POSITIVE SOLUTIONS OF
SINGULAR INTEGRAL EQUATIONS

MARIA MEEHAN AND DONAL O'REGAN

ABSTRACT. Continuous, positive solutions of singular integral equations of the form $y(t) = h(t) + \int_0^T k(t,s) \, [f(y(s)) + g(y(s))] \, ds$ are sought. Here $f : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing while $g : (0, \infty) \to [0, \infty)$ is continuous, nonincreasing and possibly singular. The case when $T = \infty$ is also discussed.

1. Introduction. In the first half of this paper, Schauder's fixed point theorem is used to obtain the existence of continuous, positive solutions of

$$(1.1) \qquad y(t) = h(t) + \int_0^T k(t,s)[f(y(s)) + g(y(s))] \, ds, \quad t \in [0, T].$$

It is assumed that $f : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing, while $g : (0, \infty) \to [0, \infty)$ is continuous, nonincreasing and possibly singular, that is, the possibility of $g(0)$ being undefined is allowed. In Section 2, by placing appropriate conditions on h, k, f and g, we use Schauder's fixed point theorem to prove the existence of a solution $y \in C[0, T]$ such that $0 < \beta < y(t) < \alpha$, $t \in [0, T]$ for some $0 < \beta < \alpha$. In addition a special case of this result, which occurs when $h \in C[0, T]$ is such that $h(t) > 0$, $t \in [0, T]$, is stated for completeness.

In Section 3 we extend the results of Section 2 and consider the possibly singular equation

$$(1.2) \qquad y(t) = h(t) + \int_0^\infty k(t,s)[f(y(s)) + g(y(s))] \, ds, \quad t \in [0, \infty).$$

Schauder's fixed point theorem and the Schauder-Tychonoff fixed point theorem are used to establish the existence of a positive solution $y \in C_l[0, \infty)$ and $y \in BC[0, \infty) \subset C[0, \infty)$ respectively of (1.2). (Here

$BC[0, \infty)$ denotes the space of bounded, continuous functions on $[0, \infty)$, while $C_l[0, \infty)$ denotes the space of bounded, continuous functions on $[0, \infty)$ whose limit exists at infinity.)

We conclude the introduction by stating the two fixed point theorems that will be used in this paper.

Theorem 1.1 Schauder fixed point theorem. *Let K be a convex subset of a Banach space E and $N : K \to K$ a compact, continuous map. Then N has at least one fixed point in K.*

Theorem 1.2 Schauder-Tychonoff fixed point theorem. *Let K be a closed, convex subset of a locally convex, Hausdorff space E. Assume that $N : K \to K$ is continuous and $N(K)$ is relatively compact in E. Then N has at least one fixed point in K.*

2. Singular integral equations on compact intervals. Prompted by the application of the nonlinear integral equation

$$(2.1) \qquad 1 = u(t) + u(t) \int_\alpha^\beta \frac{R(t, s)}{t^2 - s^2} \, u(s) \, ds, \quad t \in [\alpha, \beta]$$

to nuclear physics, this equation and generalizations have provoked some interest in the literature [**1**], [**3**], [**5**], [**8**], [**9**]. Specifically equations of the form (1.1) have been studied with a view to obtaining the existence of positive solutions. In keeping with the nature of (2.1), we assume that (1.1) is a singular integral equation, by which we mean that the possibility of $g(0)$ being undefined is permitted.

The following result uses Schauder's fixed point theorem to guarantee the existence of a positive solution $y \in C[0, T]$ of (1.1).

Notation. Throughout this section we let

$$(2.2) \quad K_1 := \sup_{t \in [0,T]} \int_0^T k(t, s) \, ds \quad \text{and} \quad K_2 := \inf_{t \in [0,T]} \int_0^T k(t, s) \, ds.$$

In addition the norm on $C[0, T]$ will be denoted by $||.||_\infty$, that is, for $y \in C[0, T]$, $||y||_\infty := \sup_{t \in [0,T]} |y(t)|$.

Theorem 2.1. *Suppose that*

(2.3) $$h \in C[0,T] \quad with \ h(t) \geq 0, \quad t \in [0,T],$$

(2.4) $$0 < k_t = k(t,s) \in L^1[0,T] \quad for \ each \ t \in [0,T],$$

(2.5) $$the \ map \quad t \mapsto k_t \quad is \ continuous \ from \ [0,T] \ to \ L^1[0,T],$$

(2.6) $$\begin{cases} f : [0,\infty) \to [0,\infty) & is \ continuous \ and \ nondecreasing, \\ g : (0,\infty) \to [0,\infty) & is \ continuous \ and \ nonincreasing, \\ and \ f + g : (0,\infty) \to (0,\infty) \end{cases}$$

and

(2.7) $$\begin{cases} 0 < \beta < \alpha \ exists \ such \ that \\ \dfrac{\beta}{\min_{t \in [0,T]} h(t) + K_2[f(\beta)+g(\alpha)]} < 1 \\ < \dfrac{\alpha}{||h||_\infty + K_1[f(\alpha)+g(\beta)]} \\ (here \ K_1 \ and \ K_2 \ are \ as \ defined \ in \ (2.2)) \end{cases}$$

hold. Then (1.1) has at least one positive solution $y \in C[0,T]$ such that $0 < \beta < y(t) < \alpha$, $t \in [0,T]$.

Proof. Define $f^\star : \mathbf{R} \to [f(\beta), f(\alpha)]$ and $g^\star : \mathbf{R} \to [g(\alpha), g(\beta)]$ by

$$f^\star(y) := \begin{cases} f(\alpha) & y \geq \alpha, \\ f(y) & \beta \leq y \leq \alpha, \\ f(\beta) & y \leq \beta, \end{cases} \quad and \quad g^\star(y) := \begin{cases} g(\alpha) & y \geq \alpha, \\ g(y) & \beta \leq y \leq \alpha, \\ g(\beta) & y \leq \beta, \end{cases}$$

respectively. In addition let the operator $K^\star$ be given by

$$K^\star y(t) := h(t) + \int_0^T k(t,s)[f^\star(y(s)) + g^\star(y(s))] \, ds, \quad t \in [0,T],$$

and define a convex subset $C \subset C[0,T]$ by

$$C := \{y \in C[0,T] : ||y||_\infty \le \alpha\}.$$

We will use Schauder's fixed point theorem to show that $K^\star$ has a fixed point in C, and consequently we must ensure that

(2.8) $K^\star : C \to C$ is a continuous, compact operator

holds.

Let $y \in C$. Then for $t, t' \in [0,T]$ we have that

$$|K^\star y(t') - K^\star y(t)| \le |h(t') - h(t)| + \int_0^T |k_{t'}(s) - k_t(s)|$$
$$\cdot \left[f^\star(y(s)) + g^\star(y(s))\right] ds.$$

Furthermore (2.6) yields
(2.9)
$$|K^\star y(t') - K^\star y(t)| \le |h(t') - h(t)| + \left(\int_0^T |k_{t'}(s) - k_t(s)| \, ds\right)[f(\alpha) + g(\beta)].$$

From this inequality and (2.5) we see that

$$|K^\star y(t') - K^\star y(t)| \to 0 \quad \text{as } t \to t',$$

and hence $K^\star : C \to C[0,T]$ is well-defined. In addition note that for $y \in C$, (2.6) and (2.7) imply

(2.10) $$||K^\star y||_\infty \le ||h||_\infty + K_1[f(\alpha) + g(\beta)] < \alpha,$$

and consequently $K^\star : C \to C$ is well-defined.

We next show that $K^\star : C \to C$ is a continuous operator. Let $y_n \to y$ in $C[0,T]$. Then for any $t \in [0,T]$,

$$|K^\star y_n(t) - K^\star y(t)|$$
$$\le K_1 \left[\sup_{t \in [0,T]} |f^\star(y_n(t)) - f^\star(y(t))| + \sup_{t \in [0,T]} |g^\star(y_n(t)) - g^\star(y(t))| \right]$$

and hence by (2.6) we have that $K^\star : C \to C$ is a continuous operator.

Finally we use the Arzéla-Ascoli theorem to show that $K^\star : C \to C$ is compact. Immediately we obtain from (2.10) that $K^\star(C)$ is uniformly bounded, while the equicontinuity of $K^\star(C)$ follows from (2.9). The Arzéla-Ascoli theorem thus guarantees that $K^\star : C \to C$ is a compact operator.

We have therefore shown that (2.8) is true and hence by Schauder's fixed point theorem, $K^\star : C \to C$ has a fixed point $y \in C$, that is,

$$y(t) = h(t) + \int_0^T k(t,s)[f^\star(y(s)) + g^\star(y(s))]\,ds, \quad t \in [0,T],$$

has a solution $y \in C[0,T]$. From (2.7) however we see that

$$||y||_\infty \le ||h||_\infty + K_1[f(\alpha) + g(\beta)] < \alpha$$

and

$$y(t) \ge \min_{t \in [0,T]} h(t) + K_2[f(\beta) + g(\alpha)] > \beta, \quad t \in [0,T],$$

that is, $\beta < y(t) < \alpha$, $t \in [0,T]$. Consequently, by the definition of $f^\star$ and $g^\star$, we have that $y \in C[0,T]$ is a solution of (1.1). $\qquad\square$

Example 2.1. Suppose for simplicity that $h \equiv 0$ in Theorem 2.1, and let $f(y) = y^p$, $0 \le p < 1$ and $g(y) = y^{-q}$, $0 \le q < 1$. Since it is possible to find (for any positive $K_2 \le K_1$) $0 < \beta < \alpha$ such that

$$\frac{\alpha^q \beta}{\alpha^q \beta^p + 1} < K_2 \le K_1 < \frac{\alpha \beta^q}{\alpha^p \beta^q + 1}$$

holds, then it is clear that this choice of f and g satisfies (2.6) and (2.7).

If $h \in C[0,T]$ in Theorem 2.1 is strictly positive, that is, $h(t) > 0$, $t \in [0,T]$, then defining $m > 0$ by

$$m := \min_{t \in [0,T]} h(t) > 0,$$

we note that, provided

(2.11) $\alpha > 0$ exists such that $1 < \dfrac{\alpha}{||h||_\infty + K_1[f(\alpha) + g(m)]}$

holds, the hypotheses of Theorem 2.1 are satisfied with $\beta = m$. We thus state the following special case of Theorem 2.1 that occurs when $h \in C[0, T]$ is strictly positive.

Theorem 2.2. *Suppose that* $(2.4) - (2.6)$ *and* (2.12) *hold along with*

$$(2.12) \qquad h \in C[0, T] \quad with \ h(t) > 0 \quad for \ t \in [0, T].$$

Then (1.1) *has at least one positive solution* $y \in C[0, T]$. *In addition* $0 < h(t) < y(t) < \alpha, \ t \in [0, T]$.

3. Singular integral equations on the half-line. We now want to extend the ideas of the previous section to obtain analogous results for singular integral equations of the form (1.2). Denote by $BC[0, \infty)$ the normed space of bounded, continuous functions defined on $[0, \infty)$ with norm given by

$$(3.1) \qquad ||y||_\infty := \sup_{t \in [0, \infty)} |y(t)|.$$

In addition we denote by $C_l[0, \infty)$ the subset of $BC[0, \infty)$ which consists of all $y \in BC[0, \infty)$ such that $\lim_{t \to \infty} y(t)$ exists. $C_l[0, \infty)$, equipped with norm $||.||_\infty$, is a Banach space and compactness criteria for this space is readily available (see, for example, [**2**]). Therefore by revising the conditions imposed on h, k, f and g in Theorem 2.1, one can once again apply Schauder's fixed point theorem, this time to obtain the existence of at least one positive solution $y \in C_l[0, \infty)$ of (1.2).

Theorem 3.1. *Suppose that* (2.6),

$$(3.2) \qquad h \in C_l[0, \infty) \quad with \ h(t) \geq 0, \quad t \in [0, \infty),$$

$$(3.3) \qquad 0 < k_t(s) = k(t, s) \in L^1[0, \infty) \quad for \ each \ t \in [0, \infty),$$

$$(3.4) \qquad the \ map \quad t \mapsto k_t \quad is \ continuous \ from \ [0, \infty) \ to \ L^1[0, \infty),$$

$$(3.5) \quad \tilde{k} \in L^1[0, \infty) \quad exists \ such \ that \quad k_t \to \tilde{k} \ in \ L^1[0, \infty) \ as \ t \to \infty$$

and

$$(3.6) \quad \begin{cases} 0 < \beta < \alpha \text{ exists such that} \\ \dfrac{\beta}{\inf_{t \in [0,\infty)} h(t) + K_4[f(\beta) + g(\alpha)]} < 1 \\ \quad < \dfrac{\alpha}{\|h\|_\infty + K_3[f(\alpha) + g(\beta)]} \\ \text{(here } K_3 \text{ and } K_4 \text{ are as defined in } (3.7) \text{ below)} \end{cases}$$

hold. Then (1.2) has at least one positive solution $y \in C_l[0,\infty)$ such that $0 < \beta < y(t) < \alpha$, $t \in [0,\infty)$.

Notation. Throughout this section let

$$(3.7) \quad K_3 := \sup_{t \in [0,\infty)} \int_0^\infty k(t,s)\,ds \quad \text{and} \quad K_4 := \inf_{t \in [0,\infty)} \int_0^\infty k(t,s)\,ds.$$

(From (3.3)–(3.5) one can show that K_3 and K_4 are well defined.)

Proof. Let $f^\star$ and $g^\star$ be as defined in the proof of Theorem 2.1. Let the operator $K^\star$ be given by

$$K^\star y(t) := h(t) + \int_0^\infty k(t,s)[f^\star(y(s)) + g^\star(y(s))]\,ds, \quad t \in [0,\infty).$$

Defining the convex subset C of $C_l[0,\infty)$ by

$$C := \{y \in C_l[0,\infty) : \|y\|_\infty \leq \alpha\},$$

one can show, with the aid of compactness criteria from [**2**, p. 62], that

$$K^\star : C \to C \quad \text{is a continuous, compact operator.}$$

The analysis is similar to that in Theorem 2.1; therefore, we omit the details. The theorem now follows from Schauder's fixed point theorem. Once again we omit the detail since it is almost identical to that in the proof of Theorem 2.1.   □

Rather than require the solution of (1.2) to lie in $C_l[0,\infty)$, suppose we seek solutions that are in the space $BC[0,\infty) \subseteq C[0,\infty)$. Here we denote the set of all continuous functions on $[0,\infty)$ by $C[0,\infty)$. This is not a normed space, rather it is a Fréchet space, and for $y \in C[0,\infty)$ we define, for each $m \in \{1, 2, \dots\}$, the seminorm $\rho_m(y)$ by

$$\rho_m(y) := \sup_{t \in [0,m]} |y(t)|.$$

Recall that a subset C of $C[0,\infty)$ is bounded if a positive function $\alpha \in C[0,\infty)$ exists such that $|y(t)| \leq \alpha(t)$, for all $t \in [0,\infty)$ and $y \in C$. In addition recall that a subset C of $C[0,\infty)$ is compact if it is compact on each compact subinterval of $[0,\infty)$.

We conclude the section by presenting a result which relies on the Schauder-Tychonoff theorem to ensure that (1.2) has a positive solution in the Fréchet space $C[0,\infty)$. Conditions on h, k, f and g will then further imply that in fact the solution belongs to $BC[0,\infty)$.

Theorem 3.2. *Suppose that* (2.6)*,* (3.3)*,* (3.4)*,* (3.6)*,*

$$(3.8) \qquad h \in BC[0,\infty) \quad with\ h(t) \geq 0, \quad t \in [0,\infty)$$

and

$$(3.9) \qquad the\ map \quad t \mapsto k_t \quad is\ bounded\ from\ [0,\infty)\ to\ L^1[0,\infty)$$

hold. Then (1.2) *has at least one positive solution* $y \in C[0,\infty)$ *such that* $0 < \beta < y(t) < \alpha$, $t \in [0,\infty)$.

Remark 3.1. It is easy to check that conditions (3.3), (3.4) and (3.9) ensure that K_3 and K_4 as given in (3.7) are well defined.

Proof. The idea behind the proof is quite similar to the idea in the proof of Theorem 3.1 except that now we use the Schauder-Tychonoff fixed point theorem.

Define $f^\star$, $g^\star$ and $K^\star$ as in the proof of Theorem 3.1. Let the closed, convex subset C, required by the Schauder-Tychonoff theorem, be given by

$$C := \{y \in C[0,\infty) : y \in BC[0,\infty)\ \text{and}\ ||y||_\infty < \alpha\}.$$

Firstly note that $K^\star : C \to C$ is well-defined. Let $y \in C$. Then $K^\star y \in C[0,\infty)$ since for any $t, t' \in [0,\infty)$, we have from (3.3), (3.4) and (3.8) that

(3.10)
$$|K^\star y(t) - K^\star y(t')| \le |h(t) - h(t')| + \left(\int_0^\infty |k(t,s) - k(t',s)| \, ds \right)$$
$$\cdot [f(\alpha) + g(\beta)] \to 0 \quad \text{as } t \to t'$$

holds. In addition, it is clear from (3.6) that

(3.11) $\quad ||K^\star y||_\infty \le ||h||_\infty + K_3[f(\alpha) + g(\beta)] < \alpha \quad$ for all $y \in C[0,\infty)$

is true; therefore, we see that $K^\star : C \to C$ is well defined. (Note in fact that $K^\star : C[0,\infty) \to C$ is well defined.)

Secondly we show that $K^\star(C)$ is relatively compact in $C \subset C[0,\infty)$. To do this we must show that $K^\star(C)$ is uniformly bounded and equicontinuous on each compact subinterval of $[0,\infty)$. It is immediate from (3.11) that $K^\star(C)$ is in fact uniformly bounded on $[0,\infty)$, while the equicontinuity of $K^\star(C)$ on each compact subinterval of $[0,\infty)$ follows directly from (3.10). Hence $K^\star(C)$ (and also $K^\star(C[0,\infty))$) is relatively compact in $C[0,\infty)$.

Lastly we are required to show that $K^\star : C \to C$ is continuous. Suppose that $y_n \to y$ in $C \subseteq C[0,\infty)$, that is, $y_n \to y$ in $C[0,m]$ for each $m \in \{1, 2, \dots\}$. Clearly this implies the pointwise convergence of y_n to y on $[0,\infty)$. Coupling this fact with (2.6) we obtain for each $t \in [0,\infty)$ that

$$k_t(s)[f(y_n(s)) + g(y_n(s))] \to k_t(s)[f(y(s)) + g(y(s))] \quad \text{a.e. } s \in [0,\infty).$$

In addition for each $t \in [0,\infty)$

$$0 \le k_t(s)[f(y_n(s)) + g(y_n(s))] \le k_t(s)[f(\alpha) + g(\beta)],$$
$$\text{a.e. } s \in [0,\infty), \quad \text{for all } n \in \mathbf{N}.$$

Consequently by the Lebesgue dominated convergence theorem we have

(3.12) $\qquad K^\star y_n(t) \to K^\star y(t), \quad$ for each $t \in [0,\infty) \quad$ as $n \to \infty$.

Now fix $m \in \{1, 2, \ldots\}$. Since $[0, m]$ is compact, combining (3.10) and (3.12) yields

$$K^\star y_n \to K^\star y \quad \text{in } C[0, m] \quad \text{as } n \to \infty.$$

Obviously this is true for any $m \in \{1, 2, \ldots\}$, and therefore $K^\star y_n \to K^\star y$ in $C[0, \infty)$ as $y_n \to y$ in C.

In summary we have that

$$K^\star : C \to C \quad \text{is a continuous and compact operator,}$$

and therefore, by the Schauder-Tychonoff theorem, $K^\star$ has a fixed point $y \in C$. Clearly $0 < \beta < y(t) < \alpha$, $t \in [0, \infty)$, and we have the desired result.   □

REFERENCES

1. P.J. Bushell, *On a class of Volterra and Fredholm nonlinear integral equations*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 329–335.

2. C. Corduneanu, *Integral equations and stability of feedback systems*, Academic Press, New York, 1973.

3. L. Erbe, D. Guo and X. Liu, *Positive solutions of a class of nonlinear integral equations and applications*, J. Integral Equations Appl. **4** (1992), 179–195.

4. M. Meehan and D.O'Regan, *Multiple nonnegative solutions of nonlinear integral equations on compact and semi-infinite intervals*, Appl. Anal., to appear.

5. M. Meehan and D. O'Regan, *Positive solutions of singular and nonsingular Fredholm integral equations*, J. Math. Anal. Appl., **240** (1999), 416–432.

6. D. O'Regan, *Existence results for nonlinear integral equations*, J. Math. Anal. Appl. **192** (1995), 705–726.

7. D. O'Regan and M. Meehan, *Existence theory for nonlinear integral and integrodifferential equations*, Kluwer Academic Publ., Dordrecht, 1998.

8. C.A. Stuart, *Positive solutions of a nonlinear integral equation*, Math. Ann. **192** (1971), 119–124.

9. C.A. Stuart, *Existence theorems for a class of nonlinear integral equations*, Math. Z. **137** (1974), 49–66.

Department of Mathematics, National University of Ireland, Dublin, Ireland

Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: Donal.ORegan@nuigalway.ie