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ANALYSIS AND NUMERICS OF AN INTEGRAL EQUATION MODEL FOR SLENDER BODIES IN LOW REYNOLDS-NUMBER FLOWS

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ABSTRACT. The interaction of particular slender bodies with low Reynolds-number flows is in the limit "slenderness to zero" described by the linear Fredholm integral equation of the second kind

$$c\phi(s) = F(s) + \int_{-1}^{1} \frac{\phi(t) - \phi(s)}{|t-s|}\, dt, \quad s \in [-1, 1],$$

where c is a real number, F is a continuous function and ϕ is unknown. The integral operator T of this equation is symmetric on certain subsets of its domain. T has a denumerable set of eigenvalues of logarithmic growth. The respective eigenspaces contain the Legendre-polynomials. A theorem similar to a classical result of Plemelj-Privalov for integral operators with Cauchy kernels is proved. In contrast to Cauchy kernel operators, T maps no α-Hölder space into itself. A spectral analysis of the restriction $\tilde{T}$ of T to the space of all polynomials is performed. $\tilde{T}$ has a self-adjoint extension $\tilde{T}^{sa}$ in $\mathcal{L}^2([-1,1])$. The spectrum of $\tilde{T}^{sa}$ is a pure point spectrum. The respective eigenspaces are spanned by Legendre-polynomials. A spectral method based on expansions in terms of the Legendre polynomials is presented and stability and convergence properties are proved. The results are illustrated by several numerical simulations. In case of sufficiently smooth functions F a modified spectral method is proposed. For that method uniform stability and convergence results are proved.

1. Introduction. The starting point of the subsequent investigations is a model for the shape of a long, slender body (e.g., a fiber) exposed to a normal flow [**6**]. Applying the singularity method for linearized fluid dynamics [**1**], [**15**] and under several assumptions (fluid velocity approaches a constant value as the spatial variable tends to

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infinity, the fiber lies in a plane, has a circular cross-section, the fiber's curvature is $O(1)$ and the center-line of the fiber does not reapproach itself) one deduces from asymptotical expansions a Fredholm integral equation model for the force acting on the fiber [**2**], [**8**], [**6**]:

$$c\phi(s) = F(s) + \int_{-1}^{1} \frac{\phi(t) - \phi(s)}{|t-s|}\, dt, \quad s \in [-1,1]. \tag{1.1}$$

Here $s \in [-1,1]$ is the spatial variable of the one-dimensional fiber. $c \in \mathbf{R}$ is a negative constant which depends on the fiber's geometry. (Loosely speaking, the smaller the radius of the real fiber, the more negative c becomes, e.g., if the real fiber is an ellipsoid whose s-dependent radius $r(s)$ equals $\varepsilon\sqrt{1-s^2}$, $\varepsilon > 0$, then $c = 2\ln(2/\varepsilon) - 1$.) In (1.1) the scaled force per unit length $\phi : [-1,1] \to \mathbf{R}$ is unknown. The continuous function $F : [-1,1] \to \mathbf{R}$ is the scaled difference of the fiber's velocity and the limiting velocity of the fluid.

Some of the assumptions that are met in deriving (1.1) can be relaxed, e.g., allowing nonconstant velocity profiles at infinity [**8**], [**6**], noncircular cross-section of the fiber [**9**]. These models are applied in various fields ranging from fiber-spinning [**12**], [**6**] to biofluiddynamics [**10**], [**13**].

Seemingly, (1.1) has not been treated in the literature yet [**14**].

Heuristically, (1.1) is in between the *Abelian integral equations*

$$c(s)\phi(s) + \int_{-1}^{1} \frac{\phi(t)}{|s-t|^{\alpha}}\, dt = F(s), \quad s \in [-1,1], \tag{1.2}$$

where $c : [-1,1] \to \mathbf{R}$ is continuous and $0 < \alpha < 1$, and integral equations with *Cauchy kernels*

$$c\phi(s) + \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s-t}\, dt = F(s), \quad s \in [-1,1]. \tag{1.3}$$

The Abelian integral equation is certainly one of the best-known integral equations, see, e.g., [**5**] and the references therein. However, the analysis of Abelian integral equations heavily relies on the integrability of the kernel $|s-t|^{-\alpha}$, $0 < \alpha < 1$. A straightforward application of the respective theory is therefore out of sight.

The available theory for integral equations with Cauchy kernels $(s-t)^{-1}$ is settled on complex methods which rely on the fact that the kernel of (1.3) allows for a meromorphic extension to the complex plane, see, e.g., [**11**]. However, in our case the kernel is not meromorphic in $\mathbf{C}$. A straightforward application of the theory for (1.3) is therefore not possible.

The integral equations with Cauchy kernels appear in applications like airfoil theory [**3**]. The solution ϕ of (1.3) has a series expansion in terms of Chebyshev polynomials [**16**], [**18**] converging *uniformly* to ϕ as the number of the series' terms tend to infinity.

A frequently used procedure for integral equations of the second kind is the *method of successive approximations*, compare [**8**]. In order to solve an integral equation

$$\phi = T[\phi] + F, \tag{1.4}$$

where T is a linear integral operator, one considers for $n \in \mathbf{N}_0$,

$$\phi_{n+1} = T[\phi_n] + F \quad \text{with } \phi_0 \text{ suitable.} \tag{1.5}$$

If the operator T is bounded with spectral radius less than 1, then the sequence $(\phi_n)_{n\in\mathbf{N}}$ will converge to the unique solution of (1.4) for any initial value ϕ_0. However, this method *cannot* be applied to (1.1); as will be shown later on, the integral operator of (1.1) is unbounded on reasonable normed spaces.

As a summary, a theory for (1.1) is not available yet. It is one of the purposes of the present paper to perform that analysis.

The paper is organized as follows. In Section 2 we give a precise definition of the integral operator T arising in (1.1). The domain of T is a subset of the space of all continuous functions whose domain is $[-1,1]$. The model equation (1.1) is rewritten as operator equation $c\phi = F + T(\phi)$. In Section 2.1 the operator T is restricted to spaces of α-Hölder continuous functions. Each α-Hölder continuous, $\alpha \in (0,1]$, function is in the domain of T. Furthermore, T restricted to the set of all α-Hölder continuous functions, $\alpha \in (0,1]$, is symmetric with respect to the canonical inner product on the space $\mathcal{L}^2([-1,1])$ of all square integrable functions with domain $[-1,1]$. An analogon of a classical result of Plemelj-Privalov is proven. However, T maps no space of α-Hölder continuous functions into itself. This result exhibits, on the one hand, an important difference between (1.1) and the integral equations with Cauchy kernels.

In Section 2.2 the restriction $\tilde{T}$ of T to the space of all polynomials is introduced. It readily follows from respective properties of $\tilde{T}$ that T has denumerable many eigenvalues

$$-L_k := -2\sum_{l=1}^{k} \frac{1}{l}, \quad k \in \mathbf{N}_0.$$

The eigenspace of $-L_k$, $k \in \mathbf{N}$, contains the kth Legendre polynomial P_k. Due to the logarithmic growth of the eigenvalues the operator T is not bounded on any reasonable infinite dimensional subspace of its domain.

The authors are indebted to one of the anonymous referees for initializing via many suggestions the investigations of Section 3. There a spectral analysis of $\tilde{T}$ is performed. It is shown that $\tilde{T}$ is essentially self-adjoint in $\mathcal{L}^2([-1,1])$. The adjoint operator of $\tilde{T}$ is the self-adjoint closure $\tilde{T}^{sa}$ of $\tilde{T}$. It is shown that $\tilde{T}^{sa}$ has a pure point spectrum $\sigma(\tilde{T}^{sa}) = \sigma_p(\tilde{T}^{sa}) = \{-L_k : k \in \mathbf{N}_0\}$. The eigenspace of $\tilde{T}^{sa}$ of $-L_k$, $k \in \mathbf{N}_0$, is spanned by the kth Legendre polynomial P_k. As a consequence, the operator $\tilde{T}^{sa}$ is diagonalizable and a rather complete theory of (1.1) in $\mathcal{L}^2([-1,1])$ is available. Moreover, a spectral method to treat (1.1) numerically is immediately available. Stability and convergence properties, in $\mathcal{L}^2([-1,1])$, are deduced in Section 4. Several numerical results are presented.

From a theoretical point of view, the investigations of T as performed in Sections 3 and 4 are rather convincing.

Due to practical demands, however, the expansion of the solution of (1.1) in terms of Legendre polynomials and the corresponding convergence results in $\mathcal{L}^2([-1,1])$ is not entirely satisfying. In applications one is rather interested in computing the force acting on the fiber within certain bounds rather than obtaining averaged accuracy results in $\mathcal{L}^2([-1,1])$. Furthermore, the numerical results of Section 4 indicate uniform convergence of the approximations in the $\|\cdot\|_\infty$-norm.

Hence a theory of uniform convergence is required. Such a theory is developed in Section 5. The core of the investigations is the introduction of the function space $\mathsf{dom}\,(\hat{T})$ consisting of all analytic functions whose power series expansions with respect to the monomials $(s/2)^k$, $k \in \mathbf{N}_0$, has uniformly bounded coefficients. The operator $\hat{T}$, which is the restriction of T to $\mathsf{dom}\,(\hat{T})$, maps $\mathsf{dom}\,(\hat{T})$ into $\mathcal{L}_w^\infty$ which, loosely speaking, consists of all analytic functions whose power series expansions with respect to $(s/2)^k$, $k \in \mathbf{N}_0$, grow at most logarithmically. A spectral analysis of $\hat{T}$ is performed in Section 5.1. The spectrum of $\hat{T}$ is the same as the spectrum of $\tilde{T}^{sa}$, i.e., it is a pure point spectrum and $\sigma_p(\hat{T}) = \{-L_k : k \in \mathbf{N}_0\}$. Estimates on the norms of the resolvents of $\hat{T}$ are derived by means of the upper triangular structure of $\hat{T}$. In Section 5.2 convergence and stability (with respect to uniform convergence) of a corresponding modified spectral method is investigated.

The proofs of the results are deferred to the Appendix.

2. The integral operator T**.** We shall make use of the following notations. I is the interval $[-1, 1]$. Let C denote the (real) vector space of all continuous functions $f : \mathsf{I} \to \mathbf{R}$. C is equipped with its canonical norm,

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathsf{I}\}, \quad f \in \mathsf{C}\,.$$

For $\alpha \in (0, 1]$ we denote by $\mathsf{C}^{\,0,\alpha}$ the (real) vector space of all α-Hölder continuous functions $g : \mathsf{I} \to \mathbf{R}$. We equip $\mathsf{C}^{\,0,\alpha}$ with its canonical norm,

$$\|g\|_{0,\alpha} = \|g\|_\infty + \sup\left\{\frac{|g(s) - g(t)|}{|s - t|^\alpha} : s, t \in \mathsf{I}, s \neq t\right\}, \quad g \in \mathsf{C}^{\,0,\alpha}.$$

Furthermore, let

$$\mathcal{P} = \{p \downharpoonright \mathsf{I} : p \in \mathbf{R}^{\mathbf{R}} \text{ is a polynomial}\}.$$

We shall also make use of the real sequence spaces

$$\ell^p := \left\{(x_k)_{k\in\mathbf{N}_0} \in \mathbf{R}^{\mathbf{N}_0} : \sum_{k=1}^\infty |x_k|^p < \infty\right\}, \quad 1 \le p < \infty,$$
$$\ell^\infty := \{(z_k)_{k\in\mathbf{N}_0} \in \mathbf{R}^{\mathbf{N}_0} : \sup\{|z_k| : k \in \mathbf{N}_0\} < \infty\},$$

equipped with the respective canonical norms

$$|(x_k)_{k\in\mathbf{N}_0}|_p := \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}, \quad (x_k)_{k\in\mathbf{N}_0} \in \ell^p, \quad 1 \le p < \infty,$$
$$|(z_k)_{k\in\mathbf{N}_0}|_\infty := \sup\{|z_k| : k \in \mathbf{N}_0\}, \quad (z_k)_{k\in\mathbf{N}_0} \in \ell^\infty.$$

Our first goal is to properly define the integral operator arising in (1.1). The physical model from which (1.1) originates requires continuous solutions. The investigations are therefore settled on subspaces of C.

Considering the integrand of (1.1), we introduce for $\Psi \in \mathsf{C}$ and $s \in \mathsf{I}$ the function

$$Q_s[\Psi] : \mathsf{I} \to \mathbf{R}, \quad Q_s[\Psi](t) = \begin{cases} \dfrac{\Psi(t) - \Psi(s)}{|t-s|}, & t \neq s \\ 0, & \text{else}. \end{cases}$$

The integral of (1.1) has a well-defined meaning if and only if $Q_s[\Psi]$ is Lebesgue-integrable over I, i.e.,

$$T[\Psi](s) := \int_{-1}^{1} Q_s[\Psi](t)\, dt,$$

well-defined if and only if

$$\Psi \in \mathsf{dom} := \{\psi \in \mathsf{C} \mid \forall\, s \in \mathsf{I} : Q_s[\psi] \text{ is Lebesgue-integrable over } \mathsf{I}\}.$$

This observation suggests defining T via

$$T : \mathsf{dom} \to \mathbf{R}^{\mathsf{I}}, \begin{cases} T[\psi] : \mathsf{I} \to \mathbf{R}, \\ \qquad s \mapsto T[\psi](s) = \displaystyle\int_{-1}^{1} \frac{\psi(s) - \psi(t)}{|s-t|}\, dt. \end{cases}$$

Obviously dom is the largest subspace of C on which T is well-defined.

For later reference we rewrite integral equation (1.1) as

$$(c - T)[\phi] = F. \tag{2.6}$$

2.1. T **restricted to** $\mathsf{C}^{0,\alpha}$**.** We easily verify

- Each α-Hölder continuous function belongs to dom, i.e., $\cup_{\alpha\in(0,1]}\mathsf{C}^{0,\alpha} \subseteq \mathsf{dom}$.

• dom is a proper subset of C, i.e., $\mathsf{dom} \subseteq \mathsf{C}$, but $\mathsf{dom} \neq \mathsf{C}$ because, e.g., the continuous function

$$g : \mathsf{I} \to \mathbf{R}, \quad g(t) = \begin{cases} 0\,, & t = -1, \\ -\left[\log\left(\dfrac{t+1}{2}\right)\right]^{-1}, & -1 < t \leq -\dfrac{e-2}{e} \\ 1\,, & -\dfrac{e-2}{e} \leq t \leq 1 \end{cases}$$

does not belong to dom.

• we have $\mathsf{C}^{\,0,1} \subseteq \mathsf{dom} \subseteq \mathsf{C}$ and $\mathsf{C}^{\,0,1}$ is $\|\cdot\|_\infty$-dense in C. Hence dom is $\|\cdot\|_\infty$-dense in C, too.

Another interesting property of T when acting on Hölder-continuous functions is its symmetry with respect to the inner product on $\mathcal{L}^2$, the vector space of all square integrable functions defined on I.

Theorem 1. *For all $\psi, \phi \in \cup_{\alpha\in(0,1]}\mathsf{C}^{\,0,\alpha}$, the function $\psi T[\phi]$ is integrable over I and*

$$\int_{\mathsf{I}} \psi(s)T[\phi](s)\,ds = \int_{\mathsf{I}} \phi(s)T[\psi](s)\,ds.$$

Since T is defined on any space $\mathsf{C}^{\,0,\alpha}$, $\alpha \in (0,1]$, the question arises whether T also leaves any of these spaces invariant. (As it is the case, e.g., for the Cauchy kernel operator on a smooth **closed** contour). But this is *not* true.

Proposition 2. *For all $\alpha \in (0,1] : T[\mathsf{C}^{\,0,\alpha}]$ is not contained in $\mathsf{C}^{\,0,\alpha}$.*

On the other hand, we have the following extension of a classical result of Plemelj-Privalov (see, e.g., [**7**]) which originally applies to the operator $\psi \mapsto \int_{-1}^{1}(\psi(t) - \psi(s))/(t-s)\,dt$:

Lemma 3. *For all $\alpha \in (0,1]$ and all $\alpha' \in [0,\alpha)$:*

$$T[\mathsf{C}^{\,0,\alpha}] \subseteq \mathsf{C}^{\,0,\alpha'}.$$

We draw the following conclusions from Lemma 3. The operator T maps $\cup_{\alpha\in(0,1]}\mathsf{C}^{0,\alpha}$ into itself and therefore

$$\bigcup_{\alpha\in(0,1]}\mathsf{C}^{0,\alpha}\subseteq \mathsf{dom}_{\mathsf{C}} := \{\psi\in\mathsf{dom} : T[\psi]\in\mathsf{C}\}.$$

Hence $\mathsf{dom}_{\mathsf{C}}$ is a $\|\cdot\|_\infty$-dense subset of C. Since $T[\mathsf{dom}_{\mathsf{C}}]\subseteq\mathsf{C}$ one may try to apply the bounded linear transformation theorem [**17**] on $T\downharpoonleft\mathsf{dom}_{\mathsf{C}}$ to extend T to C. However, this procedure is not performable: The operator $T\downharpoonleft\mathsf{dom}_{\mathsf{C}}$ is not continuous with respect to the norm $\|\cdot\|_\infty$, see Corollary 1 below.

Since T is not $\|\cdot\|_\infty$-bounded, one may ask whether $T\downharpoonleft\mathsf{C}^{0,\alpha}:\mathsf{C}^{0,\alpha}\to\mathsf{C}$, $\alpha\in(0,1]$, where $\mathsf{C}^{0,\alpha}$ is equipped with the α-Hölder norm, is bounded. A simple calculation gives

$$\|T\downharpoonleft\mathsf{C}^{0,\alpha}\|_{\mathsf{C}^{0,\alpha}\to\mathsf{C}}\le\frac{2^{1-\alpha}}{\alpha}.$$

2.2. $\tilde{T}=T$ **restricted to the space of polynomials.** The investigations of the previous section indicate that spaces of Hölder continuous functions provide no appropriate framework for an analysis of T.

On the other hand well-known properties of seemingly related integral operators with Cauchy kernels suggest a close look at T acting on $\mathcal{P}$, the space of all polynomials (restricted to I).

We recall

$$L_k:=\sum_{l=1}^{k}\frac{2}{l},\quad k\in\mathbf{N}_0$$

where we agree upon $L_0=0$. For later reference we note, according to well-known properties of the harmonic series,

$$\forall\, k\in\mathbf{N}_0 : 0<L_k-2\log(1+k)<2. \tag{2.7}$$

Then we have

Lemma 4. (i) *Each $-L_k$, $k\in\mathbf{N}_0$, is an eigenvalue of T. The eigenspace of each eigenvalue $-L_k$, $k\in\mathbf{N}$, of T contains the kth Legendre-polynomial P_k.*

(ii) *Each* $-L_k$, $k \in \mathbf{N}_0$, *is an eigenvalue of* $\tilde{T} : \mathcal{P} \to \mathcal{P}$. *The eigenspace of each eigenvalue* $-L_k$, $k \in \mathbf{N}$, *of* $\tilde{T}$ *is one-dimensional and spanned by the* k*th Legendre polynomial* P_k.

We deduce

Corollary 1. *For all linear subspaces* U *of* C *with* $T[U] \subseteq \mathsf{C}$ *and* $\mathcal{P} \subseteq U$:

$$T \upharpoonright U : U \to \mathsf{C} \text{ is not continuous (with respect to the norm } \|\cdot\|_\infty).$$

Remark 1. a) Corollary 1 shows that T is not $\|\cdot\|_\infty$-bounded on any reasonable infinite dimensional subspace of C.

b) Concerning the solvability of (2.6), we obtain: If $F \in \mathcal{P}$, then (2.6) has a unique solution in $\mathcal{P}$ if and only if $c \notin \{-L_0, -L_1, -L_2, \dots\}$.

3. A spectral analysis of $\tilde{T}$. The spectral properties of $\tilde{T}$ as described in Lemma 4 suggest looking for expansions in terms of the Legendre polynomials.

To put the investigations into the appropriate framework of Hilbert space theory, let us introduce $(\mathcal{L}^2, \langle .,. \rangle)$, the Hilbert space of all real-valued square integrable functions whose domain is I, in particular,

$$\langle G_1, G_2 \rangle = \int_{\mathsf{I}} G_1(s) G_2(s)\, ds, \quad G_1, G_2 \in \mathcal{L}^2$$

with corresponding norm $\|\cdot\|_2$.

Then one can prove

Theorem 5. *The operator* $\tilde{T} : \mathcal{L}^2 \supset \mathcal{P} \to \mathcal{L}^2$ *is essentially self-adjoint, i.e.,* $\tilde{T}$ *is* $\|\cdot\|_2$*-densely defined, symmetric and its closure* $\tilde{T}^{sa} : \mathcal{L}^2 \supset \mathsf{dom}\,(\tilde{T}^{sa}) \to \mathcal{L}^2$ *is self-adjoint.*

Remark 2. It follows from the proof of Theorem 3.8 below that $\tilde{T}^{sa}$ is the adjoint operator $\tilde{T}^*$ of $\tilde{T}$.

The operator $\tilde{T}^{sa}$ allows for a rather complete spectral theory in terms of the orthonormal basis $\{P_k^{\mathrm{norm}} : k \in \mathbf{N}\}$ of $\mathcal{L}^2$, where P_k^{norm} is the $\|\cdot\|_2$-normalized Legendre polynomial, i.e.,

$$P_k^{\mathrm{norm}} = \frac{P_k}{\|P_k\|_2}, \quad k \in \mathbf{N}_0.$$

We recall there is for each $G \in \mathcal{L}^2$ a unique sequence $(g_k)_{k\in\mathbf{N}_0}$ such that

$$\lim_{N\to\infty} \left\| G - \sum_{k=0}^{N} \langle P_k^{\mathrm{norm}}, G\rangle . P_k^{\mathrm{norm}} \right\|_2, \tag{3.8}$$

where $(\langle P_k^{\mathrm{norm}}, G\rangle)_{k\in\mathbf{N}_0} \in \ell^2$. By a slight abuse of notation we shall henceforth refer to the statement (3.8) as

$$G = \sum_{k\in\mathbf{N}_0} \langle P_k^{\mathrm{norm}}, G\rangle . P_k^{\mathrm{norm}}.$$

Theorem 6.

(i)

$$\mathsf{dom}\,(\tilde{T}^{sa}) = \left\{ G = \sum_{k\in\mathbf{N}_0} \alpha_k . P_k^{\mathrm{norm}} : (L_k . \alpha_k)_{k\in\mathbf{N}_0} \in \ell^2 \right\}.$$

(ii) $\tilde{T}^{sa}$ *has a pure point spectrum, namely,*

$$\sigma(\tilde{T}^{sa}) = \sigma_p(\tilde{T}^{sa}) = \{-L_k : k \in \mathbf{N}_0\}.$$

(iii) *The eigenspace of each eigenvalue* $-L_k$, $k \in \mathbf{N}$, *of* $\tilde{T}^{sa}$ *is one-dimensional and spanned by the* k*th Legendre polynomial* P_k.

(iv) *The operator equation*

$$(c - \tilde{T}^{sa})[\phi] = F, \quad F \in \mathcal{L}^2 \tag{3.9}$$

has a unique solution $\phi \in \mathcal{L}^2$ *if and only if* $c \notin \sigma_p(\tilde{T}^{sa}) = \{-L_k : k \in \mathbf{N}_0\}$.

(v) *If* $c + L_k \neq 0$ *for all* $k \in \mathbf{N}_0$, *then*

$$\phi = \sum_{k \in \mathbf{N}_0} \frac{\langle P_k^{\text{norm}}, F\rangle}{c + L_k} P_k^{\text{norm}} \tag{3.10}$$

is the unique solution, in $\mathcal{L}^2$, *of* (3.9).

(vi) *If* $c + L_k \neq 0$, *for all* $k \in \mathbf{N}_0$, *then*

$$\left\|(c - \tilde{T}^{sa})^{-1} : \mathcal{L}^2 \to \mathcal{L}^2\right\| = \max\left\{\frac{1}{c + L_k} : k \in \mathbf{N}_0\right\}.$$

Remark 3. Theorem 6 suggests a spectral method to treat (3.9) numerically. Corresponding analytical investigations and numerical examples are given in the subsequent section.

Remark 4. Assume $c + L_N \neq 0$ for all $N \in \mathbf{N}_0$. According to (3.10), the kth expansion coefficient (with respect to the orthonormal base $\{P_k^{\text{norm}} : k \in \mathbf{N}_0\}$) of the unique solution ϕ of (3.9) is given by

$$\phi_k = \frac{\langle P_k^{\text{norm}}, F\rangle}{c + L_k}, \quad k \in \mathbf{N}_0.$$

It is quite interesting that this representation of ϕ is closely related to an iterative procedure proposed in [**8**] where a recursion $(\phi^{(n)})_{n \in \mathbf{N}_0}$ is defined via

$$c\phi^{(n+1)} = F(s) + T[\phi^{(n)}], \quad n \in \mathbf{N}_0$$

with initial guess $\phi^{(0)} = 0$. When rewriting this iteration scheme in terms of expansions in P_k^{norm}, $k \in \mathbf{N}_0$, we obtain for the kth Legendre coefficients $\phi_k^{(n+1)}$ of $\phi^{(n+1)}$,

$$\phi_k^{(n+1)} = \frac{\langle P_k^{\text{norm}}, F\rangle}{c} - \frac{L_k}{c}\phi_k^{(n)}, \quad n \in \mathbf{N}_0,$$

hence by resolving the recursion

$$\phi_k^{(n)} = \left(\frac{-L_k}{c}\right)^n \phi_k^{(0)} + \frac{\langle P_k^{\text{norm}}, F\rangle}{c}\left(1 + \cdots + \left(\frac{-L_k}{c}\right)^{n-1}\right).$$

Indeed, this iteration converges as $n \to \infty$ if and only if

$$\left|\frac{-L_k}{c}\right| < 1, \quad \text{if and only if} \quad L_k < |c|,$$

i.e., for all small $k \in \mathbf{N}_0$, and it does not converge for all $k \in \mathbf{N}_0$ with $|-L_k/c| \geq 1$, i.e., for all but finitely many $k \in \mathbf{N}_0$.

We conclude: The iteration scheme does not usually converge as $n \to \infty$. Nevertheless, for all $k \in \mathbf{N}_0$ with $L_k < |c|$, we have

$$\begin{aligned}\lim_{n\to\infty} \phi_k^{(n)} &= \lim_{n\to\infty} \left(\frac{-L_k}{c}\right)^n \phi_k^{(0)} + \frac{\langle P_k^{\text{norm}}, F\rangle}{c} \sum_{l\in\mathbf{N}_0} \left(\frac{-L_k}{c}\right)^l \\ &= \frac{\langle P_k^{\text{norm}}, F\rangle}{c} \frac{1}{1+(L_k/c)} = \frac{\langle P_k^{\text{norm}}, F\rangle}{c+L_k} = \phi_k,\end{aligned}$$

i.e., these expansion coefficients converge as $n \to \infty$ to the correct limiting value, namely, the respective expansion coefficient of ϕ.

As a consequence, when F of (3.9) is a polynomial of sufficiently small degree (depending on c), then the recursion proposed in [**8**] converges as $n \to \infty$ in $\mathcal{L}^2$ to the unique solution of (3.9).

Remark 5. As mentioned above, (3.9) has a unique solution if and only if $c \notin \{-L_k : k \in \mathbf{N}_0\}$.

However, if $c = -L_k$ for some $k \in \mathbf{N}$, then (3.9) has either no solution or the set of all solutions of (3.9) is a one-dimensional affine manifold.

The question arises whether (5.16) is a well-posed problem or not. From a theoretical point of view this is certainly not the case: the number of values for c in, let's say, intervals $[-l-1, -l]$, $l \in \mathbf{N}$, for which (3.9) is not uniquely solvable, increases exponentially with l. Hence a slight perturbation of $c \approx -l$, l "large," may entirely change the solvability of (3.9).

From a numerical point of view, however, the situation is much less dramatic. Due to practical limitations only finitely many digits of the number c can be handled. But such c (unless in $\{0, -2, -3\}$), never equals $-L_k$ for any $k \in \mathbf{N}$, i.e., for "realistic" values of c, $c \notin \{0, -2, -3\}$, (3.9) is always uniquely solvable, i.e., (3.9) is for practical reasons a well-posed problem.

4. Numerics. The spectral properties of $\tilde{T}^{sa}$ as described in Theorem 6 suggest the employment of a spectral method to solve (3.9) numerically. We assume $c + L_k \neq 0$ for all $k \in \mathbf{N}_0$ in this section.

`The spectral method.`

0. Choose $K \in \mathbf{N}$.

1. Determine the first $K+1$ Fourier coefficients $F_0, \dots, F_K$ of F with respect to the orthonormal basis $\{P_k^{\mathrm{norm}} : k \in \mathbf{N}_0\}$, i.e., compute

$$F_j = \int_{\mathsf{I}} F(s) P_j^{\mathrm{norm}}(s)\, ds, \quad j = 0, \dots, K.$$

2. Calculate $\phi_0, \dots, \phi_K$ via

$$\phi_j = \frac{F_j}{c + L_j}.$$

3. Visualize

$$\phi^K = \sum_{j=0}^{K} \phi_j P_j^{\mathrm{norm}}.$$

It is easy to deduce the following error estimate from Theorem 6.

Corollary 2. *Let $\phi \in \mathcal{L}^2$ be the unique solution of* (3.9) (*we recall the assumption $c + L_k \neq 0$ for all $k \in \mathbf{N}_0$*). *For $K \in \mathbf{N}$ let ϕ^K be as described above, and let*

$$F^K = \sum_{j=0}^{K} \langle P_j^{\mathrm{norm}}, F \rangle . P_j^{\mathrm{norm}}.$$

Then

$$\|\phi - \phi^K\|_2 \leq \max\{(c + L_k)^{-1} : k \in \mathbf{N}_0\} . \|F - F^K\|_2,$$

in particular, $\lim_{K \to \infty} \|\phi - \phi^K\|_2 = 0$.

Remark 6. The convergence result of Corollary 2 does not allow for a conclusion about the pointwise behavior of the approximating series.

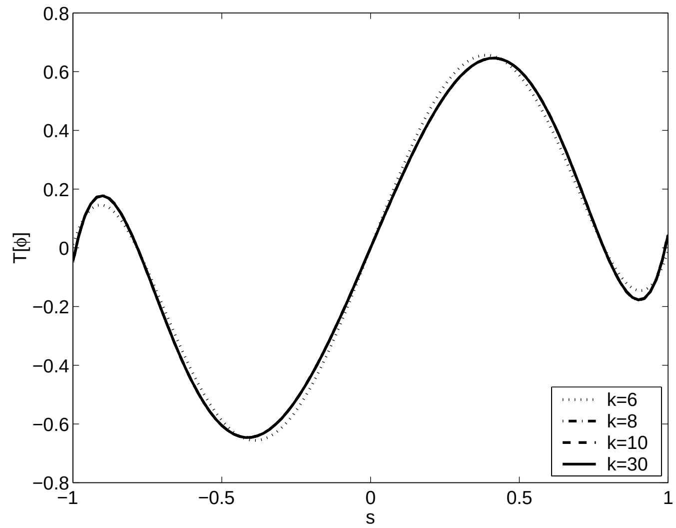

FIGURE 1. Solutions to (1.1) for $c = -5$ and $F(s) = -\sin(\pi\ s)$.

Since the derivative of the kth Legendre polynomial P_k at ± 1 tends to $\pm\infty$ as $k \to \infty$, one has to expect (numerical) singularities at the endpoints of 1.

Now we give some numerical examples for the spectral method.

The implementations are based on the package `FORTRAN Routines for Spectral Methods` [**4**] and use Legendre polynomials up to order $K = 30$.

We set $c = -5$.

As a first example we consider the problem (1.1) with $F(s) = -\sin(\pi\ s)$, see Figure 1.

The presented results were obtained for different degrees K of the approximating Legendre polynomials. As the inhomogeneity function $F(s) = -\sin(\pi\ s)$ is odd and since the integral equation preserves parity the solution is odd, too. The numerical scheme conserves this symmetry. The Legendre coefficients of the even polynomials equal zero up to machine precision.

A very good agreement of the low order solutions and the most

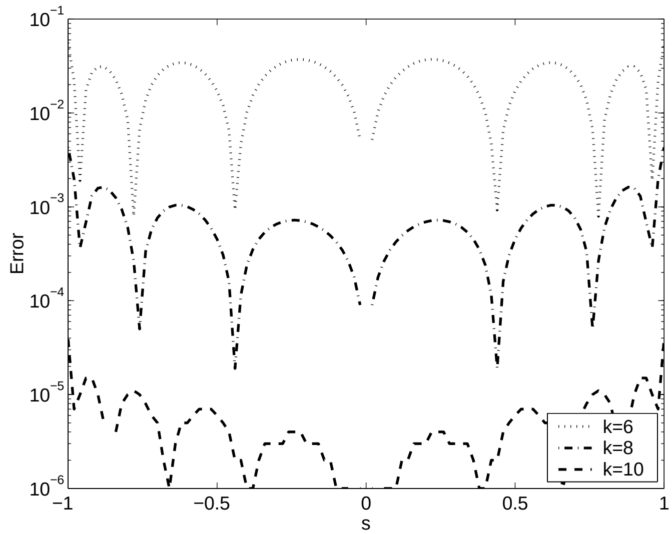

FIGURE 2. Differences to the solution for $K = 30$.

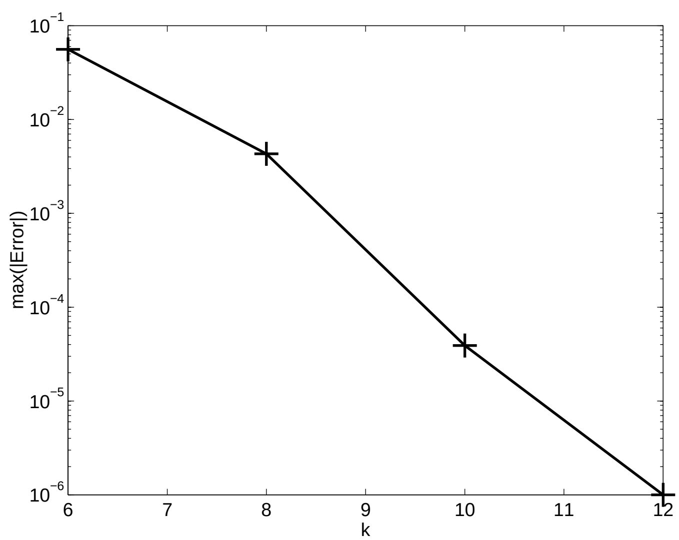

FIGURE 3. ∞-norm of the difference in Figure 2.

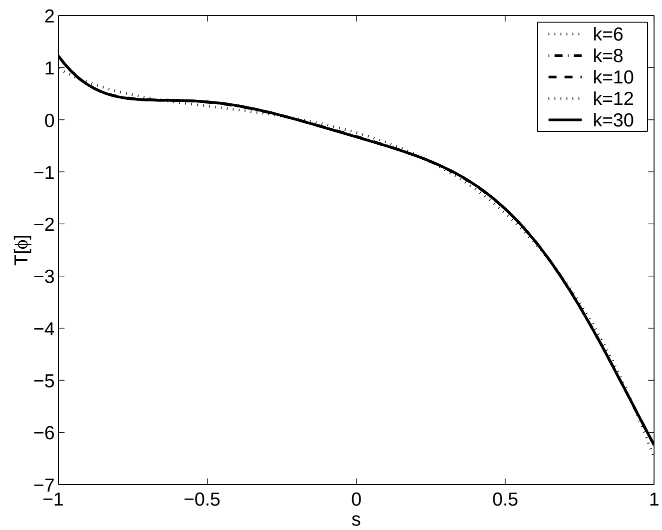

FIGURE 4. Solutions to (1.1) for $c = -5$ and $F(s) = e^{3(s+1)/2} + \ln(\varepsilon + (s+1)/2)(\varepsilon + (1-s)/2)$, $\varepsilon = 0.1$.

accurate one ($K = 30$) is apparent, see Figure 2. The $\|\cdot\|_\infty$-norm of these differences is given (in logarithmic scaling) in Figure 3, exhibiting an exponential dependence of the error and the maximal degree of the polynomials used. Now we choose

$$F(s) = e^{3(s+1)/2} + \ln\left(\varepsilon + \frac{s+1}{2}\right)\left(\varepsilon + \frac{1-s}{2}\right), \quad \varepsilon = \frac{1}{10}.$$

The results are given in Figures 4, 5 and 6. Since the function $F(s)$ is neither odd nor even, the solution is neither odd nor even, too. An analogous property holds for the approximations of the solutions.

Finally, let us consider $F(s) = \ln((1-s^2)/4)$ with $K = 50$. In spite of the unboundedness of F, the algorithm is rather stable (see Figures 8 and 9). The solutions computed for polynomial degrees up to $K = 50$ are given in Figure 7.

5. Uniform convergence of a modified spectral method. The calculated differences of the numerical approximations in terms of the $\|\cdot\|_\infty$-norm (see Figures 2, 5, 8, 3, 6 and 9) suggest not only $\mathcal{L}^2$-convergence but also uniform convergence.

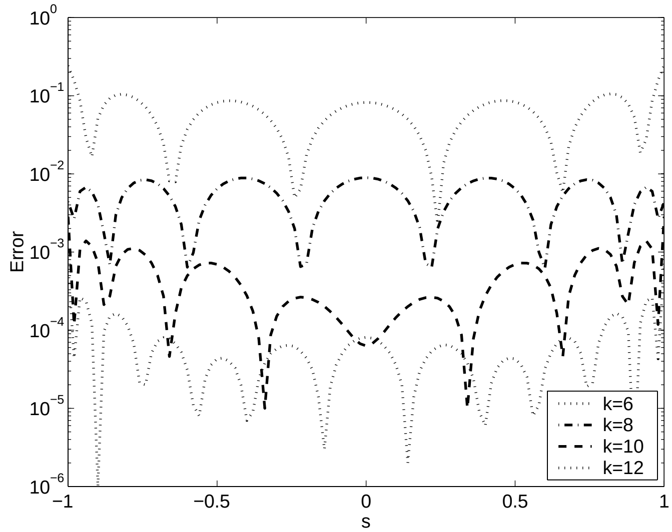

FIGURE 5. Differences to the solution for $K = 30$.

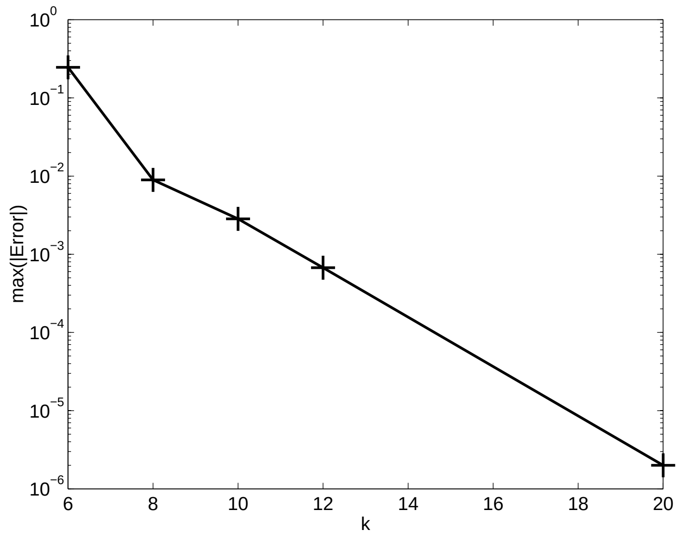

FIGURE 6. ∞-norm of the difference in Figure 5.

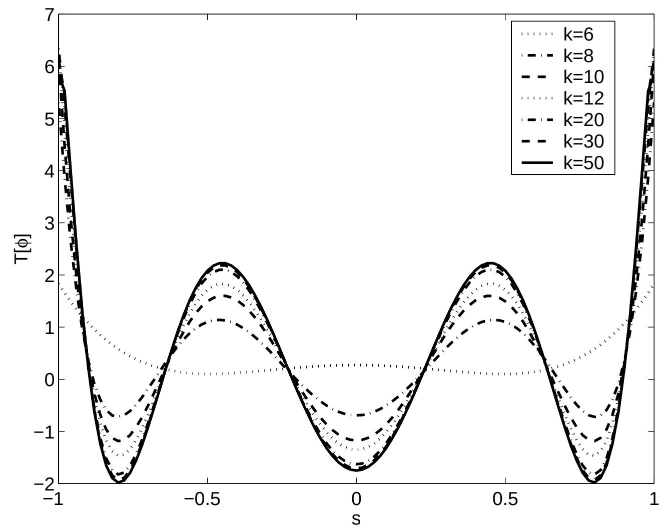

FIGURE 7. Solutions to (1.1) for $c = -5$ and $F(s) = \ln \frac{1-s^2}{4}$.

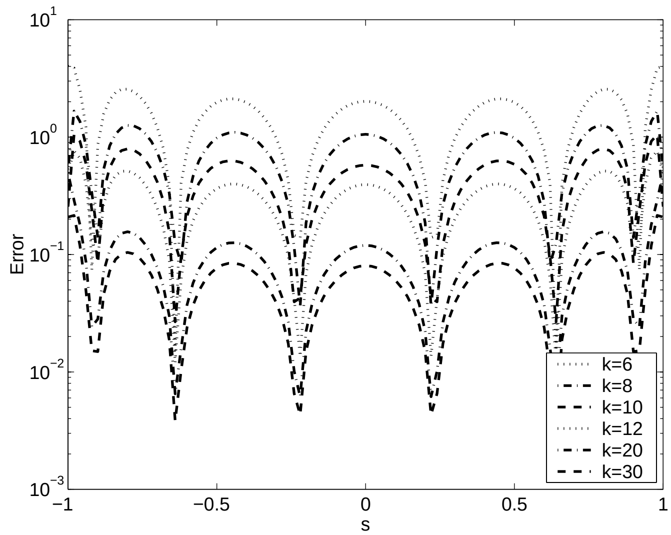

FIGURE 8. Differences to the solution for $K = 50$.

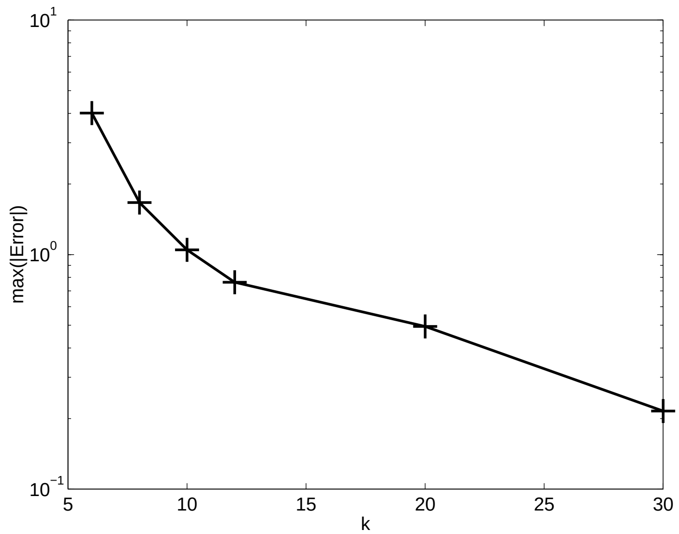

FIGURE 9. ∞-norm of the difference in Figure 8.

This observation is extremely important from an applicational point of view. In applications one is much more interested in computing the force acting on the fiber (i.e., ϕ) within certain bounds than in averaged accuracy results.

However, investigations on uniform convergence are difficult (seemingly even impossible) to perform by means of the spectral theory of the previous section, simply because of $\|P_k^{\mathrm{norm}}\|_\infty \to \infty$ as $k \to \infty$.

Consequently, the investigations have to be settled on grounds of another function space, call it "$\mathsf{dom}\,(\hat{T})$" for the moment, rather than $\mathcal{L}^2$.

$\mathsf{dom}\,(\hat{T})$ shall be chosen such that the spectral properties of $\tilde{T}$ are of immediate use, i.e., the restriction $\hat{T}$ of T to $\mathsf{dom}\,(\hat{T})$ shall have a transparent representation. This suggests the employment of a polynomial Schauder basis of C more tractable than the Legendre-polynomials. Seemingly a good choice are monomials, σ^k, $k \in \mathbf{N}_0$:

$$\sigma^k : \mathsf{I} \to \mathbf{R}, \qquad \sigma^k(s) := \left(\frac{s}{2}\right)^k.$$

But one has to be careful with expansions in terms of this Schauder basis. T is not continuous with respect to the $\| \cdot \|_\infty$-norm. Hence one may lose one of the most promising features of such expansions, namely, the possibility of interchanging limits.

One can expect that the interchange of "applying T" with the Schauder expansion will go through if one imposes growth conditions on the expansions' coefficients.

Hence $\mathsf{dom}\,(\hat{T})$ shall consist of functions whose Schauder coefficients (with respect to $\{\sigma^k : k \in \mathbf{N}_0\}$) satisfy some growth conditions.

Indeed there are several possibilities to choose $\mathsf{dom}\,(\hat{T})$ in accordance with these requirements.

The peculiar choice of $\mathsf{dom}\,(\hat{T})$ we consider here is taken for the sake of transparency. Other choices may also be possible.

We set

$$\mathsf{dom}\,(\hat{T}) := \left\{ \sum_{l=0}^{\infty} \psi_l \sigma^l : (\psi_l)_{l \in \mathbf{N}_0} \in \ell^\infty \right\},$$

where we make use of the fact that the series arising in that definition converge uniformly on I. As a consequence, the elements of $\mathsf{dom}\,(\hat{T})$ are analytic functions whose domain is I.

Remark 7. dom contains nonanalytic functions (recall that $\mathsf{C}^{0,1} \subseteq \mathsf{dom}$). Therefore, $\mathsf{dom}\,(\hat{T}) \subseteq \mathsf{dom}$ but $\mathsf{dom}\,(\hat{T}) \neq \mathsf{dom}$.

We set

$$\hat{T} := T \downharpoonleft \mathsf{dom}\,(\hat{T})$$

and equip $\mathsf{dom}\,(\hat{T})$ with its canonical norm

$$\left\| \sum_{l=0}^{\infty} \psi_l \sigma^l \right\| := |(\psi_l)_{l \in \mathbf{N}_0}|_\infty .$$

It is left to the reader to verify that $\|\cdot\|$ is well defined. Furthermore, it is easy to see that $(\mathsf{dom}\,(\hat{T}), \|\cdot\|)$ is a Banach space which is continuously embedded in $(\mathsf{C}, \|\cdot\|_\infty)$. Hence convergence in $\mathsf{dom}\,(\hat{T})$ implies uniform convergence on I.

As it may be anticipated from the spectral properties of $\tilde{T}$, the operator $\hat{T}$ does not map $\mathsf{dom}\,(\hat{T})$ into itself but into

$$\mathcal{L}_w^\infty := \left\{ \sum_{l=0}^{\infty} g_l \sigma^l : (g_l)_{l\in\mathbf{N}_0} \in \ell_w^\infty \right\},$$

where

$$\ell_w^\infty := \left\{ (g_l)_{l\in\mathbf{N}} \in \mathbf{R}^{\mathbf{N}_0} : \sup \left\{ \frac{|g_l|}{3+2\log(1+l)} : l \in \mathbf{N}_0 \right\} < \infty \right\}.$$

We equip ℓ_w^∞ with its canonical norm

$$|(g_l)_{l\in\mathbf{N}_0}|_{w,\infty} := \sup \left\{ \frac{|g_l|}{3+2\log(1+l)} : l \in \mathbf{N}_0 \right\}, \quad (g_l)_{l\in\mathbf{N}_0} \in \ell_w^\infty,$$

and introduce the corresponding norm $\|\cdot\|_w$ on $\mathcal{L}_w^\infty$,

$$\left\| \sum_{l=0}^{\infty} g_l \sigma^l \right\|_w := |(g_l)_{l\in\mathbf{N}_0}|_{w,\infty}, \quad \sum_{l=0}^{\infty} g_l \sigma^l \in \mathcal{L}_w^\infty.$$

It is left to the reader to verify that $(\ell_w^\infty, |\cdot|_{w,\infty})$ and $(\mathcal{L}_w^\infty, \|\cdot\|_w)$ are (isometric) Banach spaces.

Remark 8. a) One immediately realizes that $(g_l)_{l\in\mathbf{N}_0} \in \ell_w^\infty$ implies convergence with respect to the $\|\cdot\|_\infty$-norm of the series $(\sum_{l=0}^{n} g_l \sigma^l)_{n\in\mathbf{N}_0}$. Hence the elements of $\mathcal{L}_w^\infty$ are well defined analytic functions whose domain is I and whose power series (centered at 0) has coefficients of at most logarithmic growth. Furthermore, one can argue as in b) of Remark 7 to deduce $\mathcal{L}_w^\infty \subseteq \mathsf{dom}$ but $\mathcal{L}_w^\infty \neq \mathsf{dom}$.

b) Certainly, $\mathsf{dom}\,(\hat{T}) \subseteq \mathcal{L}_w^\infty$ but $\mathsf{dom}\,(\hat{T}) \neq \mathcal{L}_w^\infty$.

Occasionally, we shall make use of the projection operators, $n \in \mathbf{N}_0$,

$$\mathsf{P}_0^n : \mathcal{L}_w^\infty \to \mathcal{L}_w^\infty, \qquad \mathsf{P}_0^n \left[\sum_{l=0}^{\infty} \psi_l \sigma^l \right] = \sum_{l=0}^{n} \psi_l \sigma^l,$$

$$\mathsf{P}_{n+1}^\infty : \mathcal{L}_w^\infty \to \mathcal{L}_w^\infty, \quad \mathsf{P}_{n+1}^\infty \left[\sum_{l=0}^{\infty} \psi_l \sigma^l \right] = \sum_{l=n+1}^{\infty} \psi_l \sigma^l.$$

We certainly have: $\mathsf{P}_0^n + \mathsf{P}_{n+1}^\infty$ is the identity on $\mathcal{L}_w^\infty$.

Although $\hat{T}$ has no diagonal structure with respect to the Schauder basis $\{\sigma^k : k \in \mathbf{N}_0\}$, a rather transparent representation of $\hat{T}$ with respect to $\{\sigma^k : k \in \mathbf{N}_0\}$ is available. We observe

Proposition 7. *For all* $l \in \mathbf{N}_0$:

$$\hat{T}[\sigma^l] = \sum_{\nu=0}^{l} B_\nu^l \sigma^\nu,$$

where

$$\begin{cases} B_l^l := -L_l \\ B_\nu^l := \dfrac{1+(-1)^{l-\nu}}{2^{l-\nu}(l-\nu)}, \quad \nu = 0, \dots, l-1. \end{cases}$$

Furthermore, for all $\nu \in \mathbf{N}_0$,

$$\sum_{l=\nu+1}^{\infty} B_\nu^l = \sum_{k=1}^{\infty} \frac{1+(-1)^k}{2^k k} = \log(4/3). \tag{5.11}$$

With the aid of Proposition 7 we obtain

Theorem 8. *Let* $\psi = \sum_{l=0}^\infty \psi_l \sigma^l \in \mathsf{dom}\,(\hat{T})$. *Then*

$$\hat{T}[\psi] = \sum_{l=0}^{\infty} \left[\sum_{\nu=l}^{\infty} B_l^\nu \psi_\nu\right] \sigma^l \in \mathcal{L}_w^\infty,$$

and

$$\lim_{n\to\infty} \|\hat{T}[\psi] - \mathsf{P}_0^n[\hat{T}[\psi]]\|_\infty = \lim_{n\to\infty} \left\| \hat{T}[\psi] - \sum_{l=0}^{n} \left[\sum_{\nu=l}^{\infty} B_l^\nu \psi_\nu\right] \sigma^l \right\|_\infty = 0,$$

as well as

$$\lim_{n\to\infty} \|\hat{T}[\psi] - (\mathsf{P}_0^n \circ \hat{T} \circ \mathsf{P}_0^n)[\psi]\|_\infty = \lim_{n\to\infty} \left\| \hat{T}[\psi] - \sum_{l=0}^{n} \left[\sum_{\nu=l}^{n} B_l^\nu \psi_\nu\right] \sigma^l \right\|_\infty = 0.$$

Remark 9. With the notations of Theorem 8 we have, for $n \in \mathbf{N}_0$,

$$\psi^{(n)} := (\mathsf{P}_0^n \circ \hat{T} \circ \mathsf{P}_0^n)[\psi] = (\hat{T} \circ \mathsf{P}_0^n)[\psi] = \sum_{l=0}^{n} \Bigg[\sum_{\nu=l}^{n} B_l^\nu \psi_\nu \Bigg] \sigma^l,$$

and

$$\lim_{n\to\infty} \|\psi - \psi^{(n)}\|_\infty = 0, \qquad \lim_{n\to\infty} \|T[\psi] - T[\psi^{(n)}]\|_\infty = 0.$$

Nevertheless, T is not $\|\cdot\|_\infty$-$\|\cdot\|_\infty$-continuous.

According to Theorem 8 we have for each $\psi = \sum_{l=0}^\infty \psi_l \sigma^l \in \mathsf{dom}\,(\hat{T})$ the identity

$$\hat{T}[\psi] = \sum_{l=0}^{\infty} \Gamma_l[\psi] \sigma^l, \qquad \Gamma_l[\psi] := \sum_{\nu=l}^{\infty} B_l^\nu \psi_\nu,$$

where

$$\forall\, l \in \mathbf{N}_0 : |\Gamma_l[\psi]\| \leq (3 + 2\log(1+l))\|\psi\|.$$

We readily deduce

Corollary 3. *$\hat{T}$ maps $\mathsf{dom}\,(\hat{T})$ into $\mathcal{L}_w^\infty$ with operator norm*

$$\|\hat{T} : \mathsf{dom}\,(\hat{T}) \to \mathcal{L}_w^\infty\| = 1.$$

Remark 10. The estimate $\|\hat{T} : \mathsf{dom}\,(\hat{T}) \to \mathcal{L}_w^\infty\| \leq 1$ is obvious. To obtain equality let us consider the coefficients $\Gamma_l[\psi]$ of the series expansion of $\hat{T}[\psi]$, $\psi = (1,1,1,\dots) \in \mathsf{dom}\,(\hat{T})$. Then

$$\lim_{l\to\infty} \frac{|\Gamma_l[(1,1,1,\dots)]|}{2\log(1+l)} = 1,$$

such that $\|\hat{T} : \mathsf{dom}\,(\hat{T}) \to \mathcal{L}_w^\infty\| \geq 1$ as well.

We are now in a position to pass to a matrix representation for $\hat{T}$ which will be of great importance for the numerical approximations of (1.1). We give

Definition 1. For $l, m \in \mathbf{N}_0$, let $A_{l,m} \in \mathbf{R}$. Let $(F_l)_{l\in\mathbf{N}_0} \in \mathbf{R}^{\mathbf{N}_0}$, and let $(x_l)_{l\in\mathbf{N}_0} \in \mathbf{R}^{\mathbf{N}_0}$. Then $(x_l)_{l\in\mathbf{N}_0}$ is said to be a "solution of the matrix equation"

$$\begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & \cdots \\ A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots \\ A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & \cdots \\ A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \end{pmatrix},$$

if and only if for all $l \in \mathbf{N}_0$:

$$\left(\sum_{\nu=0}^{n} A_{l,\nu} x_\nu \right)_{n\in\mathbf{N}_0} \in \ell^1, \quad \text{and} \quad \sum_{\nu=0}^{\infty} A_{l,\nu} x_\nu = F_l.$$

Combining Definition 1 with Theorem 8 it is easy to deduce

Corollary 4. *For all $c \in \mathbf{R}$, all $\phi = \sum_{l=0}^{\infty} \phi_l \sigma^l \in \mathsf{dom}\,(\hat{T})$ and all $F = \sum_{l=0}^{\infty} \mathrm{F}_l \sigma^l \in \mathcal{L}_w^\infty$ the propositions* (i) *and* (ii) *are equivalent.*

(i) $c\phi = F + \hat{T}[\phi]$.

(ii) $(\phi_l)_{l\in\mathbf{N}_0}$ *is a solution of the matrix equation*

(5.12)

$$\begin{pmatrix} c{+}L_0 & -B_0^1 & -B_0^2 & -B_0^3 & -B_0^4 & -B_0^5 & \cdots \\ 0 & c{+}L_1 & -B_1^2 & -B_1^3 & -B_1^4 & -B_1^5 & \cdots \\ 0 & 0 & c{+}L_2 & -B_2^3 & -B_2^4 & -B_2^5 & \cdots \\ 0 & 0 & 0 & c{+}L_3 & -B_3^4 & -B_3^5 & \cdots \\ 0 & 0 & 0 & 0 & c{+}L_4 & -B_4^5 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \end{pmatrix}$$

5.1 The spectrum of $\hat{T}$. We shall investigate the spectrum of $\hat{T}$ now. It will turn out that the spectrum of $\hat{T}$ is exactly the spectrum of $\tilde{T}^{sa}$, i.e., $\hat{T}$ has a pure point spectrum, namely $\{-L_k : k \in \mathbf{N}_0\}$. For $\rho \in \mathbf{R}$, $\rho + L_k \neq 0$ for all $k \in \mathbf{N}_0$, an estimate of the norm of $[\rho - \hat{T}]^{-1} : \mathcal{L}_w^\infty \to \mathsf{dom}\,(\hat{T})$ will be given in terms of an appropriately defined norm of the inverse of a matrix $A_{n;\rho}$ representing the canonical finite-dimensional approximation $\mathsf{P}_0^n \circ (\rho - \hat{T}) \circ \mathsf{P}_0^n = (\rho - \hat{T}) \circ \mathsf{P}_0^n$ of $\rho - \hat{T}$. We put $n \in \mathbf{N}$, and $\rho + L_k \neq 0$ for all $k \in \mathbf{N}_0$,

(5.13)

$$A_{n;\rho} := \begin{pmatrix} \rho + L_0 & -B_0^1 & -B_0^2 & -B_0^3 & \cdots & -B_0^n \\ 0 & \rho + L_1 & -B_1^2 & -B_1^3 & \cdots & -B_1^n \\ 0 & 0 & \rho + L_2 & -B_2^3 & \cdots & -B_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \rho + L_{n-1} & -B_{n-1}^n \\ 0 & 0 & 0 & \cdots & 0 & \rho + L_n \end{pmatrix}.$$

By a slight abuse of notation, we identify the matrix $A_{n;\rho}$ with the finite-dimensional operator induced by $A_{n;\rho}$ which maps $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+1}$. Certainly, for $\rho + L_k \neq 0$ for all $k \in \mathbf{N}_0$, this operator has a bounded inverse $A_{n;\rho}^{-1}$ whose norm is

$$\begin{aligned} p_1(n;\rho) &:= \sup\left\{ \left|A_{n;\rho}^{-1}[\eta]\right|_{n+1} : |\eta|_{w,n+1} \leq 1 \right\} \\ &= \left\| \left[(\rho - \hat{T}) \circ \mathsf{P}_0^n\right]^{-1} : \mathsf{P}_0^n\left[\mathcal{L}_w^\infty\right] \to \mathsf{P}_0^n\left[\mathsf{dom}\,(\hat{T})\right] \right\|, \end{aligned} \tag{5.14}$$

where for $n \in \mathbf{N}_0$ and $\eta = (\eta_0, \ldots, \eta_n) \in \mathbf{R}^{n+1}$,

$$|\eta|_{n+1} := \max\{|\eta_0|, \ldots, |\eta_n|\},$$

and

$$|\eta|_{w,n+1} := \max\left\{ \frac{|\eta_0|}{3 + 2\log(1+0)}, \ldots, \frac{|\eta_n|}{3 + 2\log(1+n)} \right\}.$$

Corollary 4 contains most of the important information to prove

Theorem 9. $\hat{T} : \mathcal{L}_w^\infty \supseteq \mathsf{dom}\,(\hat{T}) \to \mathcal{L}_w^\infty$ *has a pure point spectrum,*

$$\sigma(\hat{T}) = \sigma_p(\hat{T}) = \{-L_k : k \in \mathbf{N}_0\}.$$

Furthermore,

a) *the eigenspace of* $-L_k$, $k \in \mathbf{N}_0$, *is spanned by the* k*th Legendre polynomial* P_k;

b) *for all* $\rho \in \mathbf{R}\backslash\sigma_p(\hat{T})$ *and for all* $n \in \mathbf{N}$ *with*

$$n \geq \min\{k \in \mathbf{N} : L_{k+1} > \log(4/3) - \rho\},$$

one has

$$\|(\rho - \hat{T})^{-1} : \mathcal{L}_w^\infty \to \mathsf{dom}\,(\hat{T})\| \leq p_1(n;\rho)(1 + p_2(n;\rho)), \tag{5.15}$$

where $p_1(n;\rho)$ *is as in* (5.14) *and*

$$p_2(n;\rho) := \frac{\log(4/3)}{1-(\log(4/3)/(\rho+L_{n+1}))}\sup\left\{\frac{3+2\log(1+l)}{\rho + L_{l+1}} : l\in\mathbf{N}, n\leq l\right\}.$$

5.2 A modified spectral method. In this section we are concerned with the integral equation

$$c\phi = F + T[\phi], \quad \phi \in \mathsf{dom}\,(\hat{T}), \quad F \in \mathcal{L}_w^\infty, \tag{5.16}$$

with $c < 0$.

We deduce from Theorem 9: (5.16) has a unique solution if and only if $c \notin \{-L_k : k \in \mathbf{N}_0\}$ and if $c \notin \{-L_k : k \in \mathbf{N}_0\}$, then

$$\phi = (c - \hat{T})^{-1}[F]$$

with $\mathcal{L}_w^\infty - \mathsf{dom}\,(\hat{T})$ continuous $(c - \hat{T})^{-1}$.

We assume the latter case, i.e., $c + L_k \neq 0$ for all $k \in \mathbf{N}_0$ henceforth.

Let us turn our attention to a numerical approximation of (5.16) now. Due to the spectral properties of $\hat{T}$ it is convenient to project (5.16) to the space of all polynomials of degree less than or equal to $n \in \mathbf{N}$ and let n tend to ∞ then.

The corresponding $n+1$-dimensional matrix approximation of (5.16) reads

$$A_{n;-c}[(\phi_0^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]})^t] = (F_0, F_1, \dots, F_n)^t, \tag{5.17}$$

where $A_{n;-c}$ is the matrix as in (5.13), $(\phi_0^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]}) \in \mathbf{R}^{n+1}$ is unknown and

$$F = \sum_{l=0}^{\infty} F_l \sigma^l.$$

Then we easily deduce the following approximation result.

Theorem 10. *Let $F = \sum_{l=0}^{\infty} F_l \sigma^l \in \mathcal{L}_w^{\infty}$. Let $c + L_k \neq 0$ for all $k \in \mathbf{N}_0$. Then we have, for all $n \in \mathbf{N}$,*

(i) *the operator equation*

$$\Big((c - \hat{T}) \circ \mathsf{P}_0^n\Big)[\phi^{[n]}] = \mathsf{P}_0^n[F]$$

has a unique solution $\phi^{[n]} = \sum_{l=0}^{n} \phi_l^{[n]} \sigma^l \in P_0^n[\mathsf{dom}\,(\hat{T})]$.

(ii) $(\phi_0^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]}) \in \mathbf{R}^{n+1}$ *is the unique solution of* (5.17).

(iii) *If $\phi = \sum_{l=0}^{\infty} \phi_l \sigma^l \in \mathsf{dom}\,(\hat{T})$ is the unique solution of* (5.16)*, then*

$$\phi - \phi^{[n]} = (c - \hat{T})^{-1} \left[\sum_{l=n+1}^{\infty} F_l \sigma^l \right],$$

i.e., if

$$\lim_{n \to \infty} |(F_{n+1}, F_{n+2}, F_{n+3}, \dots)|_{w,\infty} = 0,$$

then

$$\lim_{n \to \infty} \|\phi - \phi^{[n]}\|_{1/2} = 0,$$

in particular,

$$\lim_{n \to \infty} \phi_l^{[n]} = \phi_l,$$

uniformly in $l \in \mathbf{N}_0$.

APPENDIX

6. Proofs.

Theorem 1. Let $\phi, \psi \in \cup_{\alpha \in (0,1]} \mathsf{C}^{0,\alpha}$. Then the mapping

$$H : \mathsf{I} \times \mathsf{I} \to \mathbf{R}, \quad H(s,t) = \begin{cases} \dfrac{\psi(s)\phi(t) - \psi(t)\phi(s)}{|t-s|} & \text{if } s \neq t \\ 0 & \text{if } s = t, \end{cases}$$

is Lebesgue-integrable over $\mathsf{I}\times\mathsf{I}$. By symmetry we have

$$\int_{\mathsf{I}\times\mathsf{I}} H(s,t)\,d(s,t) = 0.$$

We furthermore set

$$H_1 : \mathsf{I}\times\mathsf{I}\to\mathbf{R},\quad H_1(s,t) = \begin{cases} \psi(s)\dfrac{\phi(t)-\phi(s)}{|t-s|} & \text{if } s\neq t\\ 0 & \text{if } s=t\end{cases}$$

and

$$H_2 : \mathsf{I}\times\mathsf{I}\to\mathbf{R},\quad H_2(s,t) = \begin{cases} \phi(s)\dfrac{\psi(t)-\psi(s)}{|t-s|} & \text{if } s\neq t,\\ 0 & \text{if } s=t.\end{cases}$$

We certainly have $H=H_1-H_2$ and the functions H_1, H_2 are obviously Lebesgue-integrable over $\mathsf{I}\times\mathsf{I}$. hence

$$\psi T[\phi] = \int_{\mathsf{I}} H_1(\cdot,t)\,dt,\qquad \phi T[\psi] = \int_{\mathsf{I}} H_2(\cdot,t)\,dt$$

are Lebesgue-integrable over I, and we deduce from the Fubini-Tonelli Theorem,

$$\begin{aligned}\left(\int_{\mathsf{I}}\psi(s)T[\phi](s)\,ds\right) - \left(\int_{\mathsf{I}}\phi(s)T[\psi](s)\,ds\right)&\\ &= \int_{\mathsf{I}\times\mathsf{I}} H_1(s,t)\,d(s,t) - \int_{\mathsf{I}\times\mathsf{I}} H_2(s,t)\,d(s,t)\\ &= \int_{\mathsf{I}\times\mathsf{I}} H(s,t)\,d(s,t) = 0.\end{aligned}$$

□

Proposition 2. If $\alpha\in(0,1)$ consider $\psi(s) = ((s+1)/2)^\alpha$, $s\in\mathsf{I}$. Then $\psi\in\mathsf{C}^{0,\alpha}$ and we obtain for all $s\in(-1,1]$ with the notation $u=u(s)=(s+1)/2$,

$$\begin{aligned}\frac{T[\psi](s)-T[\psi](-1)}{(s+1)^\alpha} &= \frac{1}{2^\alpha u^\alpha}\left(\int_0^1 \frac{v^\alpha-u^\alpha}{|v-u|}\,dv - \int_0^1 \frac{v^\alpha-1}{v}\,dv\right)\\ &= \frac{1}{2^\alpha}\left(-\int_0^1 \frac{1-\sigma^\alpha}{1-\sigma}\,d\sigma + \int_1^{1/u}\frac{\sigma^\alpha-1}{\sigma-1}\,d\sigma - \frac{1}{\alpha u^\alpha}\right)\\ &= 2^{-\alpha}\left(\frac{1}{\alpha} - \int_0^1\frac{1-\sigma^\alpha}{1-\sigma}\,d\sigma + \int_1^{(2/(1+s))}\frac{\sigma^{\alpha-1}-1}{\sigma-1}\,d\sigma\right),\end{aligned}$$

such that

$$\lim_{s\to -1}\left|\frac{T[\psi](s)-T[\psi](-1)}{(s+1)^\alpha}\right| = \infty,$$

i.e., $T[\psi]$ is *not* α-Hölder continuous.

In case of $\alpha = 1$, consider $\psi(s) := |s|/2 \in \mathsf{C}^{0,1}$. We immediately obtain

$$T[\psi](s) = \begin{cases} 1+s+s\log\dfrac{s-1}{s} & -1 \le s < 0 \\ 1 & s = 0 \\ 1-s-s\log\dfrac{s+1}{s} & 0 < s \le 1, \end{cases}$$

i.e., $T[\psi]$ is *not* Lipschitz-continuous. □

Lemma 3. We can assume $\alpha \in (0,1)$. It suffices to consider $\alpha' \in (0,\alpha)$. Let $s, s' \in \mathsf{I}$ with $s \neq s'$ and $\delta = 2|s-s'|$. We observe $0 < \delta \le 4$. Let $U_1 = \mathsf{I} \cap [s-\delta, s+\delta]$ and $U_2 = \mathsf{I}\backslash[s-\delta, s+\delta]$. For $t \neq s, s'$, let $q(t) := [(\psi(t)-\psi(s))/|t-s|] - [(\psi(t)-\psi(s'))/|t-s'|]$. We set without loss of generality $q(s) = q(s') = 0$.

By $K_1, K_2, \dots$, we denote positive constants which may depend on ϕ but are independent of s, s'.

We have

$$|T[\psi](s) - T[\psi](s')| = \left|\int_{-1}^{1} q(t)\,dt\right| \le \int_{U_1} |q(t)|\,dt + \int_{U_2} |q(t)|\,dt.$$

Since $\psi \in \mathsf{C}^{0,\alpha}$, we have

$$\int_{U_1} |q(t)|\,dt \le K_1 \int_{U_1} |t-s|^{\alpha-1}\,dt + K_1 \int_{U_1} |t-s'|^{\alpha-1}\,dt,$$

where we note that

$$\int_{U_1} |t-s|^{\alpha-1}\,dt \le \int_{s-\delta}^{s+\delta} |t-s|^{\alpha-1}\,dt = \frac{2}{\alpha}\delta^\alpha.$$

On the other hand, we have $U_1 \subseteq [s-\delta, s+\delta] \subseteq [s'-2\delta, s'+2\delta]$ such that

$$\int_{U_1} |t-s'|^{\alpha-1}\,dt \le \int_{s'-2\delta}^{s'+2\delta} |t-s'|^{\alpha-1}\,dt = \frac{2^{1+\alpha}}{\alpha}\delta^\alpha,$$

and we obtain, due to $|s-s'|^{\alpha-\alpha'} \leq 2$,

$$\int_{U_1} |q(t)|\,dt \leq K_2|s-s'|^{\alpha} \leq K_3|s-s'|^{\alpha'}.$$

Now we consider the integration over U_2. We introduce for $t \neq s, s'$, $q_1(t) := (\psi(s') - \psi(s))/|t-s|$ and $q_2(t) := (\psi(t) - \psi(s'))((1/|t-s|) - (1/|t-s'|))$ and put without loss of generality $q_{1,2}(s) = q_{1,2}(s') = 0$ such that $q(t) = q_1(t) + q_2(t)$. We have, due to Hölder-continuity and due to $\delta \leq 4$,

$$\begin{aligned}
\int_{U_2} |q_1(t)|\,dt &= |\psi(s') - \psi(s)| \int_{U_2} \frac{dt}{|t-s|} \\
&\leq K_4|s-s'|^{\alpha} 2 \int_{\delta}^{\max(s+1,1-s)} \frac{dt}{t} \\
&\leq K_4|s-s'|^{\alpha} 2 \int_{\delta}^{4} \frac{dt}{t} \\
&= 2K_4|s-s'|^{\alpha}(\log(4) - \log(\delta)) \\
&= 2K_4|s-s'|^{\alpha}(\log(4) - \log(4|s-s'|)) \\
&\leq K_5|s-s'|^{\alpha'}.
\end{aligned}$$

Concerning $\int_{U_2} |q_2(t)|\,dt$ we have for all $t \in U_2$ the estimate

$$\begin{aligned}
|q_2(t)| &\leq |\psi(t) - \psi(s')| \frac{|s-s'|}{|t-s||t-s'|} \\
&\leq K_6|s-s'||t-s'|^{\alpha-1}|t-s|^{-1}.
\end{aligned}$$

Since $t \in U_2$ we have $|t-s| \geq \delta = 2|s-s'|$. Hence $|t-s'| \geq ||t-s| - |s-s'|| = |t-s| - |s-s'| \geq |t-s|/2$. We obtain, for all $t \in U_2$, $|t-s'|^{\alpha-1} \leq 2^{1-\alpha}|t-s|^{\alpha-1}$ and therefore, due to $\delta \leq 4$,

$$\begin{aligned}
\int_{U_2} |q_2(t)|\,dt &\leq K_7|s-s'| \int_{U_2} |t-s|^{\alpha-2}\,dt \\
&\leq 2K_7|s-s'| \int_{\delta}^{5} |t|^{\alpha-2}\,dt \\
&= K_8|s-s'|(-5^{\alpha-1} + \delta^{\alpha-1}) \\
&\leq K_9|s-s'| + K_{10}|s-s'|^{1+\alpha-1} \\
&\leq K_{11}|s-s'|^{\alpha'}.
\end{aligned}$$

By addition we obtain

$$|T[\psi](s) - T[\psi](s')| \leq (K_3 + K_5 + K_{11})|s - s'|^{\alpha'}.$$

Since $K_1, \ldots, K_{11}$ are independent of s, s' we have $T[\psi] \in \mathsf{C}^{0,\alpha'}$ with

$$\sup_{s,s' \in \mathsf{I}, s \neq s'} \frac{|T[\psi](s) - T[\psi](s')|}{|s - s'|^{\alpha'}} \leq K_3 + K_5 + K_{11}. \qquad \square$$

Lemma 4. We define for $k \in \mathbf{N}_0$ by recursion the coefficients

(6.18)
$$a_k^k := 1$$

(6.19)
$$a_j^k := \frac{1}{L_k - L_j} \sum_{l=j+1}^{k} \frac{a_l^k((-1)^{l-j+1} - 1)}{l - j}, \quad j = k-1, \ldots, 1$$

(6.20)
$$a_0^k := \frac{1}{L_k} \sum_{l=1}^{k} \frac{a_l^k((-1)^{l+1} - 1)}{l}, \quad k \neq 0,$$

and introduce for $k \in \mathbf{N}_0$ the polynomials

$$q_k : \mathsf{I} \to \mathbf{R}, \quad q_k(s) = s^k + a_{k-1}^k s^{k-1} + \cdots + a_1^k s + a_0^k.$$

Furthermore we set, for $k \in \mathbf{N}_0$,

$$\phi_k : \mathsf{I} \to \mathbf{R}, \quad \phi_k(s) = s^k$$
$$p_k : \mathsf{I} \to \mathbf{R}, \quad p_k(s) := \sum_{l=0}^{k} \frac{s^{k-l}}{l+1}(1 + (-1)^{l+1}),$$

and we put

$$p_{-1} : \mathsf{I} \to \mathbf{R}, \quad p_{-1}(s) = 0.$$

We calculate for all $k \in \mathbf{N}$, and for all $s \in \mathsf{I}$,

$$\begin{aligned}
T[\phi_k](s) &= -\int_{-1}^{s} \frac{t^k - s^k}{t-s}\,dt + \int_{s}^{1} \frac{t^k - s^k}{t-s}\,dt \\
&= -\sum_{l=0}^{k-1} s^{k-1-l} \int_{-1}^{s} t^l\,dt + \sum_{l=0}^{k-1} s^{k-1-l} \int_{s}^{1} t^l\,dt \qquad (6.21)\\
&= -2s^k \sum_{l=1}^{k} \frac{1}{l} + \sum_{l=0}^{k-1} \frac{s^{k-1-l}}{l+1}(1+(-1)^{l+1}),
\end{aligned}$$

while for all $s \in \mathsf{I}$, $T[\phi_0](s) = p_{-1}(s)$. Hence for all $k \in \mathbf{N}_0$,

$$T[\phi_k] = -L_k\phi_k + p_{k-1}. \tag{6.22}$$

We observe for all $k \in \mathbf{N}$,

$$q_k = \phi_k + a_{k-1}^k \phi_{k-1} + \cdots + a_1^k \phi_1 + a_0^k \phi_0.$$

Hence, due to (6.22),

$$\begin{aligned}
T[q_k] &= \sum_{l=0}^{k} a_l^k T[\phi_l] = \sum_{l=0}^{k} a_l^k(-L_l\phi_l + p_{l-1}) \\
&= -\sum_{l=0}^{k} a_l^k L_l \phi_l + \sum_{l=0}^{k} a_l^k p_{l-1} = -\sum_{l=1}^{k} a_l^k L_l\phi_l + \sum_{l=1}^{k}\sum_{\nu=0}^{l-1} \frac{a_l^k \phi_\nu}{l-\nu} \\
&= -\sum_{l=1}^{k} a_l^k L_l\phi_l + \sum_{l=1}^{k}\sum_{\nu=1}^{l} \frac{a_l^k\phi_{\nu-1}}{l+1-\nu} = -\sum_{l=1}^{k} a_l^k L_l\phi_l + \sum_{\nu=1}^{k}\sum_{l=\nu}^{k} \frac{a_l^k\phi_{\nu-1}}{l+1-\nu} \\
&= -L_k\phi_k + \sum_{l=1}^{k-1}\left[-L_l a_l^k + \sum_{\nu=l+1}^{k} \frac{a_\nu^k}{\nu - l}\right]\phi_l + \sum_{\nu=1}^{k} \frac{a_\nu^k}{\nu}\phi_0 \\
&= -L_k\phi_k + \sum_{l=1}^{k-1}[-L_l a_l^k + (L_l - L_k)a_l^k]\phi_l - L_k a_0^k\phi_0 \\
&= -L_k(\phi_j + a_{j-1}^j\phi_{j-1} + \cdots + a_0^k\phi_0) \\
&= -L_k q_k,
\end{aligned}$$

i.e., $-L_k$ is for each $k \in \mathbf{N}_0$ an eigenvalue of T. We deduce from the symmetry of T (Theorem 1): The set $\{q_k : k \in \mathbf{N}_0\}$ is $\mathcal{L}^2$-orthogonal.

Since the degree of each polynomial q_k, $k \in \mathbf{N}_0$, is k we have $q_k = \alpha_k P_k$, $\alpha_k \in \mathbf{R}$. The respective properties of $\tilde{T}$ follow immediately. $\square$

Theorem 5. We wish to apply the basic criterion of essential self-adjointness as given in the corollary on page 257 of [**17**]. This statement requires a complex Hilbert space theory. We introduce

$$\mathcal{P}_{\mathbf{C}} := \{p_1 + i.p_2 : p_1, p_2 \in \mathcal{P}\},$$

where i is the imaginary unit. Obviously, $\mathcal{P}_{\mathbf{C}}$ is the complex vector space of all complex-valued polynomials restricted to I. In similarity, we introduce the complex vector space $\mathcal{L}^2_{\mathbf{C}}$ of all complex-valued square-integrable functions whose domain is I. We equip $\mathcal{L}^2_{\mathbf{C}}$ with its canonical inner product

$$(G_1, G_2) = \int_{\mathsf{I}} \overline{G_1}(s) G_2(s)\, ds, \quad G_1, G_2 \in \mathcal{L}^2_{\mathbf{C}},$$

where $\overline{G_1}$ is the complex-conjugate function of G_1. The corresponding norm is $\|\cdot\|_2$.

Next we introduce the complex extension of $\tilde{T}$, i.e., we set

$$S : \mathcal{P}_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}}, \quad S(p_1 + i.p_2) = T(p_1) + i.T(p_2).$$

It remains to prove: S is essentially self-adjoint.

We easily obtain: The domain of S is $\|\cdot\|_2$-dense in $\mathcal{L}^2_{\mathbf{C}}$ and $S : \mathcal{L}^2_{\mathbf{C}} \supset \mathcal{P}_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}}$ is symmetric with respect to the inner product defined above. Following the corollary on page 257 of [**17**] we have to prove: $\mathsf{ran}\,(S \pm i)$ in $\|\cdot\|_2$-dense in $\mathcal{L}^2_{\mathbf{C}}$. Hence it suffices to prove that $\mathsf{ran}\,(S \pm i) = \mathcal{P}_{\mathbf{C}}$. Indeed, let $q_1 + i.q_2 \in \mathcal{P}_{\mathbf{C}}$. Then there are $N \in \mathbf{N}$ and $\alpha_0, \dots, \alpha_N, \beta_1, \dots, \beta_N \in \mathbf{R}$ such that

$$q_1 = \alpha_0.P_0 + \cdots + \alpha_N.P_N, \qquad q_2 = \beta_0.P_0 + \cdots + \beta_N.P_N,$$

where P_k, $k \in \mathbf{N}$, is the kth Legendre polynomial, restricted to I.

Following Lemma 4 we are looking for $\gamma_0, \dots, \gamma_N, \delta_0, \dots \delta_N \in \mathbf{R}$ such that

$$\begin{aligned}(S \pm i)(p_1 + i.p_2) &= (S \pm i)((\gamma_0.P_0 + \cdots + \gamma_N.P_N) \\ &\qquad \pm (\delta_0.P_0 + \cdots + \delta_N.P_N)) \\ &= q_1 + i.q_2 = (\alpha_0.P_0 + \cdots + \alpha_N.P_N) \\ &\qquad + i.(\beta_0.P_0 + \cdots + \beta_N.P_N).\end{aligned} \tag{6.23}$$

Following Lemma 4, equation(6.23) has a solution if and only if γ_j, δ_j satisfies for all $j \in \{0, \dots, N\}$ the linear equation

$$\begin{pmatrix} -L_j & \mp 1 \\ \pm 1 & -L_j \end{pmatrix} \begin{pmatrix} \gamma_j \\ \delta_j \end{pmatrix} = \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}.$$

The determinant of the matrix equals $1 + L_j^2 \neq 0,\ j \in \{0, \dots, N\}$. Hence, it is always possible to determine $\gamma_0, \dots, \gamma_N, \delta_0, \dots, \delta_N \in \mathbf{R}$ such that (6.23) holds. $\square$

Theorem 6.

Step 1. Let $\tilde{T}^*$ be the adjoint of $\tilde{T}$. Since $\mathsf{dom}\,(\tilde{T})$ is $\|\cdot\|_2$-dense in $\mathcal{L}^2$ and since $\tilde{T}$ is symmetric, $\tilde{T}^*$ is a closed extension of $\tilde{T}$, see [**17**]. We recall

$$\mathsf{dom}\,(\tilde{T}^*) = \{G \in \mathcal{L}^2 : (\exists F \in \mathcal{L}^2 : (\forall\, H \in \mathcal{P} : \langle \tilde{T}[H], G\rangle = \langle H, F\rangle))\},$$

and for $G \in \mathsf{dom}\,(\tilde{T}^*)$ there is exactly one $\tilde{T}^*[G] \in \mathcal{L}^2$ such that

$$\forall\, H \in \mathcal{P} : \langle \tilde{T}[H], G\rangle = \langle H, \tilde{T}^*[G]\rangle.$$

We wish to prove:

$$\mathsf{dom}\,(\tilde{T}^*) = D := \left\{ G = \sum_{k \in \mathbf{N}_0} \alpha_k . P_k^{\mathrm{norm}} : (L_k \cdot \alpha_k)_{k \in \mathbf{N}_0} \in \ell^2 \right\},$$

where we note $D \subseteq \mathcal{L}^2$. Let $G = \sum_{k \in \mathbf{N}_0} \alpha_k . P_k^{\mathrm{norm}} \in D$. We introduce

$$F = \sum_{k \in \mathbf{N}_0} (-L_k . \alpha_k) . P_k^{\mathrm{norm}} \in \mathcal{L}^2.$$

Employing the fact that $\{P_k^{\mathrm{norm}} : k \in \mathbf{N}_0\}$ is an orthonormal base of $\mathcal{L}^2$, we have for all $H = \sum_{k=0}^{M} \gamma_k . P_k^{\mathrm{norm}}$, $M \in \mathbf{N}$, $\gamma_0, \dots, \gamma_M \in \mathbf{R}$, in $\mathcal{P}$ the identity

$$\begin{aligned} \langle \tilde{T}[H], G\rangle &= \sum_{k=0}^{\min\{N,M\}} (-L_k . \gamma_k) . \alpha_k \\ &= \sum_{k=0}^{\min\{N,M\}} \gamma_k . (-L_k . \alpha_k) = \langle H, F_N\rangle \\ &= \langle H, F\rangle. \end{aligned}$$

Hence $G \in \mathsf{dom}\,(\tilde{T}^*)$ with $F = \tilde{T}^*[G]$. This settles $D \subseteq \mathsf{dom}\,(\tilde{T}^*)$. Now let $G \in \mathsf{dom}\,(\tilde{T}^*)$. Then $G \in \mathcal{L}^2$, hence $G = \sum_{k \in \mathbf{N}_0} \langle P_k^{\mathrm{norm}}, G \rangle . P_k^{\mathrm{norm}}$ with $(\langle P_k^{\mathrm{norm}}, G \rangle)_{k \in \mathbf{N}_0} \in \ell^2$. Due to the fact that $\{P_k^{\mathrm{norm}} : k \in \mathbf{N}_0\}$ is an orthonormal base of $\mathcal{L}^2$, we calculate for all $N \in \mathbf{N}_0$

$$\begin{aligned}
\langle P_N^{\mathrm{norm}}, \tilde{T}^*[G] \rangle &= \langle \tilde{T}[P_N^{\mathrm{norm}}], G \rangle \\
&= \Big\langle -L_N . P_N^{\mathrm{norm}}, \sum_{k \in \mathbf{N}_0} \langle P_k^{\mathrm{norm}}, G \rangle . P_k^{\mathrm{norm}} \Big\rangle \\
&= \sum_{k \in \mathbf{N}_0} \langle P_k^{\mathrm{norm}}, G \rangle . \langle -L_N . P_N^{\mathrm{norm}}, P_k^{\mathrm{norm}} \rangle \\
&= -L_N . \langle P_N^{\mathrm{norm}}, G \rangle .
\end{aligned}$$

Since $\tilde{T}^*[G] = \sum_{N \in \mathbf{N}_0} \langle P_N^{\mathrm{norm}}, \tilde{T}^*[G] \rangle . P_N^{\mathrm{norm}} \in \mathcal{L}^2$, we obtain $(\langle P_N^{\mathrm{norm}}, \tilde{T}^*[G] \rangle)_{N \in \mathbf{N}} \in \ell^2$, hence $(L_N . \langle P_N^{\mathrm{norm}}, G \rangle)_{N \in \mathbf{N}_0}$, which settles $G \in D$ and therefore $\mathsf{dom}\,(\tilde{T}^*) \subseteq D$.

We conclude: $\mathsf{dom}\,(\tilde{T}^*) = D$.

Step 2. The next step is to prove that $\tilde{T}^*$ is the closure $\tilde{T}^{sa}$ of $\tilde{T}$. Since $\tilde{T}^*$ is a closed extension of $\tilde{T}$, see [**17**], we have $\tilde{T}^{sa} \subseteq \tilde{T}^*$. It remains to prove $\tilde{T}^* \subseteq \tilde{T}^{sa}$, i.e., $\mathsf{dom}\,(\tilde{T}^*) \subseteq \mathsf{dom}\,(\tilde{T}^{sa})$. Let $G \in \mathsf{dom}\,(\tilde{T}^*)$. Since $\hat{T}^{sa}$ is the closure of $\tilde{T}$, we have to prove that there is a sequence $(G_N)_{N \in \mathbf{N}_0}$ in $\mathcal{P}$ such that

$$\lim_{N \to \infty} \|G - G_N\|_2 = \lim_{N \to \infty} \|\tilde{T}^*[G] - \tilde{T}^*[G_N]\|_2 .$$

It is left to the reader to verify that the sequence

$$(G_N)_{N \in \mathbf{N}_0} = \left(\sum_{k=0}^{N} \langle P_k^{\mathrm{norm}}, G \rangle . P_k^{\mathrm{norm}} \right)_{N \in \mathbf{N}_0}$$

has the desired properties.

Step 3. As already outlined in Step 1, we have for all $G \in \mathsf{dom}\,(\tilde{T}^{sa})$,

$$\tilde{T}^{sa}(G) = \sum_{k \in \mathbf{N}_0} -L_k . \langle P_k^{\mathrm{norm}}, G \rangle . P_k^{\mathrm{norm}} .$$

It is left to the reader to deduce the remaining statements of the theorem from that representation. □

Proposition 7. For all $l \in \mathbf{N}_0$ and all $s \in \mathsf{I}$ we have, due to (6.21),

$$\begin{aligned} T[\sigma^l](s) &= \int_{-1}^{1} \frac{(t/2)^l - (s/2)^l}{|t-s|}\, dt \\ &= \frac{1}{2^l} \int_{-1}^{1} \frac{t^l - s^l}{|t-s|}\, dt \\ &= \frac{1}{2^l}\Bigg(- L_l s^l + \sum_{j=0}^{l-1} s^{l-1-j}\, \frac{1+(-1)^{j+1}}{j+1}\Bigg) \\ &= -L_l \sigma^l + \sum_{\nu=0}^{l-1} \frac{1+(-1)^{l-\nu}}{2^{l-\nu}(l-\nu)} \sigma^\nu. \end{aligned}$$

The verification of (5.11) is left to the reader. □

Theorem 8. We have, due to (5.11), (2.7) for all $l \in \mathbf{N}_0$ and all $n \in \mathbf{N}_0$ the estimate

$$\begin{aligned} \sum_{\nu=l}^{l+n} |B_l^\nu \psi_\nu| \le \|\psi\| \sum_{\nu=l}^{l+n} B_l^\nu &\le \|\psi_1\|(L_l + \log(4/3)) \\ &\le \|\psi\|(L_l+1) \le \|\psi\|(3 + 2\log(1+l)). \end{aligned}$$

We note that this estimate is independent of n. Hence for all $l \in \mathbf{N}_0$,

$$(*) \qquad \sum_{\nu=l}^{\infty} |B_l^\nu \psi_\nu| \le (3 + 2\ln(1+l))\|\psi\|.$$

For $n \in \mathbf{N}_0$, let $\psi^{(n)} := \mathsf{P}_0^n[\psi]$. We immediately verify

$$\hat{T}[\psi^{(n)}] = \sum_{l=0}^{n} \left[\sum_{\nu=l}^{n} B_l^\nu \psi_\nu \right] \sigma^l = (\mathsf{P}_0^n \circ \hat{T})[\psi^{(n)}].$$

This settles $\mathsf{P}_0^n \circ \hat{T} \circ \mathsf{P}_0^n = \hat{T} \circ \mathsf{P}_0^n$.

Now we introduce for $s \in \mathsf{I}$ and $n \in \mathbf{N}$,

$$g_s^{(n)} : \mathsf{I} \to \mathbf{R}, \quad g_s^{(n)}(t) = \sum_{l=0}^{n-1} \left(\sum_{\nu=l+1}^{n} \psi_\nu (\sigma(s))^{\nu-l-1} \right) (\sigma(t))^l,$$

i.e.,

$$g_s^{(n)}(t) = \begin{cases} \dfrac{\psi^{(n)}(s) - \psi^{(n)}(t)}{s-t}, & t \neq s \\ \dfrac{d\psi^{(n)}}{ds}(t), & t = s, \end{cases}$$

such that for all $s \in \mathsf{I}$ and for all $n \in \mathbf{N}$,

$$\hat{T}[\psi^{(n)}](s) = -\int_{-1}^{s} g_s^{(n)}(t)\, dt + \int_{s}^{1} g_s^{(n)}(t)\, dt.$$

We need an auxiliary result.

Proposition 11. *For $n \in \mathbf{N} \cup \{\infty\}$, let $(\gamma_l^{(n)})_{l \in \mathbf{N}_0}$ be a real sequence. Assume* a) *There is a $K \in \mathbf{R}^+$ such that for all $l \in \mathbf{N}_0$ and for all $n \in \mathbf{N}$*:

$$|\gamma_l^{(n)} - \gamma_l^{(\infty)}| \leq K,$$

b)

$$\limsup_{l \to \infty} \sqrt[l]{|\gamma_l^{(\infty)}|} < 2.$$

c) *For all $l \in \mathbf{N}_0$: $\lim_{n\to\infty} \gamma_l^{(n)} = \gamma_l^{(\infty)}$.*

Then:

1. *The series*

$$\left(\sum_{l=0}^{m} \gamma_l^{(\infty)} \sigma^l \right)_{m \in \mathbf{N}_0}$$

converges uniformly on I.

2.

$$\lim_{n\to\infty} \| \sum_{l=0}^{n} \gamma_l^{(n)} \sigma^l - \sum_{l=0}^{\infty} \gamma_l^{(\infty)} \sigma^l \|_\infty = 0.$$

Proof. 1. follows from b).

2. We observe that a) and b) imply for all $n \in \mathbf{N}_0$:

$$\limsup_{l\to\infty} \sqrt[l]{|\gamma_l^{(n)}|} \le \max\left\{1, \limsup_{l\to\infty} \sqrt[l]{|\gamma_l^{(\infty)}|}\right\} < 2,$$

such that $(\sum_{l=0}^{n} \gamma_l^{(n)} \sigma^l)_{n\in\mathbf{N}_0}$ converges uniformly on I. Let $\varepsilon > 0$. Then there is $N_1(\varepsilon) \in \mathbf{N}$ such that, for all $n \ge N_1(\varepsilon)$:

$$\left\| \sum_{l=n+1}^{\infty} \gamma_l^{(\infty)} \sigma^l \right\|_\infty \le \frac{\varepsilon}{3}.$$

Furthermore, we can take $N_2(\varepsilon) \in \mathbf{N}$, $N_1(\varepsilon) \le N_2(\varepsilon)$, such that for all $n \ge N_2(\varepsilon)$

$$\sup_{s\in\mathsf{I}} \sum_{l=n+1}^{\infty} |\sigma(s)|^l \le \frac{\varepsilon}{3K}.$$

Finally we can take $N_3(\varepsilon) \in \mathbf{N}$, $N_2(\varepsilon) \le N_3(\varepsilon)$, using assumption c) such that for all $n \ge N_3(\varepsilon)$

$$\left\| \sum_{l=0}^{N_2(\varepsilon)} (\gamma_l^{(n)} - \gamma_l^{(\infty)}) \sigma^l \right\|_\infty \le \frac{\varepsilon}{3}.$$

We therefore obtain for all $s \in \mathsf{I}$ and for all $n \in \mathbf{N}$, $n \ge N_3(\varepsilon)$,

$$\begin{aligned}
&\left| \sum_{l=0}^{n} \gamma_l^{(n)} \sigma^l - \sum_{l=0}^{\infty} \gamma_l^{(\infty)} \sigma^l \right| \\
&\quad \le \left| \sum_{l=0}^{N_2(\varepsilon)} (\gamma_l^{(n)} - \gamma_l^{(\infty)}) \sigma^l \right| + \left| \sum_{l=N_2(\varepsilon)+1}^{n} (\gamma_l^{(n)} - \gamma_l^{(\infty)}) \sigma^l \right| + \left| \sum_{l=n+1}^{\infty} \gamma_l^{(\infty)} \sigma^l \right| \\
&\quad \le \frac{\varepsilon}{3} + \sum_{l=N_2(\varepsilon)+1}^{n} |\gamma_l^{(n)} - \gamma_l^{(\infty)}| |\sigma|^l + \frac{\varepsilon}{3} \le \varepsilon. \qquad \square
\end{aligned}$$

We apply Proposition 11, for fixed $s \in \mathsf{I}$, with

$$\gamma_l^{(n)} := \sum_{\nu=l+1}^{n} \psi_\nu(\sigma(s))^{\nu-l-1} \quad \text{for } 0 \le l \le n-1$$

and

$$\gamma_l^{(n)} := 0 \quad \text{for } l \geq n$$

and

$$\gamma_l^{(\infty)} := \sum_{\nu=l+1}^{\infty} \psi_\nu (\sigma(s))^{\nu-l-1}$$

to obtain

$$\lim_{n\to\infty} \sup_{t\in \mathsf{I}} |g_s^{(n)}(t) - g_s^{(\infty)}(t)| = 0,$$

where

$$g_s^{(\infty)} : \mathsf{I} \to \mathbf{R}, \quad g_s^{(\infty)}(t) = \sum_{l=0}^{\infty} \biggl(\sum_{\nu=l+1}^{\infty} \psi_\nu (\sigma(s))^{\nu-l-1} \biggr) (\sigma(t))^l,$$

i.e.,

$$g_s^{(\infty)}(t) = \begin{cases} \dfrac{\psi(s)-\psi(t)}{s-t}, & t \neq s \\ \dfrac{d\psi}{ds}(t), & t = s. \end{cases}$$

Hence, for all $s \in \mathsf{I}$,

$$\begin{aligned} \hat{T}[\psi](s) &= -\int_{-1}^{s} \frac{\psi(s)-\psi(t)}{s-t}\, dt + \int_{s}^{1} \frac{\psi(s)-\psi(t)}{s-t}\, dt \\ &= -\lim_{n\to\infty} \int_0^s g_s^{(n)}(t)\, dt + \lim_{n\to\infty} \int_s^1 g_s^{(n)}(t)\, dt \\ &= \lim_{n\to\infty} \hat{T}[\psi^{(n)}](s). \end{aligned}$$

We furthermore have, for each $n \in \mathbf{N}$,

$$-\int_0^s g_s^{(n)}(t)\, dt + \int_s^1 g_s^{(n)}(t)\, dt = \sum_{l=0}^{n} \biggl[\sum_{\nu=l}^{n} B_l^\nu \psi_\nu \biggr] (\sigma(s))^l,$$

such that we can apply Proposition 11 again to obtain

$$\lim_{n\to\infty}\left\| \sum_{l=0}^{n}\left[\sum_{\nu=l}^{n} B_l^\nu \psi_\nu\right]\sigma^l - \sum_{l=0}^{\infty}\left[\sum_{\nu=l}^{\infty} B_l^\nu \psi_\nu\right]\sigma^l \right\|_\infty = 0.$$

We set for $l \in \mathbf{N}_0$, $\gamma_l := \sum_{\nu=l}^\infty B_l^\nu \psi_\nu$. Then we have due to $(*)$

$$\limsup_{l\to\infty} \sqrt[l]{|\gamma_l|} \le 1,$$

such that the (formal) power series $(\sum_{l=0}^n \gamma_l \sigma^l)_{n\in\mathbf{N}_0}$ converges for each $\theta \in]0,1[$ uniformly on $[-2+\theta, 2-\theta]$. This implies convergence of

$$\left(\sum_{l=0}^{n}\left[\sum_{\nu=l}^{\infty} B_l^\nu \psi_\nu\right]\sigma^l\right)_{n\in\mathbf{N}_0}$$

in C with respect to $\|\cdot\|_\infty$ with limiting function $\sum_{l=0}^\infty[\sum_{\nu=l}^\infty B_l^\nu\psi_\nu]\sigma^l = \hat{T}[\psi]$. □

Theorem 9. We shall discuss the operator equation $(\lambda - \hat{T})[\psi] = F$ with unknown $\psi = \sum_{l=0}^\infty \psi_l \sigma^l \in \mathsf{dom}\,(\hat{T})$ and given $F = \sum_{l=0}^\infty F_l \sigma^l \in \mathcal{L}_w^\infty$, $\lambda \in \mathbf{R}$.

We observe that $\psi \in \mathsf{dom}\,(\hat{T})$ satisfies

$$\hat{T}_\lambda[\psi] := (\lambda - \hat{T})[\psi] = F \tag{6.24}$$

if and only if $\psi \in \mathsf{dom}\,(\hat{T})$ satisfies

$$\begin{aligned}(\mathsf{P}_0^n \circ \hat{T}_\lambda \circ \mathsf{P}_0^n)[\psi] + (\mathsf{P}_0^n \circ \hat{T}_\lambda \circ \mathsf{P}_{n+1}^\infty)[\psi] &= \mathsf{P}_0^n[F],\\ (\mathsf{P}_{n+1}^\infty \circ \hat{T}_\lambda \circ \mathsf{P}_{n+1}^\infty)[\psi] &= \mathsf{P}_{n+1}^\infty[F],\end{aligned}$$

where we make use of the fact that $\mathsf{P}_{n+1}^\infty \circ \hat{T}_\lambda \circ \mathsf{P}_0^n$ is the zero operator on $\mathsf{dom}\,(\hat{T})$.

Hence (6.24) has a solution $\psi \in \mathsf{dom}\,(\hat{T})$ if and only if $\psi = \eta + \xi$, $\eta \in \mathsf{P}_0^n[\mathsf{dom}\,(\hat{T})]$, $\xi \in \mathsf{P}_{n+1}^\infty[\mathsf{dom}\,(\hat{T})]$ and

$$\begin{aligned}(\mathsf{P}_0^n \circ \hat{T}_\lambda)[\eta] + (\mathsf{P}_0^n \circ \hat{T}_\lambda)[\xi] &= \mathsf{P}_0^n[F],\\ (\mathsf{P}_{n+1}^\infty \circ \hat{T}_\lambda)[\xi] &= \mathsf{P}_{n+1}^\infty[F].\end{aligned} \tag{6.25}$$

This observation suggests to introduce for $n \in \mathbf{N}$ the operators

$$\mathsf{A}_n := \mathsf{P}_0^n \circ \hat{T}_\lambda \downharpoonleft \mathsf{P}_0^n[\mathsf{dom}\,(\hat{T})],$$
$$\mathsf{B}_n := \mathsf{P}_0^n \circ \hat{T}_\lambda \downharpoonleft \mathsf{P}_{n+1}^\infty[\mathsf{dom}\,(\hat{T})],$$
$$\mathsf{D}_n := \mathsf{P}_{n+1}^\infty \circ \hat{T}_\lambda \downharpoonleft \mathsf{P}_{n+1}^\infty[\mathsf{dom}\,(\hat{T})],$$

such that $(\eta, \xi) \in \mathsf{P}_0^n[\mathsf{dom}\,(\hat{T})] \times \mathsf{P}_{n+1}^\infty[\mathsf{dom}\,(\hat{T})]$ is a solution of (6.25) if and only if

$$\mathsf{A}_n[\eta] + \mathsf{B}_n[\xi] = \mathsf{P}_0^n[F], \tag{6.26}$$
$$\mathsf{D}_n[\xi] = \mathsf{P}_{n+1}^\infty[F]. \tag{6.27}$$

It is easy to verify that, for $n \in \mathbf{N}_0$,

$$\sum_{l=0}^{n} \eta_l \sigma^l \in \mathsf{P}_0^n[\mathsf{dom}\,(\hat{T})], \qquad \sum_{l=n+1}^{\infty} \xi_l \sigma^l \in \mathsf{P}_{n+1}^\infty[\mathsf{dom}\,(\hat{T})],$$

the matrix representation of $\mathsf{A}_n[\eta]$ is

(6.28)

$$\begin{pmatrix} \lambda+L_0 & -B_0^1 & -B_0^2 & -B_0^3 & \cdots & -B_0^n & 0 & \cdots \\ 0 & \lambda+L_1 & -B_1^2 & -B_1^3 & \cdots & -B_1^n & 0 & \cdots \\ 0 & 0 & \lambda+L_2 & -B_2^3 & \cdots & -B_2^n & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \lambda+L_{n-1} & -B_{n-1}^n & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda+L_n & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \\ \eta_n \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

the matrix representation of $\mathsf{B}_n[\xi]$ is

$$\begin{pmatrix} 0 & \cdots & 0 & -B_0^{n+1} & -B_0^{n+2} & -B_0^{n+3} & \cdots \\ 0 & \cdots & 0 & -B_1^{n+1} & -B_1^{n+2} & -B_1^{n+3} & \cdots \\ 0 & \cdots & 0 & -B_2^{n+1} & -B_2^{n+2} & -B_2^{n+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & -B_{n-1}^{n+1} & -B_{n-1}^{n+2} & -B_{n-1}^{n+3} & \cdots \\ 0 & \cdots & 0 & -B_n^{n+1} & -B_n^{n+2} & -B_n^{n+3} & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \xi_{n+4} \\ \vdots \end{pmatrix}, \tag{6.29}$$

and the matrix representation of $\mathsf{D}_n[\xi]$ is
(6.30)

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \lambda+L_{n+1} & -B_{n+1}^{n+2} & -B_{n+1}^{n+3} & -B_{n+1}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & \lambda+L_{n+2} & -B_{n+2}^{n+3} & -B_{n+2}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \lambda+L_{n+3} & -B_{n+3}^{n+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \vdots \end{pmatrix}.$$

Due to (6.30) we have $\mathsf{D}_n = \Lambda_n + \mathsf{E}_n$, where the operators

$$\Lambda_n : \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}],$$
$$\mathsf{E}_n : \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})] \to \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}]$$

are defined by their respective matrix representations, namely, for

$$\xi = \sum_{l=n+1}^{\infty} \xi_l \sigma^l \in \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})],$$

the matrix representation of $\Lambda_n[\xi]$ is
(6.31)

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \lambda+L_{n+1} & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \lambda+L_{n+2} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \lambda+L_{n+3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \vdots \end{pmatrix},$$

and the matrix representation of $\mathsf{E}_n[\xi]$ is

(6.32)

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & -B_{n+1}^{n+2} & -B_{n+1}^{n+3} & -B_{n+1}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & -B_{n+2}^{n+3} & -B_{n+2}^{n+4} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & -B_{n+3}^{n+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_{n+1} \\ \xi_{n+2} \\ \xi_{n+3} \\ \vdots \end{pmatrix}.$$

It is left to the reader to verify

Proposition 12. *Let* $\lambda \in \mathbf{R}$. *Let*

$$N_1(\lambda) := \min\{k \in \mathbf{N} : L_{k+1} > \log(4/3) - \lambda\}.$$

Then

(i) $0 < \lambda + L_{N(\lambda)+1} < \lambda + L_{N(\lambda)+2} < \cdots,$

(ii) *and for all* $n \in \mathbf{N}_0$ *with* $N_1(\lambda) \leq n$, *the linear operator* Λ_n *has a bounded inverse with*

$$\|\Lambda_n^{-1} : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})]\| \leq \sup\left\{\frac{3 + 2\log(1+l)}{\lambda + L_l} : l \in \mathbf{N}, n+1 \leq l\right\} < \infty,$$

(iii) *and for all* $n \in \mathbf{N}_0$ *with* $N_1(\lambda) \leq n$,

$$\|\Lambda_n^{-1} \circ \mathsf{E}_n : \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})] \to \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})]\| \leq \frac{\log(4/3)}{\lambda + L_{n+1}} < 1.$$

Furthermore,

$$\limsup_{n\to\infty} \|\Lambda_n^{-1} : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})]\| \leq 2.$$

Due to Proposition 12, the operator Λ_n has a bounded inverse Λ_n^{-1} and

$$\|\Lambda_n^{-1} \circ \mathsf{E}_n : \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})] \to \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})]\| < 1. \tag{6.33}$$

Hence we can rewrite equation (6.27) for $n \in \mathbf{N}$ with $N_1(\lambda) \le n$ as

$$[1 + \Lambda_n^{-1} \circ E_n][\xi] = \Lambda_n^{-1}[\mathsf{P}_{n+1}^{\infty}[F]]. \tag{6.34}$$

Now it readily follows from (6.33) that (6.34) has for all $n \in \mathbf{N}$ with $N_1(\lambda) \le n$ a unique solution $\xi \in \mathsf{dom}\,(\hat{T})$, i.e., equation (6.27) has for all $n \in \mathbf{N}$ with $N_1(\lambda) \le n$ a unique solution $\xi \in \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})]$. Furthermore, we deduce from (6.34) for all $n \in \mathbf{N}_0$ with $N_1(\lambda) \le n$ the estimate

$$\|\xi\|_{1/2} \le \frac{\|\Lambda_n^{-1} : \mathsf{P}_{n+1}^{\infty}[\mathcal{L}_w^{\infty}] \to \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})]\|}{1 - (\log(4/3)/(\lambda + L_{n+1}))}\|F\|_w. \tag{6.35}$$

Inserting ξ in equation (6.26), we obtain

$$\mathsf{A}_n[\eta] = \mathsf{P}_0^n[F] - \mathsf{B}_n[\xi], \quad n \in \mathbf{N},\ n \ge N_1(\lambda), \tag{6.36}$$

where we note that the righthand side of this equation belongs to the finite-dimensional space $\mathsf{P}_0^n[\mathcal{L}_w^{\infty}]$. Due to the diagonal structure of A_n, two possibilities arise:

1) $\lambda \notin \{-L_k : k \in \mathbf{N}_0\}$. Then A_n is invertible for any $n \in \mathbf{N}_0$ and we obtain for all $n \in \mathbf{N}$ with $N_1(\lambda) \le n$ the estimate

$$\|\eta\|_{1/2} \le \|\mathsf{A}_n^{-1} : \mathsf{P}_0^n[\mathcal{L}_w^{\infty}] \to \mathsf{P}_0^n[\mathsf{dom}\,(\hat{T})]\|(\|F\|_w + \log(4/3)\|\xi\|),$$

where we made use of the fact that

$$\|\mathsf{B}_n : \mathsf{P}_{n+1}^{\infty}[\mathsf{dom}\,(\hat{T})] \to \mathsf{P}_0^n[\mathcal{L}_w^{\infty}]\| \le \log(4/3).$$

(5.14) follows from the estimates given so far.

2) $\lambda = -L_k$ for some $k \in \mathbf{N}_0$. In this case the defect of the operator A_n is, due to the diagonal structure of its matrix representation, one. Since $\tilde{T}^{sa}$ is an extension of $\hat{T}$ it readily follows from Theorem 6 that λ is an eigenvalue of $\hat{T}$ whose eigenspace contains P_k. $\square$

Theorem 10. We employ the notations of the proof of Theorem 9. We observe that $\phi^{[n]} \in \mathsf{P}_0^n[\mathsf{dom}\,(\hat{T})]$ for any $n \in \mathbf{N}$ and

$$\mathsf{A}_n[\phi^{[n]}] = \mathsf{P}_0^n[F].$$

Hence, due to (6.26) and (6.27),

$$\mathsf{A}_n[\eta - \phi^{[n]}] - \mathsf{B}_n[\xi] = 0, \qquad \mathsf{D}_n[\xi] = \mathsf{P}_{n+1}^{\infty}[F],$$

where $\eta = \mathsf{P}_0^n[\phi]$ and $\xi = \mathsf{P}_{n+1}^{\infty}[\phi]$. Hence

$$\phi - \phi^{[n]} = (\eta - \phi^{[n]}) + \xi$$

solves

$$(c - \hat{T})[\phi - \phi^{[n]}] = \mathsf{P}_{n+1}^{\infty}[F].$$

The verification of several statements of the theorem can now be left to the reader. □

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