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ON WELL-POSEDNESS OF ONE-SIDED NONLINEAR BOUNDARY VALUE PROBLEMS FOR ANALYTIC FUNCTIONS

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ABSTRACT. We consider two model "one-sided" nonlinear boundary value problems for analytic functions, namely, the power type Riemann-Hilbert problem and the modulus problem.

Our main question is how to make the problems well-posed, i.e., to find classes of functions in which these problems possess a unique solution. These classes are those with prescribed collections of zeros in the domains and/or on their boundaries.

1. Introduction. Linear boundary value problems for analytic functions are well-studied due to numerous applications in different branches of mathematics, mechanics, queueing theory, etc. (background expositions can be found in [**1**], [**6**]). The corresponding nonlinear problems which also occur in a lot of applications are less investigated because of the much more complicated technique that needs to be used. For a description of the results in the area, we refer to the surveys [**7**], [**9**], [**11**] and to the books [**3**], [**5**], [**12**] and to the literature cited there. Among the approaches presented are those of a constructive nature (see e.g. [**5**, [**7**], [**9**] where the analytic methods applied in the linear case are generalized). The latter methods cannot always be generalized for the nonlinear case especially if we consider so-called "one-sided" problems posed for one unknown function analytic in the domain satisfying certain conditions on the boundary.

This article is connected with the paper [**10**] in which the classes of analytic functions were found in order for the nonlinear conjugation problem to be uniquely solvable. These are classes of functions with

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prescribed zeros in the domains or on their boundaries (cf. also [**2**]). The solutions of one-sided problems do not always have an explicit form. Therefore, in order to describe main ideas rather than all situations that occur, we mainly restrict our study to the problems posed on the unit disc.

We show, in particular, that solutions of nonlinear one-sided problems can have a denumerable collection of zeros (this is not the case for the nonlinear conjugation problem (cf. [**10**]).

We develop, in part, the methods proposed in [**8**], [**9**].

Lastly, we have to note that our results can be reformulated for nonlinear integral equations equivalent to the problems studied in the paper.

2. Homogeneous nonlinear Riemann-Hilbert problem of power type. Let $\lambda : \mathbf{T} \to \mathbf{C}$, $\mathbf{T} := \{t \in \mathbf{C} : |t| = 1\}$ be a given Hölder-continuous function, $p \neq 0$ a given, in general complex, constant. The homogeneous nonlinear Riemann-Hilbert problem of power type consists in finding a function $\phi \in \mathcal{C}_+(\mathbf{D}) := \mathcal{A}(\mathbf{D}) \cap \mathcal{C}(\mathrm{cl}\,\mathbf{D})$, analytic in the unit disc $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, continuous up to its boundary $\mathbf{T}$, satisfying the following boundary condition:

$$\Re\{\overline{\lambda(t)}\phi^p(t)\} = 0, \quad t \in \mathbf{T}. \tag{2.1}$$

We suppose that

$$\lambda(t) \neq 0, \quad t \in \mathbf{T}. \tag{2.2}$$

Hence, dividing the condition (2.1) on $|\lambda(t)|$ one can always assume that $|\lambda(t)| \equiv 1$ on $\mathbf{T}$.

We are looking for solutions of (2.1) with the prescribed zeros in $\mathbf{D}$ and/or on $\mathbf{T}$. It should be noted that, if

$$\Re p < 0, \tag{2.3}$$

then the problem (2.1) has no solution, neither with zeros on the boundary nor with an infinite set of zeros in the domain $\mathbf{D}$. In the case of a finite set of zeros in $\mathbf{D}$, one can use absolutely the same arguments for p having negative and positive real part (with appropriate change of the coefficient). Therefore, we can suppose that

$$\Re p \geq 0. \tag{2.4}$$

The solvability of (2.1) has to be described in the following three principally different cases for p:

(i) p is a positive integer,

(ii) p is positive irrational,

(iii) p is purely imaginary.

All other cases can be reduced to these cases.

Case (a) $p = m \in \mathbf{N}$. Following [**8**] we introduce the following classes of analytic functions (cf. also [**5**]).

$\mathcal{A}^k \subset \mathcal{C}_+$ is the subclass of functions from $\mathcal{C}_+$ with k zeros in the domain and nonvanishing boundary function;

$\tilde{\mathcal{A}}^k \subset \mathcal{C}_+$ is the subclass of functions from $\mathcal{A}_+$ with k zeros in the domain whose boundary function can have zeros on $\mathbf{T}$.

Let the solution ϕ of the problem (2.1) have exactly k zeros in $\mathbf{D}$ with $k \in \mathbf{N}_0$ being a subject for further determination. Then, by selecting k arbitrary points $z_1, \dots, z_k \in \mathbf{D}$, one can reduce the problem (2.1) to the following

$$\Re\{\overline{\mu(t)}\psi^p(t)\} = 0, \quad t \in \mathbf{T}, \tag{2.5}$$

with respect to a new unknown function

$$\psi(z) := \phi(z)\left(\prod_{j=1}^{k} \frac{z - z_j}{1 - \overline{z_j}z}\right)^{-1} := \phi(z)(B(z))^{-1} \in \mathcal{A}^0, \quad z \in \mathbf{D}. \tag{2.6}$$

The coefficient of the problem (2.5) has the form

$$\mu(t) = \lambda(t)(B(t))^{-p}, \quad t \in \mathbf{T}. \tag{2.7}$$

The function

$$\omega(z) := \psi^p(z), \quad z \in \mathbf{D}, \tag{2.8}$$

is also analytic and nonvanishing into the unit disc $\mathbf{D}$. Therefore, one can consider the following auxiliary linear Riemann-Hilbert problem for the unit disc in the class $\mathcal{A}^0$ (or $\tilde{\mathcal{A}}^0$):

$$\Re\{\overline{\mu(t)}\omega(t)\} = 0, \quad t \in \mathbf{T}, \tag{2.9}$$

or, which is equivalent, the problem

$$\Re\left\{\frac{\omega \exp\{-i\mathbf{T}(\arg\mu)(t)\}}{t^{\chi_0}}\right\} = 0, \quad t \in \mathbf{T}, \tag{2.10}$$

where

$$\mathbf{T}(\nu)(z) := \frac{1}{2\pi}\int_0^{2\pi} \nu(\sigma)\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma. \tag{2.11}$$

The solvability of the latter problem depends on its index (cf., e.g., [**1**]):

$$\chi_0 := \chi - kp = wind_{\mathbf{T}}\lambda - kp. \tag{2.12}$$

(i) if $\chi_0 < 0$, then the problem (2.9) has no solution in $\mathcal{A}^0$;

(ii) if $\chi_0 = 0$, then the unique solution of the problem (2.9) in the class $\mathcal{A}^0$ can be delivered by the formula

$$\omega(z) = iC_0 \exp\{i\mathbf{T}(\arg\mu)(z)\}, \tag{2.13}$$

where $C_0 \neq 0$ is an arbitrary real constant;

(iii) if $\chi_0 > 0$, then the analytic solutions of (2.9) are given by the formula (cf., e.g., [**1**])

$$\omega(z) = iC_0 \exp\{i\mathbf{T}(\arg\mu(\sigma) - \chi_0\sigma)(z)\}Q_{2\chi_0}(z), \tag{2.14}$$

where

$$Q_{2\chi_0}(z) = z^{\chi_0}\left(P_{\chi_0} - \overline{P_{\chi_0}\left(\frac{1}{\bar{z}}\right)}\right),$$

and P_{χ_0} is an arbitrary polynomial, $\deg P_{\chi_0} = \chi_0$. As $Q_{2\chi_0}$ has certain symmetry, namely,

$$Q_{2\chi_0}\left(\frac{1}{\bar{z}}\right) = 0 \Longleftrightarrow Q_{2\chi_0}(z) = 0,$$

then $Q_{2\chi_0}$ has no zero in $\mathbf{D}$ if and only if all its zeros lie on the unit circle $\mathbf{T}$, i.e.,

$$Q_{2\chi_0}(z) = C_{\chi_0} \prod_{j=\mathbf{1}}^{2\chi_0} (z - t_j), \quad t_j = e^{is_j}, j = 1, \dots, 2\chi_0; C_{\chi_0} = \rho e^{i\theta}.$$

Substituting it into the boundary condition (2.9), one gets the following identity to be satisfied:

$$\Re\left\{ \overline{t}^{\chi_0} \rho e^{i\theta} \prod_{j=\mathbf{1}}^{2\chi_0} (t - e^{is_j}) \right\} = \rho 2^{2\chi_0} \cos\left(\theta + \frac{1}{2}\sum_{j=\mathbf{1}}^{2\chi_0} s_j\right) \prod_{j=\mathbf{1}}^{2\chi_0} \sin\frac{s - s_j}{2} \equiv 0.$$

The latter takes place if and only if θ is chosen as (cf. [**8**]),

$$\theta := -\frac{1}{2}\sum_{j=\mathbf{1}}^{2\chi_0} s_j + \left(n + \frac{1}{2}\right)\pi, \quad \text{for certain } n \in \mathbf{Z}.$$

Hence, the polynomial $Q_{2\chi_0}$ should be taken in the form

$$Q_{2\chi_0}(z) = \pm\rho \prod_{j=\mathbf{1}}^{2\chi_0} (z - t_j)\overline{t_j}^{1/2}. \tag{2.15}$$

At last, the exponential term in (2.14) can be rewritten on the basis of the following identity

$$\mathbf{T}(\arg\lambda(\sigma) - p\arg B(\sigma) - \chi_0\sigma)(z) \equiv \mathbf{T}(\arg\lambda(\sigma) - \chi_0\sigma)(z) - 2ip\log\prod_{j=\mathbf{1}}^{k}(1 - \overline{z_j}z).$$

Combining these results with the previous notations, one gets the description of solvability of the initial problem (2.1) in the classes $\mathcal{A}^k$ and $\tilde{\mathcal{A}}^k$.

Proposition 1. *Let p be a positive integer number.*

(i) *If the winding number of the coefficient λ is negative, i.e., $\chi < 0$, then the problem* (2.1) *has no solution in any class $\mathcal{A}^k$ and $\tilde{A}^k$.*

(ii) *If $\chi = 0$, then the problem* (2.1) *has a solution only in the class $\mathcal{A}^0$. It is delivered by the formula*

$$\phi(z) = \rho e^{\{(4m\pm1)/(2p)\pi i\}} \exp\left\{\frac{i}{p}\mathbf{T}(\arg\lambda)(z)\right\}, \quad z \in \mathbf{D}, \tag{2.16}$$

$\rho > 0$ is an arbitrary positive constant, $m \in \mathbf{Z}$. The branch of corresponding multi-valued function can be chosen arbitrarily.

(iii) *If $\chi > 0$, then the problem* (2.1) *has a solution in the class $\mathcal{A}^k$ with $k := (\chi/p)$ if and only if $(\chi/p) \in \mathbf{N}$. This solution is presented by the formula*

$$\begin{aligned} \phi(z) &= \rho e^{\{(4m\pm1)/(2p)\pi i\}} \exp\left\{\frac{i}{p}\mathbf{T}(\arg\lambda)(z)\right\} \\ &\cdot \prod_{j=\mathbf{1}}^{k}(z - z_j)(1 - \overline{z_j}z), \quad z \in \mathbf{D}, \end{aligned} \tag{2.17}$$

where z_j are certain fixed points in $\mathbf{D}$ and the branch of exponential function can be chosen arbitrarily.

(iv) *If $\chi > 0$, then the problem* (2.1) *has a solution in any class $\tilde{\mathcal{A}}^k$ with $0 \leq k \leq [\chi/p]$ if $(\chi/p) \notin \mathbf{N}$. This solution is presented by the formula*

$$\begin{aligned} \phi(z) &= \rho e^{\{(4m\pm1)/(2p)\pi i\}} \exp\left\{\frac{i}{p}\mathbf{T}(\arg\lambda(\sigma) - (\chi - kp)\sigma)(z)\right\} \\ &\cdot \prod_{j=\mathbf{1}}^{k}(z-z_j)(1-\overline{z_j}z) \cdot \prod_{j=\mathbf{1}}^{2(\chi-kp)} (z-t_j)^{1/p}\overline{t_j}^{(1/2p)}, \quad z \in \mathbf{D}, \end{aligned} \tag{2.18}$$

where t_j are certain fixed points on $\mathbf{T}$, the branch of exponential function can be chosen arbitrarily, the branch of any function $(z-t_j)^{1/p}$ is chosen arbitrarily in the plane which is cut along the ray $\{z \in \mathbf{C} : z = re^{is_j}, 1 < r < \infty\}$.

(v) *If $\chi > 0$, then the problem* (2.1) *has no solution either in any class $\mathcal{A}^k$ or in any class $\tilde{\mathcal{A}}^k$ with $k > [\chi/p]$.*

Case (b) $p \in \mathbf{R}_+ \backslash \mathbf{Q}$. Let us choose a number $k \in \mathbf{N}$ and k arbitrary points $z_1, \dots, z_k \in \mathbf{D}$. Now we have to change our scheme used in case (a) since the function $B^p(t)$ in (2.7) is necessarily discontinuous on $\mathbf{T}$ for p being the irrational number. Thus, we introduce a new unknown function ψ in the following way:

$$\phi(z) := \prod_{j=1}^{k} (z - z_j)\psi(z) =: \tilde{B}(z)\psi(z), \quad z \in \mathbf{D}, \tag{2.19}$$

and rewrite the boundary condition (2.1) in the form

$$\Re\left\{ t^{kp}\overline{\lambda(t)}\left(\prod_{j=1}^{k} \left(1 - \frac{z_j}{t}\right)\right)^p \psi^p \right\} = 0, \quad t \in \mathbf{T}. \tag{2.20}$$

In the latter equality ψ is zero-free in the domain. Hence, one can take $\psi^p(t)$ to be boundary values of any single-valued branch of $\psi^p(z)$. The product has all its branching points $0, z_1, \dots, z_k$ inside $\mathbf{D}$. Therefore, its pth power is single-valued outside certain lines connecting these points. Thus the restriction of $(\prod_{j=1}^{k}(1-(z_j/z)))^p$ to $\mathbf{T}$ is a continuous function. Besides,

$$wind_{\mathbf{T}}\left(\prod_{j=1}^{k} \left(1 - \frac{z_j}{t}\right)\right)^p = 0. \tag{2.21}$$

At last, the function t^{kp} is by no doubt discontinuous at a certain point of $\mathbf{T}$ because the cut connecting the branching point of corresponding multi-valued functions evidently intersects $\mathbf{T}$. Consequently, the problem (2.1) in the class $\tilde{\mathcal{A}}^k$ is equivalent to the problem (2.20) in the class $\tilde{\mathcal{A}}^0$ with an additional condition: its solution $\psi(z)$ ought to have zeros at all points of discontinuity of the function t^{kp}. These points will be specified later on.

Denoting by

$$\omega(z) := \psi^p(z), \quad z \in \mathbf{D},$$

any fixed branch of multi-valued function ψ^p in $\mathbf{D}$, and exploiting the regularizing factor's technique, one can reduce the boundary condition (2.20) to the following one:

$$\Re\left\{ \frac{\omega(t)\exp\{-i\mathbf{T}(\arg\tilde{\mu})(t)\}}{t^{\chi_0}} \right\} = 0, \quad t \in \mathbf{T}, \tag{2.22}$$

where $\chi_0 = \chi - kp \in \mathbf{R}\backslash\mathbf{Q}$ is now an irrational number,

$$\tag{2.23} \begin{aligned} \tilde{\mu}(t) &= t^{-\chi}\lambda(t)\bigg(\prod_{j=1}^{k}\Big(1 - \frac{z_j}{t}\Big)\bigg)^p \\ &= \lambda_0\bigg(\prod_{j=1}^{k}\Big(1 - \frac{z_j}{t}\Big)\bigg)^p, \quad t \in \mathbf{T}. \end{aligned}$$

Let us consider in the class $\tilde{\mathcal{A}}^0$ an auxiliary problem

$$\tag{2.24} \Re\bigg\{\frac{F(t)}{t^\alpha}\bigg\} = 0, \quad t \in \mathbf{T},$$

where $\alpha \in (0,1)$. It follows from [**9**] that the problem (2.24) has partial solutions of the type $iF_\alpha(z)$ where

$$F_\alpha(z) := (z - \tau_j)^\alpha(1 - \bar{\tau}_j z)^\alpha, \quad z \in \mathbf{D}.$$

The function F_α is the analytic branch of the multi-valued function in the plane which is cut along the ray $\{z : z = re^{i\arg\tau_j}, 1 < r < +\infty\}$ chosen under condition

$$\tag{2.25} F_\alpha(z)_{|_{z=t\in\mathbf{T}}} = t^\alpha|t - \tau_j|^{2\alpha}.$$

Here $\tau_j \in \mathbf{T}$ are arbitrary points on $\mathbf{T}$ (we can choose them as the points of discontinuity of the function t^{kp}). Using (2.25) one can rewrite the boundary condition (2.22) in the form

$$\tag{2.26} \Re\bigg\{\frac{\widetilde{\omega(t)}}{t^{[\chi_0]}}\bigg\} = 0, \quad t \in \mathbf{T},$$

where $\widetilde{\omega(z)} = \omega(z)\exp\{-i\mathbf{T}(\arg\tilde{\mu})(z)\}\prod_{j=1}^{s} F_{\alpha_j}^{-1}(z)$, $0 < \alpha_j < (1/2)$, $\sum\alpha_j = \chi_0 - [\chi_0]$. The problem (2.26) has to be solved in a subclass of functions from $\tilde{\mathcal{A}}^0$ satisfying additional conditions

$$\tag{2.27} \exists \lim_{t\to\tau_j} \frac{\widetilde{\omega(t)}}{(t - \tau_j)^{2\alpha_j}}.$$

It is not hard to see that (cf. solution of (2.10)),

(i) if $[\chi_0] < 0$, i.e., if $\chi_0 < 0$, then no solution to (2.26) of such a type exists;

(ii) if $[\chi_0] \geq 0$, i.e., if $\chi_0 > 0$, then the solution to (2.26) can be represented in the form

$$\widetilde{\omega(t)} = \pm i\rho \prod_{j=\mathbf{1}}^{2[\chi_0]} (z - t_j)\bar{t}_j^{1/2},$$

where t_j are arbitrarily chosen points on $\mathbf{T}$. Therefore, the solution of the problem (2.1) has in this case the form

$$\begin{aligned} \phi(z) = \rho \exp&\left\{\frac{4m \pm (1 - 2\alpha_0)}{2p}\pi i\right\} \exp\{i\mathbf{T}(\arg\tilde{\mu})(z)\} \\ &\cdot \prod_{j=\mathbf{1}}^{s} (z - \tau_j)^{(2\alpha_j/p)} \tau_j^{-\alpha_j/p} \\ &\cdot \prod_{j=\mathbf{1}}^{k} (z - z_j) \cdot \prod_{j=\mathbf{1}}^{2[\chi_0]} (z - t_j)^{1/p} t_j^{-1/(2p)}, \end{aligned} \tag{2.28}$$

where $\rho > 0$; $m = 0, \pm 1, \ldots$; $\alpha_0 = \chi_0 - [\chi_0] = \sum_{j=\mathbf{1}}^{s} \alpha_j$, $0 < \alpha_j < (1/2)$; $\tilde{\mu}$ is given in the formula (2.23). Proposition 2 follows immediately.

Proposition 2. *Let p be a positive irrational number.*

(i) *The problem* (2.1) *has no solution in any class $\mathcal{A}^k$, $k \neq 0$.*

(ii) *If the winding number of the coefficient $\chi < 0$, then the problem* (2.1) *has no solution in any class $\tilde{\mathcal{A}}^k$, $k \in \mathbf{N}_0$, as well as in $\mathcal{A}^0$.*

(iii) *If $\chi = 0$, then the problem* (2.1) *is solvable only in the class $\mathcal{A}^0$; the unique solution is delivered by the formula* (2.16).

(iv) *If $\chi > 0$, then the problem* (2.1) *is solvable in any class $\tilde{\mathcal{A}}^k$ with $0 \leq k \leq [\chi/p]$; the solution in these classes is represented in the form* (2.28).

(v) *If $\chi > 0$, then the problem* (2.1) *has no solution in any class $\tilde{\mathcal{A}}^k$ with $k > [\chi/p]$.*

Case (c) $p = i\beta$, $\beta \in \mathbf{R}$. As was already said, the case $p = \alpha + i\beta$ with $\alpha \neq 0$ can be reduced to the case of positive α, and then repeating the argument of (2a) and (2b) to the case of the purely imaginary exponent p.

Let us consider first the problem (2.1) in the class $\mathcal{A}^k$, or $\tilde{\mathcal{A}}^k$, with certain nonnegative $k \in \mathbf{N}_0$. Following Case (a) we choose k points $z_j \in \mathbf{D}$ and introduce a new unknown function (cf. (2.6)),

$$\psi(z) := \phi(z)\left(\prod_{j=1}^{k} \frac{z - z_j}{1 - \overline{z_j} z}\right)^{-1} := \phi(z), \quad (B(z))^{-1}, z \in \mathbf{D}. \tag{2.29}$$

Then the boundary condition (2.1) can be rewritten in the form ($\omega(z) := \{\psi(z)\}^{i\beta}$):

$$\Re\{\overline{\lambda(t)}(B(t))^{i\beta}\omega(t)\} = 0, \quad t \in \mathbf{T}. \tag{2.30}$$

Here the function $(B(t))^{i\beta}$ is the restriction to $\mathbf{T}$ of a certain single-valued branch of the corresponding multi-valued function. Since the latter has branching points $(z_1, \dots, z_k)$ in $\mathbf{D}$ as well as $(\bar{z}_1^{-1}, \dots, \bar{z}_k^{-1}, \infty)$ in $\hat{\mathbf{C}} \backslash \mathrm{cl}\, \mathbf{D}$, hence the cut for determining a single-valued branch necessarily crosses the unit circle $\mathbf{T}$. Therefore, there exists at least one point, say t_0, on $\mathbf{T}$ at which $(B(t))^{i\beta}$ has to be discontinuous. The values of $(B(t))^{i\beta}$ on $\mathbf{T} \backslash \{t_0\}$ are real positive numbers, as $|B(t)| \equiv 1$ on $\mathbf{T}$. It immediately implies that the problem (2.30) in $\tilde{\mathcal{A}}^0$ is equivalent to the problem

$$\Re\{\overline{\lambda(t)}\omega(t)\} = 0, \quad t \in \mathbf{T}, \tag{2.31}$$

considered in the subclass of functions from $\tilde{\mathcal{A}}^0$ having zero at $t = t_0$ (of any positive order).

If $\chi = wind_{\mathbf{T}} \lambda < 0$, then there is no analytic solution to the problem (2.31). Hence, there is no solution in the above-mentioned subclass. If $\chi = 0$, then the only analytic solution has the form

$$\omega(z) = iC_0 \exp\{i\mathbf{T}(\arg \lambda)(z)\}, \quad z \in \mathbf{D}, \tag{2.32}$$

which is nonvanishing in $\mathrm{cl}\, \mathbf{D}$. Therefore, it does not belong to the desired subclass. At last, if $\chi > 0$, then the solution of the problem

is determining up to polynomials whose zeros t_j lie on $\mathbf{T}$. It is simply sufficient to choose t_0 to be one of these points. The solution of (2.31) in the above described subclass is presented in the formula:
(2.33)

$$\omega(z) = iC_0 \exp\{i\mathbf{T}(\arg\lambda(\sigma) - \chi\sigma)(z)\}(z - t_0)\bar{t}_0^{1/2} \prod_{j=\mathbf{1}}^{2\chi-1} (z - t_j)\bar{t}_j^{1/2},$$

where $C_0 \neq 0$ is an arbitrary real constant, $t_0, t_1, \dots, t_{2\chi-1} \in \mathbf{T}$.

Returning to the problem (2.1), we get the following

Proposition 3. *Let $p \neq 0$ be a purely imaginary number, i.e., $p = i\beta$, $\beta \neq 0$.*

(i) *The problem* (2.1) *has no solution in any class $\mathcal{A}^k$, $k \neq 0$.*

(ii) *If $\chi < 0$, then the problem* (2.1) *has no solution in any class $\tilde{\mathcal{A}}_k$, $k \in \mathbf{N}_0$, as well as in $\mathcal{A}^0$.*

(iii) *If $\chi = 0$, then the problem* (2.1) *is solvable only in the class $\mathcal{A}^0$; the unique solution is delivered by the formula*

$$\phi(z) = e^{(\pi(4m\pm1)-2i\log\rho)/2\beta} \exp\left\{\frac{1}{\beta}\mathbf{T}(\arg\lambda)(z)\right\}, \quad z \in \mathbf{D}, \tag{2.34}$$

where $\rho > 0$ is any positive real number; $m = 0, \pm1, \dots$; *the branch of exponential function is chosen arbitrarily in* $\mathbf{D}$.

If $\chi > 0$, then the problem (2.1) *is solvable in the class $\tilde{\mathcal{A}}^0$; the unique solution is delivered by the formula*

$$\begin{aligned}\phi(z) &= e^{(\pi(4m\pm1)-2i\log\rho)/2\beta} \exp\left\{\frac{1}{\beta}\mathbf{T}(\arg\lambda(\sigma) - \chi\sigma)(z)\right\} \\ &\quad \cdot \prod_{j=\mathbf{1}}^{2\chi} (z - t_j)^{-i/\beta} t_j^{i/(2\beta)}, \quad z \in \mathbf{D},\end{aligned} \tag{2.35}$$

where $t_j \in \mathbf{T}$, $r > 0$, $m = 0, \pm, \dots$.

(v) *If $\chi > 0$, then the problem* (2.1) *is solvable in any class $\tilde{\mathcal{A}}^k$ with*

$0 < k < \infty$; *the solution is delivered by the formula*

$$\begin{aligned}\phi(z) = e^{(\pi(4m\pm 1)-2i\log\rho)/2\beta} \exp\left\{\frac{1}{\beta}\mathbf{T}(\arg\lambda(\sigma)-\chi\sigma)(z)\right\}\\ \cdot\prod_{j=\mathbf{1}}^{k}\left(\frac{z-z_j}{1-\overline{z_j}z}\right)\cdot(z-t_0)^{-i/\beta}t_0^{i/(2\beta)}\\ \cdot\prod_{j=\mathbf{1}}^{2\chi-1}(z-t_j)^{-i/\beta}t_j^{i/(2\beta)}, \quad z\in\mathbf{D}.\end{aligned} \tag{2.36}$$

Remark 1. It follows from Proposition 3 that the only real part of the exponent p controls a number of internal zeros of the solution to the nonlinear problem (2.1).

In this connection it is natural to ask the question whether it is possible for the solution of (2.1) in the case $\rho = i\beta$ to have an infinite collection of internal zeros? Clearly, it suffices to consider only case $\chi > 0$.

Let us suppose that $\phi(z)$ has an infinite, hence denumerable, collection of internal zeros $z_1, z_2, \dots, z_n, \dots$; $z_n \in \mathbf{D}$, with certain accumulation points on $\mathbf{T}$. Then the Blaschke product $B(z)$ in (2.29) becomes an infinite one. It converges if and only if (cf., e.g., [**4**]),

$$\sum_j (1-|z_j|) < \infty. \tag{2.37}$$

Under this condition $B(z)$ has boundary limit at almost all points $t \in \mathbf{T}$. Is there a condition which guarantees an existence of the boundary limit everywhere on $\mathbf{T}$? It is known (cf., e.g., [**4**]) that if the sequence (z_j) satisfies the condition

$$\sum_j \frac{1-|z_j|}{e^{i\theta_q}-z_j} < \infty, \tag{2.38}$$

where $e^{i\theta_q}$ are all accumulation points of (z_j), then the radial boundary function

$$\hat{B}(t) := \lim_{\rho\to 1-0} B(\rho e^{i\theta})$$

is continuous on $\mathbf{T}$. Hence $(\hat{B}(t))^{i\beta}$ is positive for any $t \in \mathbf{T}$ (it is valid even for so-called nontangential boundary functions). But it is no longer true for the standard boundary function $B(t)$. More of the asymptotic values of $|B(t)|$ at the points $e^{i\theta_q}$ almost fill in the segment $[0,1]$. Therefore, the function

$$\lambda(t)\overline{B(t)}^{i\beta}, \quad t \in \mathbf{T},$$

can be continuous at $t = e^{i\theta_q}$ if and only if the initial coefficient λ vanishes at $e^{i\theta_q}$. Besides, the (asymptotic) values of $\overline{B(t)}^{i\beta}$ are not positive on $\mathbf{T}$. Hence, the problem (2.30) could not in general be reduced to (2.31). Of course, those solutions of (2.31) which vanish at all points $e^{i\theta_q}$ do satisfy condition (2.30).

Remark 2. Following the same line as in Proposition 3 one can describe some particular cases of the distribution of sequence (z_j), satisfying condition (2.38), which can form a denumerable collection of internal zeros of the solution of the problem (2.1).

Remark 3. The conclusions of Proposition 1 and Remark 2 lead us to the following conjecture: *conditions on the boundary behavior of solutions of the problem* (2.1) *determining uniqueness classes have to be nonlocal.*

3. Inhomogeneous nonlinear Riemann-Hilbert problem of power type. Let us now consider the following problem: given two Hölder-continuous functions $\lambda : \mathbf{T} \to \mathbf{C}$, $\lambda(t) \neq 0$, $f : \mathbf{T} \to \mathbf{R}$, and a (complex) number $\rho \neq 0$, find a function $\phi \in \mathcal{A}_+(\mathbf{D})$, analytic in the unit disc $\mathbf{D}$, continuous up to its boundary $\mathbf{T}$, satisfying the following nonlinear boundary condition

$$\Re\{\overline{\lambda(t)}\phi^p(t)\} = f(t), \quad t \in \mathbf{T}. \tag{3.1}$$

One can suppose additionally that p has nonnegative real part and consider, as in Section 2, only cases (a), (b) and (c) for the exponent p.

(a) $p = m \in \mathbf{N}$. Introducing a new unknown function

$$\omega(z) = (\phi(z)B^{-1}(z))^p \in \mathcal{A}^0 \quad (\text{or } \tilde{\mathcal{A}}^0)$$

we reduce the problem (3.1) to the following linear boundary value problem

$$\Re\{\overline{\mu(t)}\omega(t)\} = f(t), \quad t \in \mathbf{T}. \tag{3.2}$$

The latter is solved in the standard way (see, e.g., [**1**]). Determining $\gamma(z)$ by using the Schwarz operator $\mathbf{T}$ (cf. (2.11)) for the unit disc $\mathbf{D}$:

$$\gamma(z) := u(z) + iv(z) = \mathbf{T}(\arg\mu(\sigma) - (\chi - kp)\sigma)(z) \tag{3.3}$$

one can rewrite the boundary condition (3.2) in the form

$$\Re\left\{\frac{\omega(t)}{t^{\chi-kp}e^{i\gamma(t)}}\right\} = e^{v(t)}f(t), \quad t \in \mathbf{T}. \tag{3.4}$$

If $\chi_0 := \chi - kp = 0$, then the solution of the latter problem is given by the formula:

$$\omega(z) = e^{i\gamma(t)}[\mathbf{T}(e^{v(\sigma)}f(\sigma))(z) + iC_0]. \tag{3.5}$$

In order to have in (3.5) the solution of class $\mathcal{A}^0$ or $\tilde{\mathcal{A}}^0$, one ought to choose the real constant C_0 in the following way

$$\begin{array}{rl} \text{(i)} & \text{for } \mathcal{A}^0 : C_0 \notin i\mathbf{T}(e^{v(\sigma)}f(\sigma))(\mathrm{cl}\,\mathbf{D}), \\ \text{(ii)} & \text{for } \tilde{\mathcal{A}}^0 : C_0 \notin i\mathbf{T}(e^{v(\sigma)}f(\sigma))(\mathbf{D}). \end{array} \tag{3.6}$$

Both choices can be realized because the function $\mathbf{T}(\cdot)$ maps the unit disc $\mathbf{D}$ onto the bounded domain on the complex plane and is continuous up to the boundary.

If $\chi_0 > 0$, then the same consideration as in the homogeneous case gives us the following solution of (3.4):

$$\omega(z) = e^{i\gamma(t)}[z^{\chi_0}\mathbf{T}(e^{v(\sigma)}f(\sigma))(z) + iC_0Q(z)], \tag{3.7}$$

where $Q(z) = \prod_{j=\mathbf{1}}^{2\chi_0}(z - t_j)\bar{t}_j^{1/2}$. The real constant $C_0 \neq 0$ should be chosen in such a way:

$$\begin{array}{rl} \text{(i)} & \text{for } \mathcal{A}^0 : C_0 \neq i\mathbf{T}(e^{v(\sigma)}f(\sigma))(z)R(z), \quad z \in \mathrm{cl}\,\mathbf{D}, \\ \text{(ii)} & \text{for } \tilde{A}^0 : C_0 \neq i\mathbf{T}(e^{v(\sigma)}f(\sigma))(z)R(z), \quad z \in \mathbf{D}, \end{array} \tag{3.8}$$

where $R(z) = z^{-\chi_0}Q(z)$. It is not hard to see that both choices are possible too (see [**10**]).

If $\chi_0 < 0$, then the only solution to the problem (3.4) is given by the formula:

$$\omega(z) = z^{\chi_0} e^{i\gamma(t)} \mathbf{T}(e^{v(\sigma)} f(\sigma))(z), \quad z \in \mathbf{D}. \tag{3.9}$$

It has in general a pole at the point $z = 0$. The solution (3.9) becomes analytic if and only if the following necessary and sufficient solvability conditions are satisfied:

$$\mathbf{T}^{(l)}(e^{v(\sigma)} f(\sigma))(0) = 0, \quad l = 0, \dots, -\chi_0 - 1. \tag{3.10}$$

The solution satisfying (3.10) belongs to those classes considered if
(3.11)

(i) for $\mathcal{A}^0 : 0 \notin \mathbf{T}(e^{v(\sigma)} f(\sigma))(\mathrm{cl}\,\mathbf{D}\backslash\{0\}); \mathbf{T}^{(-\chi_0)}(e^{v(\sigma)} f(\sigma))(0) \neq 0$,

(ii) for $\tilde{\mathcal{A}}^0 : 0 \notin \mathbf{T}(e^{v(\sigma)} f(\sigma))(\mathbf{D}\backslash\{0\}); \mathbf{T}^{(-\chi_0)}(e^{v(\sigma)} f(\sigma))(0) \neq 0$.

Remark 4. To make the problem (3.2) uniquely and unconditionally solvable in the case $\chi_0 < 0$, one has to pose it in the classes of meromorphic functions with the prescribed poles either in $\mathbf{D}$ or on $\mathbf{T}$.

(b) $p \in \mathbf{R}\backslash\mathbf{Q}$. As in Section 2(b), we introduce a new unknown function

$$\omega(z) = \left(\phi(z)\left[\prod_{j=\mathbf{1}}^{k}(z - z_j)\right]^{-1}\right)^p =: (\phi(z)\tilde{B}^{-1}(z))^p, \tag{3.12}$$

and rewrite the boundary condition (3.1) in the form
(3.13)

$$\Re\left\{\frac{\omega(t)}{t^{[\chi_0]} e^{i\gamma(t)} \prod_{j=\mathbf{1}}^{s} F_{\alpha_j}(t)}\right\} = e^{v(t)} \prod_{j=\mathbf{1}}^{s} |t - \tau_j|^{-2\alpha_j} f(t), \quad t \in \mathbf{T},$$

where $\chi_0 := \chi - kp \in \mathbf{R}\backslash\mathbf{Q}$; $\sum_{j=\mathbf{1}}^{s} \alpha_j = \chi_0 - [\chi_0]$, $0 < \alpha_j < (1/2)$; $F_{\alpha_j}(z) := (z - \tau_j)^{\alpha_j}(1 - \bar{\tau}_j z)^{\alpha_j}$, $\tau_j \in \mathbf{T}$ are different arbitrary points on the unit circle, $\gamma(z) := u(z) + iv(z) = \mathbf{T}(\arg\mu(\sigma) - (\chi - kp)\sigma)(z)$.

This linear problem should be solved in the subclass of function ω from $\tilde{\mathcal{A}}^0$ satisfying the following asymptotic relation:

$$\exists \lim_{t\to\tau_j} \frac{\omega(t)}{(t-\tau_j)^{2\alpha_j}}. \tag{3.14}$$

If $[\chi_0] = 0$, then the unique solution of such a problem is given by the formula:

$$\omega(z) = e^{i\gamma(t)} \prod_{j=1}^{s} F_{\alpha_j}(z) \cdot [\mathbf{T}(e^{v(\sigma)}|\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(z) + iC_0]. \tag{3.15}$$

In order to have this solution in the desired classes, one has to choose C_0 as follows:
(3.16)

(i) for $\mathcal{A}^0 : C_0 \notin i\mathbf{T}(e^{v(\sigma)}|\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(\mathrm{cl}\,\mathbf{D}\backslash\{\tau_1,\dots,\tau_s\})$,
and $\forall\, j = 1,\dots,s$, $\lim\limits_{t\to\tau_j}(t-\tau_j)^{2\alpha_j}\mathbf{T}(e^{v(\sigma)}|\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(t) \neq 0$;

(ii) for $\tilde{\mathcal{A}}^0 : C_0 \notin i\mathbf{T}(e^{v(\sigma)}|\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(\mathrm{cl}\,\mathbf{D}\backslash\{\tau_1,\dots,\tau_s\})$,
and $\forall\, j = 1,\dots,s$, $\lim\limits_{t\to\tau_j}(t-\tau_j)^{2\alpha_j}\mathbf{T}(e^{v(\sigma)}|\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(t) \neq 0$.

It is not hard to see that the second conditions in (3.16)(i) and (3.16)(ii) are equivalent to the condition $f(\tau_j) \neq 0$, for all $j = 1,\dots,s$, (cf., e.g., [**1**]).

If $[\chi_0] > 0$, then the solutions to the problem (3.13) are given by the formula:
(3.17)

$$\omega(z) = e^{i\gamma(t)} z^{[\chi_0]} \prod_{j=\mathbf{1}}^{s} F_{\alpha_j}(z) \cdot [\mathbf{T}(e^{v(\sigma)}|\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(z) + iC_0 Q(z)].$$

In order to have this solution in the desired classes, one has to choose C_0 as follows:

$$\begin{aligned} &\text{(i)}\quad \text{for } \mathcal{A}^0 : C_0 \neq i\mathbf{T}\Big(e^{v(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma-\tau_j|^{-2\alpha_j} f(\sigma)\Big)(z)R(z),\\ &\qquad z \in \mathrm{cl}\,\mathbf{D}\backslash\{\tau_1,\dots,\tau_s\}, \quad \text{and } \forall\, j=1,\dots,s,\ f(\tau_j)\neq 0;\\ &\text{(ii)}\quad \text{for } \tilde{\mathcal{A}}^0 : C_0 \neq i\mathbf{T}(e^{v(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma-\tau_j|^{-2\alpha_j} f(\sigma))(z)R(z),\\ &\qquad z \in \mathbf{D}\backslash\{\tau_1,\dots,\tau_s\}, \quad \text{and } \forall\, j=1,\dots,s,\ f(\tau_j)\neq 0. \end{aligned} \tag{3.18}$$

If $[\chi_0] < 0$, then the only solution to the problem (3.13) is given by the formula:
(3.19)

$$\omega(z) = z^{[\chi_0]} e^{i\gamma(t)} \prod_{j=\mathbf{1}}^{s} F_{\alpha_j}(z) \cdot \mathbf{T}(e^{v(\sigma)}|\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(z), \quad z \in \mathbf{D}.$$

It has in general a pole at the point $z = 0$. The solution (3.19) becomes analytic if and only if the following necessary and sufficient solvability conditions satisfy:

$$\mathbf{T}^{(l)}(e^{v(\sigma)}|\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(0) = 0, \quad l = 0, \dots, -\chi_0 - 1. \tag{3.20}$$

The solution satisfying (3.20) belongs to the classes considered if

$$\begin{aligned} &\text{(i)} \quad \text{for } \mathcal{A}^0 : 0 \notin \mathbf{T}(e^{v(\sigma)}|\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(\mathrm{cl}\,\mathbf{D}\backslash\{0\}); \\ &\qquad \mathbf{T}^{(-\chi_0)}(e^{v(\sigma)}|\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(0) \neq 0, \\ &\text{(ii)} \quad \text{for } \tilde{\mathcal{A}}^0 : 0 \notin \mathbf{T}(e^{v(\sigma)}|\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(\mathbf{D}\backslash\{0\}); \\ &\qquad \mathbf{T}^{(-\chi_0)}(e^{v(\sigma)}|\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(0) \neq 0. \end{aligned} \tag{3.21}$$

(c) $p = i\beta$, $\beta \in \mathbf{R}_+$. Choosing a nonnegative number k and k points $z_1, \dots, z_k \in \mathbf{D}$, one can reduce the problem (3.1) to the linear inhomogeneous problem

$$\Re\{\overline{\lambda(t)}\omega(t)\} = e^{\beta \arg B(t)} f(t), \quad t \in \mathbf{T}, \tag{3.22}$$

where

$$\omega(z) := \left[\left(\prod_{j=1}^{k} \frac{z - z_j}{1 - \bar{z}_j z}\right)^{-1} \phi(z)\right]^{i\beta} =: [(B(z))^{-1}\phi(z)]^{i\beta}.$$

The function $e^{\beta \arg B(t)}$ could not be defined continuously on the whole circle $\mathbf{T}$ because the increment of $\arg B(t)$ along $\mathbf{T}$ is equal to $2\pi k$. Hence the righthand side of (3.22) has at least one point of jump discontinuity on $\mathbf{T}$. There are in principle two ways out. If $f(t)$ vanishes at least at one point of $\mathbf{T}$, say $t = \tau_0$, then we draw the cut for determining a single-valued branch of $\arg B(z)$ intersecting $\mathbf{T}$ at τ_0. Therefore, the righthand side of (3.22) remains continuous and it is possible to apply the standard considerations. If $f(t) \neq 0$, then we are forced to lose one degree of freedom. Choosing s arbitrary points $\tau_1, \dots, \tau_s \in \mathbf{T}$ and s positive numbers $\alpha_1, \dots, \alpha_s \in (0, (1/2))$, $\sum_{j=\mathbf{1}}^{s} \alpha_j = 1$, we introduce s functions analytic in $\mathbf{D}$:

$$F_{\alpha_j}(z) := (z - \tau_j)^{\alpha_j} (1 - \bar{\tau}_j z)^{\alpha_j}, \quad z \in \mathbf{D}, \tag{3.23}$$

which are single-valued in $\mathbf{D}$ and satisfy the following condition on $\mathbf{T}$:

$$F_{\alpha_1}(t) \cdot \dots \cdot F_{\alpha_s}(t) = t \prod_{j=\mathbf{1}}^{s} |t - \tau_j|^{2\alpha_j}, \quad t \in \mathbf{T}. \tag{3.24}$$

Using these functions one can rewrite the boundary condition (3.22) in the form

$$\Re\left\{ \frac{\omega(t)}{t^{\chi-\mathbf{1}} e^{i\gamma(t)} \prod_{j=\mathbf{1}}^{s} F_{\alpha_j}(t)} \right\} = e^{v(t)+\beta \arg B(t)} \prod_{j=\mathbf{1}}^{s} |t - \tau_j|^{-2\alpha_j} f(t), \quad t \in \mathbf{T}, \tag{3.25}$$

where $\gamma(z) = u(z) + iv(z) = \mathbf{T}(\arg \lambda(\sigma) - \chi\sigma)(z)$.

If $\chi > 0$, then the analytic solutions of the problem (3.25) are given by the formula:

$$\begin{aligned} \omega(z) = e^{i\gamma(z)} \prod_{j=\mathbf{1}}^{s} F_{\alpha_j}(z) \Big[& z^{\chi-\mathbf{1}} \cdot \mathbf{T}(e^{v(\sigma)+\beta \arg B(\sigma)} \\ & \cdot \prod_{j=\mathbf{1}}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(z) + iC_0 Q_{2\chi-2}(z) \Big], \end{aligned} \tag{3.26}$$

where $Q_{2\chi-2}(z) = \prod_{j=\mathbf{1}}^{2\chi-2} (z - t_j) \bar{t}_j^{1/2}$ is a polynomial with only zeros on the unit circle. This solution belongs to the classes considered if the following "branching" conditions are satisfied:

(3.27)

(i) for $\mathcal{A}^0 : C_0 \neq i\mathbf{T}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=1}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(z) R(z)$,

$z \in \operatorname{cl} \mathbf{D} \backslash \{\tau_1, \dots, \tau_s\}$, and $\forall\, j = 1, \dots, s, f(\tau_j) \neq 0$;

(ii) for $\tilde{\mathcal{A}}^0 : C_0 \neq i\mathbf{T}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(z) R(z)$,

$z \in \mathbf{D} \backslash \{\tau_1, \dots, \tau_s\}$, and $\forall\, j = 1, \dots, s, f(\tau_j) \neq 0$, where $R(z) := z^{\chi-1} \cdot Q_{2\chi-2}^{-1}(z)$.

If $\chi \le 0$, then the formula (3.26) with $Q_{2\chi-2}(z) \equiv 0$ represents the solution of the problem (3.25) with the pole of order $-\chi+1$ at $z=0$. It becomes analytic if and only if the following solvability conditions hold:

(3.28)
$$\mathbf{T}^{(l)}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(0) = 0, \quad l = 0, \dots, -\chi.$$

It belongs to desired classes if additionally

(3.29)

(i) for $\mathcal{A}^0 : 0 \notin \mathbf{T}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=1}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(\mathrm{cl}\,\mathbf{D}\backslash\{0\})$;

$$\mathbf{T}^{(-\chi+\mathbf{1})}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(0) \ne 0,$$

(ii) for $\tilde{\mathcal{A}}^0 : 0 \notin \mathbf{T}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(\mathbf{D}\backslash\{0\})$;

$$\mathbf{T}^{(-\chi+\mathbf{1})}(e^{v(\sigma)+\beta \arg B(\sigma)} \prod_{j=\mathbf{1}}^{s} |\sigma - \tau_j|^{-2\alpha_j} f(\sigma))(0) \ne 0.$$

We are now in a position to formulate the final result on solvability of the nonlinear inhomogeneous problem (3.1).

Proposition 4. (a) *Let $p \in \mathbf{N}$ be a positive integer number.*

(i) *If $\chi < 0$, then the number of internal zeros is determined by the conditions* (3.10); *the solution belongs to $\mathcal{A}^k$ (to $\tilde{\mathcal{A}}^k$) if and only if the given functions satisfy the branching conditions* (3.11)(i), (3.11)(ii), *respectively. If so, then the solution of the problem* (3.1) *has the form*

$$\phi(z) = B(z)(\omega(z))^{1/p}, \quad z \in \mathbf{D}, \tag{3.30}$$

where $(\omega)^{1/p}$ is any analytic branch of the corresponding multi-valued function with ω given in (3.9).

(ii) *If $\chi = 0$, then the unique solution in $\mathcal{A}^0$ (in $\tilde{\mathcal{A}}^0$) is given by the formula* (3.30) *with $B(z) \equiv 1$ and ω given in* (3.5) *with the choice of* *the constant* C_0 *in accordance to* (3.6)(i), *to* (3.6)(ii), *respectively. The solution in* $\mathcal{A}^k$ (*in* $\tilde{\mathcal{A}}^k$) *exists under the same conditions as in* a_i).

(iii) *If* $\chi > 0$, *then for all* $0 \le k \le (\chi/p)$, *the formula* (3.30) *with* ω *in* (3.7) *determines the solution of the problem* (3.1) *of the class* $\mathcal{A}^k$ (*of* $\tilde{\mathcal{A}}^k$) *under the choice of* C_0 *in accordance to* (3.8)(i), *to* (3.8)(ii), *respectively. A number of conditions* (3.10) *determines the class* $\mathcal{A}^k$ (*or* $\tilde{\mathcal{A}}^k$), *with* $k > \chi/p$, *in which* (3.1) *is solvable. The solution is given by the formula* (3.30) *with* ω *in* (3.9). *It really belongs to the corresponding class if the branching conditions* (3.11)(i), *to* (3.11)(ii), *respectively, are valid.*

(b) *Let* $p \in \mathbf{R}_+ \backslash \mathbf{Q}$ *be a positive irrational number.*

(i) *If* $\chi < 0$, *then a possible number* k *of internal zeros of the solution of* (3.1) *is determined by the number of conditions* (3.20). *This solution belongs to* $\mathcal{A}^k$ (*or* $\tilde{\mathcal{A}}^k$) *if the condition* (3.21)(i) (*or* (3.21)(ii)) *are satisfied. Under these conditions the solution of the problem* (3.1) *has the form*

$$\phi(z) = \tilde{B}(z)(\omega(z))^{1/p}, \quad z \in \mathbf{D}, \tag{3.31}$$

where $(\omega)^{1/p}$ *is any analytic branch of the corresponding multi-valued function with* ω *given in* (3.19) *and* $\tilde{B}$ *as in* (3.12);

(ii) *If* $\chi = 0$, *then the unique solution in* $\mathcal{A}^0$ (*or* $\tilde{\mathcal{A}}^0$) *is given by* (3.31) *with* $\tilde{B}(z) \equiv 1$, ω *given in* (3.15), *with* $\alpha_1 = \cdots = \alpha_s = 0$, *and with the choice of the constant* C_0 *in accordance to* (3.16)(i) (*or* (3.16)(ii)). *This solution belongs to* $\mathcal{A}^k$ (*or* $\tilde{\mathcal{A}}^k$), *for certain* $k \ge 1$, *under the same conditions as in* b_i).

(iii) *If* $\chi > 0$, *then for all* k, $0 \le k \le [\chi/p]$, *the formula* (3.31) *with* ω *given in* (3.17) *determines a solution of the problem* (3.1) *of the class* $\mathcal{A}^k$ (*or* $\tilde{\mathcal{A}}^k$) *under the choice of* C_0 *in accordance to* (3.18)(i) (*or* (3.18)(ii)). *A number of conditions* (3.20) *determines the class* $\mathcal{A}^k$ (*or* $\tilde{\mathcal{A}}^k$), $k > [\chi/p]$ *in which the formula* (3.31) *with* ω *given in* (3.19) *represents the solution of such type.*

(c) *Let* $p = i\beta$, $\beta \in \mathbf{R}\backslash\{0\}$ *be a purely imaginary number. Then*

(i) *If* $\chi \le 0$, *then a possible number of internal zeros of the solution to* (3.1) *is determined by the number of conditions* (3.28). *This solution really belongs to* $\mathcal{A}^K$ (*or* $\tilde{\mathcal{A}}^k$) *if conditions* (3.29)(i) (*or* (3.29)(ii)) *are satisfied. Under these conditions the solution of the problem* (3.1) *has the form* (3.30) *with* ω *given by* (3.26), *and* $Q_{2\chi-2}(z) \equiv 0$;

(ii) *if* $\chi > 0$*, then for any positive integer number* k *the formula* (3.30) *with* ω *in* (3.26) *determines a solution of the problem* (3.1) *of class* $\mathcal{A}^k$ (*or* $\tilde{\mathcal{A}}^k$) *under the choice of* C_0 *in accordance to* (3.27)(i) (*or* (3.27)(ii)).

4. The modulus problem. Let L be a simple closed smooth curve encircling a domain $D = \operatorname{int} L \notin \infty$, find an analytic function w in D via its given modulus on the boundary L, i.e.,

$$|w(t)| = a(t), \quad t \in L, \tag{4.1}$$

where $a(t) \geq 0$ is Hölder-continuous on L. We study this problem in two situations: (i) $a(t) > 0$, $t \in L$; and (ii) $a(t) \geq 0$, vanishing at some points of L.

It should be noted that the problem (4.1) is conformal invariant, i.e., does not change its type under conformal mapping of the domain D onto another simply connected domain of the complex plane $\mathbf{C}$. Hence, without loss of generality one can suppose that $D \ni 0$. Besides the problem (4.1) is equivalent to the following one:

$$|w(\omega^{-1}(\tau))| = a(\omega^{-1}(\tau)), \quad \tau \in \mathbf{T},$$

where ω is the Riemann map of the domain D onto the unit disc $\mathbf{D}$. Thus, the problem (4.1) can be considered only for the case of the unit disc. Anyway, we give the results for the general situation with corresponding comments concerning the case of the unit disc.

(i) $a(t) > 0$, $t \in L$. The uniqueness theorem for analytic functions shows us that the unknown function w does not have more than a finite number of zeros in D. Therefore, we have to consider problem (4.1) in one of the classes $\mathcal{A}^k(D)$, where k is a prescribed number of zeros of the solution in D. Let us fix some, not necessarily different, points $z_j \in D$, $j = 1, \dots, k$, supposing that they are zeros of the solution w (for simplicity, suppose additionally that $(1/\overline{z_j}) \notin \operatorname{cl}(D)$). Any function $w \in \mathcal{A}^k(D)$ with the given zeros can be represented in the form:

$$w(z) = \prod_{j=1}^{k} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z} w_0(z), \tag{4.2}$$

where $w_0 \in \mathcal{A}^0(D)$, i.e., is analytic and nonvanishing in $\mathrm{cl}\, D$. Then the boundary condition (4.1) can be rewritten as:

$$|w_0(t)| = a(t) \prod_{j=\mathbf{1}}^{k} \left| \frac{1-\overline{z_j}t}{z_j - t} \right| =: b(t), \quad t \in L. \tag{4.3}$$

It is evident that in the case of the unit disc the product on the righthand side of (4.3) is identically equal to 1, hence $a(t) \equiv b(t)$. But it is in general not the case for an arbitrary domain D, although b remains positive and Hölder-continuous on L. Taking the logarithm of both sides of (4.3) we obtain the following boundary condition equivalent to (4.3) (for any choice of the branch of logarithmic function in D):

$$\log |w_0(t)| = \log b(t), \quad t \in L. \tag{4.4}$$

As every branch of logarithmic function $\log w_0(z)$ is analytic in D, so one can consider (4.4) as the boundary condition of the Schwarz problem for $\log w_0(z)$. Its solution has the form:

$$\log w_0(z) = \mathbf{T}(\log b(t))(z), \quad z \in D,$$

where $\mathbf{T}$ is the Schwarz operator for the domain D. Hence the solutions of the starting problem (4.1) in the class $w \in \mathcal{A}^k(D)$ is given by the formula

$$w(z) = \prod_{j=\mathbf{1}}^{k} \frac{|z_j|}{z_j} \frac{z_j - z}{1-\overline{z_j}z} \exp\left\{ \mathbf{T}\left(\log \left(a(t) \prod_{j=\mathbf{1}}^{k} \left| \frac{1-\overline{z_j}t}{z_j - t} \right| \right) \right)(z) \right\}, \quad z \in D. \tag{4.5}$$

(ii) $a(t) \geq 0$ is vanishing at some points of L. Suppose additionally that a has a finite collection of zeros on L. Let, for definiteness,

$$a(t) = \prod_{s=1}^{m} (t - t_s)^{d_s} a_0(t); \quad t_s \in L, \quad d_s \in \mathbf{R}_+; \quad a_0(t) \neq 0, \quad t \in L. \tag{4.6}$$

We can consider the problem (4.1) under condition (4.6) or in the class $\tilde{\mathcal{A}}^k(D)$ (of analytic functions in D with k zeros there and with admissible zeros on L), or in the class $\tilde{\mathcal{A}}^\infty(D)$. Besides, we can note that the problems (4.1) and (4.6) can be studied "locally," taking into account an influence of every point t_s independently. It means that one can investigate the model problem

$$|w_s(t)| = |t - t_s|^{d_s}, \quad s \in \{1, \dots, m\}, \quad t \in L, \tag{4.7}$$

and then construct the general solution of (4.1) and (4.6) as the product of solutions of the problem (4.7), and solution (4.5) with corresponding changing of data. Let us consider the problem (4.7) for different values of d_s. First let

$$0 < d_s < 1.$$

Then the problem (4.7) has solutions in any class $\tilde{\mathcal{A}}^k(D)$, but not in $\tilde{\mathcal{A}}^\infty(D)$. It follows from the uniqueness theorem for analytic functions and asymptotic behavior of such functions near the boundary L. Any solution of the class $\tilde{\mathcal{A}}^k(D)$ is delivered by the formula

$$w_s(z) = (z - t_s)^{d_s} \prod_{j=1}^{k} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z} w_{0,s}(z) \tag{4.8}$$

where $w_{0,s}$ is analytic and nonvanishing in D, the points z_j are arbitrary fixed points in D. The branch of multi-valued function $(z - t_s)^{d_s}$ is chosen arbitrarily in $\hat{\mathbf{C}} \backslash L_s$ where L_s is a smooth arc connecting the point $z = t_s$ and $z = \infty$, $L_s \cap L = t_s$. Substituting (4.8) into the boundary condition (4.7), one gets the following problem to be solved with respect to $w_{0,s}$ in $\tilde{\mathcal{A}}^0(D)$:

$$|w_{0,s}| = \prod_{j=\mathbf{1}}^{k} \left| \frac{1 - \overline{z_j} t}{z_j - t} \right|, \quad t \in L. \tag{4.9}$$

Hence, the solution of the problem (4.7) in $\tilde{\mathcal{A}}^k(D)$ has the form

$$\begin{aligned} w_s(z) = (z - t_s)^{d_s} &\prod_{j=\mathbf{1}}^{k} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z} \\ &\cdot \exp\left\{ \mathbf{T}\biggl(\log\left(\prod_{j=1}^{k} \left| \frac{1 - \overline{z_j} t}{z_j - t} \right| \right) \biggr)(z) \right\}, \quad z \in D. \end{aligned} \tag{4.10}$$

If the exponent d_s is a positive integer, i.e.,

$$d_s \in \mathbf{N},$$

then the problem (4.7) can have the solution as in any class $\tilde{\mathcal{A}}^k(D)$, delivered by the formula (4.10), in which the first factor is now single-valued, as in the class $\tilde{\mathcal{A}}^\infty(D)$. Let us show it only in the case of unit disc $\mathbf{D}$. If we choose the points $z_j \in \mathbf{D}$ such that

$$\sum_{j=\mathbf{1}}^{\infty} \frac{1-|z_j|}{|t_s - z_j|} < \infty, \tag{4.11}$$

then the Blaschke product

$$B(z) := \prod_{j=\mathbf{1}}^{\infty} \left(\frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z} \right)^{d_s} \tag{4.12}$$

has the radial, and even nontangential, limit at any point of $\mathbf{T}$, but $B(z)$ does not have the limit of t_s anyway. Besides,the asymptotic values of radial boundary function $\hat{B}(t)$ of $B(z)$ are such that $|\hat{B}(t)| = 1$ for $t \in \mathbf{T} \backslash \{t_s\}$, and $0 \le |\hat{B}(t)| \le 1$ for all $t \in \mathbf{T}$. Therefore, the solution of the problem (4.7) in $\mathcal{A}^\infty(\mathbf{D})$ has the following form

$$\begin{aligned} w_s(z) = (z - t_s)^{d_s} \prod_{j=1}^{\infty} \left(\frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z} \right)^{d_s} \\ \cdot \exp \left\{ \mathbf{T} \Bigg(\log \Bigg(\prod_{j=\mathbf{1}}^{\infty} \left| \left(\frac{1 - \overline{z_j} t}{z_j - t} \right) \right|^{d_s} \Bigg) \Bigg)(z) \right\}, \quad z \in D. \end{aligned} \tag{4.13}$$

If, at last, $d_s > 1$ is an arbitrary noninteger positive real number, then the problem (4.7) can have solutions as in the class $\tilde{\mathcal{A}}^k(D)$, delivered by the formula of the type (4.10), as in the class $\tilde{\mathcal{A}}^\infty(D)$. In the last case one has to represent first the number d_s in the form

$$d_s = d_{s,1} + d_{s,2},$$

where $d_{s,\mathbf{1}} \in \mathbf{N}_0$, $d_{s,2} \in \mathbf{R}_+ \backslash N$. The problem (4.7) is then reduced to two problems of the same type but with $d_{s,1}, d_{s,2}$ instead of d_s. Both problems were solved before. The general solution is then the product of the solutions of corresponding problems.

Summarizing the above results we can formulate the following

Proposition 5. *Let $\alpha(t)$ be a Hölder-continuous positive function on ∂D. Then the problem* (4.1) *has a solution in any class $A^k(D)$, $k = 0, 1, \dots$. This solution is delivered by the formula* (4.5). *There is no solution in $\tilde{\mathcal{A}}^k(D)$ for any $k = 0, 1, \dots$.*

Let $\alpha(t)$ be a Hölder-continuous nonnegative function on ∂D, represented in the form (4.6). *Then the problem* (4.1) *has a solution in any class $\tilde{\mathcal{A}}^k(D)$, $k = 0, 1, \dots$. This solution is given by the formula* (4.10). *There is no solution in $\mathcal{A}^k(D)$ for any $k = 0, 1, \dots$.*

If at least one number d_s, $s = 1, \dots, m$, is greater than or equal to 1, *then the problem* (4.1) *has a solution in $\tilde{\mathcal{A}}^\infty(D)$, delivered by the formula* (4.13). *If not, i.e., if $0 < d_s < 1$, $s = 1, \dots, m$, then there is no solution of the problem* (4.1) *in $\tilde{\mathcal{A}}^\infty(D)$.*

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