](s))\,ds \Big\|_w \\
&\qquad = \sup_{t\in I} \frac{1}{w(t)} \Big\| \int_I F(t,s,f(s),\mathcal{W}[S[f]](s))\,ds \Big\|_{\mathbf{R}^n} \\
&\qquad \le \sup_{t\in I} \int_I \frac{1}{w(t)} \|F(t,s,f(s),\mathcal{W}[S[f]](s))\|_{\mathbf{R}^n}\,ds \\
&\qquad \le (1+L) \sup_{t\in I} \int_I \frac{1}{w(t)}\, m(t,s) \sup_{s\in I} \|f(s)\|_{\mathbf{R}^n}\,ds \\
&\qquad \le (1+L) \Big\{ \sup_{t\in I} \int_I \frac{w(s)}{w(t)}\, m(t,s)\,ds \Big\} \|f\|_w \\
&\qquad \le \frac{1}{2}\, b.
\end{aligned} \tag{3.6}
$$

Also we have the estimate

$$
\begin{aligned}
&\Big\| \int_I [F(t,s,y(s),\mathcal{W}[S[y]](s)) - F(t,s,f(s),\mathcal{W}[S[f]](s))]\,ds \Big\|_w \\
&\quad \le \sup_{t\in I} \int_I \frac{1}{w(t)} m\,(t,s) [\|y(s) - f(s)\|_{\mathbf{R}^n} + |\mathcal{W}[S[y]](s) - \mathcal{W}[S[f]](s)|]\,ds \\
&\quad \le (1+L) \sup_{t\in I} \int_I \frac{1}{w(t)}\, m(t,s) \sup_{s\in I} \|y(s) - f(s)\|_{\mathbf{R}^n}\,ds \\
&\quad \le (1+L) \sup_{t\in I} \int_I \frac{w(s)}{w(t)}\, m(t,s)\,ds \|y-f\|_w \\
&\quad \le \frac{1}{2}\, b.
\end{aligned} \tag{3.7}
$$

From (3.5)–(3.7), we obtain

$$
\|\mathcal{F}y - f\|_w \le b. \tag{3.8}
$$

Consequently, $\mathcal{F}$ maps $D$ into itself. To prove (ii), we have

$$
\begin{aligned}
\|\mathcal{F}y_2 - \mathcal{F}y_1\|_w &\le (1+L) \sup_{t\in I} \int_I \frac{w(s)}{w(t)}\, m(t,s)\,ds \|y_2 - y_1\|_w \\
&\le \frac{1}{2} \|y_2 - y_1\|_w.
\end{aligned} \tag{3.9}
$$

Then $\mathcal{F}$ is a contraction on $D$. Therefore, by the contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $D$, i.e., the integral equation (3.3) has a unique solution in $D$. However this does not prove the uniqueness in $C_w$. In the next theorem we will discuss the uniqueness in $C_w$.

**Theorem 3.2** (Uniqueness). *Let assumptions* (H1)–(H4) *and* (H6) *be satisfied. Then equation* (3.3) *has at most one solution in* $C_w$.

*Proof.* Let $y$ and $\tilde{y}$ be two solutions of equation (3.3) in $C_w$. Then

$$\|y(t)-\tilde{y}(t)\|_{\mathbf{R}^n} \leq (1+L)\int_I m(t,s)\|y(s)-\tilde{y}(s)\|\,ds. \tag{3.10}$$

Let $z(t) = \|y(t)-\tilde{y}(t)\|_{\mathbf{R}^n}/w(t)$ and $p(s) = (1+L)\sup_{t\in I}\{(w(s)/w(t))|m(t,s)|\}$, $s \in I$. It is clear, by the aid of (H6), that $p(.)$ is integrable on $I$. Also the function $z(.)$ is bounded. Consequently, (3.10) implies that

$$z(t) \leq \int_I p(s)z(s)\,ds. \tag{3.11}$$

The iteration arguments of (3.11) implies, for any positive integer $n$, that

$$z(t) \leq \Big(\int_I p(r)\,dr\Big)^n \int_I p(s)z(s)\,ds, \tag{3.12}$$

which by assumption (H6) gives

$$z(t) \leq \delta^n \int_I p(s)z(s)\,ds. \tag{3.13}$$

Thus for all $t \in I$

$$z(t) \leq \sup_{z\in I} z(s)\int_I p(s)\,ds \leq \delta \sup_{z\in I} z(s)$$

so that

$$\sup_{t\in I} z(t) \leq \delta \sup_{z\in I} z(s). \tag{3.14}$$

Since $\delta < 1$ it follows that $\sup_{z\in I} z(s) = 0$. This proves the uniqueness of the solution of equation (3.3) in $C_w$.

We gather together the results of Theorem 3.1 and Theorem 3.2; keeping in mind $E \leq \tilde{E}$, we obtain the sufficient conditions for existence and uniqueness.

**Theorem 3.3** (Existence and uniqueness). *Let assumptions* (H1)–(H5) *be satisfied and, in addition,* $\tilde{E} \leq 1/2$. *Then equation* (3.3) *has a unique solution in* $C_w$.

**4. Equations of the second kind.** In this section we restrict ourselves to the unique solvability of the Urysohn-Volterra equation of the second kind with hysteresis, namely,

$$y(t) = f(t) + \int_a^t F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds, \tag{4.1}$$

where $a$ can be finite or $-\infty$. Let $I = [a, \infty)$ or $\mathbf{R}$. In what follows we shall need the following fact, see [**8**]:

Let $g : [a, t] \to \mathbf{R}^n$ be an absolutely integrable function, then

(4.2)
$$\int_a^t g(t_1)\, dt_1 \int_a^{t_1} g(t_2)\, dt_2 \cdots \int_a^{t_{n-1}} g(t_n)\, dt_n = \frac{1}{n!}\Big(\int_a^t g(s)\, ds\Big)^n.$$

Now we are in a position to state and prove our main result in this section.

**Theorem 4.1.** *Let* $f \in C_w(I; \mathbf{R}^n)$. *Assume that* $F$ *satisfies all the assumptions as in Theorem* 3.1 *on* $a \leq s \leq t$, $t \in I$, *and, instead of* (H5), *let us assume that*

$$0 < \tilde{E} = (1 + L) \int_a^t \sup_{t>s} \Big\{ \frac{w(s)}{w(t)}\, m(t, s) \Big\}\, ds < \infty. \tag{4.3}$$

*In addition assume* (H6) *holds. Then equation* (4.1) *has a unique solution in* $C_w(I; \mathbf{R}^n)$.

*Proof*. Define a sequence of successive approximations $\{y_n(t)\}$ by

$$\begin{aligned} y_0(t) &= f(t), \\ y_{n+1}(t) &= f(t) + \int_a^t F(t,s,y_n(s),\mathcal{W}[S[y_n]](s))\,ds. \end{aligned} \tag{4.4}$$

Let $p(s) = (1+L)\sup_{t>s}\{(w(s)/w(t))m(t,s)\}$, $s \in I$. Thus (4.3) takes the form $\int_I p(s)\,ds < \infty$. We have the estimate

$$\begin{aligned} \frac{\|y_1(t)-y_0(t)\|}{w(t)} &\le (1+L)\int_a^t \sup_{t>s}\Big\{\frac{w(s)}{w(t)}\,m(t,s)\Big\}\,ds\|y_0\|_w \\ &= \int_a^t p(s)\,ds\|y_0\|_w. \end{aligned}$$

In general we have

$$\begin{aligned} \frac{\|y_{n+1}(t)-y_n(t)\|}{w(t)} &\le \|y_0\|_w \int_a^t p(t_1)\,dt_1 \int_a^{t_1} p(t_2)\,dt_2 \cdots \int_a^{t_{n-1}} p(t_n)\,dt_n \\ &\le \frac{\|y_0\|_w}{n!}\Big(\int_a^t p(s)\,ds\Big)^n, \end{aligned}$$

where we have used (4.2). Taking the supremum over $I$, we obtain

$$\begin{aligned} \|y_{n+1}-y_n\|_w &\le \frac{\|y_0\|_w}{n!}\Big(\int_I p(s)\,ds\Big)^n \\ &\le \frac{\tilde{E}^n}{n!}\|y_0\|_w. \end{aligned}$$

Therefore the sequence $\{y_n(t)\}$ converges in norm in $C_w(I;\mathbf{R}^n)$ to a function $y(t)$. To see that $y(t)$ is a solution of equation (4.1), let $J$ be an arbitrary compact subinterval of $I$ and let $M = \max_J w(t)$. The sequence $\{y_n(t)\}$ converges uniformly to $y(t)$ on $J$, passing the limit in (4.4). Then $y(t)$ is a solution of equation (4.1) on any compact subinterval of $I$ and hence on all of $I$.

To prove the uniqueness in $C_w(I;\mathbf{R}^n)$, let $y(t)$ and $\tilde{y}(t)$ be any two solutions of equation (4.1) in $C_w(I;\mathbf{R}^n)$. Then

$$y(t)-\tilde{y}(t) = \int_a^t [F(t,s,y(s),\mathcal{W}[S[y]](s)) - F(t,s,\tilde{y}(s),\mathcal{W}[S[\tilde{y}]](s))]\,ds,$$

which implies

$$\|y(t) - \tilde{y}(t)\| \le (1 + L) \int_I m(t,s) \sup_{s \in I} \|y(s) - \tilde{y}(s)\| \, ds.$$

Define $z(t) = \|y(t) - \tilde{y}(t)\|/w(t)$. Then

$$z(t) \le \int_a^t p(s) z(s) \, ds, \tag{4.5}$$

which implies, by the previously used inductive procedure

$$z(t) \le \frac{1}{n!} \Big( \int_a^t p(s) \, ds \Big)^n \sup_{z \in I} z(s) \le \frac{\tilde{E}^n}{n!} \sup_{z \in I} z(s). \tag{4.6}$$

If we let $n \to \infty$, it follows that $z(t) = 0$ for all $t \in I$. This completes the proof.

**5. Equations of the first kind.** In this section we will extend the results in the above section to the Urysohn-Volterra equation of the first kind with hysteresis, namely,

$$\int_a^t F(t, s, y(s), \mathcal{W}[S[y]](s)) \, ds = f(t), \tag{5.1}$$

where $a \ge -\infty$. Let $w$ be a weight function on $I = [a, \infty)$ or $\mathbf{R}$. Let us first state the following assumptions:

(h1) $f \in C_w(\mathbf{R}; \mathbf{R}^n)$ such that $f(a) = 0$ and $f' \in C_w(\mathbf{R}; \mathbf{R}^n)$,

(h2) $F(t, s, y, w)$ and $(\partial/\partial t) F(t, s, y, w)$ are continuous for $a \le s \le t$, $t \in I$, $y \in \mathbf{R}^n$ and $w \in \mathbf{R}$; moreover, there exists a measurable function $m(t, s)$ such that

$$\begin{aligned} \Big\| \frac{\partial}{\partial t} F(t, s, y_2, w_2) - \frac{\partial}{\partial t} F(t, s, y_1, w_1) \Big\| & \\ \le m(t, s) \{ \|y_2 - y_1\| + |w_2 - w_1| \}&, \end{aligned} \tag{5.2}$$

(h3) the equation $F(t, t, y, w) = z$ has a unique solution $y$ for all $z \in \mathbf{R}^n$, $w \in \mathbf{R}$ and $t \in I$, (h4) there exists a $\delta > 0$ such that

$$\|F(t,t,y_2,w_2) - F(t,t,y_1,w_1)\| \geq \delta(\|y_2 - y_1\| + |w_2 - w_1|), \tag{5.3}$$

for all $y_i \in \mathbf{R}^n$, $w_i \in \mathbf{R}$, $i = 1, 2$ and all $t \in I$,

(h5) $\mathcal{W} \circ S : C(I; \mathbf{R}^n) \to C(I)$ satisfies the Lipschitz condition

$$|\mathcal{W}[S[y_2]](t) - \mathcal{W}[S[y_1]](t)| \leq L \sup_{t \in I} \|y_2(s) - y_1(s)\|_{\mathbf{R}^n},$$

for some $L > 0$, and

$$\tilde{E} = (1 + L) \int_I \sup_{t>s} \left\{ \frac{w(s)}{w(t)}\, m(t,s) \right\} ds < \infty. \tag{h6}$$

Differentiating equation (5.1) with respect to $t$, we obtain

$$\begin{aligned} &F(t,t,y(t),\mathcal{W}[S[y]](t)) \\ &\qquad + \int_a^t \frac{\partial}{\partial t} F(t,s,y(s),\mathcal{W}[S[y]](s))\, ds = f'(t). \end{aligned} \tag{5.4}$$

Let us define now a sequence $\{y_n(.)\}$ in $C_w(\mathbf{R}; \mathbf{R}^n)$ as follows.

Let $y_0(.) \in C_w(\mathbf{R}; \mathbf{R}^n)$ be arbitrary and

$$\begin{aligned} &F(t,t,y(t),\mathcal{W}[S[y_{n+1}]](t)) \\ &\qquad + \int_a^t \frac{\partial}{\partial t} F(t,s,y(s),\mathcal{W}[S[y_n]](s))\, ds = f'(t). \end{aligned} \tag{5.5}$$

By the aid of (h3), the function $y_{n+1}(.)$ is well defined. To prove that $y_{n+1}$ remains in $C_w(\mathbf{R}; \mathbf{R}^n)$ and that the sequence converges in $C_w(\mathbf{R}; \mathbf{R}^n)$, we have the estimate

$$\begin{aligned} &\|F(t,t,y(t),\mathcal{W}[S[y_{n+1}]](t)) - F(t,t,f(t),\mathcal{W}[S[y_n]](t))\| \\ &\qquad \leq \int_a^t \Big\| \frac{\partial}{\partial t} F(t,s,y(s),\mathcal{W}[S[y_n]](s)) \\ &\qquad\qquad - \frac{\partial}{\partial t} F(t,s,y(s),\mathcal{W}[S[y_{n-1}]](s)) \Big\|\, ds \\ &\qquad \leq (1 + L) \int_0^t m(t,s) \sup_{s \in I} \|y_n(s) - y_{n-1}(s)\|\, ds. \end{aligned} \tag{5.6}$$

But from (h4), we have

$$\begin{aligned}\frac{\|y_{n+1}(t)-y_n(t)\|}{w(t)} \\ \leq \frac{(1+L)}{\delta}\int_a^t \sup_{t>s}\Big\{m(t,s)\frac{w(s)}{w(t)}\Big\}\frac{\|y_n(s)-y_{n-1}(s)\|}{w(s)}\,ds.\end{aligned}$$

Define

$$z_{n+1}(t)=\|y_{n+1}(t)-y_n(t)\|/w(t)$$

and

$$p(s)=\sup_{t>s}\{m(t,s)(w(s)/w(t))\},\quad s\in I.$$

Then

$$z_{n+1}(t)\leq\int_a^t p(s)z_n(s)\,ds.$$

By inductive arguments, it follows that

$$z_{n+1}(t)\leq\frac{1}{n!}\Big(\int_a^t p(s)\,ds\Big)^n\tilde{E}\|y_0\|_w\leq\frac{\tilde{E}^{n+1}}{n!}\|y_0\|_w$$

or

$$\|y_{n+1}-y_n\|_w\leq\frac{\tilde{E}^{n+1}}{n!}\|y_0\|_w.$$

Thus $y_{n+1}(.)\in C_w(\mathbf{R};\mathbf{R}^n)$ for all $n\geq 0$ and the sequence $\{y_{n+1}(.)\}$ is convergent in $C_w(\mathbf{R};\mathbf{R}^n)$. Therefore equation (5.1) has a solution in $C_w(\mathbf{R};\mathbf{R}^n)$. The proof of the uniqueness goes as in Theorem 4.1.

**Applications.** In this section, we apply the existence and uniqueness theorems established in the previous section to the following convolution equation

$$y(t)=f(t)+\int_{-\infty}^t k(t-s)\,F(s,y(s),\mathcal{W}[S[y]](s))\,ds. \tag{6.1}$$

Here, we assume that $f$, $k$ and $F$ are continuous, and there exists a measurable function $m_0$ such that

$$\|F(s,y_2,w_2)-F(s,y_1,w_1)\|\leq m_0(s)\{\|y_2-y_1\|+|w_2-w_1|\}. \tag{6.2}$$

The following theorem and its corollary are applications of Theorem 4.1.

**Theorem 6.1.** *Let* $f \in C_w(\mathbf{R}; \mathbf{R}^n)$ *and* $F$ *satisfies* (6.1). *In addition*

$$\int_{\mathbf{R}} m_0(s) \sup_{t>s} \left\{ \frac{w(s)}{w(t)}\, k(t-s) \right\} ds < \infty.$$

*Then equation* (6.1) *has a unique solution in* $C_w(\mathbf{R}; \mathbf{R}^n)$.

If the weight function is monotone nondecreasing, we have the following interesting special case

**Corollary 6.1.** *Let the same assumptions as in Theorem* 6.1 *be satisfied. In addition assume that the weight function* $w(.)$ *is monotone nondecreasing. Then, if*

$$\int_{\mathbf{R}} m_0(s) \sup_{t>s} \{|k(t-s)|\}\, ds < \infty,$$

*equation* (6.1) *has a unique solution in* $C_w(\mathbf{R}; \mathbf{R}^n)$.

*Proof.* Notice that $(w(s)/w(t)) \leq 1$ for all $s < t$. Hence the proof follows.

**Acknowledgments.** The author thanks the editor and the referee for their valuable corrections.

## REFERENCES

**1.** M. Brokate and J. Sprekels, *Hysteresis and phase transitions*, Appl. Math. Sci. **121**, Springer-Verlag, New York, 1996.

**2.** M.A. Darwish, *Global existence and uniqueness of continuous solution of Urysohn-Volterra equation*, P.U.M.A. Pure Math. Appl. **1** (1999), 1–6.

**3.** ———, *On a system of nonlinear integral equations with hysteresis*, Korean J. Comp. Appl. Math. **6** (1999), 305–313.

**4.** ———, *Hysteresis in Urysohn-Volterra systems*, Electron. J. Qualitative Theory Differential Equations **4** (1999), 1–8.

**5.** M.A. Golberg, *Solution methods for integral equations theory and applications*, Math. Concepts Meth. Sci. Engrg. **18**, Plenum Press, New York, 1979.

**6.** M.A. Krasnosel'skii and A.V. Pokrovskii, *Systems with hysteresis*, Springer-Verlag, Heidelberg, 1989; Nauka, Moscow, 1983 (in Russian).

**7.** C.E. Love, *Singular integral equations of Volterra type*, Trans. Amer. Math. Soc. **15** (1914), 467–476.

**8.** P. Rejto and M. Taboada, *Unique solvability of nonlinear Volterra equations in weighted spaces*, J. Math. Anal. Appl. **167** (1992), 368–381.

**9.** A. Visintin, *Differential models of hysteresis*, Springer-Verlag, New York, 1994.

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