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A QUENCHING PROBLEM FOR THE HEAT EQUATION

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ABSTRACT. A nonlinear partial differential equation of parabolic type is investigated for quenching behavior. Quenching occurs when the solution of the equation remains bounded while the first order time derivative becomes unbounded in finite time. We examine a quenching problem for the heat equation in a one-dimensional strip of finite width with special nonlinear boundary conditions. Specifically, the boundary condition at one end represents nonlinear absorption of heat and the boundary condition at the other end represents nonlinear heat loss. The interactions between the diffusion, the heat behavior at the boundaries and the length of the domain are studied to determine conditions under which the phenomenon of quenching will or will not occur.

1. Background. The study of specialized behavior for nonlinear parabolic partial differential equations has been an active area of research for decades. Two types of specialized behavior, solution explosion and solution quenching, have been of particular interest more recently (see [**2**], [**3**], [**10**], [**11**], [**13**]). In explosion problems, the solution becomes unbounded in finite time. In quenching problems, the solution remains bounded while the first order time derivative becomes unbounded in finite time.

The thrust of the research is generally to establish sufficient conditions for the existence of a unique solution that exhibits either explosion or quenching behavior. Other research characterizes the solution by describing, for example, the asymptotic behavior in certain key limits. The body of research on these types of problems typically formulates them as parabolic (sometimes hyperbolic) partial differential equations with nonlinear source terms or boundary conditions (see [**1**]–[**4**], [**10**], [**11**]). In some cases, the analysis proceeds best when the problem is recast in terms of a nonlinear integral equation (see [**6**], [**8**], [**9**], [**12**]–[**14**]).

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The analysis for explosion of nonlinear Volterra equations has quite recently been modified and expanded to accommodate issues of quenching phenomena in related problems (see [**6**], [**12**], [**14**]). Because of the intrinsic mathematical similarities between explosion problems and quenching problems, this is a natural evolution of the theory of nonlinear Volterra integral equations. The results given here represent another contribution to this expanding body of work. In section (7), the results of this paper will be analyzed in the context of existing literature.

2. Introduction. We consider a quenching problem for the heat equation in a one-dimensional strip of finite width with nonlinear boundary conditions of a particular type. We examine

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad 0 < x < l,\ t > 0 \tag{2.1}$$

$$v(x,0) = v_0(x), \quad 0 \le v_0(x) < 1 \tag{2.2}$$

$$\frac{\partial v}{\partial x}(0,t) = f[v] \tag{2.3}$$

$$\frac{\partial v}{\partial x}(l,t) = g[v] \tag{2.4}$$

where

$$f[v] > 0, \quad \frac{\partial f}{\partial v}[v] > 0 \quad \text{for } v > 0 \tag{2.5}$$

$$g[v] > 0, \quad \frac{\partial g}{\partial v}[v] > 0, \quad \frac{\partial^2 g}{\partial v^2}[v] > 0 \quad \text{for } v > 0 \tag{2.6}$$

and

$$g[v] \to +\infty \quad \text{as } v \to 1^-. \tag{2.7}$$

Physically, these boundary conditions represent cooling of the strip at $x = 0$ and heating at $x = l$. The quenching value is $v(x,\hat{t}\,) = 1$ as $t \to \hat{t} < \infty$. Note that (2.2) guarantees that the solution starts below the quenching value. We will establish conditions for the existence and non-existence of a unique quenching solution.

3. Conversion to an integral equation. Our solution approach involves finding lower and upper bound estimates on the critical quenching time. Using the standard Green's function for this problem yields a system that is not amenable to our techniques for establishing an upper bound. Instead, we will consider an alternative Green's function and use the appropriate Green's identity to express the solution of our problem.

Recall that $f[v]$ is the cooling nonlinearity at $x = 0$ and that $v \to 1$ is the quenching value. Assume that there exist an m and an M such that

$$0 < m \le f'(v) \le M < \infty \quad \text{and} \quad f(v) \le Mv \quad \text{for } 0 \le v < 1. \tag{3.1}$$

Consider the following Green's function problem:

$$G_t - G_{xx} = \delta(x-\xi)\delta(t-s) \tag{3.2}$$

$$G\,|_{t=s^-} = 0 \tag{3.3}$$

$$G_x\,|_{x=0} = MG\,|_{x=0} \tag{3.4}$$

$$G_x\,|_{x=l} = 0. \tag{3.5}$$

The Green's function is given by

$$G(x,t \mid \xi, s) = H(t-s)\,\Sigma_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\int_0^l \phi_n^2(s)ds}\, e^{-\lambda_n(t-s)} \tag{3.6}$$

where

$$\phi_n(x) = \cos\left[\sqrt{\lambda_n}\,(x-l)\right] \tag{3.7}$$

and where λ_n satisfies

$$\frac{M}{\sqrt{\lambda_n}} = \tan\left(\sqrt{\lambda_n}\,l\right). \tag{3.8}$$

The solution to the problem (2.1)–(2.6) for $v(x,t)$ can be expressed in terms of $G(x,t|\xi,s)$:

$$\begin{aligned} v(x,t) &= \int_0^l G(x,t \mid \xi, 0)v_0(\xi)\,d\xi + \int_0^t G(x,t \mid l, s)g[v(l,s)]\,ds \\ &\quad - \int_0^t G(x,t \mid 0,s)\big\{f[v(0,s)] - Mv(0,s)\big\}\,ds. \end{aligned} \tag{3.9}$$

We will consider only the temperature at $x = 0$ and $x = l$. The maximum principle guarantees that quenching can not occur elsewhere. In particular, it can be shown that if quenching occurs, it will occur at $x = l$ (see [**7**], [**14**]). This makes intuitive sense as well, given the physical description of the scenario under consideration. Heating at $x = l$ and cooling at $x = 0$ implies that if quenching is to occur, it will occur on the heated end. In such a case, the solution will remain bounded while the first order time derivative $(\partial v/\partial t)(l,t)$ becomes infinite in finite time. Let

$$u_1(t) \equiv v(0,t), \qquad u_2(t) \equiv v(l,t) \tag{3.10}$$

and

$$h_1(t) \equiv \int_0^l G(0,t \mid \xi,0)v_0(\xi)\,d\xi \tag{3.11}$$

$$h_2(t) \equiv \int_0^l G(l,t \mid \xi,0)v_0(\xi)\,d\xi. \tag{3.12}$$

The equations for $v(x,t)$ at $x = 0$ and $x = l$ are, respectively:

$$\begin{aligned} u_1(t) &= h_1(t) + \int_0^t G(0,t \mid l,s)g[u_2(s)]\,ds \\ &\quad + \int_0^t G(0,t \mid 0,s)\big\{Mu_1(s) - f[u_1(s)]\big\}\,ds \end{aligned} \tag{3.13}$$

$$\begin{aligned} u_2(t) &= h_2(t) + \int_0^t G(l,t|l,s)g[u_2(s)]\,ds \\ &\quad + \int_0^t G(l,t|0,s)\big\{Mu_1(s) - f[u_1(s)]\big\}\,ds. \end{aligned} \tag{3.14}$$

Introduce the following notation:

$$k_{ab} \equiv G(a,t \mid b,s) \equiv G_{ab}(t-s), \tag{3.15}$$

$$I_{ab}(t) \equiv \int_0^t k_{ab}(s)ds. \tag{3.16}$$

Our goal is to examine the system of nonlinear Volterra integral equations (3.13) and (3.14) to establish the existence or non-existence of quenching behavior. Some useful properties of (3.15)–(3.16) will be stated here, for use in the forthcoming theorems.

Note that since $\phi_n(0) \leq \phi_n(l)$, we have

$$k_{0l}(t) = k_{l0}(t) \leq k_{ll}(t) \tag{3.17}$$

and

$$k_{00}(t) \leq k_{ll}(t). \tag{3.18}$$

Then it follows that

$$I_{0l}(t) = I_{l0}(t) \leq I_{ll}(t) \tag{3.19}$$

and

$$I_{00}(t) \leq I_{ll}(t). \tag{3.20}$$

Also note that $k_{11}(t)$ is decreasing. It can be shown that $h_1(t)$ and $h_2(t)$ are continuous. Moreover, they are bounded:

$$\underline{h}_1 \leq h_1(t) \leq \bar{h}_1 \quad \text{and} \quad \underline{h}_2 \leq h_2(t) \leq \bar{h}_2. \tag{3.21}$$

This can be seen by noticing that

$$V(x,t) = \int_0^l G(x,t \mid \xi, 0)\, d\xi \tag{3.22}$$

is a solution of the one-dimensional heat equation with $V(x,0) = 1$; with a homogeneous Neumann condition at $x = l$; and the homogeneous condition $V_x \mid_{x=0} -MV$ at $x = 0$. Thus, by the maximum principle, $0 \leq V(x,t) \leq 1$. It then follows that $h_1(t) < 1$ and $h_2(t) < 1$.

4. Lower bound result. In this section we will establish the existence of a continuously differentiable solution of (3.13)–(3.14). Rewrite the system (3.1)–(3.14) as

$$\bar{u} = A[\bar{u}] \tag{4.1}$$

where $\bar{u}(t) \equiv (u_1(t), u_2(t))$ and the operator $A = [A_1, A_2]$ is defined by

$$\begin{aligned} A_1[\bar{u}(t)] \;&\equiv\; h_1(t) + \int_0^t k_{0l}(t-s)g[u_2(s)]\,ds \\ &\quad + \int_0^t k_{00}(t-s)\big\{Mu_1(s) - f[u_1(s)]\big\}\,ds \end{aligned} \tag{4.2}$$

$$\begin{aligned} A_2[\bar{u}(t)] \;&\equiv\; h_2(t) + \int_0^t k_{ll}(t-s)g[u_2(s)]\,ds \\ &\quad + \int_0^t k_{l0}(t-s)\big\{Mu_1(s) - f[u_1(s)]\big\}\,ds. \end{aligned} \tag{4.3}$$

Consider the space:

$$\begin{aligned} B \;\equiv\; \Big\{\bar{u} \;=\; [u_1(t), u_2(t)] :\; 0 < u_i(t) \;\le\; L \;<\; 1,\; u_i(t) \text{ continuous}, \\ 0 \;\le\; t \;<\; t^*,\; i = 1,2\Big\}. \end{aligned} \tag{4.4}$$

Define the norm as

$$||\bar{u}|| \;\equiv\; \Sigma_{n=1}^2 \sup_{0 \le t < t^*} |u_n(t)|. \tag{4.5}$$

The existence result given by the following theorem is based on a contraction mapping argument.

Theorem 4.1. *There exists a unique solution of* (4.1) *that is continuously differentiable and satisfies* $0 < u(t) \le L < 1, 0 \le t < t^*$ *where* t^* *is determined by*

$$I_{ll}(t^*) \;\equiv\; \int_0^{t^*} k_{ll}(s)\,ds \;=\; \min\Big\{\frac{1}{2g'(L)}, \frac{1}{2(M-m)}, \frac{L - \bar{h}}{g(L)+L(M-m)}\Big\}. \tag{4.6}$$

Proof. The goal is to establish the existence of a unique fixed point of (4.1). First show that the operator A maps B into B. Clearly, both $A_1[\bar{u}]$ and $A_2[\bar{u}]$ are continuous and nonnegative. Next show that $A_1[\bar{u}] \le L$ and that $A_2[\bar{u}] \le L$. Note that

$$
\begin{aligned}
A_1[\bar{u}(t)] \;&=\; h_1(t) + \int_0^t k_{0l}(t-s)\, g[u_2(s)]\, ds \\
&\quad + \int_0^t k_{00}(t-s)\big\{Mu_1(s) - (f[u_1(s)] - f(0))\big\} ds \\
&\quad - \int_0^t k_{00}(t-s) f(0)\, ds \\
&= h_1(t) + \int_0^t k_{0l}(t-s)\, g[u_2(s)]\, ds \\
&\quad + \int_0^t k_{00}(t-s) u_1(s)\big\{M - f'[\tilde{u}(s)]\big\}\, ds \\
&\quad - \int_0^t k_{00}(t-s) f(0)\, ds
\end{aligned}
\tag{4.7}
$$

where $0 \le \tilde{u}(t) \le u_1(t)$. Then

$$A_1[\bar{u}(t)] \;\le\; \bar{h}_1 + g(L)\, I_{0l}(t) + L(M-m)\, I_{00}(t). \tag{4.8}$$

Similarly,

$$A_2[\bar{u}(t)] \;\le\; \bar{h}_2 + g(L)\, I_{ll}(t) + L(M-m)\, I_{l0}(t). \tag{4.9}$$

Define $\bar{h} \equiv \max\{\bar{h}_1, \bar{h}_2\}$. Then, since $I_{00}(t) \le I_{ll}(t)$ and since $I_{0l}(t) \le I_{ll}(t)$, we obtain

$$A_1[\bar{u}(t)] \;\le\; \bar{h} + I_{ll}(t)[g(L) + L(M-m)] \tag{4.10}$$

$$A_2[\bar{u}(t)] \;\le\; \bar{h} + I_{ll}(t)[g(L) + L(M-m)]. \tag{4.11}$$

Therefore, a sufficient condition to guarantee that A maps B into B is

$$I_{ll}(t) \;\le\; \frac{L - \bar{h}}{g(L) + L(M-m)}. \tag{4.12}$$

In order to show that A is a contraction mapping, we need

$$||A[\bar{u}] - A[\bar{v}]|| \;<\; ||\bar{u} - \bar{v}||. \tag{4.13}$$

From our given equations, we have

$$
\begin{aligned}
A_1[\bar{u}] - A_1[\bar{v}] \;=\; & \int_0^t k_{0l}(t-s)\,[g[u_2(s)] - g[v_2(s)]\,]\,ds \\
& + \int_0^t k_{00}(t-s)\,[Mu_1(s) - f[u_1(s)]\,]\,ds \\
=\; & \int_0^t k_{0l}(t-s) g'\,[\tilde{u}_2(s)]\,(u_2(s) - v_2(s))\,ds \\
& + \int_0^t k_{00}(t-s)\,[M - f'[\tilde{u}_1(s)]]\,(u_1(s) - v_1(s))\,ds.
\end{aligned} \tag{4.14}
$$

Then it follows that

$$
\begin{aligned}
|A_1[\bar{u}] - A_1[\bar{v}]| \;\le\; & \int_0^t k_{0l}(t-s) g'[L]\,|u_2(s) - v_2(s)|\,ds \\
& + \int_0^t k_{00}(t-s)|M - f'[\tilde{u}_1(s)]|\,|u_1(s) - v_1(s)|\,ds \\
\le\; & g'(L)\,I_{0l}(t) \sup_{0\le t\le t^*} |u_2(t) - v_2(t)| \\
& + (M-m)\,I_{00}(t) \sup_{0\le t\le t^*} |u_1(t) - v_1(t)|
\end{aligned} \tag{4.15}
$$

where $\tilde{u}_1(t)$ lies between $u_1(t)$ and $v_1(t)$ and where $\tilde{u}_2(t)$ lies between $u_2(t)$ and $v_2(t)$. Similarly

$$
\begin{aligned}
|A_2[\bar{u}] - A_2[\bar{v}]| \;\le\; & g'(L)\,I_{ll}(t) \sup_{0\le t\le t^*} |u_2(t) - v_2(t)| \\
& + (M-m)\,I_{l0}(t) \sup_{0\le t\le t^*} |u_1(t) - v_1(t)|.
\end{aligned} \tag{4.16}
$$

Then

$$
\begin{aligned}
& \sup_{0\le t\le t^*} |A_1[\bar{u}] - A_1[\bar{v}]| + \sup_{0\le t\le t^*} |A_2[\bar{u}] - A_2[\bar{v}]| \\
& \qquad \le\; g'(L)\,[I_{0l}(t) + I_{ll}(t)] \sup_{0\le t\le t^*} |u_2(t) - v_2(t)| \\
& \qquad\quad + (M-m)\,[I_{00}(t) + I_{l0}(t)] \sup_{0\le t\le t^*} |u_1(t) - v_1(t)| \\
& \qquad \le\; 2\,I_{ll}\,(t)[g'(L) \sup_{0\le t\le t^*} |u_2(t) - v_2(t)| \\
& \qquad\quad + (M-m) \sup_{0\le t\le t^*} |u_1(t) - v_1(t)|].
\end{aligned} \tag{4.17}
$$

So a sufficient condition for A to be a contraction mapping is

$$I_{ll}(t) < \min\Big\{\frac{1}{2g'(L)}, \frac{1}{2(M-m)}\Big\}. \tag{4.18}$$

Combine conditions (4.12) and (4.18) to obtain

$$I_{ll}(t) < \min\Big\{\frac{1}{2g'(L)}, \frac{1}{2(M-m)}, \frac{L-\bar{h}}{g(L)+L(M-m)}\Big\}. \tag{4.19}$$

There will exist a unique fixed point of (4.1) whenever (4.19) holds. To push the existence of a unique, bounded solution to hold for the largest interval of time possible, we choose the largest value t^* for which (4.19) holds.

Next we will demonstrate that the continuous solution is also continuously differentiable. Consider the derivatives of $u_1(t)$ and $u_2(t)$:

$$\begin{aligned} u_1'(t) \;=\; & h_1'(t) + k_{0l}(t)g(u_2(0)) + \int_0^t k_{0l}(t-s)g'[u_2(s)]u_2'(s)\,ds \\ & + k_{00}(t)[Mu_1(0) - f[u_1(0)]\,] \\ & + \int_0^t k_{00}(t-s)[M - f'(u_1(s))]u_1'(s)\,ds \end{aligned} \tag{4.20}$$

$$\begin{aligned} u_2'(t) \;=\; & h_2'(t) + k_{ll}(t)g(u_2(0)) + \int_0^t k_{ll}(t-s)g'[u_2(s)]u_2'(s)\,ds \\ & + k_{l0}(t)[Mu_1(0) - f[u_1(0)]\,] \\ & + \int_0^t k_{l0}(t-s)[M - f'(u_1(s))]u_1'(s)\,ds. \end{aligned} \tag{4.21}$$

With the knowledge that there exists a t^* such that $u_1(t)$ and $u_2(t)$ are continuous for $0 \le t < t^*$, define the following:

$$a_1(t) \equiv h_1'(t) + k_{0l}(t)\,g(u_2(0)) + k_{00}(t)[Mu_1(0) - f[u_1(0)]\,] \tag{4.22}$$

$$a_2(t) \equiv h_2'(t) + k_{ll}(t)\,g(u_2(0)) + k_{l0}(t)[Mu_1(0) - f[u_1(0)]\,] \tag{4.23}$$

$$k_{22}(t-s) \equiv k_{ll}(t-s)g'[u_2(s)] \tag{4.24}$$

$$k_{21}(t-s) \equiv k_{l0}(t-s)[M - f'(u_1(s))] \tag{4.25}$$

$$k_{12}(t-s) \equiv k_{0l}(t-s)g'[u_2(s)] \tag{4.26}$$

$$k_{11}(t-s) \equiv k_{00}(t-s)[M - f'(u_1(s))]. \tag{4.27}$$

Then we have

$$\begin{aligned} u_1'(t) = a_1(t) &+ \int_0^t k_{11}(t-s)u_1'(s)\,ds \\ &+ \int_0^t k_{12}(t-s)u_2'(s)\,ds \end{aligned} \tag{4.28}$$

$$\begin{aligned} u_2'(t) = a_2(t) &+ \int_0^t k_{21}(t-s)u_1'(s)\,ds \\ &+ \int_0^t k_{22}(t-s)u_2'(s)\,ds. \end{aligned} \tag{4.29}$$

Equations (4.28) and (4.29) can be viewed as a system of linear Volterra equations for $u_1'(t)$ and $u_2'(t)$. The existence of continuous $u_1'(t)$ and $u_2'(t)$ for $0 \le t \le t^*$ then follows (see [**5**]).

This establishes Theorem 4.1. Since t^* is the lower bound on the quenching time, $t^* \le \hat{t}$. If $t^* = \infty$, then quenching is avoided for all time.

5. Upper bound result. In this section we obtain an upper bound on the time of quenching. Recall that if quenching does occur, it will do so at $x = l$, the end which is absorbing heat. Since quenching occurs at $x = l$, attention will be focused on $u_2(t)$. We can show that, under appropriate conditions, it is sufficient to have $u_2(t) \to 1$ as $t \to \hat{t} < \infty$ in order for quenching to occur.

Theorem 5.1. *Let* (4.1) *have a continuously differentiable solution* $u(t)$ *for* $0 \le t < \hat{t}$, *where* $u_2(t) \to 1$ *as* $t \to \hat{t}$. *If* $h_2'(t) + k_{l0}(t)[Mu_1(0) - f[u_1(0)]] > 0$, *then* $u_2'(t) \to \infty$ as $t \to \hat{t}$.

Proof. We have

$$u_2'(t) > 0 \quad \text{for } 0 < t < \hat{t} \tag{5.1}$$

from applying the theory of linear Volterra equations to (4.29) (see [**5**]).

$$\begin{aligned} u_2'(t) \;\geq\; & h_2'(t) + k_{ll}(t)g(u_2(0)) + k_{l0}(t)[Mu_1(0) - f[u_1(0)]\,] \\ & + \int_0^t k_{l0}(t-s)[M - f'(u_1(s))]u_1'(s)\,ds \\ & + k_{ll}(t)\int_0^t \frac{d}{ds}\,g[u_2(s)]\,ds. \end{aligned} \tag{5.2}$$

Since $\int_0^t (d/ds)g[u_2(s)]\,ds = g[u_2(t)] - g[u_2(0)]$, we obtain

$$u_2'(t) \;\geq\; h_2^{\omega}(t) + k_{l0}(t)[Mu_1(0) - f[u_1(0)]\,] + k_{ll}(t)g[u_2(t)]. \tag{5.3}$$

Recall that as $u_2 \to 1$, $g[u_2] \to \infty$. Hence, if $h_2'(t) + k_{l0}(t)[Mu_1(0) - f[u_1(0)]\,] > 0$, then as $u_2 \to 1$ (as $t \to \hat{t}$), $u_2'(t) \to \infty$. Theorem 5.1 is established.

Theorem 5.2. *Whenever there exists a t^{**} such that*

$$\int_{\underline{h}_2}^1 \frac{dz}{g[z]} \;=\; I_{ll}(t^{**}), \quad \underline{h}_2 \;\equiv\; \min_{0\leq t\leq t^{**}} h_2(t) \tag{5.4}$$

then (3.13) *and* (3.14) *cannot have continuous, bounded solutions for $t > t^{**}$.*

Proof. Attention is still focused on $u_2(t)$. Assume that there exist continuous, bounded solutions to (3.13) and (3.14) for $0 \leq t \leq t_1$. Then

$$\begin{aligned} u_2(t) \;=\; & h_2(t) + \int_0^t k_{ll}(t-s)g[u_2(s)]\,ds \\ & + \int_0^t k_{l0}(t-s)\big\{Mu_1(s) - f[u_1(s)]\big\}\,ds \\ \geq\; & \underline{h}_2 + \int_0^t k_{ll}(t-s)g[u_2(s)]\,ds \\ \geq\; & \underline{h}_2 + \int_0^t k_{ll}(t_1-s)g[u_2(s)]\,ds. \end{aligned} \tag{5.5}$$

The previous inequalities rely on the fact that the kernel $k_{ll}(t)$ is decreasing and that $Mu_1(t) - f[u_1(t)] > 0$. Define

$$J(t) \;\equiv\; \int_0^t k_{ll}(t_1-s)g[u_2(s)]\,ds \tag{5.6}$$

so that

$$u_2(t) \geq \underline{h}_2 + J(t). \tag{5.7}$$

Then note that

$$J'(t) = k_{ll}(t_1 - t)g[u_2(t)] \geq k_{ll}(t_1 - t)g[\underline{h}_2 + J(t)]. \tag{5.8}$$

Integrate both sides to obtain

$$\int_0^{J(t_1)} \frac{dJ}{g[J + \underline{h}_2]} \geq \int_0^{t_1} k_{ll}(t_1 - t)\, dt \tag{5.9}$$

$$\int_{\underline{h}_2}^{J(t_1 + \underline{h}_2)} \frac{dz}{g[z]} \geq \int_0^{t_1} k_{ll}(s)\, ds \tag{5.10}$$

$$\int_{\underline{h}_2}^{1} \frac{dz}{g[z]} > \int_0^{t_1} k_{ll}(s)\, ds \equiv I_{ll}(t_1). \tag{5.11}$$

If there exists a t^{**} such that

$$\int_{\bar{h}_2}^{1} \frac{dz}{g[z]} = I_{ll}(t^{**}) \tag{5.12}$$

then a contradiction occurs. Hence, (3.13) and (3.14) cannot have continuous, bounded solutions for $t > t^{**}$. Quenching will have occurred. This proves Theorem 5.2.

Note that this result makes intuitive sense. If a large amount of heat is input into the system, then $g(z)$ will tend to be large. Due to the inverse relation in (5.12), if $g(z)$ is large, t^{**} will tend to be small. Hence quenching will tend to occur sooner.

6. Quenching and nonquenching results.

6.1 If $g > f$ and $g' > f'$, then quenching will always occur. In this section we will show that if $g(z) > f(z)$ and if $g'(z) > f'(z)$, then quenching will always occur in finite time. This result is a generalization of a result in [**4**]. (See Section (7) for details.) Note that we will employ a different Green's function ($\tilde{G}$) in this section. In particular, the requirement that $Mv(t) - f[v(t)] > 0$ will be unnecessary.

Consider the original problem (2.1)–(2.6) for $v(x,t)$. We can obtain the following system of integral equations:

$$u_1(t) = h_1(t) + \int_0^t \tilde{G}(0,t \mid l,s) g[u_2(x)]\, ds - \int_0^t \tilde{G}(0,t \mid 0,s) f[u_1(x)]\, ds \tag{6.1}$$

$$u_2(t) = h_2(t) + \int_0^t \tilde{G}(l,t \mid l,s) g[u_2(x)]\, ds - \int_0^t \tilde{G}(l,t \mid 0,s) f[u_1(x)]\, ds. \tag{6.2}$$

In this case the Green's function $\tilde{G}$ satisfies the following:

$$\tilde{G} = \tilde{G}(x,t \mid \xi, s) \tag{6.3}$$

$$\tilde{G}_t - \tilde{G}_{xx} = \delta(t-s)\delta(x-\xi), \quad 0 < x < l,\ t > s^- \tag{6.4}$$

$$\tilde{G}\ |_{t=s^-} =\ 0 \tag{6.5}$$

$$\tilde{G}_x\ |_{x=0} =\ 0, \qquad \tilde{G}_x\ |_{x=l} =\ 0 \tag{6.6}$$

The Green's function is given by

$$\tilde{G}(x,t \mid \xi, s) = H(t-s)\Big[\frac{1}{l} + \frac{2}{l}\,\Sigma_{n=1}^{\infty} \cos\Big(\frac{n\pi\xi}{l}\Big)\cos\Big(\frac{n\pi x}{l}\Big) \cdot \exp\Big(\frac{-n^2\pi^2}{l^2}\,(t-s)\Big)\Big]. \tag{6.7}$$

Define $\tilde{k}_{ab}(t-s) \equiv \tilde{G}(a,t|b,s)$. Note that

$$\tilde{k}_{00}(t) = \tilde{k}_{ll}(t) \tag{6.8}$$

$$\tilde{k}_{0l}(t) = \tilde{k}_{l0}(t) \tag{6.9}$$

$$\tilde{k}_{l0}(t) < \tilde{k}_{ll}(t). \tag{6.10}$$

Each $\tilde{k}_{ab}(t)$ is positive and decreasing. Define

$$\tilde{I}_{ab}(t) \equiv \int_0^t \tilde{k}_{ab}(s)\,ds. \tag{6.11}$$

The behavior of $\tilde{I}_{ll}(t)$ will be essential in the results of this section:

$$\begin{aligned} \tilde{I}_{ll}(t) &= \int_0^t \tilde{k}_{ll}(s)\,ds \\ &= \int_0^t \Big[\frac{1}{l} + \frac{2}{l}\,\Sigma_{n=1}^{\infty} \exp\Big(\frac{-n^2\pi^2}{l^2}\,(t-s)\Big)\Big]\,ds \\ &= \frac{t}{l} + \frac{2l}{\pi^2}\,\Sigma_{n=1}^{\infty}\,\frac{1}{n^2}\Big[\exp\Big(\frac{-n^2\pi^2}{l^2}\,(t)\Big)\Big] \end{aligned} \tag{6.12}$$

Then

$$\frac{t}{l} \le \tilde{I}_{ll}(t) \le \frac{t}{l} + \frac{2l}{\pi^2}\Big(\frac{\pi^2}{6}\Big). \tag{6.13}$$

Hence

$$\lim_{t\to\infty} \tilde{I}_{ll}(t) = \infty. \tag{6.14}$$

Since heat enters at $x = l$ and leaves at $x = 0$, it makes intuitive sense that $u_2(t) \ge u_1(t)$. This fact is proved in the following lemma.

Lemma 6.1. *Consider the system of equations* (6.1) *and* (6.2). *If* $h_2(t) > h_1(t)$ *and if* $g(z) \ge f(z)$, *then* $u_2(t) \ge u_1(t)$.

Proof. Note that

(6.15)

$$\begin{aligned} u_2(t)-u_1(t) &= h_2(t)-h_1(t) + \int_0^t \big[\tilde{k}_{0l}(t-s) + \tilde{k}_{ll}(t-s)\big] g[u_2(s)]\,ds \\ &\quad - \int_0^t [\tilde{k}_{00}(t-s) + \tilde{k}_{l0}(t-s)] f[u_1(s)]\,ds \\ &= h_2(t) - h_1(t) + \int_0^t \big[\tilde{k}_{00}(t-s) + \tilde{k}_{l0}(t-s)\big] \\ &\qquad \cdot \big[g[u_2(s)] - f[u_1(s)]\big]\,ds. \end{aligned}$$

The previous equality holds since $\tilde{k}_{00}(t) = \tilde{k}_{ll}(t)$ and $\tilde{k}_{0l}(t) = \tilde{k}_{l0}(t)$. Let

$$b(t) \equiv h_2(t) - h_1(t) \tag{6.16}$$

so that $b(t) > 0$. Also define

$$\tilde{k}(t) = \tilde{k}_{00}(t) + \tilde{k}_{l0}(t) \tag{6.17}$$

so that

$$\begin{aligned} u_2(t) - u_1(t) &= b(t) + \int_0^t \tilde{k}(t-s)\big[g[u_2(s)] - f[u_1(s)]\big]\,ds \\ &\geq b(t) + \int_0^t \tilde{k}(t-s)\big[f[u_2(s)] - f[u_1(s)]\big]\,ds \\ &= b(t) + \int_0^t \tilde{k}(t-s)f'[\tilde{u}(s)][u_2(s) - u_1(s)]\,ds \end{aligned} \tag{6.18}$$

where $\tilde{u}(s)$ lies between $u_1(s)$ and $u_2(s)$. Let

$$w(t) = u_2(t) - u_1(t) \tag{6.19}$$

and

$$\hat{k}^{**}(t,s) = \tilde{k}(t-s)f'[\tilde{u}(s)] \tag{6.20}$$

so that

$$w(t) \geq b(t) + \int_0^t \hat{k}(t,s)w(s)\,ds. \tag{6.21}$$

Note that $w(0) > 0$ since $b(0) > 0$. Also $w(t)$ is continuous and can never equal zero (since $\hat{k}(t,s) > 0$ and $b(t) > 0$). Therefore $w(t) > 0$ and so $u_2(t) > u_1(t)$. This establishes Lemma 6.1.

Now a similar argument to that of Section (4) can be used here to establish the existence of a unique, continuous, bounded solution to (6.1) and (6.2) up to some t^*. With a lower bound established, we can now seek an upper bound on the quenching time of the system

(6.1)–(6.2). Note that we will focus only on $u_2(t)$ since quenching will occur at $x = l$ (since $u_2 \geq u_1$):

(6.22)

$$\begin{aligned} u_2(t) &= h_2(t) + \int_0^t \tilde{G}(l, t \mid l, s) g[u_2(s)]\, ds - \int_0^t \tilde{G}(l, t \mid 0, s) f[u_1(s)]\, ds \\ &= h_2(t) + \int_0^t \tilde{k}_{ll}(t) g[u_2(s)]\, ds - \int_0^t \tilde{k}_{l0}(t) f[u_1(s)]\, ds \\ &\geq h_2(t) + \int_0^t \tilde{k}_{ll}(t) g[u_2(s)]\, ds - \int_0^t \tilde{k}_{ll}(t) f[u_2(s)]\, ds. \end{aligned}$$

This inequality holds since $u_2(t) \geq u_1(t)$, $g(z) > f(z)$, and $\tilde{k}_{ll}(t) > \tilde{k}_{l0}(t)$. Then obtain

$$\begin{aligned} u_2(t) &\geq h_2(t) + \int_0^t \tilde{k}_{ll}(t) \big\{ g[u_2(s)] - f[u_2(s)] \big\}\, ds \\ &\geq \underline{h}_2 + \int_0^t \tilde{k}_{ll}(t_1 - s) \big\{ g[u_2(s)] - f[u_2(s)] \big\}\, ds. \end{aligned} \tag{6.23}$$

Define

$$\tilde{J}(t) \equiv \int_0^t \tilde{k}_{ll}(t_1 - s) \big\{ g[u_2(s)] - f[u_2(s)] \big\}\, ds \tag{6.24}$$

so that

$$u_2(t) \geq \beta h_2 + \tilde{J}(t). \tag{6.25}$$

Use the fact that $g'(z) > f'(z)$ to obtain

$$\begin{aligned} \tilde{J}'(t) &= \tilde{k}_{ll}(t_1 - t) \big\{ g[u_2(s)] - f[u_2(s)] \big\} \\ &\geq \tilde{k}_{ll}(t_1 - t) \big\{ g[\underline{h}_2 + \tilde{J}(t)] - f[\underline{h}_2 + \tilde{J}(t)] \big\}. \end{aligned} \tag{6.26}$$

Integrate both sides to obtain

$$\int_0^{\tilde{J}(t_1)} \frac{d\tilde{J}}{g[\tilde{J} + \underline{h}_2] - f[\tilde{J} + \beta h_2]} \geq \int_0^{t_1} \tilde{k}_{ll}(t_1 - t)\, dt \equiv \tilde{I}_{ll}(t_1). \tag{6.27}$$

If there exists a $t^{**} < \infty$ such that

$$\int_{\underline{h}_2}^{1} \frac{dz}{g[z] - f[z]} = I_{ll}(t^{**}) \tag{6.28}$$

then a contradiction occurs. Hence, (6.1) and (6.2) cannot have continuous solutions for $t > t^{**}$. Thus t^{**} provides an upper bound on the time of quenching.

Recall that

$$\lim_{t\to\infty} \tilde{I}_{ll}(t) = \infty. \tag{6.29}$$

Thus, condition (6.28) will always be met for some $t^{**} < \infty$. Quenching will always occur in finite time. The conditions $g(z) > f(z)$ and $g'(z) > f'(z)$ can be interpreted to mean that more heat is being put into the system than is being removed. Hence it makes sense that quenching would occur in this case.

6.2 Avoid quenching by controlling l and M. In this section we will obtain conditions that, when met, guarantee that quenching can be avoided for all time. We will show that quenching can be avoided by making l sufficiently small and M sufficiently large. Recall that l is the length of the domain and that M is, in a sense, a bound on the amount of heat leaving the system: $m \le f'(u) \le M$ and $f(u) \le Mu$. Physically, this means that if the length of the rod is small and if the heat loss is large, then quenching can be avoided. We rely upon the results from sections (2)–(4) employing the Green's function G.

Since the condition (4.19) for obtaining the lower bound on the time of quenching depends only on $I_{ll}(t)$, we need only address the behavior of $I_{ll}(t)$ as $t \to \infty$. Recall that $I_{ll}(t) = \int_0^t G(l,t|l,s)\,ds$. We will analyze the following comparison problem in order to determine $\lim_{t\to\infty} I_{ll}(t)$. Consider the problem:

$$\tilde{v}_t = \tilde{v}_{xx} \quad 0 \le x \le l,\ 0 \le t \tag{6.30}$$

$$\tilde{v}(x,0) = \tilde{v}_0(x) = 0 \tag{6.31}$$

$$\tilde{v}_x|_{x=0} = M\tilde{v}|_{x=0} \tag{6.32}$$

$$\tilde{v}_x|_{x=l} = 1. \tag{6.33}$$

Then $\tilde{v}(x,t) = \int_0^t G(x,t|l,s)(1)\,ds$. Evaluate $\tilde{v}(x,t)$ at $x = l$ to obtain:

$$\tilde{v}(l,t) \;=\; \int_0^t G(l,t \mid l,s)\,ds. \tag{6.34}$$

The steady state solution of (6.30)–(6.33)is

$$\lim_{t\to\infty} \tilde{v}(x,t) \;=\; x + \frac{1}{M}. \tag{6.35}$$

Since $\tilde{v}(l,t) = I_{ll}(t)$, we have

$$\lim_{t\to\infty} I_{ll}(t) \;=\; \lim_{t\to\infty} \tilde{v}(l,t) \;=\; l + \frac{1}{M}. \tag{6.36}$$

Recall that in order to avoid quenching entirely, we need:

$$I_{ll}(t) \;<\; \min\Big\{\frac{1}{2g^{\omega}(L)},\, \frac{1}{2(M-m)},\, \frac{L-\bar{h}}{g(L)+L(M-m)}\Big\} \tag{6.37}$$

or, from (6.36):

$$l + \frac{1}{M} \;<\; \min\Big\{\frac{1}{2g'(L)},\, \frac{1}{2(M-m)},\, \frac{L-\bar{h}}{g(L)+L(M-m)}\Big\}. \tag{6.38}$$

From this expression, it can be seen how lowering l sufficiently and increasing M sufficiently can prevent quenching. From (6.38), it can be seen that $(M/2) < m$ must hold.

6.3 Guarantee quenching by controlling l and M. In order to guarantee that quenching occurs in finite time, we need:

$$\int_{\underline{h}_2}^{1} \frac{dz}{g[z]} \;<\; \lim_{t\to\infty} I_{ll}(t). \tag{6.39}$$

This is obtained from condition (5.12). Then (6.36) gives

$$\int_{\underline{h}_2}^{1} \frac{dz}{g[z]} \;<\; l + \frac{1}{M}. \tag{6.40}$$

Clearly, quenching can be guaranteed to occur by increasing l and decreasing M sufficiently. Also, increasing $g(z)$ tends to encourage the onset of quenching.

7. Analysis of results in the context of the existing literature. The literature has treated two similar problems in [**4**], [**14**]. A comparison between these papers and the results included here helps place this work into a broader context.

In [**4**], Chan and Yeun examine a similar problem: a heat equation with cooling at one end and heating at the other end. They specify that $f[v]$ and $g[v]$ (given in our (2.3)–(2.4)) must be of the particular form $(1-v(t))^{-p}$.

Our results establish that quenching will always occur if $g > f$ and $g' > f'$ for a more general f and g. We can show quenching will always occur if l is large enough and M small enough. These results represent a generalization of analogous statements in [**4**]. (Specifically, for these particular results we do not require the extra restrictions on f, such that $Mv - f(v) > 0$ or $m \leq f'(v) \leq M$.)

Additionally, we show quenching can be avoided if l is small enough and M is large enough. This result is not a complete generalization of the analogous results in [**4**] since we require additional conditions on f, namely: $Mv - f(v) > 0$ and $m \leq f'(v) \leq M$. Nonetheless, our function g is more general and we allow for a larger class of functions for f. Moreover, our results provide bounds on the time of quenching and show how to control those bounds by changing l, f and g.

In [**14**], Roberts and Olmstead examine quenching problems with a nonlinear boundary condition that is of either local or nonlocal type. For their local case with Neumann boundary conditions, heat enters one end of the rod through a nonlinear, Neumann boundary condition: $u_x \mid_{x=l} = g(u)$. The boundary condition at the left end is insulated: $u_x \mid_{x=0} = 0$. The result in [**14**] is that quenching always occurs.

In our problem, the boundary condition at $x = 0$ is generalized as $u_x \mid_{x=0} = f[u] > 0$, where $f[u]$ is nonlinear. Since heat loss is allowed at this end, we can control whether or not quenching occurs. This represents a generalization of the results established in [**14**].

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