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ON THE RANGE OF THE STRUVE $\mathbf{H}_\nu$-TRANSFORM

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ABSTRACT. The range of the $\mathbf{H}_\nu$-transform on some spaces of functions is described.

1. Introduction. The Struve $\mathbf{H}_\nu$-transform as an example of an asymmetric Watson transform is defined as [**8**], [**9**]

$$f(x) = (\mathbf{H}_\nu g)(x) = \int_0^\infty \sqrt{xy}\mathbf{H}_\nu(xy)g(y)\,dy, \qquad x \in (0,\infty) = R_+, \tag{1}$$

if the integral converges in some sense (absolutely, improper or mean convergence). Here $\mathbf{H}_\nu(x)$ is the Struve function [**1**]. The boundedness and range of the Struve $\mathbf{H}_\nu$-transform on the space $\mathcal{L}_{\mu,p}$ of functions f, measurable on R_+, and such that

$$\|f\|_{\mu,p} = \left\{ \int_0^\infty |x^\mu f(x)|^p \frac{dx}{x} \right\}^{1/p} < \infty, \quad 1 \le p < \infty, \tag{2}$$

have been considered in [**2**], [**4**], [**5**]. It has been proved there that, under some restrictions on parameters ν, μ, p, the range of the Struve $\mathbf{H}_\nu$-transform (1) coincides with the range of the Hankel transform

$$f(x) = (\mathcal{H}_{\nu+1}g)(x) = \int_0^\infty \sqrt{xy}J_{\nu+1}(xy)g(y)\,dy, \quad x \in R_+, \tag{3}$$

on the space $\mathcal{L}_{\mu,p}$. It is well known that the Hankel transform (3) is an automorphism on the space $L_2(R_+) = \mathcal{L}_{1/2,2}$, hence in the strip $-2 < \mathcal{R}e\,\nu < 0$ the Struve $\mathbf{H}_\nu$-transform is bounded on $L_2(R_+)$, and moreover, if $\mathcal{R}e\,\nu \ne -1$, its range is the whole space $L_2(R_+)$:

$$\|\mathbf{H}_\nu g\|_{L_2(R_+)} \le C\|g\|_{L_2(R_+)}, \quad -2 < \mathcal{R}e\,\nu < 0, \tag{4}$$

$$\|g\|_{L_2(R_+)} \le C\|\mathbf{H}_\nu g\|_{L_2(R_+)}, \quad |1 + \mathcal{R}e\,\nu| < 1, \tag{5}$$

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where $C \in [1,\infty)$ is an independent constant.

When $-1 < \mathcal{R}e\,\nu < 0$ the inverse of the Struve $\mathbf{H}_\nu$-transform on $L_2(R_+)$ is the so-called Y_ν-transform, defined by [**8**], [**9**]

$$g(x) = (Y_\nu f)(x) = \int_0^\infty \sqrt{xy}\, Y_\nu(xy) f(y)\, dy, \quad x \in R_+. \tag{6}$$

Here $Y_\nu(x)$ is the Bessel function of the second kind [**1**]. The Y_ν-transform is a bounded operator on $L_2(R_+)$ if $|\mathcal{R}e\,\nu| < 1$. In the strip $-2 < \mathcal{R}e\,\nu < -1$ the inverse of the $\mathbf{H}_\nu$-transform should be modified to

$$g(x) = \int_0^\infty \left[\sqrt{xy}\, Y_\nu(xy) - \frac{\cot(\pi\nu)(xy)^{\nu+1/2}}{2^\nu \Gamma(\nu+1)}\right] f(y)\, dy, \quad x \in R_+. \tag{7}$$

The $\mathbf{H}_\nu$- and Y_ν-transforms are useful in many axially-symmetric potential problems when solutions singular on the symmetric axis are required (see, for example, [**4**]).

In this work we characterize the range of the $\mathbf{H}_\nu$-transform on some spaces of functions. On the spaces considered in this paper, the Y_ν-transform and its modified form (7) are the inverse of the $\mathbf{H}_\nu$-transform, hence their respective ranges can be easily derived.

2. H_ν**-transform of rapidly decreasing functions.** We describe the range of the $\mathbf{H}_\nu$-transform on a subspace of the space of functions $g(y)$ such that $y^n g(y)$, $n = 1, 2, \dots$, are square integrable.

Theorem 1. *A function $f(x)$ is the Struve $\mathbf{H}_\nu$ transform ($-1/2 < \mathcal{R}e\,\nu < 0$) of a function $g(y)$ such that $y^n g(y)$, $n = 1, 2, \dots$, are square integrable and*

$$\int_0^\infty y^{\nu+2n+3/2} g(y)\, dy = 0, \quad n = 0, 1, \dots, \tag{8}$$

if and only if

(i) *$f(x)$ is infinitely differentiable on R_+;*

(ii) *$((d^2/dx^2) + (1/x^2)((1/4) - \nu^2))^n f(x)$, $n = 0, 1, \dots$, belong to $L_2(R_+)$;*

(iii) $x^{\mathcal{R}e\,\nu-1/2}((d^2/dx^2)+(1/x^2)((1/4)-\nu^2))^n f(x)$, $n=0,1,\dots$, *tend to* 0 *as* $x\to 0$;

(iv) $((d^2/dx^2)+(1/x^2)((1/4)-\nu^2))^n f(x)$, $n=0,1,\dots$, *tend to zero as* x *approaches infinity*;

(v) $(d/dx)((d^2/dx^2)+(1/x^2)((1/4)-\nu^2))^n f(x)$, $n=0,1,\dots$, *tend to* 0 *as* $x\to 0$;

(vi) $(d/dx)((d^2/dx^2)+(1/x^2)((1/4)-\nu^2))^n f(x)$, $n=0,1,\dots$, *tend to zero as* x *approaches infinity.*

Proof. Necessity. Let $y^n g(y)$ belong to $L_2(R_+)$ for all $n=0,1,2,\dots$; then $y^n g(y)$ belongs to $L_1(R_+)$ for all $n=0,1,2,\dots$. The Struve function $\mathbf{H}_\nu(y)$ has the order $O(y^{1+\mathcal{R}e\,\nu})$ at 0 and grows no faster than polynomials at infinity [**1**]. Therefore, integral (1) converges absolutely if $\mathcal{R}e\,\nu>-5/2$. Let $f(x)$ be the $\mathbf{H}_\nu$-transform $(-1/2<\mathcal{R}e\,\nu<0)$ of the function $g(y)$.

(i) We have [**1**]

$$\mathbf{H}_\nu(x)=\frac{2^{1-\nu}x^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^1(1-t^2)^{\nu-1/2}\sin(xt)\,dt,\quad \mathcal{R}e\,\nu>-1/2. \tag{9}$$

Therefore,

$$\frac{\partial^n}{\partial x^n}(\sqrt{xy}\,\mathbf{H}_\nu(xy)) =\frac{2^{1-\nu}y^{\nu+1/2}}{\sqrt{\pi}\Gamma(\nu+1/2)}\sum_{k=0}^n(-1)^k(k-\nu-3/2)_k\binom{n}{k}x^{\nu+1/2-k}y^{n-k} \cdot\int_0^1(1-t^2)^{\nu-1/2}t^{n-k}\sin(xyt+\pi(n-k)/2)\,dt, \tag{10}$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol [**1**]. Consequently, $(\partial^n/\partial x^n)[\sqrt{xy}\,\mathbf{H}_\nu(xy)]$, $\mathcal{R}e\,\nu>-1/2$, as a function of y has the asymptotics $O(y^{1/2+\mathcal{R}e\,\nu})$ in a neighborhood of zero and $O(y^{1/2+\mathcal{R}e\,\nu+n})$ at infinity. Hence,

$$\frac{\partial^n}{\partial x^n}[\sqrt{xy}\,\mathbf{H}_\nu(xy)]g(y),\quad \mathcal{R}e\,\nu>-1/2,$$

as a function of y belongs to $L_1(R_+)$ for all $n = 0, 1, 2, \dots$. Therefore, the function $f(x)$ is infinitely differentiable on R_+.

(ii) As the Struve function $\mathbf{H}_\nu(x)$ satisfies the nonhomogeneous Bessel differential equation [**1**]

$$x^2 u'' + x u' + (x^2 - \nu^2) u = \frac{2^{1-\nu} x^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu + 1/2)}, \tag{11}$$

we have

$$\begin{aligned}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right](\sqrt{xy}\,\mathbf{H}_\nu(xy)) = {} & \frac{2^{1-\nu} x^{\nu-1/2} y^{\nu+3/2}}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \\ & - y^2 \sqrt{xy}\,\mathbf{H}_\nu(xy).\end{aligned} \tag{12}$$

Consequently,

$$\begin{aligned}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right] f(x) = {} & \frac{2^{1-\nu} x^{\nu-1/2}}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_0^\infty y^{\nu+3/2} g(y)\, dy \\ & - \int_0^\infty \sqrt{xy}\,\mathbf{H}_\nu(xy) y^2 g(y)\, dy, \\ & |\mathcal{R}e\,\nu| < 1/2.\end{aligned} \tag{13}$$

Now using condition (8) we obtain

$$\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right] f(x) = - \int_0^\infty \sqrt{xy}\,\mathbf{H}_\nu(xy) y^2 g(y)\, dy. \tag{14}$$

Applying the same procedure and condition (8) n times we get

$$\begin{aligned}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x) = (-1)^n \int_0^\infty \sqrt{xy}\,\mathbf{H}_\nu(xy) y^{2n} g(y)\, dy, \\ -1/2 < \mathcal{R}e\,\nu < 0.\end{aligned} \tag{15}$$

From inequality (4) for the $\mathbf{H}_\nu$-transform we see that $[(d^2/dx^2) + (1/x^2)((1/4) - \nu^2)]^n f(x)$, $-1/2 < \mathcal{R}e\,\nu < 0$, $n = 0, 1, \dots$, belong to $L_2(R_+)$.

(iii) The Struve function $\mathbf{H}_\nu(y)$, $\mathcal{R}e\,\nu < 1/2$, has the asymptotics [**1**]

$$\mathbf{H}_\nu(y) = \begin{cases} \sqrt{\dfrac{2}{\pi y}}\left[\sin\left(y - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right) + \dfrac{4\nu^2-1}{8y}\cos\left(y - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right)\right] \\ + \dfrac{2^{1-\nu}y^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+(1/2))} + O\left(y^{-5/2}\right), & y \to \infty, \\ O\left(y^{\mathcal{R}e\,\nu+1}\right), & y \to 0. \end{cases} \tag{16}$$

Therefore, the function $\sqrt{xy}\,\mathbf{H}_\nu(xy)$, $|\mathcal{R}e\,\nu| < 1/2$, is uniformly bounded on R_+. As $y^{2n}g(y) \in L_1(R_+)$, applying the dominated convergence theorem and formula (16) we obtain

$$\begin{aligned} &\lim_{x\to 0} x^{\mathcal{R}e\,\nu-(1/2)}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x) \\ &\qquad = (-1)^n \int_0^\infty \lim_{x\to 0}[x^{\mathcal{R}e\,\nu}\,\mathbf{H}_\nu(xy)]y^{2n+1/2}g(y)\,dy = 0, \\ &\qquad\qquad -1/2 < \mathcal{R}e\,\nu < 0. \end{aligned} \tag{17}$$

(iv) The function $\sqrt{y}\,\mathbf{H}_\nu(y)$ can be expressed in the following form by virtue of formula (16)

$$\begin{aligned} \sqrt{y}\,\mathbf{H}_\nu(y) = \sqrt{\frac{2}{\pi}}\sin\left(y - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \varphi(y), \\ -3/2 < \mathcal{R}e\,\nu < 1/2, \end{aligned} \tag{18}$$

where $\varphi(y)$ is uniformly bounded on R_+ and $\lim_{y\to\infty}\varphi(y) = 0$. Since $y^n g(y) \in L_1(R_+)$, applying the Riemann-Lebesgue lemma and the dominated convergence theorem we obtain

$$\begin{aligned} &\lim_{x\to\infty}\int_0^\infty \sqrt{xy}\,\mathbf{H}_\nu(xy)y^n g(y)\,dy \\ &\qquad = \lim_{x\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty \sin\left(xy - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)y^n g(y)\,dy \\ &\qquad\quad + \int_0^\infty \lim_{x\to\infty}\varphi(xy)y^n g(y)\,dy = 0, \\ &\qquad\qquad -3/2 < \mathcal{R}e\,\nu < 1/2. \end{aligned} \tag{19}$$

Hence

$$\lim_{x\to\infty}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)=0, \tag{20}$$
$$n=0,1,\dots,\ -1/2<\mathcal{R}e\,\nu<0.$$

(v) Using the formula [**1**]

$$\frac{\partial}{\partial x}(\sqrt{xy}\,\mathbf{H}_\nu(xy))=(1/2-\nu)\sqrt{\frac{y}{x}}\,\mathbf{H}_\nu(xy)+y\sqrt{xy}\,\mathbf{H}_{\nu-1}(xy), \tag{21}$$

we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)\\ =(-1)^n\int_0^\infty \sqrt{xy}\,\mathbf{H}_{\nu-1}(xy)y^{2n+1}g(y)\,dy,\\ +\frac{(-1)^n}{x}\left(\frac{1}{2}-\nu\right)\int_0^\infty \sqrt{xy}\,\mathbf{H}_\nu(xy)y^{2n}g(y)\,dy.\end{aligned} \tag{22}$$

The functions $x^{-1/2}\mathbf{H}_\nu(x)$ and $x^{1/2}\mathbf{H}_{\nu-1}(x)$, $|\mathcal{R}e\,\nu|<1/2$, are uniformly bounded on R_+ and tend to 0 as x approaches 0. Hence, applying again the dominated convergence theorem we obtain

$$\lim_{x\to 0}\frac{1}{x}\int_0^\infty \sqrt{xy}\,\mathbf{H}_\nu(xy)y^{2n}g(y)\,dy = \int_0^\infty \lim_{x\to 0}[(xy)^{-1/2}\mathbf{H}_\nu(xy)]y^{2n+1}g(y)\,dy=0,$$

$$\lim_{x\to 0}\int_0^\infty \sqrt{xy}\,\mathbf{H}_{\nu-1}(xy)y^{2n+1}g(y)\,dy = \int_0^\infty \lim_{x\to 0}[\sqrt{xy}\,\mathbf{H}_{\nu-1}(xy)]y^{2n+1}g(y)\,dy=0. \tag{23}$$

From formulas (22) and (23) we get

$$\lim_{x\to 0}\frac{d}{dx}\left[\frac{d^2}{dx^2}+\frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^n f(x)=0, \tag{24}$$
$$n=0,1,\dots,\ -1/2<\mathcal{R}e\,\nu<0.$$

(vi) If $-1/2 < \mathcal{R}e\,\nu < 0$, then $-3/2 < \mathcal{R}e\,\nu - 1 < -1$. Hence, one can apply formula (19) to obtain

$$\lim_{x\to\infty} \int_0^\infty \sqrt{xy}\,\mathbf{H}_{\nu-1}(xy) y^{2n+1} g(y)\,dy = 0, \quad -1/2 < Re\nu < 0. \tag{25}$$

Now using formulas (22) and (25) we have

$$\lim_{x\to\infty} \frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x) = 0, \tag{26}$$
$$n = 0, 1, \dots,\ -1/2 < \mathcal{R}e\,\nu < 0.$$

Sufficiency. Suppose now that f satisfies conditions (i)–(vi) of Theorem 1. Then $[(d^2/dx^2) + (1/x^2)((1/4) - \nu^2)]^n f(x)$, $n = 0, 1, \dots$, belong to $L_2(R_+)$. Let $g_n(y)$, $n = 0, 1, \dots$, be their Y_ν-transforms

$$g_n(y) = \int_0^\infty \sqrt{xy}\,Y_\nu(xy)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x)\,dx, \tag{27}$$
$$-1/2 < \mathcal{R}e\,\nu < 0,\ n = 0, 1, 2, \dots,$$

where the integral is considered in the L_2 sense. Set

$$g_n^N(y) = \int_{1/N}^N \sqrt{xy}\,Y_\nu(xy)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^n f(x)\,dx, \tag{28}$$
$$n = 0, 1, 2, \dots,$$

we see that $g_n^N(y)$ tends to $g_n(y)$ in L_2 norm as $N \to \infty$. Let $n \geq 1$; integrating (28) by parts twice we obtain

$$\begin{aligned} g_n^N(y) = {} & \left\{\sqrt{xy}\,Y_\nu(xy)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(x)\right\}\Big|_{x=1/N}^{x=N} \\ & - \left\{\frac{\partial}{\partial x}(\sqrt{xy}\,Y_\nu(xy))\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(x)\right\}\Big|_{x=1/N}^{x=N} \\ & + \int_{1/N}^N \left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right] \\ & \qquad \cdot (\sqrt{xy}\,Y_\nu(xy))\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \nu^2\right)\right]^{n-1} f(x)\,dx. \end{aligned} \tag{29}$$

Using the formulas [**1**]

$$(30)\qquad \begin{aligned} &\frac{\partial}{\partial x}(\sqrt{xy}\,Y_\nu(xy)) = (1/2-\nu)\sqrt{\frac{y}{x}}Y_\nu(xy) + y\sqrt{xy}\,Y_{\nu-1}(xy),\\ &\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right](\sqrt{xy}\,Y_\nu(xy)) = -y^2\sqrt{xy}\,Y_\nu(xy),\end{aligned}$$

we have

$$(31)\qquad g_n^N(y) = \sqrt{Ny}\,Y_\nu(Ny)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(N)$$

$$(32)\qquad -\sqrt{\frac{y}{N}}\,Y_\nu(y/N)\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(1/N)$$

$$(33)\qquad +\left(\nu-\frac{1}{2}\right)\sqrt{\frac{y}{N}}\,Y_\nu(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(N)$$

$$(34)\qquad -y\sqrt{Ny}\,Y_{\nu-1}(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(N)$$

$$(35)\qquad +\left(\frac{1}{2}-\nu\right)\sqrt{Ny}\,Y_\nu(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(1/N)$$

$$(36)\qquad +y\sqrt{\frac{y}{N}}\,Y_{\nu-1}(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(1/N)$$

$$(37)\qquad -y^2\int_{1/N}^{N}\sqrt{xy}\,Y_\nu(xy)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4}-\nu^2\right)\right]^{n-1} f(x)\,dx.$$

Here $P(d/dx)f(N)$ means $P(d/dx)f(x)|_{x=N}$.

Applying the following asymptotic formula for the Bessel function of the second kind [**1**]
(38)

$$Y_\nu(y) = \begin{cases} \sqrt{\dfrac{2}{\pi y}}\left[\sin\left(y-\dfrac{\nu\pi}{2}-\dfrac{\pi}{4}\right) + \dfrac{4\nu^2-1}{8y}\cos\left(y-\dfrac{\nu\pi}{2}-\dfrac{\pi}{4}\right)\right] & \\ +O(y^{-5/2}) & y\to\infty,\\ O(y^{-|\mathcal{R}e\,\nu|}) & y\to 0,\end{cases}$$

we conclude that the function $\sqrt{Ny}\,Y_\nu(Ny)$, $|\mathcal{R}e\,\nu| < 1/2$, is uniformly bounded. The function $(d/dx)[(d^2/dx^2) + (1/x^2)((1/4)-\nu^2)]^{n-1}f(N)$

tends to zero as N approaches infinity (property (vi)); therefore, the expression on the righthand side of (31) tends to zero as N approaches infinity.

From (v) we see that $(d/dx)[(d^2/dx^2)+(1/x^2)((1/4)-\nu^2)]^{n-1}f(1/N)$ has the order $o(1)$, whereas the function $\sqrt{y/N}\,Y_\nu(y/N)$ has the order $O(N^{-\mathcal{R}e\,\nu-1/2})$. Hence expression (32) approaches zero as N tends to infinity.

The function $\sqrt{y/N}\,Y_\nu(Ny)$ has the order $O(N^{-1})$, whereas the expression $[(d^2/dx^2) + (1/x^2)((1/4) - \nu^2)]^{n-1}f(N)$ is $o(1)$ (property (iv)), therefore expression (33) is $o(1)$.

The function $y\sqrt{Ny}\,Y_{\nu-1}(Ny)$ is $O(1)$, hence property (iv) shows that expression (34) is $o(1)$.

Since the function $\sqrt{Ny}\,Y_\nu(y/N)$ has the order $O(N^{1/2-\mathcal{R}e\,\nu})$, and $[(d^2/dx^2) + (1/x^2)((1/4) - \nu^2)]^{n-1}f(1/N)$ is $o(N^{-1/2+\mathcal{R}e\,\nu})$ (property (iii)) expression (35) is also $o(1)$.

The function $y\sqrt{y/N}\,Y_{\nu-1}(y/N)$ has the order $O(N^{1/2-\mathcal{R}e\,\nu})$, hence expression (36) is $o(1)$ by virtue of property (iii).

Therefore, we observe that the righthand side of formula (31), as well as all functions (32)–(36), vanish as N tends to infinity, whereas expression (37) converges to $-y^2 g_{n-1}(y)$. Consequently, $g_n(y) = -y^2 g_{n-1}(y)$, and hence $g_n(y) = (-y^2)^n g_0(y)$, $n = 0, 1, \dots$. Thus $g(y) = g_0(y)$ with $y^{2n}g(y) \in L_2(R_+)$, $n = 0, 1, \dots$, is the Y_ν-transform of a function f. As the Y_ν-transform is the inverse of the $\mathbf{H}_\nu$-transform, the function f is the Struve $\mathbf{H}_\nu$-transform $(-1/2 < \mathcal{R}e\,\nu < 0)$ of a function g such that $y^n g(y) \in L_2(R_+)$, $n = 0, 1, \dots$.

We have proved that the function $(-y^2)^n g(y)$ is the Y_ν-transform $(-1/2 < \mathcal{R}e\,\nu < 0)$ of the function $[(d^2/dx^2)+(1/x^2)((1/4)-\nu^2)]^n f(x)$, $n = 0, 1, \dots$. Hence, $[(d^2/dx^2) + (1/x^2)((1/4) - \nu^2)]^n f(x)$ is the Struve $\mathbf{H}_\nu$ transform, $-1/2 < \mathcal{R}e\,\nu < 0$, of $(-y^2)^n g(y)$, $n = 0, 1, \dots$. Consequently, formula (14) holds. We recall that formula (13) is valid if $(-y^2)^n g(y) \in L_2(R_+)$, $n = 0, 1, \dots$, hence comparing it with formula (14) we get formula (8) for $n = 0$. Applying the same procedure with $[(d^2/dx^2) + (1/x^2)((1/4) - \nu^2)]^{n-1} f(x)$ instead of $f(x)$ and $(-y^2)^{n-1}g(y)$ instead of $g(y)$ we obtain formula (8) for other values of n.

Theorem 1 is thus proved. □

Remark 1. Let $\mathcal{S}(R)$ be the Schwartz space of infinitely differentiable and rapidly decreasing functions on $R = (-\infty, \infty)$ [**12**]. The Lisorkin space [**3**] $\Phi(R) \subset \mathcal{S}(R)$ is the set of Schwartz functions φ with zero moments

$$\int_{-\infty}^{\infty} y^n \varphi(y)\, dy = 0, \quad n = 0, 1, 2, \dots . \tag{39}$$

The Lisorkin space $\Phi(R)$ plays an important role in fractional integrals, potential theory [**6**] and singular integrals [**7**], for example, the Weyl fractional integral and derivative, and the Riesz potential are automorphisms on $\Phi(R)$ [**6**]. It is easy to see that the restrictions of the Lisorkin odd functions on R_+, multiplied by $y^{-\nu-1/2}$, belong to the class of functions considered in Theorem 1.

3. H_ν-transform of functions analytic in an angle. Let $\mathcal{G}$ be the space of functions $g(z)$ that are (i) regular in an angle $-\alpha < \arg z < \beta$ where $0 < \alpha$, $\beta \le \pi$, (ii) of the order $O(|z|^{-a-\varepsilon})$ for small z and $O(|z|^{-b+\varepsilon})$ for large z uniformly in any angle interior to the above, for every positive ε, where $a < 1/2 < b$, (iii) satisfying the following conditions

$$\begin{gathered} \int_0^{\infty} y^{\nu-2n-1/2} g(y)\, dy = 0, \\ n \in (-b/2 + \mathcal{R}e\,\nu/2 + 1/4, -a/2 + \mathcal{R}e\,\nu/2 + 1/4), \\ \int_0^{\infty} y^{\nu+2n+3/2} g(y)\, dy = 0, \\ n \in (a/2 - \mathcal{R}e\,\nu/2 - 5/4, b/2 - \mathcal{R}e\,\nu/2 - 5/4), \end{gathered} \tag{40}$$

for all nonnegative integers n (if such an n exists).

Let $\mathcal{F}$ be the space of functions $f(z)$, which are (i) regular in the angle $-\beta < \arg z < \alpha$, (ii) of the order $O(|z|^{1-b-\varepsilon})$ for small z and $O(|z|^{1-a+\varepsilon})$ for large z uniformly in any angle interior to the above for every positive ε and (iii) satisfying the following conditions

$$\begin{gathered}\int_0^\infty x^{\nu+2n+1/2} f(x)\,dx = 0,\\ n \in (-b/2 - \mathcal{R}e\,\nu/2 - 1/4, -a/2 - \mathcal{R}e\,\nu/2 - 1/4),\\ \int_0^\infty x^{-\nu+2n+1/2} f(x)\,dx = 0,\\ n \in (-b/2 + \mathcal{R}e\,\nu/2 - 1/4, -a/2 + \mathcal{R}e\,\nu/2 - 1/4),\end{gathered} \tag{41}$$

for all nonnegative integers n if such an n exists; for example, if $\mathcal{R}e\,\nu = -1$, then $n = 0$ always belongs to the interval $(-b/2-1/2, -a/2-1/2)$.

Theorem 2. *The* $\mathbf{H}_\nu$*-transform,* $-2 < \mathcal{R}e\,\nu < 0$*, maps the space* $\mathcal{G}$ *one-to-one onto the space* $\mathcal{F}$*.*

Proof. Let $g(z)$ belong to the space $\mathcal{G}$. Then the restriction of the function $g(z)$ on R_+ belongs to $L_2(R_+)$ and its Mellin transform $g^*(s)$

$$g^*(s) = \int_0^\infty x^{s-1} g(x)\,dx \tag{42}$$

is regular in the strip $a < \mathcal{R}es < b$ and has the asymptotics

$$g^*(s) = \begin{cases} O(e^{-(\beta-\varepsilon)\mathcal{I}ms}) & \mathcal{I}ms \to \infty,\\ O(e^{(\alpha-\varepsilon)\mathcal{I}ms}) & \mathcal{I}ms \to -\infty,\end{cases} \tag{43}$$

uniformly in any strip interior to $a < \mathcal{R}es < b$ for every positive ε (see [**9**]). Let $f(x)$ be the $\mathbf{H}_\nu$-transform $(-2 < \mathcal{R}e\,\nu < 0)$ of $g(y)$. Since $g(y)$ belongs to $L_2(R_+)$, the Parseval identity holds on the line $\mathcal{R}es = 1/2$ and [**2**]

$$\begin{aligned} f^*(s) &\\ = 2^{s-1} &\frac{\Gamma((1/4)-(\nu/2)-(s/2))\Gamma((3/4)+(\nu/2)+(s/2))}{\Gamma((3/4)+(\nu/2)-(s/2))\Gamma((3/4)-(\nu/2)-(s/2))} g^*(1-s).\end{aligned} \tag{44}$$

Because of condition (40) the function $g^*(1-s)$ equals 0 at the poles of the function $\Gamma((1/4)-(\nu/2)-(s/2))\Gamma((3/4)+(\nu/2)+(s/2))$ in the strip $1-b < \mathcal{R}es < 1-a$, provided there exists one. Hence, from formula (44) one can see that $f^*(s)$ is analytic in the strip $1-b < \mathcal{R}es < 1-a$. Furthermore, since the function $2^{s-1/2}[(\Gamma((1/4)-(\nu/2)-(s/2))\Gamma((3/4)+(\nu/2)+(s/2))]/[\Gamma((3/4)+(\nu/2)-(s/2))\Gamma((3/4)-(\nu/2)-(s/2))]$ is uniformly bounded on any compact subdomain of the strip $1-b < \mathcal{R}es < 1-a$ containing no poles of the function $\Gamma((1/4)-(\nu/2)-(s/2))\Gamma((3/4)+(\nu/2)+(s/2))$, and has at most only polynomial growth as $\mathcal{I}ms \to \pm\infty$, from formula (43) we see that the function $f^*(s)$ also decays exponentially

$$f^*(s) = \begin{cases} O(e^{(\beta-\varepsilon)\mathcal{I}ms}) & \mathcal{I}ms \to -\infty, \\ O(e^{-(\alpha-\varepsilon)\mathcal{I}ms}) & \mathcal{I}ms \to \infty, \end{cases} \tag{45}$$

uniformly in any strip interior to $1-b < \mathcal{R}es < 1-a$ for every positive ε. Hence its inverse Mellin transform $f(z)$ is regular in the angle $-\beta < \arg z < \alpha$ and of the order $O(|z|^{b-1-\varepsilon})$ for small z and $O(|z|^{a-1+\varepsilon})$ for large z uniformly in any angle interior to the above angle, for every positive ε [**9**]. Moreover, $f^*(s)$ has zeros at the poles of the function $\Gamma((3/4)+(\nu/2)-(s/2))\Gamma((3/4)-(\nu/2)-(s/2))$ in the strip $1-b < \mathcal{R}es < 1-a$ (provided one exists); hence (41) holds.

Conversely, let $f(z)$ belong to the space $\mathcal{F}$. Then the restriction of the function $f(z)$ on R_+ belongs to $L_2(R_+)$ and its Mellin transform (42) $f^*(s)$ is analytic in the strip $1-b < \mathcal{R}es < 1-a$ and satisfies (45). Furthermore, from condition (41) we see that $f^*(s)$ has zeros at the poles of the function $\Gamma((3/4)+(\nu/2)-(s/2))\Gamma((3/4)-(\nu/2)-(s/2))$ in the strip $1-b < \mathcal{R}es < 1-a$, provided one exists. Therefore, if we express $f^*(s)$ in the form (44), the function $g^*(s)$ is analytic in the strip $a < \mathcal{R}es < b$ and has asymptotics (43) uniformly in any strip interior to $a < \mathcal{R}es < b$ for every positive ε. Furthermore, $g^*(1-s)$ has zeros at the poles of the function $\Gamma((1/4)-(\nu/2)-(s/2))\Gamma((3/4)+(\nu/2)+(s/2))$ in the strip $1-b < \mathcal{R}es < 1-a$. Consequently, the inverse Mellin transform $g(z)$ of $g^*(s)$ satisfies the conditions of Theorem 2 and f is the Struve $\mathbf{H}_\nu$-transform of g.

If we take $\alpha = \beta$ and $0 < a < \min\{|\nu|, |\nu+1|, |\nu+2|\}$, then in the strip $1/2 - a < \mathcal{R}es < 1/2 + a$ there are no poles and zeros of the function $2^{s-1/2}[\Gamma((1/4)-(\nu/2)-(s/2))\Gamma((3/4)+(\nu/2)+(s/2))]/[\Gamma((3/4)+(\nu/2)-(s/2))\Gamma((3/4)-(\nu/2)-(s/2))]$. This leads to the following corollary.

Corollary 1. *The* $\mathbf{H}_\nu$*-transform* $(0 < |\mathcal{R}e\,\nu + 1| < 1)$ *is a bijection on the space of functions, regular in the angle* $|\arg z| < \alpha$*,* $0 < \alpha \le \pi$ *of the order* $O(|z|^{a-1/2-\varepsilon})$ *for small* z *and* $O(|z|^{-a-1/2+\varepsilon})$ *for large* z *uniformly in any angle interior to the above, for every positive* ε*, where* $0 < a < \min\{|\nu|, |\nu+1|, |\nu+2|\}$.

4. $\mathbf{H}_\nu$-transform on some other space of functions. Let Φ be any linear subspace of either $L_1(R)$ or $L_2(R)$ having properties

(i) if $\phi(t) \in \Phi$ then $\phi(-t) \in \Phi$;

(ii) the functions $\varphi(t) = (2^{it}\cosh(\pi/2)(t - i\nu)\Gamma((1/2) + (\nu/2) + (it/2)))/[\Gamma((1/2) + (\nu/2) - (it/2))]$, $0 < |1 + \mathcal{R}e\,\nu| < 1$ and $\varphi^{-1}(t)$ are multipliers of Φ.

It is easy to see that $\varphi^{-1}(-t)$ is also a multiplier of Φ. The multipliers $\varphi(t)$ and $\varphi^{-1}(t)$ are infinitely differentiable and uniformly bounded on R and their derivatives grow logarithmically; therefore, many classical spaces on R are special cases of Φ (for example, any L_1 or L_2 space with L_∞-weights, the Schwartz space $\mathcal{S}(R)$ and the space of infinitely differentiable functions with compact support [**12**]). On R_+ we define by $\mathcal{M}^{-1}(\Phi)$ the space of functions g that can be represented in the form

$$g(x) = \int_{-\infty}^{\infty} \phi(t) x^{it-1/2}\,dt \tag{46}$$

almost everywhere, where $\phi \in \Phi$ (if $\phi \notin L_1(R)$ the integral should be understood as the inverse Mellin transform in L_2 [**9**]). The space $\mathcal{M}^{-1}_{c,\gamma}(L)$ [**10**], [**11**] as well as the space of functions considered in Corollary 3.1 are special cases of $\mathcal{M}^{-1}(\Phi)$.

Theorem 3. *The* $\mathbf{H}_\nu$*-transform,* $0 < |1 + \mathcal{R}e\,\nu| < 1$*, is a bijection on* $\mathcal{M}^{-1}(\Phi)$*.*

Proof. From representation (46) we see that if $g \in \mathcal{M}^{-1}(\Phi)$ then g can be expressed in the form of the inverse Mellin transform

$$g(x) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} g^*(s) x^{-s}\,ds, \tag{47}$$

where $g^*(1/2+it) \in \Phi$. The Mellin transform (42) of the function $k(x) = \sqrt{x}\,\mathbf{H}_\nu(x)$, $-2 < \mathcal{R}e\,\nu < 0$, is $k^*(s) = -\varphi(i/2 - is)$ [**1**]. Applying the Parseval equation for the Mellin transform

$$(48) \qquad \int_0^\infty k(xy)g(y)\,dy = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} k^*(s)g^*(1-s)x^{-s}\,ds,$$

we obtain

$$(49) \qquad \begin{aligned}(\mathbf{H}_\nu g)(x) &= \int_0^\infty \sqrt{xy}\mathbf{H}_\nu(xy)g(y)\,dy \\ &= -\frac{1}{2\pi}\int_{-\infty}^\infty \varphi(t)g^*(1/2-it)x^{-it-1/2}\,dt.\end{aligned}$$

The Parseval formula (48) has been proved for $g^*(1/2+it) \in L_2(R)$ in [**9**] and $g^*(1/2+it) \in L_1(R)$ in [**10**]. Since $\varphi(t)$ and $\varphi^{-1}(-t)$ are multipliers of Φ, the function $\varphi(t)g^*(1/2-it)$ belongs to Φ if and only if $g^*(1/2+it)$ belongs to Φ. Therefore, $(\mathbf{H}_\nu g)(x) \in \mathcal{M}^{-1}(\Phi)$ if and only if $g \in \mathcal{M}^{-1}(\Phi)$. Theorem 3 is proved.

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