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ASYMPTOTIC ERROR ANALYSIS OF A QUADRATURE METHOD FOR INTEGRAL EQUATIONS WITH GREEN'S FUNCTION KERNELS

WILLIAM F. FORD, JAMES A. PENNLINE,
YUESHENG XU AND YUNHE ZHAO

ABSTRACT. We conduct an asymptotic error analysis of the trapezoidal quadrature method applied to nonlinear integral equations with Green's function kernels and, based on the asymptotic error expansion of the approximate solution, justify the Richardson extrapolation method. Following the complete error analysis, numerical examples are given to demonstrate the theory. The examples are taken from two-point boundary value problems governed by nonlinear ordinary differential equations, which can be transformed into nonlinear integral equations by using Green's functions. One of the examples involves a regular singular operator for which other well-known numerical techniques such as finite differences may not be applicable.

1. Introduction. Consider nonlinear integral equations in the form

$$u(s) - \int_0^1 G(s,t)\psi\left(t, u(t)\right) dt \;=\; f(s), \quad s \in [0,1], \tag{1.1}$$

where ψ and f are given functions and u is the unknown to be determined. In this paper, we consider *Green's function* type kernels $G(s,t)$ for equation (1.1). Specifically, we let m be a positive integer and assume that $G(s,t)$ satisfies the conditions

(1) $G(s,t) \in C([0,1]\times[0,1])$

(2) $G(s,t) \in C^{2m}((0,1)\times(0,1)\setminus D)$ where

$$D := \{(s,t) \in (0,1)\times(0,1) : s = t\}$$

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The third author is the correspondence author.

(3) $G^{(0,1)}(s, 0^+)$, $G^{(0,i)}(s, 1^-)$, $G^{(0,i)}(s, s^-)$ and $G^{(0,i)}(s, s^+)$ exist for $i = 1, 2, \ldots, 2m$, where $G^{(i,j)}(s, t)$ denotes the partial derivative $(\partial^{i+j}/(\partial s^i \partial t^j))G(s, t)$, and a similar property holds for the first argument.

If function $G : [0, 1] \times [0, 1] \to R$ satisfies these conditions we call it a Green's function kernel. We allow the derivatives of the kernels to have jump discontinuities across the diagonal. These conditions are fulfilled if $G(s, t)$ is a Green's function of a differential operator. In particular, a Green's function of a linear differential operator of order two satisfies these conditions with a jump discontinuity in the first derivatives of the kernel across the diagonal $s = t$. Although equation (1.1) is motivated from a two-point boundary value problem of a nonlinear ordinary differential equation of order two, we do not restrict ourselves to this special case. Throughout this paper, we assume $G(s, t)$ to be a function that satisfies the three conditions given above.

We assume that $\psi \in C^{2m}([0, 1] \times R)$ satisfies a Lipschitz condition with respect to the second variable and $f \in C^{2m}[0, 1]$. Under the condition on the kernel, if equation (1.1) has a solution u, this solution u is in $C^{2m}[0, 1]$. In many circumstances, equation (1.1) has a unique solution. However, to extend the scope of applications of the method developed in this paper, we will not assume that equation (1.1) has a unique solution. Rather we assume that $u_0 \in C[0, 1]$ is an isolated solution of equation (1.1) and consider its numerical approximation. This will allow the method and theory developed in this paper to be applicable both to the case when equation (1.1) has a unique solution in $C[0, 1]$ and to the case when equation (1.1) has multiple isolated solutions as well.

Integral equations of type (1.1) arise, in particular, when two-point boundary value problems governed by ordinary differential equations are converted to integral equations, one of the many known ways to solve this problem which includes finite difference techniques applied to the differential equations. Next, we present two examples to illustrate the reformulation.

As a first example, we consider the two-point boundary value problem

$$u''(s) = f(s, u(s)), \quad 0 < s < 1,$$
$$u(0) = a, \quad u(1) = b.$$

Following the theory developed in [**18**], [**19**], this boundary value problem can be converted into the integral equation

$$u(s) - \int_0^1 g_k(s,t)\,[k^2u(t) - f(t,u(t))]\,dt \;=\; h(s), \quad 0 \le s \le 1, \tag{1.2}$$

where

$$g_k(s,t) \;=\; \frac{1}{k\sinh k}\begin{cases} \sinh ks\,\sinh k(1-t) & 0 \le s < t, \\ \sinh k(1-s)\,\sinh kt & t \le s \le 1, \end{cases} \tag{1.3}$$

$$h(x) \;=\; \frac{1}{\sinh k}\,[a\sinh k(1-s) + b\sinh ks], \tag{1.4}$$

and k is a parameter chosen to guarantee convergence of Picard iteration.

The boundary value problem of a regular singular operator also leads to an integral equation of form (1.1). To explain this, we consider

$$u''(s) + \frac{m}{s}\,u'(s) \;=\; f(s,u(s)), \quad 0 < s < 1,$$
$$u'(0) = 0, \quad u(1) = \lambda.$$

Following the theory in [**20**], this problem can be converted into the integral equation of the second kind

$$u(s) \;=\; h(s) + \int_0^1 g_k(s,t)\,[k^2u(t) - f(t,u(t))]\,dt, \tag{1.5}$$

where

$$g_k(s,t) \;=\; \frac{1}{kI_l(k)}\begin{cases} (kt)^{(m+1)/2}V_l(kt)(ks)^{-(m-1)/2}I_l(ks) & 0 \le s < t, \\ (kt)^{(m+1)/2}I_l(kt)(ks)^{-(m-1)/2}V_l(ks) & t < s \le 1. \end{cases}, \tag{1.6}$$

$$V_l(ks) \;=\; I_l(k)K_l(ks) - K_l(k)I_l(ks), \quad l = \frac{m-1}{2}, \tag{1.7}$$

and

$$h(s) \;=\; \frac{\lambda I_l(ks)}{sI_l(k)}. \tag{1.8}$$

Here I_l and K_l are the lth order modified Bessel functions of the first and second kind, respectively. Again, k is a parameter chosen to guarantee convergence of Picard iteration for the integral equation. Clearly the Green's functions $g_k(s,t)$ defined by (1.3) and (1.6) satisfy conditions (1)–(3) in the definition of Green's function kernels. Therefore, the nonlinear integral equations (1.2) and (1.5) are special cases of equation (1.1).

Since equations (1.2) and (1.5) are special cases of (1.1) derived from the ordinary differential equations above, methods in this paper can be used for numerical solutions of these boundary value problems. It may be argued that approximation of the ordinary differential equations directly by finite difference methods would lead to more efficient algorithms because the resulting systems are banded; such is the case for the example of (1.2) used in Section 4. We would like to emphasize, however, that the primary purpose of the special case of equation (1.2) used in Section 6 is a demonstration of the theory. But there may be situations wherein the integral equation approach is a better approach numerically. For instance, approximation of differentiation by finite differences may lead to an unstable algorithm if u varies rapidly over the range (although we have not explored this issue here). Another case arises when the ordinary differential equation is regular singular; direct application of a uniform center difference approximation would lead to a singular system. The integral equation emerging from the Green's function approach may be perfectly regular; this is illustrated by the special case of (1.5) in Section 6, where a central difference approximation leads to a system singular near $x = 0$.

A second advantage of the integral equation method over the finite difference method applied directly to the original differential equations is that the discretization of the integral equations using the trapezoidal quadrature formula leads to a matrix having a bounded condition number (see, for example, Theorem 14.8 of [**13**]) while the finite difference method leads to a matrix with condition number that grows in the order $\mathcal{O}(n^2)$, where n is the size of the matrix.

We now return to discussion of the general case. Let $C[0,1]$ denote the usual Banach space of continuous functions on $[0,1]$ with the uniform norm $\|\cdot\|$. We define the operator $\mathcal{K} : C[0,1] \to C[0,1]$ by

$$(\mathcal{K}u)(s) := \int_0^1 G(s,t)u(t)\,dt, \quad s \in [0,1].$$

Because $G(s,t)$ is a continuous kernel, operator $\mathcal{K}$ is compact in $C[0,1]$. Let Ψ denote the Nemyckii operator for the function ψ. Specifically,

$$(\Psi u)(t) := \psi(t, u(t)), \quad t \in [0,1].$$

Using the notation given above, we write equation (1.1) in the operator form as

$$u - \mathcal{K}\Psi u = f. \tag{1.9}$$

Integral equations with Green's function kernels have been studied by many authors. Projection methods and Nyström methods were considered in [**4**], [**5**], [**6**] as numerical methods for these equations. A corrected quadrature method was applied to these equations in [**23**] to achieve higher order convergence. An extrapolation method based on the iterated piecewise linear polynomial collocation method was proposed in [**14**].

Asymptotic error analysis and extrapolation methods are classical numerical analysis topics. Extrapolation methods based on quadrature methods for integral equations with smooth kernels are found in [**16**]. Extrapolation methods were studied in [**8**], [**15**] for collocation methods and iterated collocation methods, for integral equations with smooth kernels. For the case when an integral kernel is sufficiently smooth, [**17**] gave an asymptotic error analysis for numerical solutions of linear operator equations and applied it to the Nyström, collocation and Galerkin methods for second kind Fredholm integral equations. An asymptotic analysis was provided in [**9**] for numerical methods solving nonlinear operator equations. An asymptotic error expansion was established in [**27**] for approximate solutions obtained from a quadrature method based on a quadrature formula of [**22**] for a class of boundary integral equations, which are the reformulation of third-kind boundary value problems of the Laplace equation. A generalized extrapolation method was introduced in [**21**] using multi-parameters for boundary integral equations.

The main purpose of this paper is to derive an asymptotic error expansion for approximate solutions of equation (1.9) obtained from a quadrature method using the trapezoidal rule. This asymptotic expansion will lead to an extrapolation scheme for equations (1.9). Because Green's function kernels are not smooth, standard methods developed in [**9**], [**16**], [**17**] for asymptotic error analysis for smooth kernels cannot be directly applied to this case. The analysis used to prove this asymptotic expansion is based on an asymptotic expansion established in [**23**] for the trapezoidal rule applied to integral operators with Green's function kernels. In developing the asymptotic expansion for nonlinear equations, we also have to treat the nonlinearity of the integral operators.

This paper is organized as follows. We present in Section 2 an analysis of the approximate solvability and order of convergence of the quadrature method for equation (1.1) using the trapezoidal rule. In Section 3, we establish a modified Stetter's theorem [**24**] which gives an asymptotic error expansion of the approximate solutions of nonlinear operator equations. In Section 4 we specialize this result to the quadrature method for equation (1.1). This asymptotic expansion shows that the extrapolation scheme accelerates the convergence of the approximate solution sequence produced by the quadrature method. A reconstruction of approximate solutions from the extrapolated function values of the approximate solutions at the quadrature notes is presented in Section 5. Numerical examples are presented in Section 6 to illustrate the theoretical results.

2. A quadrature scheme. In this section we describe a quadrature method for the integral equations (1.1) and provide a complete analysis of the approximate solvability and order of convergence of the quadrature method. This analysis on one hand serves as a preliminary for the asymptotic error analysis presented in Section 4 for this quadrature method and, on the other hand, it has its own independent interest.

We begin with a review of the trapezoidal rule applied to an integral operator with a Green's function kernel, upon which our quadrature method for equation (1.1) is based. To this end, we let $t_j := (j/n)$ for $j = 0, 1, \dots, n$, and let $h := (1/n)$. The trapezoidal rule T_n applied to

$\tilde{G}(s,t) := G(s,t)u(t)$ for a continuous function u gives

$$T_n(\tilde{G}(s,t)) := h \sum_{j=0}^{n}{}'' \tilde{G}(s,t_j) \tag{2.1}$$

where the double prime in the summation indicates that the first and last term of this summation are multiplied by 1/2. The points t_j are called quadrature nodes.

We now use the quadrature formula (2.1) to derive our quadrature method for equation (1.1). For this purpose, we define for each positive integer n the operator $\mathcal{K}_n : C[0,1] \to C[0,1]$ by

$$(\mathcal{K}_n u)(s) := h \sum_{j=0}^{n}{}'' G(s,t_j)u(t_j), \quad s \in [0,1]. \tag{2.2}$$

This operator is bounded and it approximates the integral operator $\mathcal{K}$. Using $\mathcal{K}_n$ to replace $\mathcal{K}$ in equation (1.1) leads to the following quadrature method for equation (1.1):

$$u_n(s) - (\mathcal{K}_n \Psi u_n)(s) = f(s), \quad s \in [0,1], \tag{2.3}$$

where

$$(\mathcal{K}_n \Psi u)(s) = h \sum_{j=0}^{n}{}'' G(s,t_j)\psi(t_j, u(t_j)), \quad s \in [0,1].$$

Upon collocating equation (2.3) at the quadrature nodes t_i, we obtain a discrete system of nonlinear equations

$$u_n(t_i) - (\mathcal{K}_n \Psi u_n)(t_i) = f(t_i), \quad i = 0,1,\dots,n. \tag{2.4}$$

Since the operator $\mathcal{K}_n$ has a finite rank, equation (2.3) is equivalent to system (2.4). Solving this system, we obtain $n+1$ values $u_n(t_i)$ for $i = 0,1,\dots,n$, which in turn give an approximate solution u_n for equation (1.1), namely,

$$u_n(s) = h \sum_{j=0}^{n}{}'' G(s,t_j)\psi(t_j, u_n(t_j)) + f(s), \quad s \in [0,1]. \tag{2.5}$$

We next analyze the quadrature method described above. We will prove that this system has a unique solution in a neighborhood of an isolated solution u_0 of equation (1.1) for sufficiently large n, and that the approximate solutions u_n have the convergence order $\mathcal{O}(h^2)$. For this purpose, we present some properties of the approximate operators $\mathcal{K}_n$.

Lemma 2.1. *The approximate operators $\mathcal{K}_n$ have the properties*:

(i) *For each $u \in C[0,1]$, $\|\mathcal{K}_n u - \mathcal{K}u\| \to 0$ as $n \to \infty$.*

(ii) *The set $\{\|\mathcal{K}_n\|\}$ is bounded.*

(iii) *The set $\{\mathcal{K}_n\}$ is collectively compact operators on $C[0,1]$.*

(iv) *For each $u \in C^2[0,1]$, $\|\mathcal{K}u - \mathcal{K}_n u\| = \mathcal{O}(h^2)$.*

Proof. The proof of (i)–(iii) follows directly from results in Chapter 2 of [**1**] because the Green's function kernel G is a continuous kernel. It remains to prove (iv). Let $s \in [0,1]$ and suppose that $s \in [t_{i-1}, t_i]$ for some $i = 0, 1, \dots, n$. Write $(\mathcal{K}u)(s)$ as the sum of four integrals of $G(s,t)u(t)$ with respect to the variable t on intervals $[0, t_{i-1}], [t_{i-1}, s], [s, t_i]$ and $[t_i, 1]$. Applying the composite trapezoidal rule to the first and last integrals with quadrature nodes t_j, for $j = 0, 1, \dots, i-1$ and for $j = i, i+1, \dots, n$, respectively, and applying the trapezoidal rule to the second and third integrals with quadrature nodes t_{i-1}, s and s, t_i, respectively, we obtain the formula

$$\begin{aligned}(\mathcal{K}u)(s) \;=\; &(\mathcal{K}_n u)(s) + \frac{t_i - s}{2}\,[G(s,s)u(s) - G(s,t_{i-1})u(t_{i-1})] \\ &+ \frac{s - t_{i-1}}{2}\,[G(s,s)u(s) - G(s,t_i)u(t_i)] + E(s),\end{aligned}$$

where

$$\begin{aligned}E(s) \;:=\; &-\frac{t_{i-1}h^2}{12}\,\frac{\partial^2}{\partial t^2}\,[G(s,t)u(t)]_{|t=\xi_1} \\ &-\frac{(1-t_i)h^2}{12}\,\frac{\partial^2}{\partial t^2}\,[G(s,t)u(t)]_{|t=\xi_2} \\ &-\frac{(s-t_{i-1})^3}{12}\,\frac{\partial^2}{\partial t^2}\,[G(s,t)u(t)]_{|t=\xi_3} \\ &-\frac{(t_i-s)^3}{12}\,\frac{\partial^2}{\partial t^2}\,[G(s,t)u(t)]_{|t=\xi_4}\end{aligned}$$

with $\xi_1 \in [0, t_{i-1}]$, $\xi_2 \in [t_i, 1]$, $\xi_3 \in [t_{i-1}, s]$ and $\xi_4 \in [s, t_i]$. The assumption on the Green's function kernel G ensures that $E(s) = \mathcal{O}(h^2)$. We use the mean value theorem for the second and third terms in the formula above to conclude that

$$(\mathcal{K}u)(s) \,=\, (\mathcal{K}_n u)(s) + \mathcal{O}(h^2), \quad s \in [0,1].$$

Thus, the statement in (iv) follows. □

To prove the unique solvability of equation (2.3), or equivalently (2.4), in a neighborhood of an isolated solution u_0 of equation (1.1), we study the invertibility of the linear operator $\mathcal{I} - (\mathcal{K}_n\Psi)'(u_0)$ where the prime notation denotes the Fréchet derivative.

Lemma 2.2. *Let u_0 be an isolated solution of equation* (1.1). *Suppose that* 1 *is not an eigenvalue of* $(\mathcal{K}\Psi)'(u_0)$. *Then, for sufficiently large n, the inverse operators* $(\mathcal{I} - (\mathcal{K}_n\Psi)'(u_0))^{-1}$ *exist and are uniformly bounded on* $C[0,1]$.

Proof. Since $\mathcal{K}$ and $\mathcal{K}_n$ are bounded linear operators, we have

$$(\mathcal{K}\Psi)'(u_0) \,=\, \mathcal{K}\Psi'(u_0) \quad \text{and} \quad (\mathcal{K}_n\Psi)'(u_0) \,=\, \mathcal{K}_n\Psi'(u_0).$$

By (i) of Lemma 2.1, the sequence of approximate operators $(\mathcal{K}_n\Psi)'(u_0)$ converges pointwise to the linear operator $(\mathcal{K}\Psi)'(u_0)$, that is, for all $u \in C[0,1]$, there holds

$$\|(\mathcal{K}_n\Psi)'(u_0)u - (\mathcal{K}\Psi)'(u_0)u\| \;\longrightarrow\; 0, \quad \text{as} \quad n \to \infty.$$

Using (iii) of Lemma 2.1, we further conclude that the set of operators $\{(\mathcal{K}_n\Psi)'(u_0)\}$ are collectively compact operators. By assumption, the operator $\mathcal{I} - (\mathcal{K}\Psi)'(u_0)$ is invertible. It follows from Theorem 1.10 of [**1**] that, for sufficiently large n, $(\mathcal{I} - (\mathcal{K}_n\Psi)'(u_0))^{-1}$ exist and are uniformly bounded in $C[0,1]$. □

We also need a result from [**26**] for the analysis of the unique solvability of quadrature scheme (2.3). We state this result in the next lemma.

Lemma 2.3. *Let $\mathcal{A}$ and $\hat{\mathcal{A}}$ be continuous operators over some open set $\Omega \subset C[0,1]$. Assume that equation $u = \mathcal{A}u$ has a solution $u_0 \in \Omega$, $\hat{\mathcal{A}}$ is Fréchet differentiable in some neighborhood of u_0 and $\mathcal{I} - \hat{\mathcal{A}}'(u_0)$ is continuously invertible. Suppose that, for some $\delta > 0$,*

$$B(u_0, \delta) := \{u \in C[0,1] : \|u - u_0\| \le \delta\} \subset \Omega$$

and for $0 < q < 1$ the following inequalities hold:

$$\sup_{\|u-u_0\|\le\delta} \|(\mathcal{I} - \hat{\mathcal{A}}'(u_0))^{-1}(\hat{\mathcal{A}}(u) - \hat{\mathcal{A}}(u_0))\| \le q,$$

and

$$\varepsilon := \|(\mathcal{I} - \hat{\mathcal{A}}'(u_0))^{-1}(\mathcal{A}(u_0) - \hat{\mathcal{A}}(u_0))\| \le \delta(1-q).$$

Then the equation $u = \hat{\mathcal{A}}u$ has a unique solution $\hat{u}_0 \in B(u_0, \delta)$. Moreover,

$$\frac{\varepsilon}{1+q} \le \|u_0 - \hat{u}_0\| \le \frac{\varepsilon}{1-q}. \tag{2.6}$$

To apply this theorem to the current situation, we identify the nonlinear operators $\mathcal{A}$ and $\hat{\mathcal{A}}$ in Lemma 2.3 by

$$\mathcal{A}u := \mathcal{K}\Psi u + f, \quad \hat{\mathcal{A}}u := \mathcal{K}_n\Psi u + f.$$

Thus the Fréchet derivatives of the nonlinear operators $\mathcal{A}$ and $\hat{\mathcal{A}}$ are given, respectively, by the formulas

$$\mathcal{A}'u_0 = \mathcal{K}\Psi'(u_0), \quad \hat{\mathcal{A}}'u_0 = \mathcal{K}_n\Psi'(u_0).$$

Using Lemma 2.3 we obtain the following results concerning the solvability of the approximate equation (2.3) and the order of convergence for the approximate solutions.

Theorem 2.4. *Let $u_0 \in C[0,1]$ be an isolated solution of (1.1). Assume that 1 is not an eigenvalue of $(\mathcal{K}\Psi)'(u_0)$. Let $\psi(\cdot, u_0(\cdot)), f \in C^2[0,1]$. Then, for sufficiently large n equation (2.3) has a unique solution u_n in the ball $B(u_0, \delta)$ for some $\delta > 0$ having property*

$$\|u_0 - u_n\| \le Ch^2 \quad \textit{for some constant} \quad C > 0.$$

Proof. The proof of this theorem is done by verifying the hypotheses of Lemma 2.3. Lemma 2.2 insures that, for sufficiently large n, inverse operators $(\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}$ exist and are uniformly bounded on $C[0,1]$ by a positive constant C_0. Set

$$\mathcal{T}_n(u) := (\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}(\mathcal{K}_n\Psi(u)-\mathcal{K}_n\Psi(u_0))$$

and let C_1 be a constant that bounds the norm of operators $\mathcal{K}_n$, that is, $\|\mathcal{K}_n\| \le C_1$ for all $n \ge 1$. It follows that, for any $u \in C[0,1]$ and sufficiently large n,

$$\begin{aligned}\|\mathcal{T}_n(u)\| &\le \|(\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}\|\,\|\mathcal{K}_n\|\,\|\Psi(u)-\Psi(u_0)\|\\ &\le C_0C_1\,\|\Psi(u)-\Psi(u_0)\|.\end{aligned}$$

Let $L > 0$ be the Lipschitz constant for ψ with respect to the second variable, that is,

$$|\psi(t,u_1)-\psi(t,u_2)| \le L\,|u_1-u_2|,$$
$$\text{for all}\quad u_1,\,u_2 \in (-\infty,\infty)\quad\text{and for all}\quad t\in[0,1].$$

Thus we have the estimate

$$\|\Psi(u)-\Psi(u_0)\| \le L\,\|u-u_0\|.$$

We choose $\delta < 1/(2C_0C_1L)$ and $q = 1/2$. Then, combining the inequalities above yields

$$\sup_{\|u-u_0\|\le\delta}\|\mathcal{T}_n(u)\| \le C_0C_1\,L\,\|u-u_0\| \le C_0C_1\,L\,\delta < \frac{1}{2} = q.$$

Moreover, by Lemma 2.1 (i), for sufficiently large n,

$$\varepsilon \le C_0\|\mathcal{K}_n\Psi(u_0)-\mathcal{K}\Psi(u_0)\| < \delta(1-q),$$

where

$$\varepsilon := \|(\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}(\mathcal{K}_n\Psi(u_0)-\mathcal{K}\Psi(u_0))\|.$$

We have verified that the conditions of Lemma 2.3 are valid. Therefore, equation (2.3) has a unique solution u_n in the ball $B(u_0,\delta)$ for sufficiently large n. Moreover, by Lemma 2.1 (iv) and the estimate (2.6) in Lemma 2.3, we conclude that

$$\|u_0-u_n\| \le 2\,C_0\,\|\mathcal{K}_n\Psi(u_0)-\mathcal{K}\Psi(u_0)\| \le Ch^2,$$

where $C > 0$ is a constant independent of h. $\square$

Theorem 2.4 insures that, for sufficiently large n, the nonlinear system (2.4) has a unique solution in a neighborhood of u_0. The argument used to prove Lemma 2.2 and Theorem 2.4 has been used in [**10**], [**11**] to deal with the nonlinearity of the operators in a different context.

3. A modified Stetter's theorem. Stetter's theorem, Theorem 1 of [**24**], presents a general result regarding asymptotic error expansions of discretization algorithms for solving nonlinear functional equations. However, the theorem is not directly applicable to our current situation, because the condition (a) of Stetter's theorem does not hold in our case and condition (e) of it is not strong enough for deriving our expansion which is somewhat in a stronger form than the expansion in that theorem. To derive an asymptotic error expansion for our approximate solutions u_n at the quadrature nodes, we are required to extend Stetter's theorem so that its modified version is applicable to our case.

For this purpose, we let E, E^1 be Banach spaces $\mathcal{F}$ be a nonlinear operator mapping from a subspace D^1 of E into E^1. We consider functional equations

$$\mathcal{F}(y) = 0 \tag{3.1}$$

and assume that equation (3.1) has multiple isolated solutions in D^1.

Let $h_0 > 0$ be a fixed real number and $h \in H := (0, h_0]$. Let E_h, E_h^1 be families of Banach-spaces, and let Δ_h and Δ_h^1 be linear transformations which map E, E^1 into E_h, E_h^1, respectively. Let $\Phi_h : E_h \to E_h^1$ be a family of nonlinear operators which discretize equation (3.1) by

$$\Phi_h(\eta) = 0. \tag{3.2}$$

Let y be an isolated solution of equation (3.1), and we assume that in a neighborhood $\tilde{D} \subset D^1$ of y, equation (3.2) has a unique solution $\eta(h)$ in $\Delta_h \tilde{D}$. Suppose that, for some positive integer N, $\mathcal{F}$ and Φ_h are M-times Fréchet differentiable at y and $\Delta_h y$, respectively, with $M \geq (N+1)$. For a nonnegative integer $\hat{n}$, we further assume that the μth Fréchet derivative of Φ_h at $\Delta_h y$ satisfies the condition

$$\|\Phi_h^{(\mu)}(\Delta_h y)\| = O(h^{\hat{n}}), \quad \mu = 1, 2, \dots, M. \tag{3.3}$$

Let $Q_p \supset Q_{2p} \supset \cdots \supset Q_{(N+1)p}$ be a sequence of subspaces in D^1 with $p \geq 1$. For integer $\mu := 1, 2, \ldots, M$ and for a subspace Q of D^1, we use Q^μ to denote the tensor product space of μ copies of Q. For integer $p \geq 1$, we let $\hat{\mu}$ be the smallest integer which is greater than or equal to μ/p, and for $1 \leq r \leq N$, we let

$$\gamma := \min\{N + 1 - \hat{\mu}, r\}.$$

Then we assume that the μth Fréchet derivative of $\mathcal{F}$ at y is an operator mapping from Q^μ_{rp} into $Q_{\gamma p}$. If $z := (z_1, z_2, \ldots, z_\mu) \in Q^\mu_{rp}$, we denote

$$\Delta_h z := (\Delta_h z_1, \Delta_h z_2, \ldots, \Delta_h z_\mu).$$

We require that there be multilinear operators $f_{\nu,r,\mu}$, $\mu = 0, 1, \ldots, M$, mapping from Q^μ_{rp} into $Q_{(\gamma-\nu)p}$ such that

$$\Phi_h(\Delta_h y) = h^{\hat{n}}\left\{\Delta^1_h\left(\sum_{\nu=1}^{N} h^{\nu p} f_{\nu,N+1,0}(y)\right) + O(h^{(N+1)p})\right\}, \tag{3.4}$$

and

$$(\Phi^{(\mu)}_h(\Delta_h y))(\Delta_h z) = h^{\hat{n}}\Delta^1_h\left\{(F^{(\mu)}(y))(z) + \sum_{\nu=1}^{\gamma-1} h^{\nu p} f_{\nu,r,\mu}(z)\right\} + O(h^{\hat{n}+\gamma p}). \tag{3.5}$$

Similarly to [**24**], we define $\hat{n}$-stability. Equation (3.2) is called $\hat{n}$-stable for the isolated solution y, if there is a constant S independent of h such that each solution $\varepsilon \in E_h$ of

$$\Phi'_h(\Delta_h y)\varepsilon = \phi$$

satisfies

$$\|\varepsilon\| \leq S h^{-\hat{n}}\|\phi\|_1, \quad h \in H,$$

where $\|\cdot\|, \|\cdot\|_1$ are the norms of E_h and E^1_h, respectively. For the unique solution η of equation (3.2), we set $\varepsilon(h) := \eta(h) - \Delta_h(y)$. We say that $\eta(h)$ in $\Delta_h\tilde{D}$ converges to y with order p, if $\varepsilon(h)$ satisfies

$$\|\varepsilon(h)\| \leq C h^p, \quad h \in H \tag{3.7}$$

where C is a constant.

We now state the following modified version of Stetter's theorem.

Theorem 3.1. *Let N be a positive integer and $M \geq N+1$. Suppose that the following conditions hold.*

(a) *$\mathcal{F}$ and Φ_h are M-times Fréchet differentiable at y and $\Delta_h y$, respectively, the μth Fréchet derivative of $\mathcal{F}$ is an operator mapping from Q^{μ}_{rp} into $Q_{\gamma p}$ and $\Phi_h^{(\mu)}$ satisfies* (3.3).

(b) *Expansions* (3.4) *and* (3.5) *hold for Φ_h and $\mathcal{F}$.*

(c) *Algorithm* (3.2) *is $\hat{n}$-stable for the isolated solution y of equation* (3.1).

(d) *The unique solution $\eta(h)$ of* (3.2) *in $\Delta_h \tilde{D}$ converges to y with order p.*

(e) *$\mathcal{F}'(y)\, e = b \in Q_{rp}$ has a unique solution $e \in Q_{rp}$ for all $r = 1, 2, \dots, N$. Then there are $e_\nu \in Q_{(N+1-\nu)p}$, $\nu = 1, 2, \dots, N$, independent of h, such that*

$$(3.8) \qquad \left\| \eta(h) - \Delta_h y - \Delta_h \sum_{\nu=1}^{N} h^{\nu p} e_\nu \right\| \leq C_N h^{(N+1)p}, \quad h \in H,$$

where C_N is a constant.

Proof. We will determine $b_\nu \in Q_{(N+1-\nu)p}$, $\nu = 1, 2, \dots, N$, such that the solutions e_ν of the equations

$$(3.9) \qquad \mathcal{F}'(y) e_\nu = b_\nu$$

satisfy the estimate (3.8). By assumption (e), the functions e_ν are uniquely determined and $e_\nu \in Q_{(N+1-\nu)p}$. We set

$$(3.10) \qquad s^N(h) := \sum_{\nu=1}^{N} h^{\nu p} e_\nu \quad \text{and} \quad \bar{\varepsilon}^N(h) := \varepsilon(h) - \Delta_h s^N(h).$$

Clearly, $\|\Delta_h s^N(h)\| = O(h^p)$, and hence by assumption (d) and equation (3.10), we conclude that

$$(3.11) \qquad \|\bar{\varepsilon}^N(h)\| = O(h^p).$$

We next compute $\Phi'_h(\Delta_h y)\bar{\varepsilon}^N(h)$. In our presentation, for notational simplicity we will drop the argument $\Delta_h y$ of the multilinear operators $\Phi_h^{(\mu)}(\Delta_h y)$ and the parameter h with $\varepsilon, \bar{\varepsilon}^N$ and s^N. Since

$$\Phi_h(\Delta_h y + \Delta_h s^N + \bar{\varepsilon}^N) \,=\, \Phi_h(\eta(h)) \,=\, 0,$$

we have that

$$\begin{aligned}\Phi'_h\bar{\varepsilon}^N \,=\, &-[\Phi_h(\Delta_h y + \Delta_h s^N + \bar{\varepsilon}^N) - \Phi_h(\Delta_h y) - \Phi'_h(\Delta_h s^N + \bar{\varepsilon}^N)] \\ &- \Phi_h(\Delta_h y) - \Phi'_h \Delta_h s^N.\end{aligned}$$

Assumption (a) allows the use of the generalized Taylor expansion. By using (3.3) and assumption (d), we obtain

$$\begin{aligned}\Psi_h \,:=\, &\Phi_h(\eta(h)) - \Phi_h(\Delta_h y) - \Phi'_h(\Delta_h s^N + \bar{\varepsilon}^N) \\ =\, &\sum_{\mu=2}^{M-1} \frac{1}{\mu!}\, \Phi_h^{(\mu)}(\Delta_h s^N + \bar{\varepsilon}^N) + O(h^{\hat{n}+MP}).\end{aligned}$$

Again, by using (3.3) and noting that $M \geq N+1$, we conclude that

$$\Psi_h \,=\, \sum_{\mu=2}^{M-1} \frac{1}{\mu!}\, \Phi_h^{(\mu)}(\Delta_h s^N)^\mu + O(h^{\hat{n}+p}\|\bar{\varepsilon}^N\|) + O(h^{\hat{n}+(N+1)p}). \tag{3.12}$$

For a multiindex $\mathbf{i}_\mu := (i_1, i_2, \ldots, i_\mu) \in \mathbf{N}_N^\mu$, where $\mathbf{N}_N^\mu := \mathbf{N}_N \times \cdots \times \mathbf{N}_N$, μ-folds, we define $|\mathbf{i}_\mu| := \sum_{\nu=1}^{\mu} i_\nu$. By definition (3.10), it follows

$$\Phi_h^{(\mu)}(\Delta_h s^N)^\mu \,=\, \Phi_h^{(\mu)}\left(\sum_{\nu=1}^{N} h^{\nu p}\Delta_h e_\nu\right)^\mu \,=\, \sum_{\mathbf{i}_\mu \in \mathbf{N}_N^\mu} h^{|\mathbf{i}_\mu| p}\, \Phi_h^{(\mu)}(\Delta_h e_{\mathbf{i}_\mu}),$$

where $e_{\mathbf{i}_\mu} := (e_{i_1}, \ldots, e_{i_\mu})$. We let $M_\mu := \max\{i_1, \ldots, i_\mu\}$ and $\hat{M}_\mu := \max\{i_1, \ldots, i_\mu, \hat{\mu}\}$. Since $e_{i_\nu} \in Q_{(N+1-i_\nu)p} \subset Q_{(N+1-M_\mu)p}$ for $1 \leq \nu \leq \mu$,

$$e_{i_1}, \ldots, e_{i_\mu} \,\in\, Q_{(N+1-M_\mu)p}. \tag{3.13}$$

Then by (3.5) with $r = N+1-M_\mu$ and noticing that $\gamma - 1 = N - \hat{M}_\mu$, we have

(3.14)

$$\begin{aligned}\Phi_h^{(\mu)}(\Delta_h e_{\mathbf{i}_\mu}) \,=\, &h^{\hat{n}}\Delta_h^1\left(\mathcal{F}^{(\mu)}(e_{\mathbf{i}_\mu}) + \sum_{\nu=1}^{N-\hat{M}_\mu} h^{\nu p}\, f_{\nu,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu})\right) \\ &+ O(h^{\hat{n}+(N+1-\hat{M}_\mu)p}),\end{aligned}$$

where $f_{\nu,N+1-M_\mu},\mu : Q^\mu_{(N+1-M_\mu)p} \to Q_{(N+1-\hat{M}_\mu-\nu)p}$. Hence, combining (3.12) and (3.14) yields

(3.15)
$$\Phi_h^{(\mu)}(\Delta_h s^N)^\mu = \sum_{\mathbf{i}_\mu \in \mathbf{N}_N^\mu} h^{|\mathbf{i}_\mu|p+\hat{n}} \Delta_h^1 \bigg(\mathcal{F}^{(\mu)}(e_{\mathbf{i}_\mu}) + \sum_{\nu=1}^{N-\hat{M}_\mu} h^{\nu p} f_{\nu,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu}) \bigg) + O(h^{\hat{n}+(N+1-\hat{M}_\mu)p}).$$

Clearly, $\hat{\mu} \leq \mu$. Hence, by noticing that $i_1, \dots, i_\mu \geq 1$, we have $|\mathbf{i}_\mu| \geq \hat{M}_\mu$. This together with (3.15) yields

(3.16)
$$\Phi_h^{(\mu)}(\Delta_h s^N)^\mu = h^{\hat{n}} \sum_{\mathbf{i}_\mu \in \mathbf{N}_N^\mu} h^{|\mathbf{i}_\mu|p} \Delta_h^1 \bigg(\mathcal{F}^{(\mu)}(e_{\mathbf{i}_\mu}) + \sum_{\nu=1}^{N-\hat{M}_\mu} h^{\nu p} f_{\nu,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu}) \bigg) + O(h^{\hat{n}+(N+1)p}).$$

Thus, from (3.12) and (3.16), we have

(3.17)
$$\Psi_h = h^{\hat{n}} \sum_{\mu=2}^{M-1} \sum_{\mathbf{i}_\mu \in \mathbf{N}_N^\mu} \frac{1}{\mu!} h^{|\mathbf{i}_\mu|p} \Delta_h^1 \bigg(\mathcal{F}^{(\mu)}(e_{\mathbf{i}_\mu}) + \sum_{\nu=1}^{N-\hat{M}_\mu} h^{\nu p} f_{\nu,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu}) \bigg) + O(h^{\hat{n}+p}\|\bar{\varepsilon}^N\|) + O(h^{\hat{n}+(N+1)p}).$$

The righthand side of (3.17) can be rewritten as

$$\sum_{\nu=2}^{N} h^{\nu p} \Delta_h^1 g_\nu(y, e_1, \dots, e_{\nu-1}) + O(h^{(N+1)p}).$$

It is easily seen that the functions g_ν depend only on e_k for $k \leq \nu - 1$. Moreover, g_ν is a linear combination of $\mathcal{F}^{(\mu)}(e_{i_1}, \dots, e_{i_\mu})$ for $|\mathbf{i}_\mu| = \nu$, $\mu \geq 2$ and $\mathbf{i}_\mu \in \mathbf{N}_N^\mu$, and $f_{l,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu})$ for $|\mathbf{i}_\mu| + l = \nu$, $\mu \geq 2$ and $\mathbf{i}_\mu \in \mathbf{N}_N^\mu$.

Notice that if $|\mathbf{i}_\mu| = \nu$, $\mu \geq 2$ and $i_1, \dots, i_\mu \geq 1$, we have $1 \leq i_1, \dots, i_\mu \leq \nu$ and $2 \leq \mu \leq \nu$. Hence, $M_\mu \leq \nu$ and $\hat{\mu} \leq \mu \leq \nu$. Then by noticing (3.13) and assumption (a) with $r = N + 1 - M_\mu$ and letting

$$\gamma_\mu := \min\{N + 1 - \hat{\mu}, N + 1 - M_\mu\}$$

we conclude that $\mathcal{F}^{(\mu)}(e_{\mathbf{i}_\mu}) \in Q_{\gamma_\mu p} \subseteq Q_{(N+1-\nu)p}$. When $|\mathbf{i}_\mu| + l = \nu$, $\mu \geq 2$ and $i_1, \dots, i_\mu \geq 1$, we have that $M_\mu + l \leq |\mathbf{i}_\mu| + l = \nu$ and $l \leq \nu - \mu$. Hence, $\hat{\mu} + l \leq \mu + l \leq \nu$. Then by noticing that $f_{l,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu})$ comes from (3.5) with $r = N + 1 - M_\mu$ and (3.13), we conclude that $f_{l,N+1-M_\mu,\mu}(e_{\mathbf{i}_\mu}) \in Q_{(\gamma_\mu - l)p} \subseteq Q_{(N+1-\nu)p}$ since $\gamma_\mu - l \geq N + 1 - \nu$. It follows that $g_\nu \in Q_{(N+1-\nu)p}$.

For the remaining parts of $\Phi_h' \bar{\varepsilon}^N$ we obtain from (3.4) that

$$\Phi_h(\Delta_h y) = h^{\hat{n}} \left(\Delta_h^1 \sum_{\nu=1}^{N} h^{\nu p} f_{\nu,N+1,0}(y) + O(h^{(N+1)p}) \right) \tag{3.21}$$

and from (3.5), we have that

$$\Phi_h'(\Delta_h s^N) = h^{\hat{n}} \Bigg(\sum_{i=1}^{N} h^{ip} \Delta_h^1 \Bigg(\mathcal{F}'(e_i) + \sum_{\nu=1}^{N-i} h^{\nu p} f_{\nu,N+1-i,1}(e_i) + O(h^{\min(N,N+1-i)p}) \Bigg) \Bigg).$$

We rearrange the terms according to the powers of h and obtain that

$$\Phi_h'(\Delta_h s^N) = h^{\hat{n}} \left(\sum_{\nu=1}^{N} h^{\nu p} \Delta_h^1 \left(b_\nu + \sum_{\lambda=1}^{\nu-1} f_{\lambda,N+1-\nu+\lambda,1}(e_{\nu-\lambda}) \right) \right) + O(h^{\hat{n}+(N+1)p}). \tag{3.22}$$

Collecting the various expressions and letting $g_1 = 0$, we have

$$\begin{aligned} \Phi_h' \bar{\varepsilon}^N = -h^{\hat{n}} \Bigg(\Delta_h^1 \sum_{\nu=1}^{N} h^{\nu p} (g_\nu(y, e_1, \dots, e_{\nu-1}) + f_{\nu,N+1,0}(y) + b_\nu \\ + \sum_{\lambda=1}^{\nu-1} f_{\lambda,N+1-\nu+\lambda,1}(e_{\nu-\lambda}) \Bigg) \Bigg) \\ + O(h^{\hat{n}+p} \|\bar{\varepsilon}^N\|) + O(h^{\hat{n}+(N+1)p}). \end{aligned} \tag{3.23}$$

Again, since the term $f_{\lambda,N+1-\nu+\lambda,1}(e_{\nu-\lambda})$ in (3.23) is from (3.5) and $e_{\nu-\lambda} \in Q_{(N+1-\nu+\lambda)p}$, we conclude that it is in $Q_{(N+1-\nu)p}$. In fact, by noticing (3.5) and $e_{\nu-\lambda} \in Q_{(N+1-\nu+\lambda)p}$ and $1 \le \lambda \le \nu - 1$, we have

$$\begin{aligned} f_{\lambda,N+1-\nu+\lambda,1}(e_{\nu-\lambda}) &\in Q_{(\min(N+1-1,N+1-\nu+\lambda)-\lambda)p} \\ &= Q_{(N+1-\nu)p}. \end{aligned} \tag{3.24}$$

Now, for $\nu = 1, 2, \dots, N$, we can recursively choose b_ν which annihilate the brackets in (3.23) since the corresponding conditions for the b_ν are in $Q_{(N+1-\nu)p}$ because of (3.20), (3.4) and (3.24) and contain only e_k's with $k < \nu$ while the ones for b_1 do not contain an e_k at all. Thus, through (3.9), all the b_ν, e_ν are uniquely determined for $\nu = 1, 2, \dots, N$. With this choice of the e_ν, (3.23) is reduced to

$$\Phi_h' \bar{\varepsilon}^N = O(h^{\hat{n}+p}\|\bar{\varepsilon}^N\|) + O(h^{\hat{n}+(N+1)p}). \tag{3.25}$$

By assumption (c) and (3.11) we conclude from (3.25) inductively $\|\bar{\varepsilon}^N\| = O(h^{jp})$, $j = 2, 3, \dots$, until j would surpass $N+1$ and final estimate $\|\bar{\varepsilon}^N\| = O(h^{\hat{n}+(N+1)p})$ is reached. The proof is now complete. □

4. An asymptotic error expansion. In this section we establish an asymptotic error expansion of the approximate solution u_n at quadrature nodes, which leads to an extrapolation scheme.

We first recall a result proved in [**23**] regarding an asymptotic error expansion of the trapezoidal rule applied to integral operators with Green's function kernels.

Theorem 4.1. *Let G be a Green's function kernel, let $u \in C^{2m}[0,1]$, and let $\tilde{G}(s,t) := G(s,t)u(t)$, $s, t \in [0,1]$. Then*

$$\begin{aligned} \int_0^1 \tilde{G}(t_i,t)\,dt &= h \sum_{j=0}^{n}{}'' \tilde{G}(t_i,t_j) + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!}\,\alpha_{2j-1}(t_i) h^{2j} \\ &\quad + \mathcal{O}(h^{2m}), \quad i = 0, 1, \dots, n, \end{aligned} \tag{4.1}$$

where B_j are Bernoulli numbers and

$$\alpha_j(t_i) := \tilde{G}^{(0,j)}(t_i,0) - \tilde{G}^{(0,j)}(t_i,1) + [\tilde{G}^{(0,j)}(t_i,t_i)], \tag{4.2}$$

with $[f(s)] := f(s^+) - f(s^-)$.

This formula can be obtained by using the classical Euler-Maclaurin formula (cf. [**7**]) on two intervals $[0, t_i]$ and $[t_i, 1]$ and then simplifying it. For details of a proof, see [**23**].

Using the operator notation, we have for $u \in C^{2r}[0,1]$, $1 \le r \le m$ and $i = 0, 1, \ldots, n$, that

$$(\mathcal{K}u)(t_i) \;=\; (\mathcal{K}_n u)(t_i) + \sum_{j=1}^{r-1} \frac{B_{2j}}{(2j)!}\, \alpha_{2j-1}(t_i)\, h^{2j} + \mathcal{O}(h^{2m}). \tag{4.3}$$

Formula (4.3) gives an asymptotic error expansion for $\mathcal{K}_n u$ approximating $\mathcal{K}u$ at the quadrature nodes. We next use this asymptotic expansion and Theorem 3.1 to derive an asymptotic expansion for the approximate solutions u_n at the quadrature nodes.

To apply Theorem 3.1, we let $D^1 = E = E^1 = C[0,1]$ and introduce nonlinear operators $\mathcal{F} : D^1 \to E^1$ by

$$\mathcal{F}(u) \;=\; \mathcal{I}u - \mathcal{K}\Psi u - f.$$

In this notation equation (1.1) can be written as

$$\mathcal{F}(u) \;=\; 0. \tag{4.4}$$

For any positive integer n, we let $h = 1/n$ and define $E_h = R^{n+1}$ with norm $\|\cdot\|_h$ defined by

$$\|\varepsilon\|_h \;=\; \max_{0 \le i \le n} |\varepsilon_i|, \quad \varepsilon \in E_h.$$

Define linear operator $\Delta_h : E \to E_h$ by

$$\Delta_h u \;=\; (u(t_0), u(t_1), \ldots, u(t_n))^T. \tag{4.5}$$

Since $E = E^1$, we let $E_h^1 = E_h$ and $\Delta_h^1 = \Delta_h$. We now choose $\Phi_h : E_h \to E_h^1$ to be

$$(\Phi_h \eta)_i \;=\; \eta_i - h \sum_{j=0}^{n}{}'' \, G(t_i, t_j)\psi(t_j, \eta_j) - f(t_i), \quad i = 0, 1, \ldots, n. \tag{4.6}$$

Hence, for any $u \in E$, we have

$$\Phi_h(\Delta_h u) \,=\, \Delta_h^1 \, (\mathcal{I}u - \mathcal{K}_n \Psi u - f), \tag{4.7}$$

and equation (2.4) is equivalent to

$$\Phi_h \, \eta \,=\, 0. \tag{4.8}$$

Then we have the following theorem which is the main result of this paper and concerns the asymptotic error expansion of the approximate solution u_n at the quadrature nodes.

Theorem 4.2. *Suppose that the assumptions on the kernel $G(s,t)$, ψ, and righthand side function f of* (1.1) *hold. Let $u_0 \in C[0,1]$ be an isolated solution of equation* (1.1). *Assume that* 1 *is not an eigenvalue of $(\mathcal{K}\Psi)'(u_0)$. Then for a sufficiently large n, equation* (2.3) *has a unique solution $u_n \in B(u_0, \delta)$ for some $\delta > 0$ which has the asymptotic expansion*

$$u_n(t_i) \,=\, u_0(t_i) + \sum_{j=1}^{m-1} e_j(t_i) \, h^{2j} + \mathcal{O}(h^{2m}), \quad i = 0, 1, \dots, n, \tag{4.9}$$

where functions $e_j \in C^{2(m-j)}[0,1]$ are independent of h.

Proof. We first notice that it follows from Theorem 2.4 that, for sufficiently large n equation, (2.3) has a unique solution u_n in the ball $B(u_0, \delta)$ for some $\delta > 0$. Hence, it remains to prove (4.9). Since equation (4.8) is equivalent to (2.4), the existence of the unique solution u_n in $B(u_0, \delta)$ for some $\delta > 0$ for sufficiently large n implies that equation (4.8) has a unique solution $\eta(h)$ in $\Delta_h B(u_0, \delta)$ for sufficiently small h and $\eta(h) = \Delta_h u_n$. Notice that u_0 is an isolated solution of equation (4.4). Thus, it suffices to show that the conditions of Theorem 3.1 hold for $M = N + 1 = m$ and $p = 2$.

Since $\psi \in C^{2m}([0,1] \times R)$, Ψ is M-times Fréchet differentiable at u_0 and, for $\mu = 1, 2, \dots, M$, $u_1, \dots, u_\mu \in E$, we have

$$\Psi^{(\mu)}(u_0)(u_1, \dots, u_\mu) \,=\, \psi^{(0,\mu)}(\cdot, u_0(\cdot))(u_1 \cdots u_\mu). \tag{4.10}$$

It follows that $\mathcal{F}$ is M-times differentiable at u_0 and

$$\mathcal{F}'(u_0) = \mathcal{I} - \mathcal{K}\,\Psi'(u_0), \tag{4.11a}$$

and

$$\mathcal{F}^{(\mu)}(u_0) = -\mathcal{K}\,\Psi^{(\mu)}(u_0), \quad \mu = 2, \dots, M. \tag{4.11b}$$

It follows again from the fact that $\psi \in C^{2m}([0,1] \times R)$ that Φ_h is M-times Fréchet differentiable at $\Delta_h u_0$ and

$$\Phi_h'(\Delta_h u_0)\,\eta = \eta - \Delta_h^1\biggl(h \sum_{j=0}^{n}{}'' \, G(s,t_j)\psi^{(0,1)}(t_j, u_0(t_j))\eta_j\biggr) \tag{4.12a}$$

and, for $\mu = 2, \dots, M$,

$$\begin{aligned}\Phi_h^{(\mu)}&(\Delta_h u_0)(\eta_1, \dots, \eta_\mu)\\ &= \Delta_h^1\biggl(-h \sum_{j=0}^{n}{}'' G(s,t_j)\psi^{(0,\mu)}(t_j, u_0(t_j))(\eta_1)_j \cdots (\eta_\mu)_j\biggr).\end{aligned} \tag{4.12b}$$

Hence, for any $u \in E = C[0,1]$, from (4.10) and (4.12a) we have

$$\Phi_h'(\Delta_h u_0)(\Delta_h u) = \Delta_h^1\,((\mathcal{I} - \mathcal{K}_n \Psi'(u_0))u) \tag{4.13a}$$

and for $\mu = 2, \dots, M$, and any $u_1, \dots, u_\mu \in E$, we have

$$\begin{aligned}\Phi_h^{(\mu)}&(\Delta_h u_0)(\Delta_h u_1, \dots, \Delta_h u_\mu)\\ &= \Delta_h^1(-\mathcal{K}_n \Psi^{(\mu)}(u_0)(u_1, \dots, u_\mu)).\end{aligned} \tag{4.13b}$$

Since $\Phi_h^{(\mu)}$, $\mu = 1, \dots, M$, are bounded operators, they satisfy (3.3) with $\hat{n} = 0$. Hence, assumption (a) of Theorem 3.1 holds. Remember that $p = 2$ and $M = N + 1 = m$ in this case. Let $Q_{rp} = C^{2r}[0,1]$, $1 \le r \le N + 1$. Notice that it follows from the hypotheses on $G(s,t), \psi(s,t)$ and f that $u_0 \in C^{2m}[0,1] = Q_{(N+1)p}$. Then, from (4.7) and (4.3) and noticing that $\mathcal{F}(u_0) = \mathcal{I}u_0 - \mathcal{K}\Psi u_0 - f = 0$, we have

$$\begin{aligned}\Phi_h(\Delta_h u_0) &= \Delta_h^1(\mathcal{I}u_0 - \mathcal{K}_n\Psi u_0 - f)\\ &= \Delta_h^1\bigg(\mathcal{I}u_0 - \mathcal{K}\Psi u_0 - f + \sum_{j=1}^{N}\beta_{(j,N+1,0)}(u_0)\, h^{2j}\bigg)\\ &\quad + O(h^{(N+1)p})\\ &= \Delta_h^1\bigg(\mathcal{F}(u_0) + \sum_{j=1}^{N}\beta_{(j,N+1,0)}(u_0)\, h^{2j}\bigg) + O(h^{(N+1)p})\\ &= \Delta_h^1\bigg(\sum_{j=1}^{N}\beta_{(j,N+1,0)}(u_0)\, h^{2j}\bigg) + O(h^{(N+1)p}),\end{aligned}$$

where

$$\begin{aligned}&\beta_{(j,N+1,0)}(u_0)\\ &= \frac{B_{2j}}{(2j)!}\big((G(s,t)\psi(t,u(t)))^{(0,2j-1)}|_{t=0} - (G(s,t)\psi(t,u(t)))^{(0,2j-1)}|_{t=1}\\ &\qquad + (G(s,t)\psi(t,u(t)))^{(0,2j-1)}|_{t=s^+}\\ &\qquad\qquad - (G(s,t)\psi(t,u(t)))^{(0,2j-1)}|_{t=s^-}\big)\end{aligned}$$

and B_{2j} are Bernoulli numbers. It follows from the assumption on $G(s,t)$ and $\psi(s,t)$ that $\beta_{(j,N+1,0)}(u_0) \in C^{2m-2j+1}[0,1] \subset C^{2m-2j}[0,1] = Q_{(N+1-j)p}$. Hence, equation (3.4) of Theorem 3.1 holds. Now we show that (3.6) and (3.5) of Theorem 3.1 also hold for this case. Note that from (4.10) we have $\Psi^{(\mu)}(u_0) : Q_{rp}^{\mu} \to C^{\min(2m-\mu,rp)}[0,1] = C^{\min(m-\mu/p,r)p}[0,1] \subset Q_{\min(N+1-\hat{\mu},r)p}$. Then it follows from (4.11) and the assumption on $G(s,t)$ that

$$\mathcal{F}^{(\mu)}(u_0) : Q_{rp}^{\mu} \longrightarrow Q_{\gamma p}, \quad 1 \le r \le N, \quad 1 \le \mu \le M.$$

That is, $\mathcal{F}^{(\mu)}(u_0)$ satisfies (3.6). To show that equation (3.5) also holds, we notice from (4.13b) and (4.3) that, for any $z_1, \dots, z_\mu \in Q_{rp} =$

$C^{2r}[0,1]$, $\mu = 2, \dots, M$,

$$\begin{aligned}
&\Phi^{(\mu)}(\Delta_h u_0)(\Delta_h z_1, \dots, \Delta_h z_\mu) \\
&= \Delta_h^1(-\mathcal{K}_n(\psi^{(0,\mu)}(t, u_0(t)) z_1 \cdots z_\mu)) \\
&= \Delta_h^1(-\mathcal{K}(\psi^{(0,\mu)}(t, u_0(t)) z_1 \cdots z_\mu) + \sum_{j=1}^{\gamma-1} h^{jp} \beta_{(j,r,\mu)}(z_1 \dots, z_\mu)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + O(h^{\gamma p}) \\
&= \Delta_h^1(\mathcal{F}^{(\mu)}(u_0)(z_1, \dots, z_\mu) + \sum_{j=1}^{\gamma-1} h^{jp} \beta_{(j,r,\mu)}(z_1, \dots, z_\mu)) + O(h^{\gamma p})
\end{aligned}$$

where

$$\begin{aligned}
&\beta_{(j,r,\mu)}(z_1, \dots, z_\mu) \\
&\quad = \frac{B_{2j}}{(2j)!} \big((G(s,t)\psi^{(0,\mu)}(t, u_0(t)) z_1(t) \cdots z_\mu(t))^{(0,2j-1)}|_{t=0} \\
&\qquad\qquad - (G(s,t)\psi^{(0,\mu)}(t, u_0(t)) z_1(t) \cdot z_\mu(t))^{(0,2j-1)}|_{t=1} \\
&\qquad\qquad + (G(s,t)\psi^{(0,\mu)}(t, u_0(t)) z_1(t) \cdots z_\mu(t))^{(0,2j-1)}|_{t=s^+} \\
&\qquad\qquad - (G(s,t)\psi^{(0,\mu)}(t, u_0(t)) z_1(t) \cdots z_\mu(t))^{(0,2j-1)}|_{t=s^-} \big).
\end{aligned}$$

Notice that it follows from the assumptions on $G(s,t)$ and $\psi(s,t)$ that $\beta_{(j,r,\mu)}(z_1, \dots, z_\mu) \in C^{\min(2m-\mu,2r)-2j+1}[0,1] \subset Q_{(\gamma-j)p}$. Hence, $\Phi_h^{(\mu)}(\Delta_h u_0)$ satisfies (3.5) for $\mu = 2, \dots, M$. Similarly, it can be shown that $\Phi_h'(\Delta_h u_0)$ also satisfies (3.5). Hence, assumption (b) of Theorem 3.1 also holds.

We will show that $\Phi_h'(\Delta_h u_0)$ satisfies assumption (c) of Theorem 3.1 with $\hat{n} = 0$ by showing that the inverse operators of $\Phi_h'(\Delta_h u_0)$ exist and are uniformly bounded for sufficiently small h. Since 1 is not an eigenvalue of $(\mathcal{K}\Psi)'(u_0)$, it follows from Lemma 2.2 that, for sufficiently small h, the inverse operators $(I - (\mathcal{K}_n\Psi)'(u_0))^{-1}$ exist and are uniformly bounded on $C[0,1] = E^1$. We first prove that $\Phi_h'(\Delta_h u_0)$ is bijective. Let $\varepsilon_1, \varepsilon_2 \in E_h$ such that $\Phi_h'(\Delta_h u_0)\varepsilon_1 = \phi \in E_h^1 = E_h$ and $\Phi_h'(\Delta_h u_0)\varepsilon_2 = \phi$. Then $\Phi_h'(\Delta_h u_0(\varepsilon_2 - \varepsilon_1) = 0$. From (4.12a), we have

$$(\varepsilon_2 - \varepsilon_1)_i - h \sum_{j=0}^{n}{}'' \, G(t_i, t_j)\, \psi^{(0,1)}(t_j, u_0(t_j))(\varepsilon_2 - \varepsilon_1)_j = 0,$$
$$i = 0, \dots, n.$$

Let

$$u_n(s) \,=\, h \sum_{j=0}^{n}{}'' \, G(s,t_j)\, \psi^{(0,1)}(t_j, u_0(t_j))(\varepsilon_2-\varepsilon_1)_j.$$

Then $\Delta_h u_n = \varepsilon_2 - \varepsilon_1$. Moreover, by noticing that $(\mathcal{K}\Psi)'(u_0)) = \mathcal{K}\Psi'(u_0)$, it can be easily checked that $u_n(t)$ is a solution of the equation $(\mathcal{I}-(\mathcal{K}_n\Psi)'(u_0))u = 0$. Since $(\mathcal{I}-\mathcal{K}_n\Psi)'(u_0))^{-1}$ exist for sufficiently large n, it follows that for sufficiently large n, that is, for sufficiently small h, that $u_n(t) = 0$. Hence, $\varepsilon_1 = \varepsilon_1$ for sufficiently small h. Thus, $\Phi_h'(\Delta_h u_0)$ is injective for sufficiently small h. Let $\phi_h \in E_h^1$ be an arbitrary element. Clearly, by the definition of Δ_h, we can find a $\phi \in E^1$ such that $\Delta_h \phi = \phi_h$. Let $\varepsilon_h = \Delta_h((\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}\phi)$ for sufficiently large n. Then by noticing (4.13a), it follows that for sufficiently large n,

$$\begin{aligned}\Phi_h'(\Delta_h u_0)\varepsilon_h \,&=\, \Delta_h\left((\mathcal{I}-(\mathcal{K}_n\Psi)'(u_0))((\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}\phi\right)\\ &=\, \Delta_h\left(\phi\right) \,=\, \phi_h.\end{aligned}$$

Hence, $\Phi_h'(\Delta_h u_0)$ is bijective for sufficiently large n, that is, for sufficiently small h. That is, the inverse operators $(\Phi_h'(\Delta_h u_0))^{-1}$ exist for sufficiently small h. In fact, they are also uniformly bounded for sufficiently small h. To see this, let $\phi_h \in E_h^1$ be an arbitrary element. It is clear from the definition of Δ_h that there exists a $\phi \in E^1 = E = C[0,1]$ such that $\Delta_h\phi = \phi_h$ and $\|\phi\| = \|\phi_h\|_h$, for example, we can let ϕ be the linear interpolation function of ϕ_h in $C[0,1]$. Let $\varepsilon = (\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}\phi$ and $\varepsilon_h = \Delta_h\varepsilon$. Then

$$(\mathcal{I}-\mathcal{K}_n\Psi'(u_0))\,\varepsilon \,=\, \phi.$$

Hence, by (4.13a) we have

$$\Phi_h'(\Delta_h u_0)(\Delta_h\varepsilon) \,=\, \Delta_h^1\left((\mathcal{I}-\mathcal{K}_n\Psi'(u_0))\varepsilon\right) \,=\, \Delta_h(\phi) \,=\, \phi_h.$$

Thus, $(\Phi_h'(\Delta_h u_0))^{-1}\phi_h = \Delta_h\varepsilon$. It follows that

$$\begin{aligned}\frac{\|(\Phi_h'(\Delta_h u_0))^{-1}\phi_h\|_h}{\|\phi_h\|_h} \,&=\, \frac{\|\Delta_h\varepsilon\|_h}{\|\phi_h\|_h}\\ &=\, \frac{\|\Delta_h((\mathcal{I}-\mathcal{K}_n\Psi'(u_0))^{-1}\phi)\|_h}{\|\phi_h\|_h}\end{aligned}$$

$$\leq \frac{\|(\mathcal{I} - \mathcal{K}_n \Psi'(u_0))^{-1}\phi)\|}{\|\phi\|}$$
$$\leq \|(\mathcal{I} - \mathcal{K}_n \Psi'(u_0))^{-1}\|$$

where we have used the fact that $\|\phi_h\| = \|\phi\|$ and $\|\Delta_h\| \leq 1$. Hence, $(\Phi_h'(\Delta_h u_0))^{-1}$ is uniformly bounded for sufficiently large n. Hence, the assumption (c) of Theorem 3.1 holds for $\hat{n} = 0$.

For the assumption (d) of Theorem 3.1, by Theorem 2.4 and noticing that $\eta(h) = \Delta_h u_n$ (see the beginning of this proof) and $\|\Delta_h\| \leq 1$, we have that for sufficiently large n,

$$\|\eta(h) - \Delta_h u_0\|_h = \|\Delta_h(u_n - u_0)\|_h \leq \|u_n - u_0\| \leq Ch^2$$

for some constant $C > 0$. Hence, assumption (d) of Theorem 3.1 holds.

Finally we show that assumption (e) of Theorem 3.1 also holds. Since 1 is not an eigenvalue of $(\mathcal{K}\Psi)'(u_0)$, $\mathcal{F}'(u_0) = \mathcal{I} - \mathcal{K}\Psi'(u_0)$ has inverse. Hence, the equation $\mathcal{F}'(u_0)e = b$ has unique solution $e = (\mathcal{F}'(u_0))^{-1}b$. By the assumption on $G(s,t)$ and $\phi(s,t)$, it is clear that if $b \in Q_{rp} = C^{2r}[0,1]$, $1 \leq r \leq N$, $e = (\mathcal{F}'(u_0))^{-1}b \in Q_{rp}$. Hence, assumption (e) of Theorem 3.1 also holds. This completes the proof. □

The asymptotic error expansion (4.9) suggests the following extrapolation scheme for the sequence $u_n(t_i)$. For each t_i, $i = 0, 1, \ldots, n$, define

$$u_{n,0}^i := u_n(t_i),$$

and

$$u_{n,l}^i := \frac{2^{2l}\, u_{2n,l-1}^{2i} - u_{n,l-1}^i}{2^{2l} - 1}, \quad l = 1, 2, \ldots, m-1. \tag{4.14}$$

Using this extrapolation scheme and Theorem 3.2, we have the next theorem regarding an asymptotic expansion of the extrapolated values $u_{n,l}^i$.

Theorem 4.3. *Suppose that the conditions of Theorem* 3.2 *hold. Then*

$$u_{n,l}^i = u_0(t_i) + \sum_{j=l+1}^{m-1} e_{l,j}(t_i)\, h^{2j} + O(h^{2m}), \quad i = 0, 1, \ldots, n,$$

where $e_{l,j}$ are functions independent of h for $l = 1, 2, \dots, m-1$.

5. A reconstruction. In this section we propose a reconstruction of a continuous function from the extrapolated discrete values at the quadrature nodes and show that it approximates the exact solution u_0 to higher order in the uniform norm.

Once the extrapolated discrete values at the quadrature nodes are obtained from the algorithm described in the last section, we can use a standard way to construct a continuous function from these discrete values by letting

$$u_{n,l}(s) \,=\, h \sum_{j=0}^{n}{}'' \, G(s, t_j)\, \psi(t_j, u_{n,l}^j) + f(s).$$

However, even though the extrapolated values $u_{n,l}^i$ approximate $u_0(t_i)$ to order $\mathcal{O}(h^{2(l+1)})$, we can only have

$$\|u_{n,l} - u_0\| \,=\, \mathcal{O}(h^2).$$

This shows that the order of $u_{n,l}$ approximating u_0 in the uniform norm is not improved, and therefore, it suggests that we should look for an alternative method of reconstruction.

We propose a reconstruction method by using interpolation. This method reconstructs an alternative continuous piecewise polynomial $\hat{u}_{n,l}$ which interpolates the extrapolated values $u_{n,l}^i$ at t_i, $i = 0, 1, \dots, n$, and has the following order of convergence

$$\|\hat{u}_{n,l} - u_0\| \,=\, \mathcal{O}(h^{2(l+1)}).$$

Specifically, when $n = (2l+1)N$ for some integer N, we construct a polynomial of degree $2l+1$ on each interval $I_i := [(i-1)/N, (i/N)]$ that interpolates the values $u_{n,l}^i$, $i = 0, 1, \dots, n$. To describe this construction, we let $k = 2l+2$ and let l_μ be the fundamental Lagrange interpolatory polynomials of degree $k-1$ on $[0, 1]$ at the interpolation nodes $\mu/(k-1)$, $\mu = 0, 1, \dots, k-1$, that is, l_μ satisfies the conditions that

$$l_\mu\left(\frac{\nu}{k-1}\right) \,=\, \delta_{\mu,\nu}, \quad \mu,\, \nu = 0, 1, \dots, k-1,$$

where $\delta_{\mu,\nu} = 1$ if $\mu = \nu$ and 0 if $\mu \neq \nu$. We will "copy" these k functions to each of the subintervals I_i. To do this, we introduce for each $i = 1, 2, \ldots, N$, an affine mapping

$$F_i(t) := \frac{1}{N}t + \frac{i-1}{N}, \quad t \in [0,1],$$

which maps $[0,1]$ bijectively onto I_i. Using these affine maps, we define for $\mu = 0, 1, \ldots, k-1$, $i = 1, 2, \ldots, N$,

$$L_{i,\mu} := l_\mu \circ F_i^{-1} \chi_{I_i},$$

and set

$$\hat{u}_{n,l}(t) = \sum_{\mu=0}^{k-1} u_{n,l}^{(i-1)k+\mu} L_{i,\mu}(t), \quad t \in I_i, \quad i = 1, 2, \ldots, N.$$

This function is the continuous piecewise polynomial of degree $k-1$ on $[0,1]$ that interpolates the discrete values $u_{n,l}^j$ at t_j, $j = 0, 1, \ldots, n$. They provide us with extrapolated approximate solutions of equation (1.1). Moreover, we define the interpolatory projection $\mathcal{P}_n$ by the formula

$$(\mathcal{P}_n f)(t) := \sum_{\mu=0}^{k-1} f(t_{(i-1)k+\mu}) L_{i,\mu}(t), \quad t \in I_i,$$

where $f \in C[0,1]$. Clearly, the uniform norm of $\mathcal{P}_n$ is bounded independent of n and

$$\|\mathcal{P}_n f - f\| = \mathcal{O}(h^{2l+2}), \tag{5.1}$$

when $f \in C^{2l+2}[0,1]$. The next theorem shows that we have a uniform norm estimate for order of convergence of $\hat{u}_{n,l}$.

Theorem 5.1. *Suppose that the conditions of Theorem* 3.2 *hold. If* $n = (2l+1)N$, *then*

$$\|\hat{u}_{n,l} - u_0\| = \mathcal{O}(h^{2(l+1)}).$$

Proof. Let

$$C := \sum_{\mu=0}^{k-1} \|l_\mu\|.$$

According to the construction of basis functions $L_{i,\mu}$, $i = 1, 2, \ldots, N$, we conclude that

$$\sup_{t \in I_i} \sum_{\mu=0}^{k-1} |L_{i,\mu}(t)| \le C, \quad i = 1, 2, \ldots, N.$$

For $t \in I_i$, for $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} |\hat{u}_{n,l}(t) - u_0(t)| &\le |\hat{u}_{n,l}(t) - (\mathcal{P}_n u_0)(t)| + |(\mathcal{P}_n u_0)(t) - u_0(t)| \\ &\le C \max_{0 \le \mu \le k-1} |u_{n,l}^{(i-1)k+\mu} - u_0(t_{(i-1)k+\mu})| \\ &\quad + \|\mathcal{P}_n u_0 - u_0\|. \end{aligned}$$

By Theorem 3.4, the first term is bounded by $\mathcal{O}(h^{2(l+1)})$ and by estimate (5.1), the second term is also bounded by the same order. This concludes the result of this theorem. □

6. Numerical examples. In this section we present two numerical examples to illustrate the theory developed in the previous sections. We also use these two examples to show the applications of the methods proposed in this paper to numerical solutions of two-point boundary value problems of ordinary differential equations.

In these two examples we use the trapezoidal rule for the quadrature scheme and Picard iteration to solve the discrete system (2.3) of nonlinear equations and obtain approximate solutions for the special cases of (1.1). Then, extrapolation scheme (4.14) with $l = 1$ is used to demonstrate the acceleration of convergence of the extrapolation. The unique solvability of the discrete systems of these two examples follows from the general theory presented in Section 2 (Theorem 2.4). The existence of a unique discrete solution to the quadrature method for these two examples was also proved in [**23**]. It was also shown in [**23**] that the Picard iterations for the corresponding nonlinear systems for these two examples converge.

Example 1. Consider the integral equation

$$u(s) - \int_0^1 g_k(s,t)\,[k^2 u(t) - 2(u(t))^3]\,dt = h(s), \quad 0 \le s \le 1, \tag{6.1}$$

where $g_k(s,t)$ and $h(s)$ are given by (1.3) and (1.4) with $a = 2$ and $b = 2/3$, respectively. This equation can be derived (cf. [**18**]) from the boundary value problem

$$u''(s) = 2u^3(s), \quad 0 < s < 1, \tag{6.2}$$

$$u(0) = 2, \quad u(1) = 2/3 \tag{6.3}$$

by subtracting k^2u from both sides and inverting the lefthand side. We note that problem (6.2)–(6.3) is easily solved by finite difference methods yielding essentially the same accuracy.

Ordinary differential equations of the form $u'' = cu^n$ with $c > 0$ and $n \geq 1$ occur in nth order reaction kinetics (see [**2**]). Normally, $u'(0) = 0$ and $u(1)$ is given rather than the Dirichlet condition (6.3). We use the Dirichlet boundary conditions because we can obtain the exact solution

$$u(s) = \frac{1}{s + (1/2)}$$

to problem (6.2)–(6.3) and hence equation (6.1) so that the error in the approximate solutions can be computed. Even without knowing the exact solution, Theorem 1 in [**18**] guarantees the existence of a unique solution to (6.1) satisfying $0 \leq u(x) \leq 2$. For $k^2 = 12$, it is given by the sequence of continuous approximation starting with $h(s)$ as the first member of the sequence (see [**18**]). Therefore, for $k^2 = 12$, we obtain the solution of the nonlinear system (2.3) for equation (6.1) by computing the Picard iterations given by the discrete sequence,

$$\begin{aligned} u_n^0(t_i) &= f(t_i), \quad i = 0, 1, \dots, n \\ u_n^{m+1}(t_i) &= f(t_i) + (\mathcal{K}_n \Psi u_n^m)(t_i), \quad i = 0, 1, \dots, n, \quad m = 0, 1, \dots . \end{aligned}$$

The following two tables give the error of the approximate solutions using different step sizes $h = (1/n)$ and of the extrapolated solutions, respectively. For each value of h, 21 Picard iterations were necessary to get the difference in successive iterates to be less than 10^{-12} in absolute value. In Table 1 we use $e_i = |u(t_i) - u_n(t_i)|$ to denote the error of the quadrature scheme solutions corresponding to the specified $h = (1/n)$. The rate of convergence guaranteed by Theorem 2.4 is of order 2.

In Table 2, we list the error of the extrapolated solutions by using the extrapolation scheme (4.14). We use e_i^1 to denote the error of the one step extrapolation obtained by using the approximate solution

TABLE 1. Error of quadrature solution for Example 1.

t_i	e_i with $h=1/20$	e_i with $h=1/40$	e_i with $h=1/80$	rate
0.1	0.1079E-02	0.2713E-03	0.6791E-04	1.9
0.2	0.1620E-02	0.4063E-03	0.1016E-03	1.9
0.3	0.1912E-02	0.4791E-03	0.1198E-03	1.9
0.4	0.2047E-02	0.5126E-03	0.1282E-03	1.9
0.5	0.2052E-02	0.5135E-03	0.1284E-03	1.9
0.6	0.1929E-02	0.4825E-03	0.1206E-03	1.9
0.7	0.1672E-02	0.4181E-03	0.1045E-03	1.9
0.8	0.1273E-02	0.3181E-03	0.7954E-04	2.0
0.9	0.7193E-03	0.1797E-03	0.4493E-04	2.0

corresponding to $h = 1/20$ and $h = 1/40$, i.e., $e_i^1 = |u(t_i) - u_{n,1}(t_i)|$ where $u_{n,1}(t_i)$ is given by (4.14) and $n = 20$. Likewise, e_i^2 denotes the error of the one step extrapolation obtained by using the approximate solution corresponding to $h = 1/40$ and $h = 1/80$. The rate of convergence guaranteed by Theorem 3.4 is of order 4.

TABLE 2. Errors of the extrapolated solution for Example 1.

t_i	e_i^1	e_i^2	rate
0.1	0.1783E-05	0.1139E-06	3.9
0.2	0.1770E-05	0.1129E-06	3.9
0.3	0.1386E-05	0.8851E-07	3.9
0.4	0.9540E-06	0.6105E-07	3.9
0.5	0.5604E-06	0.3612E-07	3.9
0.6	0.2395E-06	0.1581E-07	3.9
0.7	0.1212E-07	0.1379E-08	3.1
0.8	0.1067E-06	0.6250E-08	4.0
0.9	0.1093E-06	0.6612E-08	4.0

Clearly, the extrapolation process accelerated the order of convergence by two. This numerical result confirms the theoretic estimate given in Theorem 4.3 with $l = 1$.

Example 2. Consider the integral equation

$$u(s) - \int_0^1 g_k(s,t)\,[k^2 u(t) + 16e^{-u(t)}]\,dt \;=\; h(s), \quad 0 \le s \le 1, \tag{6.4}$$

where $g_k(s,t)$ and $h(s)$ are given by (1.6) and (1.8), respectively. This equation can be derived [**20**] from the boundary value problem of the regular singular differential operator

$$u''(s) + \frac{1}{s}\,u'(s) \;=\; -16e^{-u(s)}, \quad 0 < s < 1, \tag{6.5}$$

$$u'(0) \;=\; 0, \quad u(1) \;=\; 0 \tag{6.6}$$

by subtracting $k^2 u$ from both sides and inverting the lefthand side. This problem is an example of the second boundary value problem described in Section 1. Problems of this form are encountered in many areas of applied math such as, for example, in nonlinear diffusion problems with Michaelis-Menten kinetics and nonlinear behaviors of plane circular elastic surface under normal pressure [**25**]. The exact solution to problem (6.5)–(6.6) is given by

$$u(s) = 2\ln(2 - s^2).$$

With a proper choice of k such as $k^2 = 10$, theory in [**20**] guarantees the existence of a unique solution to (6.4) via Picard iterates. For each value of h, the number of Picard iterates performed before the absolute error between successive iterates becomes less when 10^{-12} was 19. Tables 3 and 4 display values for the same quantities as those listed in Tables 1 and 2 for Example 1.

Likewise, this numerical result confirms Theorem 4.3.

Finally, the following shows a comparison between the accuracy in quadrature and the accuracy in interpolation at off-mesh values. This was discussed in Section 5. Table 5 shows absolute error using the extrapolated values $u^i_{20,2}$, $i = 1, 2, \ldots, n$ when $n = 1/20$ for Example 1.

TABLE 3. Errors of quadrature solution for Example 2.

t_i	e_i with $h=1/20$	e_i with $h=1/40$	e_i with $h=1/80$	rate
0.0	0.1492E-01	0.4387E-02	0.1259E-00	1.8
0.1	0.1419E-02	0.3630E-03	0.9136E-04	1.9
0.2	0.7723E-03	0.1882E-03	0.4670E-04	2.0
0.3	0.1745E-02	0.4327E-03	0.1079E-03	2.0
0.4	0.2149E-02	0.5344E-03	0.1334E-03	2.0
0.5	0.2199E-02	0.5473E-03	0.1366E-03	2.0
0.6	0.1997E-02	0.4974E-03	0.1242E-03	2.0
0.7	0.1610E-02	0.4009E-03	0.1001E-03	2.0
0.8	0.1091E-02	0.2715E-03	0.6679E-04	2.0
0.9	0.5079E-03	0.1261E-03	0.3146E-04	2.0

TABLE 4. Errors of the extrapolated solution for Example 2.

t_i	e_1^1	e_i^2	rate
0.1	0.1106E-04	0.8018E-06	3.8
0.2	0.6472E-05	0.4683E-06	3.8
0.3	0.4883E-05	0.3476E-06	3.8
0.4	0.3936E-05	0.2751E-06	3.8
0.5	0.3234E-05	0.2223E-06	3.8
0.6	0.2657E-05	0.1800E-06	3.9
0.7	0.2158E-05	0.1442E-06	3.9
0.8	0.1700E-05	0.1122E-06	3.9
0.9	0.1164E-05	0.7611E-07	3.9

Table 6 shows the results for Example 2. The quantity $e_{20,2}(s) := |u_{20,2}(s) - u(s)|$ where $u_{20,2}(s)$ is the value obtained by the quadrature formula suggested in Section 5. The quantity $\hat{e}_{20,2}(s) := |\hat{u}_{20,2}(s) - u(s)|$ where $\hat{u}_{20,2}(s)$ is the interpolation suggested in Section 5. The quantity

TABLE 5. Errors in interpolation vs errors in quadrature at off mesh values for Example 1.

s	$e_{20,2}(s)$	$\hat{e}_{20,2}(s)$	$e(s)$
0.0125	0.1916E-02	0.8704D-05	0.8703E-05
0.0875	0.1515E-02	0.1414E-05	0.1417E-05
0.1625	0.1312E-02	0.1254E-05	0.1256E-05
0.2375	0.1121E-02	0.6050E-05	0.6048E-05
0.3125	0.9701E-03	0.1846E-06	0.1865E-06
0.4627	0.8154E-03	0.2725E-06	0.2712E-06
0.5375	0.8204E-03	0.6512E-07	0.5506E-07
0.6125	0.8171E-03	0.1875E-07	0.1789E-07
0.6875	0.8988E-03	0.2724E-07	0.2793E-07
0.7625	0.9829E-03	0.3031E-07	0.2978E-07
0.8375	0.1160E-03	0.4871E-08	0.5245E-08
0.9125	0.1364E-02	0.4757E-08	0.4967E-08
0.9875	0.1684E-02	0.2531E-07	0.2527E-07

$e(s) := |\hat{u}(s) - u(s)|$ where $\hat{u}(s)$ is the interpolation in Section 5 using the *exact* value of the solution u at t_i instead of $u^i_{20,2}$.

Consider Table 5. If we compare column 2 of Table 5 to column 2 of Table 1, we see that the accuracy of quadrature at off-mesh values is no better than the accuracy in $u^i_{20,0}$ at mesh values which we have already shown is order 2. Column 3 of Table 5 shows the error in the interpolation at off-mesh values which is much smaller than the error in the quadrature. It is also the same order as the interpolation error using the exact values of the solution u at t_i which we know from Lagrange theory is order 4.

This verifies Theorem 5.1. Similar analysis can be done for Example 2.

TABLE 6. Errors in interpolations vs errors in quadrature at off mesh values for Example 2.

s	$e_{20,2}(s)$	$\hat{e}_{20,2}(s)$	$e(s)$
0.0125	0.1291E-01	0.7812E-04	0.2390E-07
0.0875	0.5865E-02	0.1995E-05	0.4561E-08
0.1625	0.4394E-02	0.1073E-05	0.4683E-08
0.2375	0.3339E-02	0.4148E-05	0.2587E-07
0.3125	0.2895E-02	0.7447E-08	0.1758E-07
0.3875	0.2495E-02	0.3022E-07	0.1117E-07
0.4625	0.2368E-02	0.4870E-07	0.3408E-07
0.5375	0.2251E-02	0.1590E-06	0.1478E-06
0.6125	0.2303E-02	0.6427E-07	0.5561E-07
0.6875	0.2368E-02	0.9411E-07	0.1108E-06
0.7625	0.2576E-02	0.3067E-05	0.3061E-05
0.8375	0.2796E-02	0.6316E-06	0.6354E-06
0.9125	0.3212E-02	0.7136E-06	0.7161E-06
0.9875	0.3577E-02	0.4401E-05	0.4400E-05

All computations were done in double precision on an RS 6000.

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Computer Science Division, MS 142-5, NASA Lewis Research Center, Cleveland, Ohio 44135, U.S.A.
E-mail address: William.F.Ford@lerc.nasa.gov

Computer Science Division, MS 142-5, NASA Lewis Research Center, Cleveland, Ohio 44135, U.S.A.
E-mail address: James.A.Pennline@lerc.nasa.gov

Department of Mathematics, North Dakota State University, Fargo, ND 58105 and Academy of Mathematics and System Sciences, Academia Sinica, Beijing 100080, P.R. China
E-mail address: Yuesheng_Xu@ndsu.nodak.edu

Caelum Research Corporation, 1700 Research Blvd. Suite 100, Rockville, MD 20850, U.S.A.